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Thomas Rey

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Blow Up Analysis for Anomalous Granular Gases*

THOMAS REY[†]

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Abstract

We investigate in this article the long-time behaviour of the solutions to the energy-dependent, spatially-homogeneous, inelastic Boltzmann equation for hard spheres. This model describes a diluted gas composed of hard spheres under statistical description, that dissipates energy during collisions. We assume that the gas is “anomalous”, in the sense that energy dissipation increases when temperature decreases. This allows the gas to cool down in finite time. We study existence and uniqueness of blow up profiles for this model, together with the trend to equilibrium and the cooling law associated, generalizing the classical Haff’s Law for granular gases. To this end, we investigate the asymptotic behaviour of the inelastic Boltzmann equation with and without drift term by introducing new strongly “nonlinear” self-similar variables.

1 Introduction

We are interested in this paper in a particular model of granular gases. A granular gas is a set of particles which interact by energy dissipating contact interactions. This is a quite different model than perfect molecular gas where energy is conserved, because the particles are “microscopic” regarding the scale of the system, but “macroscopic” in the sense that there are not molecules at all, but rather grains of a given size. For example, a suspension of pollen in a fluid or a planetary ring can be seen as a granular gas when external forces are neglected, each one in a different scale.

Our goal in this article is to investigate the global behaviour of the kinetic energy of a space-homogeneous gas of inelastic hard spheres interacting *via* binary collisions (that is a granular gas), and existence and uniqueness of blow up profiles associated. The study

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[†]Université de Lyon, Université Claude Bernard Lyon 1, Institut Camille Jordan, 43 bd. du 11 Novembre 1918, 69622 Villeurbanne cedex, France (rey@math.univ-lyon1.fr).

of this kind of system started with the physics paper [11] of P.K. Haff, and has since generated a large increase of interest, both in Physics (a complete introduction of the subject can be found in the textbook [8] by N. Brilliantov and T. Pöschel) and Mathematics (cf. the review article [17] by C. Villani). It involves some complicated phenomena, such as kinetic *collapse* (cf. K. Shida [16]) where the gas freezes completely, *clustering* at hydrodynamic level for inhomogeneous gases, or even spontaneous loss of homogeneity. This article will especially deal with the case of a gas whose particle's collision rate increases with dissipation of energy. Such a gas is sometimes called *anomalous* (see article [14] of S. Mischler and C. Mouhot), because of the unusual behaviour of this collision rate.

A granular gas can be described in a purely Newtonian way, but the number of macroscopic particles involved (ranging from 10^6 to 10^{10}) leads to adopt a statistical physics' point of view. Therefore, we shall study the so-called space-homogeneous *inelastic Boltzmann* equation (also known as *granular gases* equation), given by

$$(1.1) \quad \frac{\partial f}{\partial t} = Q_e(f, f),$$

where $f = f(t, v)$ represents the particle's distribution function, depending on time $t \geq 0$ and velocity $v \in \mathbb{R}^d$. The collision operator $Q_e(f, g)$, which will be made precise in the following (particularly concerning the meaning of subscript e), models a binary inelastic collision process of hard spheres type, localized in time. This equation will be supplemented with the initial value

$$(1.2) \quad f(0, \cdot) = f_{in}.$$

The inelasticity is characterized by a collision mechanics where mass and momentum are conserved and kinetic energy is dissipated. At microscopic level, it can be described as follows: given two particles of pre-collisional velocities v and v_* , their respective post-collisional velocities, denoted by v' and v'_* , are given by

$$\begin{cases} v' = v - \frac{1+e}{2} (u \cdot \omega) \omega, \\ v'_* = v_* + \frac{1+e}{2} (u \cdot \omega) \omega, \end{cases}$$

where $u := v - v_*$ is the *relative velocity* of a pair of particles, ω is the *impact direction* and $e \in [0, 1]$ the dissipation parameter, known as *restitution coefficient* (see Figure 1 for a sketch of the collision process).

Physically, it means that energy is dissipated in the impact direction only. The parameter e can depend on relative velocity and kinetic energy of the particles. For the need of the analysis, we will simply assume that e is a constant, which is a rather good physical approximation (a complete discussion on this topic can be found in [8]). Besides, e will also be taken different from 1, since $e = 1$ concerns the classical elastic case, in

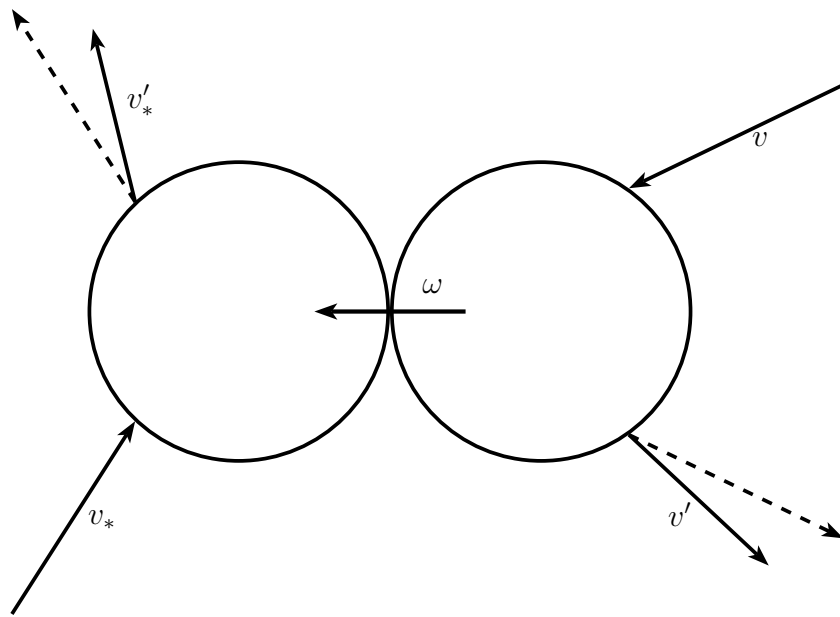


Figure 1.1: Geometry of collision (dashed lines are elastic and solid lines inelastic)

which no dissipation occurs. The inelastic case $e < 1$ can also be characterized by the fact that impact direction ω does not bisect the angle between pre and post-collisional velocities.

The parametrization of post-collisional velocities can also be found by using some properties of the model. Indeed, it is equivalent to conservation of impulsion and dissipation of energy:

$$(1.3) \quad \begin{aligned} v' + v'_* &= v + v_*, \\ |v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 &= -\frac{1-e^2}{2}|u \cdot \omega|^2 \leq 0. \end{aligned}$$

One of our goal will be the study of the macroscopic dissipation of energy, sometimes referred as cooling process of the gas. This phenomenon has been investigated mathematically by S. Mischler and C. Mouhot in [12] for the case of constant restitution coefficient e , and by R. Alonso and B. Lods in [1] for visco-elastics hard spheres, more realistic at physical level. In the latter one, energy dissipation is a power law of the relative velocity of colliding particles. We will use ideas of these two articles to write some of the results we present in the following.

Let us now describe precisely the collision operator we consider throughout the rest of this article. Let f and g be two nonnegative particles distribution functions only depending on $v \in \mathbb{R}^d$. The collision operator $Q_e(f, g)$, where $e \in [0, 1]$ is the constant restitution coefficient, can be expressed in the following weak form: given a regular test function ψ ,

$$(1.4) \quad \langle Q_e(f, g), \psi \rangle := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |u| f_* g (\psi' + \psi'_* - \psi - \psi_*) b(\hat{u} \cdot \omega, \mathcal{E}(f)) d\omega dv dv_*,$$

where we have used the usual shorthand notation $\psi' := \psi(v')$, $\psi'_* := \psi(v'_*)$, $\hat{u} := u/|u|$ and

$$\mathcal{E}(f) := \int_{\mathbb{R}^d} |v|^2 f(v) dv$$

denotes the *kinetic energy* of f . Moreover, b is a positive function known as *collisional cross section*. We will assume through this article the following properties for b :

H₁. The cross section can be written as a product

$$b(\cdot, \mathcal{E}) = b_1(\cdot) \mathcal{E}^{-a},$$

for a nonnegative constant a and a nonnegative function b_1 of mass 1 in the unit sphere.

H₂. The function b_1 involved in the cross section is bounded by below and above by positive constants β_1 and β_2 :

$$\forall x \in [-1, 1], \quad 0 < \beta_1 \leq b_1(x) \leq \beta_2 < \infty.$$

We can give a strong form of the collision operator, thanks to these hypotheses. Indeed, as it has been pointed out in [17] (the full derivation of this expression can be found *e.g.* in articles [2, 3] of A. Bobylev, J. Carrillo and I. Gamba), one can write

$$\begin{aligned} Q_e(f, g)(v) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u| \left(J \frac{|u|}{|u|} 'f 'g_* - f g_* \right) b(\hat{u} \cdot \omega, \mathcal{E}(f)) d\omega dv_*, \\ &= Q_e^+(f, g)(v) - f(v)L(g)(v). \end{aligned}$$

We have set $'v$ and $'v_*$ the pre-collisional velocities of two particles of given velocities v and v_* , and J the Jacobian of the map $(v, v_*) \mapsto ('v, 'v_*)$. Moreover, $Q_e^+(f, g)(v)$ is usually known as *gain* term because it can be seen as the number of particles of velocity v created by collisions of particles of pre-collisional velocities $'v$ and $'v_*$, whereas $f(v)L(g)(v)$ is the *loss* term, modelling loss of particles of pre-collisional velocities v . One can notice that J is not identically equal to 1, but since e is a nonzero constant,

$$J = \frac{|u|}{e^2 |'u|}.$$

Taking successively $\psi(v) = 1$, v and $|v|^2$ in (1.4) gives macroscopic conservation of mass and momentum, and dissipation of kinetic energy. This implies that equilibria of this collision operator are Dirac distributions $\rho \delta_{v=v_0}$ of prescribed mass ρ and momentum ρv_0 . In order to prevent apparition of these trivial solutions, one can add a diffusive term to give an input of energy, as was studied for example by Cercignani, R. Illner and C. Stoica in [9] and by A. Bobylev and Cercignani in [5] for Maxwell molecules. It is also possible to look for self-similar solutions to this equation thanks to a rescaling which leads to study the inelastic Boltzmann equation with drift term, as was done by M. Ernst and R. Brito in [10].

Let us now describe more precisely the asymptotic behaviour of the energy of f , assuming that this distribution is time dependent, of mass 1 and zero momentum. When there won't be any confusion possible, we will denote $\mathcal{E}(t) = \mathcal{E}(f)(t)$.

Definition 1. If f is solution to (1.1), we call *cooling process* the asymptotic behaviour of $\mathcal{E}(t)$. We say that there is a *blow up* when $\mathcal{E}(t) \rightarrow 0$ for $t \rightarrow T_c$, where T_c is the time of explosion of f .

If one multiplies equation (1.1) by $|v|^2$ and integrates in velocity, using weak formulation (1.4) of the collision operator and Hypothesis **H₁**, one gets dissipation of kinetic energy:

$$(1.5) \quad \frac{d}{dt} \mathcal{E}(t) = -\mathcal{E}(t)^{-a} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t, v) f(t, v_*) |u| D(|u|) dv dv_*,$$

where D is a nonnegative quantity usually called *dissipation rate*, given by

$$D(|u|) = \frac{1 - e^2}{4} \int_{\mathbb{S}^{d-1}} |u \cdot \omega|^2 b_1(\hat{u} \cdot \omega) d\omega.$$

Thanks to Hypothesis **H**₂ concerning the bounds on the cross section, it comes that

$$D(|u|) \geq \beta_1 \frac{1 - e^2}{4} |\mathbb{S}^{d-2}| |u|^2.$$

One can then apply Jensen's inequality to $\Psi(v) = |v|^3$ and to the probability measure $f(v_*) dv_*$, together with conservative properties of equation (1.1) in order to get

$$\int_{\mathbb{R}^d} f(v_*) \Psi(|v - v_*|) dv_* \geq \Psi \left(\left| v \int_{\mathbb{R}^d} f(v_*) dv_* - \int_{\mathbb{R}^d} v_* f(v_*) dv_* \right| \right) = \Psi(|v|).$$

Using equality (1.5) about the behaviour of the energy, it comes that

$$(1.6) \quad \frac{d}{dt} \mathcal{E}(t) \leq -\beta_e \mathcal{E}(t)^{-a} \int_{\mathbb{R}^d} f(t, v) |v|^3 dv,$$

where β_e is given by

$$\beta_e = \beta_1 \frac{1 - e^2}{4} |\mathbb{S}^{d-2}|.$$

Moreover, applying Hölder inequality with $s = 3/2$ and its conjugate $s' = 3$ to the map $v \mapsto f(v)|v|^2$, one gets thanks to mass conservation

$$(1.7) \quad \left(\int_{\mathbb{R}^d} f(v) |v|^2 dv \right)^{\frac{3}{2}} \leq \int_{\mathbb{R}^d} f(v) |v|^3 dv.$$

Using Fubini Theorem for nonnegative measurable function and gathering inequalities (1.6) and (1.7), we finally find a closed differential inequality for $\mathcal{E}(f)$, namely

$$(1.8) \quad \frac{d}{dt} \mathcal{E}(t) \leq -\beta_e \mathcal{E}(t)^{-a+\frac{3}{2}}, \text{ for } t < T_c.$$

According to this inequality, the cooling process will depend on the nonnegative coefficient a from **H**₁. We now have to describe rigorously the spaces of solutions to (1.1), in order to introduce the corresponding Cauchy problem and the main results of this article.

1.1 Functional Framework and Main Results

Let us present some functional spaces needed in the paper. We denote by L_q^1 for $q \in [1, +\infty)$ the following weighted Lebesgue space

$$L_q^1 = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable; } \|f\|_{L_q^1} := \int_{\mathbb{R}^d} |f(v)| \langle v \rangle^q dv < \infty \right\},$$

where $\langle v \rangle := \sqrt{1 + |v|^2}$. Thanks to this definition, we can introduce the sets of distribution of given mass 1, zero momentum and energy $\mathcal{E} > 0$ as

$$\mathcal{G} := \left\{ f \in L_1^1 : \int_{\mathbb{R}^d} f(v) dv = 1, \int_{\mathbb{R}^d} f(v) v dv = 0 \right\}$$

and

$$\mathcal{G}_{\mathcal{E}} := \left\{ f \in \mathcal{G} : \int_{\mathbb{R}^d} f(v) |v|^2 dv = \mathcal{E} \right\}.$$

Then, we denote by $W_q^{1,1}$ the weighted Sobolev space

$$W_q^{1,1} := \{ f \in L_q^1; \nabla f \in L_q^1 \}.$$

We also introduce the space BV_q of weighted bounded variations functions, given as the set of weak limits in \mathcal{D}' of sequences of smooth functions which are bounded in $W_q^{1,1}$. Finally, $\mathcal{M}^1(\mathbb{R}^d)$ is the space of probabilities measures on \mathbb{R}^d .

The Cauchy problem (1.1)-(1.2) has been extensively investigated in the past few years, starting with paper [2] and then by A. Bobylev, Cercignani and G. Toscani in [7], where Maxwellian molecules were considered using Fourier transform. A lot of properties of a more general model which include elastic and inelastic collisions for Maxwellian molecules as a particular case (but not hard spheres) have also been studied in the chapter of book [6] of A. Bobylev, Cercignani and I. Gamba. Concerning hard spheres kernel, S. Mischler and C. Mouhot proved existence and uniqueness of weak solutions to the associated Cauchy problem in [14]. It can be defined in the following way:

Definition 2. Given a nonnegative initial condition $f_{in} \in L_2^1 \cap \mathcal{G}$, a nonnegative function f defined on $[0, T] \times \mathbb{R}^d$ is said to be solution to the Cauchy problem (1.1)-(1.2) if

$$f \in \mathcal{C}(0, T; L_2^1),$$

and if (1.1)-(1.2) holds in weak sense, namely,

$$\int_0^T \left(\int_{\mathbb{R}^d} f(t, v) \partial_t \psi(t, v) dv + \langle Q_e(f, f)(t, \cdot), \psi \rangle \right) dt = \int_{\mathbb{R}^d} f_{in}(v) \psi(0, \cdot) dv,$$

for any $\psi \in \mathcal{C}_c^1([0, T] \times \mathbb{R}^d)$.

One can notice that it is always possible to assume the initial condition in \mathcal{G} , namely of mass 1 and zero momentum, since we may always reduce to this case by a scaling and translation argument, using invariance properties of the collision operator.

Let us now introduce the *cooling time* T_c of f as the time before blow up, namely

$$T_c := \sup \{ T > 0 : \mathcal{E}(t) > 0, \forall t < T \}.$$

For a collisional cross section given by \mathbf{H}_1 and \mathbf{H}_2 , Theorem 1.4 of [14] states well-posedness in L_3^1 of Cauchy problem for the granular gases equation, together with the existence of the cooling process:

Theorem 1.1. *Let f_{in} be a nonnegative distribution satisfying*

$$f_{in} \in L_3^1 \cap \mathcal{G} \text{ and } f_{in} \in BV_4 \cap L_5^1.$$

Then, the following results hold for a cross section given by \mathbf{H}_1 and \mathbf{H}_2 :

- (i) *the cooling time T_c is well define and positive;*
- (ii) *for each $T \in]0, T_c[$, there exists a unique solution $f \in \mathcal{C}(0, T; L_2^1) \cap L^\infty(0, T; L_3^1)$ to initial value problem (1.1)-(1.2). Such a solution is nonnegative, mass and momentum conservative, and kinetic energy dissipative;*
- (iii) *the energy of f is subject to the following asymptotic behaviour*

$$(1.9) \quad \mathcal{E}(t) \rightarrow 0 \text{ and } f(t, \cdot) \rightarrow \delta_{v=0} \text{ when } t \rightarrow T_c,$$

where the convergence of f occurs for weak- topology of $\mathcal{M}^1(\mathbb{R}^d)$.*

We now state main results of this paper concerning cooling process and blow up solutions to the anomalous model. It is known from [14] that if a is greater than $1/2$, T_c is finite, meaning that blow up of f occurs in finite time whereas it takes infinite time for $a \leq 1/2$. We give accurate asymptotic behaviour of the energy in the main theorems of this article, for all nonnegative values of the parameter a .

Theorem 1.2. *Let the collision operator be subjects to assertions \mathbf{H}_1 and \mathbf{H}_2 . If f is solution to (1.1), whose initial condition $0 \leq f_{in} \in L_3^1 \cap \mathcal{G}_1 \cap L^p$ for $p > 1$, there exist some positive constants C_i , $i \in \{1, \dots, 6\}$ and T_c , depending on e , a and f_{in} such that*

- (i) *sub-critical case: if $0 \leq a < 1/2$, and then $\alpha = 1/(2a - 1) < 0$,*

$$\frac{1}{(C_1 t + 1)^{-2\alpha}} \leq \mathcal{E}(t) \leq \frac{1}{(C_2 t + 1)^{-2\alpha}}, \forall t > 0;$$

- (ii) *critical case: if $a = 1/2$,*

$$e^{-C_3 t} \leq \mathcal{E}(t) \leq e^{-C_4 t}, \forall t > 0;$$

- (iii) *super-critical case: if $a > 1/2$, and then $\alpha = 1/(2a - 1) > 0$,*

$$(-C_5 t + 1)^{2\alpha} \leq \mathcal{E}(t) \leq (-C_6 t + 1)^{2\alpha}, \forall t < T_c.$$

Moreover, for all $a \geq 0$, there exist a function $V \in \mathcal{C}^1(0, T_c)$ and a nonnegative profile $G \in L_3^1 \cap \mathcal{G}_1$ such that the distribution

$$F(t, v) := V(t)^d G(V(t)v) \text{ for } (t, v) \in (0, T_c) \times \mathbb{R}^d,$$

is solution to (1.1), called self-similar solution.

We propose in Section 2 a proof of this Theorem based on the analysis of self-similar solutions to the inelastic Boltzmann equation with a linear drift term (widely studied in [12, 13]) and the asymptotic analysis of the time derivative of the energy. To this end, we introduce a new self-similar scaling, nonexplicit and strongly nonlinear.

Remark 1. One can check that the point (i) of this theorem is in good agreement with the one proven in [12] with $a = 0$, and which is known as *Haff's Law*[11]:

$$\frac{m_1}{(1+t)^2} \leq \mathcal{E}(f_t) \leq \frac{M_1}{(1+t)^2}.$$

We then state a theorem of existence and uniqueness (up to a translation of time) of self-similar solutions to equation (1.1) with $a \geq 0$. We also obtain convergence of solutions to (1.1) bounded in L_3^1 toward these particular solutions.

Theorem 1.3. *Let the collision operator be subjects to assertions \mathbf{H}_1 and \mathbf{H}_2 with $a \geq 0$. There exists a constructive $e_* \in (0, 1)$ such that for all $e \in [e_*, 1)$,*

(i) *the self-similar profile $G = G_e \in L_3^1 \cap \mathcal{G}_1$ is unique and if F_e and \bar{F}_e are two self-similar solutions to (1.1), there is a time $t_0 < T_c$ such that $\bar{F}_e(t, v) = F_e(t + t_0, v)$ for $t > \max\{0, -t_0\}$.*

(ii) *For any $M_0 > 0$, there exists $\eta \in (0, 1)$ such that if*

$$\begin{cases} f_{in} \in L_3^1 \cap \mathcal{G}_1, \\ \|f_{in}\|_{L_3^1} \leq M_0, \end{cases}$$

the solution f to equation (1.1) satisfies for a nonnegative constant C

$$\|f(t, \cdot) - F_e(t, \cdot)\|_{L^1} \leq \begin{cases} \frac{C}{(1 + C_2 t)^{-\alpha\mu_e}} & \text{if } a < 1/2, \\ C e^{-C_4 \mu_e t/2} & \text{if } a = 1/2, \\ C(1 - C_6 t)^{\alpha\mu_e} & \text{if } a > 1/2, \end{cases}$$

where $\alpha = 1/(2a - 1)$ and $\mu_e = (1 - \eta) + \mathcal{O}(1 - e)$.

The proof of this Theorem is also based on the analysis of self-similar solutions to the inelastic Boltzmann equation with drift term, together with the use of the explicit cooling processes of Theorem 1.2.

1.2 Plan of the Paper

The article is organized as follows. We prove Theorem 1.2 in Section 2. We give a nonlinear rescaling that allows us to treat the three different cooling process at once in Subsection 2.1. We then use the expression of this rescaling to give a relation between energy in classical and self-similar variables. Subsequently, we show in Subsection 2.2 results concerning the asymptotic behaviour of solutions to the granular equation with and without drift term, that is an estimate of the third order moment of solutions to the latter one (proven thanks to the classical Haff's Law) and that the time derivative of the energy of solutions to the former tends to 0. We finally use this result in Subsection 2.3 to prove cooling process.

We then apply in Section 3 this precise asymptotic behaviour together with a Theorem of uniqueness of self-similar profiles already proven in [13] of S. Mischler and C. Mouhot to show Theorem 1.3 about rate of convergence toward self-similar solutions.

2 Proof of Theorem 1.2

This section presents the proof of Theorem 1.2 which studies the cooling process for sub-critical ($a < 1/2$), critical ($a = 1/2$) and super-critical ($a > 1/2$) cases. To this end, we will prove some new estimates for the granular equation, with and without drift term. Let us start by introducing some nonclassical self-similar variables.

2.1 Nonlinear Self-similar Variables

We shall use a rescaling (seen *e.g.* in [10] and [12]), in order to prevent the blow up of f by “zooming” on the distribution and studying it in self-similar variables.

We will assume that the cross section follows hypotheses \mathbf{H}_1 and \mathbf{H}_2 with $a \geq 0$. Granular gases equation (1.1) reads

$$(2.1) \quad \frac{\partial f}{\partial t}(t, v) = \mathcal{E}(t)^{-a} Q_e(f, f)(t, v),$$

where the collision operator is given for test functions ψ by

$$\langle Q_e(f, g), \psi \rangle = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |u| f_* g (\psi' + \psi'_* - \psi - \psi_*) b_1(\hat{u} \cdot \omega) d\omega dv dv_*.$$

The rescaling of the distribution f will be written according to [10] as

$$(2.2) \quad \begin{cases} f(t, v) = V(t)^d g(T(t), V(t)v), \\ V(0) = 1, \quad T(0) = 0, \\ \lim_{t \rightarrow T_c} T(t) = \lim_{t \rightarrow T_c} V(t) = +\infty. \end{cases}$$

We shall determine functions T and V (the *self-similar variables*) and derive the equation followed by the distribution $g = g(s, w)$ with $s = T(t)$ and $w = V(t)v$. The term $V(t)^d$ in front of g is simply given by mass conservation. Moreover, by making a change of variables in the expression of the collision operator and thanks to hard sphere collision kernel, one gets that

$$Q_e(g(\lambda \cdot), g(\lambda \cdot))(v) = \lambda^{-d-1} Q_e(g, g)(\lambda v).$$

Especially, bilinearity of operator Q_e together with (2.2) yields

$$(2.3) \quad Q_e(f, f)(t, \cdot) = (V(t))^{d-1} Q_e(g, g)(s, \cdot).$$

One can note at this point that our approach of the problem cannot be immediately extended to the case of variable restitution coefficient. Indeed a simple parametrization of restitution coefficient is given by

$$1 - e^2(|u \cdot \omega|) = 2e_0|u \cdot \omega|^p,$$

for nonnegative p (*e.g.* [1] for more details). This is the so-called *generalized visco-elastic hard spheres* model. We can compute dissipation rate $D(|u|)$ in this case using microscopic energy dissipation (1.3), and find that the upper bound of Haff's Law is given by

$$\frac{d}{dt} \mathcal{E}(t) \leq -C \mathcal{E}(t)^{-a + \frac{3+p}{2}}, \text{ for } t < T_c.$$

for a nonnegative constant C depending on p and the dimension. This partial result agrees with the cooling law found by T. Schwager and T. Pöschel in [15] (and rigorously proved in [1], the difficult part being lower bound), namely that a gas of visco-elastic hard spheres ($a = 0$, $p = 1/5$) cools down slowly than a gas of hard spheres with constant restitution coefficient:

$$0 \leq \mathcal{E}(t) \leq \frac{M_1}{(1 + \mu_0 t)^{5/3}}, \forall t \geq 0.$$

This model arises a new difficulty: introducing self-similar variables (2.2) in relation (2.3) gives

$$Q_e(f, f)(t, \cdot) = (V(t))^2 Q_{\tilde{e}(t, \cdot)}(g, g)(s, \cdot),$$

that is collision operator becomes time dependent through a new restitution coefficient \tilde{e} . Finally, the new collision equation is not “autonomous”, which prevents us to use some previous results on this equation as we will do in the following (*e.g.* Haff's Law).

Now, if f follows rescaling (2.2), its time derivative becomes

$$(2.4) \quad \partial_t f = (V(t))^{d-1} (T'(t)V(t)\partial_s g + V'(t)\nabla_w \cdot (wg)).$$

Thanks to relations (2.3) and (2.4), if f is solution to equation (2.1), the distribution g is solution to

$$(2.5) \quad T'(t)V(t)\partial_s g + V'(t)\nabla_w \cdot (wg) = \mathcal{E}(f)(t)^{-a}Q_e(g, g).$$

We shall get rid of the term in (2.5) involving a negative power of energy to obtain a simpler equation, the classical homogeneous Boltzmann equation with an anti-drift term.

To this end, let us assume the rescaling to be “nonlinear” by asking it to depend on the energy of the solution itself:

$$\begin{cases} V'(t) = \tau \mathcal{E}(f)(t)^{-a}, \\ T'(t)V(t) = V'(t)/\tau, \end{cases}$$

where τ is a nonnegative parameter. The functions V and T are not explicit but one can see that they are well defined and agree to (2.2). Indeed, the map $t \mapsto \mathcal{E}(f)(t)^{-a}$ is increasing on $[0, T_c)$ thanks to asymptotic behaviour (1.9), is 1 when $t = 0$ given that energy of the initial distribution f_{in} has been normalized to this value and tends to infinity when $t \rightarrow T_c$. Moreover, one has $T(t) = \log(V(t))/\tau$.

With such an expression for V and T plugged in (2.5), distribution g is solution to the following equation:

$$(2.6) \quad \frac{\partial g}{\partial s} + \tau \nabla_w \cdot (wg) = Q_e(g, g).$$

Actually, this is the granular equation for inelastic hard spheres with constant restitution coefficient, complemented with an *anti-drift* term. This last term will act like an input of energy and will prevent blow up of $g(s, \cdot)$ toward a Dirac mass when $s \rightarrow \infty$. This equation has been thoroughly studied in articles [12, 13]. Especially, one has the following result.

Theorem 2.1. *Let $g_{in} \in \mathcal{G} \cap L^p$ for a fixed $p > 1$ be an initial datum for (2.6) with constant restitution coefficient e . If g is solution to the associated Cauchy problem, then*

$$(2.7) \quad 0 < c_0 \leq \mathcal{E}(g)(s) \leq c_1 < \infty, \forall s \geq 0.$$

Moreover, there exists a self-similar profile $0 \leq G \in L_2^1 \cap \mathcal{G}$:

$$\tau \nabla_w \cdot (wG) - Q_e(G, G) = 0.$$

Summarizing, the distribution f defined by

$$\begin{cases} f(t, v) = V(t)^d g(T(t), V(t)v), \\ V'(t) = \tau \mathcal{E}(f)(t)^{-a}, \\ T(t) = \log(V(t))/\tau, \end{cases}$$

is solution to granular equation (2.1) for all times as soon as the function g is solution to drift/collision equation (2.6). Conversely, if f is solution to (2.1), one can associate a solution g to rescaled equation (2.6) by setting

$$g(s, w) = e^{-d\tau s} f(V^{-1}(e^{\tau s}), e^{-\tau s} w).$$

Note that the inverse V^{-1} of V is well defined, by the discussion above.

Moreover, using change of variables $w = V(t)v$, the following relation holds between energies of f and g :

$$\mathcal{E}(g)(T(t)) = (V(t))^2 \mathcal{E}(f)(t).$$

Then function V can be written for all $0 \leq t < T_c$ as

$$V(t) = \left(\frac{\mathcal{E}(g)(T(t))}{\mathcal{E}(f)(t)} \right)^{1/2}.$$

Using this expression, the bounds (2.7) of the energy of g and the raw cooling process (1.9), one has another proof that $V(t) \rightarrow \infty$ and $T(t) = \log(V(t)) \rightarrow \infty$ when $t \rightarrow T_c$.

Finally, if G is a self-similar profile, that is a stationary solution to (2.6), we may associate a self-similar solution F to the original equation (2.1) by setting

$$F(t, v) = V(t)^d G(V(t)v), \quad \forall (t, v) \in (0, T_c) \times \mathbb{R}^d.$$

Such a G exists thanks to Theorem 2.1, which proves last assertion of Theorem 1.2.

2.2 Preliminary Results

We will show in this Subsection two asymptotic results concerning solutions to the granular equation with and without drift term, that we will need in order to prove cooling process.

Let $h = h(t, v)$ for nonnegative t and $v \in \mathbb{R}^d$ be solution to the inelastic Boltzmann equation

$$(2.8) \quad \frac{\partial h}{\partial t} = Q_\epsilon(h, h),$$

that is collision equation (2.1) with $a = 0$. According to [1] or [12], the (sub-critical) Haff's Law holds:

$$(2.9) \quad \frac{m}{(1 + \mu_0 t)^2} \leq \mathcal{E}(h)(t) \leq \frac{M}{(1 + \mu_0 t)^2}, \quad \forall t > 0.$$

An upper control of the third order moment of h can be deduced from this law. Let m_l denotes the $2l$ -th order moment of a distribution h , that is

$$m_l(h) = \int_{\mathbb{R}^d} h(v) |v|^{2l} dv.$$

Lemma 2.1. *Let h be solution to equation (2.1) for $a = 0$, with an initial condition $h_{in} \in \mathcal{G}$. There exists a nonnegative constant κ such that if $m_{3/2}(h_{in}) \leq \kappa$, then for all $t > 0$,*

$$(2.10) \quad m_{3/2}(h)(t) \leq \frac{\kappa}{(1 + \mu_0 t)^3}.$$

Proof. Setting $m_l = m_l(h)$ and multiplying (2.1) with $a = 0$ by $|v|^3$, one gets after integration in velocity space

$$\frac{d}{dt} m_{3/2}(t) = \int_{\mathbb{R}^d} Q_e(h, h)(t, v) |v|^3 dv.$$

Some more informations are then needed regarding the third order moment of the collision operator. A.V. Bobylev, I.M. Gamba and V.A. Panferov had shown in [4] (Lemma 3) the following estimate:

$$\int_{\mathbb{R}^d} Q_e(h, h)(t, v) |v|^3 dv \leq -(1 - \gamma) m_2(t) + \gamma S_{3/2}(t),$$

where $0 < \gamma < 1$ and

$$\begin{aligned} S_{3/2} &= \sum_{k=1}^{\lfloor \frac{5}{4} \rfloor} \binom{3/2}{k} (m_{k+1/2} m_{p-k} + m_k m_{p-k+1/2}), \\ &= \frac{3}{2} (m_{3/2} m_{1/2} + m_1^2). \end{aligned}$$

By definition, $m_1 = \mathcal{E}(h)$. Besides, by convexity, $m_2 \geq m_{3/2}^{4/3}$ and $m_1 \geq m_{1/2}^2$. Thus, $m_{3/2}(h_t)$ verifies thanks to sub-critical Haff's Law (2.9):

$$\begin{aligned} \frac{d}{dt} m_{3/2}(t) &\leq -(1 - \gamma) m_{3/2}(t)^{4/3} + \frac{3}{2} \gamma (\mathcal{E}(t)^2 + m_{3/2}(t) \mathcal{E}(t)^{1/2}), \\ &\leq -(1 - \gamma) m_{3/2}(t)^{4/3} + \frac{3}{2} \gamma \left(\frac{M^2}{(1 + \mu_0 t)^4} + m_{3/2}(t) \frac{M^{1/2}}{1 + \mu_0 t} \right). \end{aligned}$$

Then, one has

$$\begin{aligned} \frac{d}{dt} \left(m_{3/2}(t) - \frac{\kappa}{(1 + \mu_0 t)^3} \right) &\leq -(1 - \gamma) m_{3/2}(t)^{4/3} + \frac{3\mu_0 \kappa}{(1 + \mu_0 t)^4} \\ &\quad + \frac{3}{2} \gamma \left(\frac{M^2}{(1 + \mu_0 t)^4} + m_{3/2}(t) \frac{M^{1/2}}{1 + \mu_0 t} \right), \end{aligned}$$

where κ is a nonnegative constant such that $m_{3/2}(0) \leq \kappa$. Then inequality (2.10) is fulfilled for $t = 0$ and by continuity, the lemma is proved for $t < t_*$ with a positive time t_* . For $t = t_*$, one gets $m_{3/2}(t_*) = \kappa(1 + \mu_0 t_*)^{-3}$, and the last differential inequality reads

$$\frac{d}{dt} \left(m_{3/2}(t_*) - \frac{\kappa}{(1 + \mu_0 t_*)^3} \right) \leq \left(-(1 - \gamma)\kappa^{4/3} + C_1\kappa + \frac{3}{2}\gamma M^2 \right) \frac{1}{(1 + \mu_0 t_*)^4},$$

where $C_1 = 3 \left(\mu_0 + \gamma\sqrt{M}/2 \right)$. If the constant κ is chosen large enough, the right hand side of this inequality is negative, which concludes the proof. \square

Thanks to this estimate, one is now able to compute the limit of the time derivative of the energy of a distribution $g = g(s, w)$ solution to (2.6):

Proposition 2.1. *If g is solution to the inelastic Boltzmann equation with drift term (2.6), then*

$$\lim_{s \rightarrow +\infty} \frac{d}{ds} \mathcal{E}(g)(s) = 0.$$

Proof. It is possible to define a new distribution h by setting as in [14]

$$h(t, v) := V_0(t)^d g(T_0(t), V_0(t)v),$$

in which

$$\begin{cases} V_0(t) = 1 + \mu_0 t, \\ T_0(t) = \log(1 + \mu_0 t). \end{cases}$$

Then h is solution to the inelastic Boltzmann equation (2.8). Thanks to the expression of this rescaling,

$$\mathcal{E}(g(T_0(t))) = V_0(t)^2 \mathcal{E}(h)(t).$$

Differentiating this relation with respect to time, one gets

$$\frac{d}{dt} \mathcal{E}(g(T_0(t))) = A(t) + B(t),$$

with

$$A(t) = 2\mu_0(1 + \mu_0 t) \mathcal{E}(h)(t), \quad B(t) = (1 + \mu_0 t)^2 (\mathcal{E}(h))'(t).$$

Therefore, using once again sub-critical Haff's Law (2.9), if $t > 0$,

$$\frac{2\mu_0 m}{1 + \mu_0 t} \leq A(t) \leq \frac{2\mu_0 M}{1 + \mu_0 t}$$

and then $A(t) \rightarrow 0$ if $t \rightarrow \infty$. Moreover, by the weak expression of $Q(h, h)$, equation (2.1) with $a = 0$ and hypothesis \mathbf{H}_2 , one has on the one hand

$$(2.11) \quad \begin{aligned} B(t) &= (1 + \mu_0 t)^2 \int_{\mathbb{R}^d} Q(h, h) |v|^2 dv, \\ &\leq -\beta_e (1 + \mu_0 t)^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} h h_* |v - v_*|^3 dv dv_* \leq 0. \end{aligned}$$

On the other hand, Fubini Theorem and mass conservation yield

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} h h_* |v - v_*|^3 dv dv_* \leq 8m_{3/2}(h).$$

Then, using inequality (2.10) of Lemma 2.1 and the weak expression of $Q(h, h)$, B satisfies

$$B(t) \geq -\frac{\kappa \beta'_e}{1 + \mu_0 t}.$$

By (2.11) it comes that $B(t) \rightarrow 0$ if $t \rightarrow \infty$. The conclusion of the Proposition is finally given by the fact that the map T_0 is one-to-one. \square

We are now ready to prove Theorem 1.2.

2.3 Anomalous Cooling Process

We will prove in this Subsection the cooling process of an anomalous gas for nonnegative a , that is points (i), (ii) and (iii) of Theorem 1.2. Both upper and lower bounds for energy of f will be obtained together, thanks to the results of Subsection 2.2.

For the sake of simplicity, let us denote for $t < T_c$

$$E(t) := \mathcal{E}(f)(t) \text{ and } \bar{E}(t) := \mathcal{E}(g(T(t))),$$

and set $\tau = 1$. We have already seen that $V(t) = \bar{E}(t)^{1/2} E(t)^{-1/2}$ and that

$$(2.12) \quad \begin{aligned} E(t)^{-a} &= V'(t), \\ &= \frac{\bar{E}'(t)}{2(E(t)\bar{E}(t))^{1/2}} - \frac{\bar{E}(t)^{1/2} E'(t)}{2E(t)^{3/2}}. \end{aligned}$$

Thanks to the expression of \bar{E} and a chain rule, one has on the one hand

$$(2.13) \quad \begin{aligned} \bar{E}'(t) &= \frac{d}{dt} \left(\int_{\mathbb{R}^d} g(T(t), w) |w|^2 dw \right), \\ &= T'(t) \chi(t), \end{aligned}$$

where we defined

$$\chi := \frac{d}{ds} (\mathcal{E}(g)) \circ T.$$

On the other hand, the time derivative of T is given by

$$(2.14) \quad T'(t) = \frac{V'(t)}{V(t)} = \frac{E(t)^{-a+1/2}}{\bar{E}(t)^{1/2}}.$$

Finally, gathering relations (2.12), (2.13), (2.14) and dividing by $E(t)^{-a}$, it comes that

$$(2.15) \quad \frac{\chi(t)}{2\bar{E}(t)} - \frac{1}{2}\bar{E}(t)^{1/2}E'(t)E(t)^{a-3/2} = 1.$$

Following the result of Proposition 2.1, one has $\chi(t) \rightarrow 0$ when $t \rightarrow T_c$, because $T(t) \rightarrow \infty$ when $t \rightarrow T_c$, thanks to (1.9). Thus, using uniform bounds (2.7) on \bar{E} ,

$$\lim_{t \rightarrow T_c} \frac{\chi(t)}{2\bar{E}(t)} = 0.$$

Therefore, using equality (2.15) and the bounds of \bar{E} once more, there exists a time $t_0 > 0$ and two positive constants D_1 and D_2 such that if $t_0 < t < T_c$,

$$(2.16) \quad -D_1 \leq E'(t)E(t)^{a-3/2} \leq -D_2.$$

Then, if $0 \leq a < 1/2$ integration of the two sides of inequality (2.16) between 0 and t and the fact that $E(0) = 1$ yields

$$\frac{1}{(C_1 t + 1)^{-2\alpha}} \leq E(t) \leq \frac{1}{(C_2 t + 1)^{-2\alpha}},$$

with $\alpha = 1/(2a - 1) < 0$ and $C_i = -2\alpha D_i > 0$. This is the point (i) of Theorem 1.2. Moreover, if $a = 1/2$, the same argument gives the point (ii), namely

$$\exp(-D_1 t) \leq E(t) \leq \exp(-D_2 t).$$

Finally, taking $a > 1/2$ gives the point (iii) with $C_i = -2\alpha D_i < 0$:

$$(1 - C_1 t)^{2\alpha} \leq E(t) \leq (1 - C_2 t)^{2\alpha}.$$

Let us now show the result concerning self-similar profiles of equation (1.1).

3 Proof of Theorem 1.3

For $a \geq 0$, we will study in this Section the uniqueness of self-similar profiles of the inelastic Boltzmann equation with drift (2.6). The cooling process found in the previous Subsection will allow us to state a Theorem concerning convergence of solutions to (1.1) towards self-similar solutions.

We shall use a Theorem concerning the convergence toward self-similar profiles for small inelasticity in granular equation (2.6), which has been shown in [13]. For this, let us set

$$\tau = \tau_e := 1 - e,$$

in order to balance the dissipation of kinetic energy by the drift. Thanks to this scaling, it has been shown in [13] (Theorem 1.1.iv) that:

Theorem 3.1. *There exists a constructive $e_* \in (0, 1)$ such that for all $e \in [e_*, 1)$, the self-similar profile G_e from Theorem 2.1 is unique and globally attractive on bounded subsets of L^1_3 : for any $M > 0$, there exists $\eta \in (0, 1)$ such that if*

$$g_{in} \in L^1_3 \cap \mathcal{G}, \quad \|g_{in}\|_{L^1_3} \leq M_0,$$

solution g to equation (2.6) satisfies

$$(3.1) \quad \|g(t, \cdot) - G_e\|_{L^1_2} \leq e^{-(1-\eta)\nu_e t},$$

where $\nu_e = \tau_e + \mathcal{O}(\tau_e^2)$.

Let us show thanks to this Theorem the trend to self-similar solution of our problem. We have already seen in Subsection 2.1 that if g is solution to (2.6) then f is solution to (1.1) with $f(t, v) = V(t)^d g(T(t), V(t)v)$, where

$$(3.2) \quad \begin{cases} V(t) = \left(\frac{\mathcal{E}(g)(T(t))}{\mathcal{E}(f)(t)} \right)^{1/2}, \\ T(t) = \frac{\log(V(t))}{\tau_e}. \end{cases}$$

Thus, if G_e is the unique self-similar profile of equation (2.6), one can find a self-similar solution to equation (1.1) by setting $F_e(t, v) = V(t)^d G_e(V(t)v)$. The uniqueness of this solution up to a translation of time can be shown as in [13] to prove the first point of Theorem 1.3.

Moreover, transformation $w \rightarrow V(t)v$ and rate of convergence (3.1) give

$$(3.3) \quad \begin{aligned} \|f(t, \cdot) - F_e(t, \cdot)\|_{L^1} &= V(t)^d \int_{\mathbb{R}^d} |g(T(t), V(t)v) - G_e(V(t)v)| dv, \\ &\leq \|g(T(t), \cdot) - G_e\|_{L^1_2}, \\ &\leq e^{-(1-\eta)\nu_e T(t)}. \end{aligned}$$

Besides, thanks to the expression (3.2) of self-similar variables and the choice $\tau_e = 1 - e$, one has

$$\nu_e T(t) = \log(V(t)) + \mathcal{O}(1 - e).$$

The positive lower bound of $\mathcal{E}(g_{T(t)})$ of Theorem 2.1 together with inequality (3.3) and relation (3.2) yield

$$\|f(t, \cdot) - F_e(t, \cdot)\|_{L^1} \leq C\mathcal{E}(f)(t)^{\mu_e/2},$$

where $\mu_e = 1 - \eta + \mathcal{O}(1 - e)$ and C is a nonnegative constant. Finally, using the cooling process of Theorem 1.2 that we have shown in last Subsection, we can conclude the proof of Theorem 1.3, that is trend to self-similar solution of the solutions to (1.1) and rate of convergence depending on (weak) inelasticity $1 - e$ and a .

Conclusion and Perspectives

We have given in this paper the asymptotic behaviour of the space-homogeneous inelastic Boltzmann equation for anomalous gases. Depending on a parameter, we can observe in this model a blow up in finite time. We quantified the time of blow up and gave the associated self-similar profiles. Under a weak inelasticity hypothesis, we also obtained the uniqueness (up to a translation of the time) of the self-similar solutions to this equation, and the convergence of the classical solutions toward them.

To prove these theorems, we introduced a new energy-dependant self-similar scaling, which leads to the study of the inelastic Boltzmann equation with a linear drift term. We gave some results concerning the asymptotic behaviour of the energy of the solutions to this equation, by using some well known theorems about this equation.

Concerning the perspectives of this work, we would like to adapt our results to non-constant restitution coefficient models, such as the viscoelastic one. We also want to prove the rate of cooling using a more classical self-similar scaling, generalising the one used in [10] and [12], and moments methods. That would perhaps allow to give up the rather unphysical L^p hypothesis on the initial condition for the proof of Haff's Law.

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