



Blow up Analysis for Anomalous Granular Gases

Thomas Rey

► To cite this version:

| Thomas Rey. Blow up Analysis for Anomalous Granular Gases. 2011. hal-00588970v1

HAL Id: hal-00588970

<https://hal.science/hal-00588970v1>

Preprint submitted on 27 Apr 2011 (v1), last revised 7 May 2012 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Blow Up Analysis for Anomalous Granular Gases*

THOMAS REY[†]

Abstract

We investigate in this article the long-time behaviour of the solutions to the energy-dependent, spatially-homogeneous, inelastic Boltzmann equation for hard spheres. This model describes a diluted gas composed of hard spheres under statistical description, that dissipates energy during collisions. We assume that the gas is “anomalous”, in the sense that the energy dissipation increases when the temperature decreases. This allows the gas to cool down in finite time. We study the existence, uniqueness and attractiveness of blow up profiles for this model and the cooling law associated, generalizing the classical Haff’s Law for granular gases. To this end, we give some new estimates about the third order moment of the inelastic Boltzmann equation with drift term and we introduce new strongly “non-linear” self-similar variables.

1 Introduction

1.1 Granular Gases Equation

We are interested in this paper in a particular model of granular gases. A granular gas is a set of particles which interact by energy dissipating contact interactions. This is a quite different model than perfect molecular gas where the energy is conserved, because the particles are “microscopic” regarding the scale of the system, but “macroscopic” in the sense that there are not molecules at all, but rather grains of a given size. For example, a suspension of pollen in a fluid or a planetary ring can be seen as a granular gas, each one in a different scale.

Our goal in this article is to investigate the global behaviour of the kinetic energy of a space-homogeneous gas of inelastic hard spheres interacting *via* binary collisions (that is a granular gas), and the existence of blow up profiles associated. The study of this kind of system started with the physics paper [15] of P.K. Haff, and has since generated a large increase of interest, both in Physics (a complete introduction of the subject can be found in the textbook [8] by N. Brilliantov and T. Pöschel) and Mathematics (*cf.* the review article [21] by C. Villani). It involves some complicated phenomena, such as kinetic *collapse* (*cf.* K. Shida [20]) where the gas freezes completely, *clustering* at the hydrodynamic level for inhomogeneous gases, or even spontaneous loss of homogeneity. This article will especially deal with the case of a gas whose particle’s collision rate

*The author would like to express his gratitude for the support of the European Research Council ERC Starting Grant 2009, project 239983-NuSiKiMo;

[†]Université de Lyon, Université Claude Bernard Lyon 1, Institut Camille Jordan, 43 bd. du 11 Novembre 1918, 69622 Villeurbanne cedex, France (rey@math.univ-lyon1.fr).

increases with dissipation of energy. Such a gas is sometimes called *anomalous* (see the article [18] of S. Mischler and C. Mouhot), because of the unusual behaviour of this collision rate.

Such systems can be described in a purely Newtonian way, but the number of macroscopic particles involved (ranging from 10^6 to 10^{10}) leads to adopt a statistical physics' point of view. Therefore, we study the so-called *inelastic homogeneous Boltzmann* equation (also known as the *homogeneous granular gases* equation), given by

$$(1.1) \quad \frac{\partial f}{\partial t} = Q_e(f, f),$$

where $f = f(t, v)$ represents the distribution function of the particles, depending on time $t \geq 0$ and velocity $v \in \mathbb{R}^d$. The collision operator $Q_e(f, g)$, which will be made precise in the next section (particularly concerning the meaning of the subscript e), models a binary inelastic collision process of hard spheres type, localized in time. This equation will be supplemented with the initial value

$$(1.2) \quad f(0, \cdot) = f_{in}.$$

The inelasticity is characterized by a collision mechanics where mass and momentum are conserved and kinetic energy is dissipated. At the microscopic level, it can be described as follows: given two particles of pre-collisional velocities v and v_* , their respective post-collisional velocities, denoted by v' and v'_* , are given by

$$\begin{cases} v' = v - \frac{1+e}{2}(u \cdot \omega)\omega, \\ v'_* = v_* + \frac{1+e}{2}(u \cdot \omega)\omega, \end{cases}$$

where $u := v - v_*$ is the *relative velocity* of the pair of particles, ω is the *impact direction* and $e \in [0, 1]$ the dissipation parameter, known as *restitution coefficient* (see Figure 1.1 for a sketch of the collision process). Physically, it means that the energy is dissipated in the impact direction only. The parameter e can depend on the relative velocity and the kinetic energy of the particles. For the need of the analysis, we will simply assume that e is a constant, which is a rather good physical approximation. Besides, e will also be taken different from 1, since $e = 1$ concerns the classical elastic case, in which no dissipation occurs. The inelastic case $e < 1$ can also be characterized by the fact that the impact direction ω does not bisect the angle between the pre-collisional and post-collisional velocities.

The parametrization of the post-collisional velocities can also be found by using the properties of the model. Indeed, it is equivalent to the conservation of impulsions and the dissipation of energy:

$$(1.3) \quad \begin{aligned} v' + v'_* &= v + v_*, \\ |v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 &= -\frac{1-e^2}{2}|u \cdot \omega|^2 \leq 0. \end{aligned}$$

One of our goal will be the study of the macroscopic dissipation of the energy, sometimes referred as the *cooling process* of the gas. This phenomenon has been investigated mathematically by S. Mischler and C. Mouhot in [16] for the case of constant restitution coefficient e , and by R. Alonso and B. Lods in [1] for visco-elastics hard spheres, more realistic at the physical level. In the latter one, the energy dissipation is a power law of the relative velocity of the colliding particles. We will use ideas of these two articles to write some of the results we present in the following.

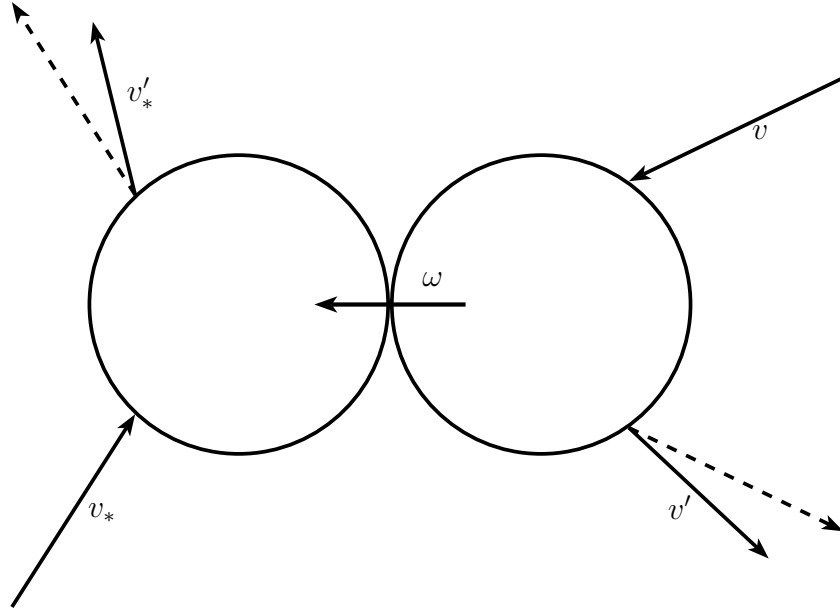


Figure 1.1: Geometry of collision (dashed lines are elastic and solid lines inelastic)

1.2 Inelastic Collision Operator and Dissipation of Energy

We shall now present the model of collision operator we consider throughout the rest of the article. Let us assume that f and g are two non-negative particles distribution functions only depending on $v \in \mathbb{R}^d$. The collision operator $Q_e(f, g)$, where $e \in [0, 1]$ is the constant restitution coefficient, can be expressed in the following weak form: given a regular test function ψ ,

$$(1.4) \quad \langle Q_e(f, g), \psi \rangle := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |u| f_* g (\psi' + \psi'_* - \psi - \psi_*) b(\hat{u} \cdot \omega, \mathcal{E}(f)) d\omega dv dv_*,$$

where we have used the usual shorthand notation $\psi' := \psi(v')$, $\psi'_* := \psi(v'_*)$, $\hat{u} := u/|u|$ and

$$\mathcal{E}(f) := \int_{\mathbb{R}^d} |v|^2 f(v) dv$$

denotes the *kinetic energy* of f . Moreover, b is a positive function known as the collisional *cross section*. We will assume through this article the following properties for the cross section:

H₁. The cross section can be written as a product

$$b(\cdot, \mathcal{E}) = b_1(\cdot) \mathcal{E}^{-a},$$

for a non-negative constant a and a non-negative function b_1 of mass 1 in the unit sphere.

H₂. The function b_1 involved in the cross section is bounded by below and above by positive constants β_1 and β_2 :

$$\forall x \in [-1, 1], \quad 0 < \beta_1 \leq b_1(x) \leq \beta_2 < \infty.$$

Taking successively $\psi(v) = 1$, v and $|v|^2$ in (1.4) gives the macroscopic conservation of mass and momentum, and the dissipation of kinetic energy. This implies that the equilibria of this collision operator are the Dirac distributions $\rho \delta_{v=v_0}$ of prescribed mass ρ and momentum v_0 . In order to prevent the apparition of these trivial solutions, one can add a diffusive term to give an input of energy, as was studied for example by Cercignani, R. Illner and C. Stoica in [9] and by A. Bobylev and Cercignani in [5] for Maxwell molecules. It is also possible to add a drift term thanks to a proper rescaling, as was done by M. Ernst and R. Brito in [12], to study self-similar profiles.

Let us now describe more precisely the cooling process of the gas, and then assume that the distribution f is time dependent, of mass 1 and zero momentum. If one multiplies the equation (1.1) by $|v|^2$ and integrates in velocity, using the weak formulation (1.4) of the collision operator and the hypothesis **H₁**, one gets the dissipation of kinetic energy:

$$(1.5) \quad \frac{d}{dt} \mathcal{E}(f_t) = -\mathcal{E}(f_t)^{-a} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_t(v) f_t(v_*) |u| D(|u|) dv dv_*,$$

where D is a non-negative quantity usually called the *dissipation rate*, given by

$$D(|u|) = \frac{1-e^2}{4} \int_{\mathbb{S}^{d-1}} |u \cdot \omega|^2 b_1(\hat{u} \cdot \omega) d\omega.$$

Thanks to the Hypothesis **H₂** concerning the bounds on the cross section, it comes that

$$D(|u|) \geq \beta_1 \frac{1-e^2}{4} |\mathbb{S}^{d-2}| |u|^2.$$

One can then apply the Jensen inequality to the function $\Psi(v) = |v|^3$ and to the probability measure $f(v_*) dv_*$ in order to get

$$\int_{\mathbb{R}^d} f(v_*) \Psi(|v - v_*|) dv_* \geq \Psi \left(\left| v \int_{\mathbb{R}^d} f(v_*) dv_* - \int_{\mathbb{R}^d} v_* f(v_*) dv_* \right| \right) = \Psi(|v|),$$

thanks to the conservative properties of equation (1.1). Using the equality (1.5) about the behaviour of the energy, it comes that

$$\frac{d}{dt} \mathcal{E}(f_t) \leq -\beta_e \mathcal{E}(f_t)^{-a} \int_{\mathbb{R}^d} f_t(v) |v|^3 dv,$$

where β_e is given by

$$\beta_e = \beta_1 \frac{1-e^2}{4} |\mathbb{S}^{d-2}|.$$

Moreover, applying the Hölder inequality with $s = 3/2$ and its conjugate $s' = 3$ to the map $v \mapsto f(v)|v|^2$, one gets

$$\left(\int_{\mathbb{R}^d} f(v) |v|^2 dv \right)^{\frac{3}{2}} \leq \int_{\mathbb{R}^d} f(v) |v|^3 dv,$$

thanks to the conservation of mass. Using the Fubini Theorem for non-negative measurable function and gathering the last two inequalities, a closed differential inequality involving the energy $\mathcal{E}(f)$ is found, namely

$$(1.6) \quad \frac{d}{dt} \mathcal{E}(f_t) \leq -\beta_e \mathcal{E}(f_t)^{-a+\frac{3}{2}}, \text{ for } t < T_c.$$

According to this inequality, the control of the cooling of f (that is its blow up) will depend on the non-negative coefficient a from **H₁**.

Remark 1. It is pointed out in [21] (and the full derivation of this expression can be found *e.g.* in the articles [2, 3] of A. Bobylev, J. Carrillo and I. Gamba) that if one denotes by $'v$ and $'v_*$ the pre-collisional velocities of two particles of given velocities v and v_* , and if J is the Jacobian of the map $(v, v_*) \mapsto ('v, 'v_*)$, then one can write the collision operator Q_e on the following strong form:

$$\begin{aligned} Q_e(f, g)(v) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u| \left(J \frac{|'u|}{|u|} 'f 'g_* - f g_* \right) b(\hat{u} \cdot \omega, \mathcal{E}(f)) d\omega dv_*, \\ &= Q_e^+(f, g)(v) - f(v) L(g)(v). \end{aligned}$$

In the last equation, $Q_e^+(f, g)(v)$ is usually known as the *gain* term because it can be seen as the number of particles of velocity v created by the collisions of particles of pre-collisional velocities $'v$ and $'v_*$, whereas $f(v)L(g)(v)$ is the *loss* term, modelling the loss of particles of pre-collisional velocities v . One can notice that J is not identically equal to 1. Since e is a non-zero constant, one has

$$J = \frac{|u|}{e^2 |'u|}.$$

In order to investigate the long-time behaviour of the kinetic energy of the granular gas, we have to describe rigorously the spaces of solutions to our model, as well as its Cauchy problem. This is the subject of the following Subsection.

1.3 The Cauchy Problem

Let us introduce some functional spaces needed in the rest of the paper. We denote by L_q^1 for $q \in [1, +\infty)$ the following weighted Lebesgue space

$$L_q^1 = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ measurable; } \|f\|_{L_q^1} := \int_{\mathbb{R}^d} |f(v)| \langle v \rangle^q dv < \infty \right\},$$

where $\langle v \rangle := \sqrt{1 + |v|^2}$. Thanks to this definition, we can introduce the sets of functions of given mass ρ , momentum $u \in \mathbb{R}^d$ and energy $\mathcal{E}$, that is

$$\begin{aligned} \mathcal{C}_{\rho, u} &:= \left\{ f \in L_1^1 : \int_{\mathbb{R}^d} f(v) dv = \rho, \int_{\mathbb{R}^d} f(v) v dv = u \right\}; \\ \mathcal{C}_{\rho, u, \mathcal{E}} &:= \left\{ f \in L_2^1 : \int_{\mathbb{R}^d} f(v) dv = \rho, \int_{\mathbb{R}^d} f(v) v dv = u, \int_{\mathbb{R}^d} f(v) |v|^2 dv = \mathcal{E} \right\}. \end{aligned}$$

Then, we denote by $W_q^{1,1}$ the weighted Sobolev space

$$W_q^{1,1} := \{ f \in L_q^1; \nabla f \in L_q^1 \}.$$

We also introduce the space BV_q of weighted bounded variations functions, given as the set of the weak limits in $\mathcal{D}'$ of sequences of smooth functions which are bounded in $W_q^{1,1}$. Finally, $\mathcal{M}^1(\mathbb{R}^d)$ is the space of probabilities measures on $\mathbb{R}^d$.

The Cauchy problem (1.1)-(1.2) has been extensively investigated in the past few years, starting with the paper [2] and then by A. Bobylev, Cercignani and G. Toscani in [7], where Maxwellian molecules were considered using Fourier transform. A lot of properties of a more general model which include elastic and inelastic collisions for Maxwellian

molecules as a particular case (but not hard spheres) have also been studied in the chapter of book [6] of A. Bobylev, Cercignani and I. Gamba. Concerning hard spheres kernel, the paper [18] deals with weak solutions to the associated Cauchy problem, which may be defined in the following way:

Definition 1. Given a non-negative initial condition $f_{in} \in L_2^1 \cap \mathcal{C}_{1,0}$, a non-negative function f defined on $[0, T] \times \mathbb{R}^d$ is said to be a solution to the Cauchy problem (1.1)-(1.2) if

$$f \in C(0, T; L_2^1),$$

and if (1.1)-(1.2) holds in the weak sense, namely,

$$\int_0^T \left(\int_{\mathbb{R}^d} f_t(v) \partial_t \psi(t, v) dv + \langle Q_e(f_t, f_t), \psi \rangle \right) dt = \int_{\mathbb{R}^d} f_{in}(v) \psi(0, \cdot) dv,$$

for any $\psi \in C_c^1([0, T] \times \mathbb{R}^d)$.

This article introduces the *cooling time* T_c of f as the time before blow up, namely

$$T_c := \sup \{T > 0 : \mathcal{E}(f_t) > 0, \forall t < T\}.$$

For a collisional cross section given by $\mathbf{H}_1$ and $\mathbf{H}_2$, they show the well-posedness in L_3^1 of the Cauchy problem for the homogeneous granular gases equation, together with the existence of the cooling process:

Theorem 1.1 ([18], Theorem 1.4). *Let f_{in} be a non-negative distribution satisfying*

$$f_{in} \in L_3^1 \cap \mathcal{C}_{1,0} \text{ and } f_{in} \in BV_4 \cap L_5^1.$$

Then, the following results hold for a cross section given by $\mathbf{H}_1$ and $\mathbf{H}_2$:

- (i) *the cooling time T_c is well define and positive;*
- (ii) *for each $T \in]0, T_c[$, there exists a unique solution $f \in C(0, T; L_2^1) \cap L^\infty(0, T; L_3^1)$ to the initial value problem (1.1)-(1.2). Such a solution is non-negative, mass and momentum conservative, and kinetic energy dissipative;*
- (iii) *the energy of f is subject to the following asymptotic behaviour*

$$(1.7) \quad \mathcal{E}(f_t) \rightarrow 0 \text{ and } f_t \rightharpoonup \delta_{v=0} \text{ when } t \rightarrow T_c,$$

where the convergence of f occurs for the weak- topology of $\mathcal{M}^1(\mathbb{R}^d)$.*

We now state the main results of this paper concerning cooling process and blow up solutions to the anomalous model.

1.4 Main Results

We know from [18] that if a is greater than $1/2$, T_c is finite, meaning that the cooling (and blow up of f) occurs in finite time whereas it takes infinite time for $a \leq 1/2$. We give the accurate asymptotic behaviour of the energy in the main theorems of this article, for all the non-negative values of the parameter a .

More precisely, we adopt another strategy than [18] by using a “non-linear” rescaling and moment estimates to first show that the energy decreases like an inverse power of the time if $0 \leq a < 1/2$, (we call this case *sub-critical*), exponentially if $a = 1/2$, (*critical* case) and polynomially if $a > 1/2$ (*super-critical* case).

Theorem 1.2. *Let the collision operator be subjects to the assertions $\mathbf{H}_1$ and $\mathbf{H}_2$. If f is a solution to (1.1), whose initial condition $0 \leq f_{in} \in L_3^1 \cap \mathcal{C}_{1,0,1}$, there exist some positive constants C_i , $i \in \{1, \dots, 6\}$ and T_c , depending on e , a and f_{in} such that*

(i) *if $0 \leq a < 1/2$, and then $\alpha = 1/(2a - 1) < 0$,*

$$\frac{1}{(C_1 t + 1)^{-2\alpha}} \leq \mathcal{E}(f_t) \leq \frac{1}{(C_2 t + 1)^{-2\alpha}}, \forall t > 0;$$

(ii) *if $a = 1/2$,*

$$e^{-C_3 t} \leq \mathcal{E}(f_t) \leq e^{-C_4 t}, \forall t > 0;$$

(iii) *if $a > 1/2$, and then $\alpha = 1/(2a - 1) > 0$,*

$$(-C_5 t + 1)^{2\alpha} \leq \mathcal{E}(f_t) \leq (-C_6 t + 1)^{2\alpha}, \forall t < T_c.$$

Moreover, for all $a \geq 0$, there exist a function $V \in C^1(0, T_c)$ and a non-negative profile $G \in L_3^1 \cap \mathcal{C}_{1,0,1}$ such that the distribution

$$F(t, v) := V(t)^d G(V(t)v) \text{ for } (t, v) \in (0, T_c) \times \mathbb{R}^d,$$

is a solution to (1.1), called self-similar solution.

Remark 2. One can check that the point (i) of this theorem is in good agreement with the one proved in [16] with $a = 0$, and which is known as the *Haff's Law*[15]:

$$\frac{m_1}{(1+t)^2} \leq \mathcal{E}(f_t) \leq \frac{M_1}{(1+t)^2}.$$

We then state a theorem of existence and uniqueness (up to a translation of time) of the self-similar solutions to equation (1.1) with $a \geq 0$, using once more the non-linear rescaling. We also obtain the convergence of the solutions to (1.1) bounded in L_3^1 toward these profiles.

Theorem 1.3. *Let the collision operator be subjects to the assertions $\mathbf{H}_1$ and $\mathbf{H}_2$ with $a \geq 0$. There exists a constructive $e_* \in (0, 1)$ such that for all $e \in [e_*, 1)$,*

(i) *the self-similar profile $G = G_e \in L_3^1 \cap \mathcal{C}_{1,0,1}$ is unique and if F_e and $\bar{F}_e$ are two self-similar solutions to (1.1), there is a time $t_0 < T_c$ such that $\bar{F}_e(t, v) = F_e(t + t_0, v)$ for $t > \max\{0, -t_0\}$.*

(ii) *For any $M_0 > 0$, there exists $\eta \in (0, 1)$ such that if*

$$\begin{cases} f_{in} \in L_3^1 \cap \mathcal{C}_{1,0,1}, \\ \|f_{in}\|_{L_3^1} \leq M_0, \end{cases}$$

the solution f to the equation (1.1) satisfies for a non-negative constant C

$$\|f_t - F_e(t, \cdot)\|_{L^1} \leq \begin{cases} \frac{C}{(1 + C_2 t)^{-\alpha\mu_e}} & \text{if } a < 1/2, \\ C e^{-C_4 \mu_e t/2} & \text{if } a = 1/2, \\ C(1 - C_6 t)^{\alpha\mu_e} & \text{if } a > 1/2, \end{cases}$$

where $\alpha = 1/(2a - 1)$ and $\mu_e = (1 - \eta) + \mathcal{O}(1 - e)$.

1.5 Sketch of the Proof

The article is organized as follows. We prove in Section 2 the Theorem 1.2, starting by giving a non-linear rescaling that allows us to treat the three different cooling process at once in Subsection 2.1. We then use the expression of this rescaling to give a relation between the energy in classical and self-similar variables. Subsequently, we show in Subsection 2.2 two new results concerning the asymptotic behaviour of a solution to the granular equation with and without drift term, that is an estimate of the third order moment of the solution to the latter one (proven thanks to the classical Haff's Law) and that the time derivative of the energy of a solution to the former tends to 0. We finally use this result in Subsection 2.3 to prove the cooling process.

We then apply in Section 3 this precise asymptotic behaviour together with a Theorem of uniqueness and attractiveness of self-similar profiles taken from the paper [17] of S. Mischler and C. Mouhot to prove Theorem 1.3 about rate of convergence toward self-similar solutions.

Finally, an alternative proof of the asymptotic behaviour in the sub-critical case can be found in Section 4. This proof is conceptually simpler than the first one, because it uses the “classical” self-similar variables and moments estimates, but allow only to deal with the sub-critical case. We give in Subsection 4.2 the proof of the cooling law (point (i) of Theorem 1.2) by computing new *a priori* estimates involving the second and third order moment of self-similar solutions. We then prove the existence of self-similar profiles by using in Subsection 4.4 an abstract Theorem concerning fixed point of evolution semi-groups. In order to apply it, we show some stability estimates concerning self-similar solutions in the the sub-critical case in Subsection 4.3.

2 Proof of Theorem 1.2

This section presents the proof of Theorem 1.2 which studies the cooling process for the sub-critical ($a < 1/2$), critical ($a = 1/2$) and super-critical ($a > 1/2$) cases. To this end, we will prove some new estimates for the homogeneous granular equation, with and without drift term. Let us start by introducing some non-classical self-similar variables.

2.1 Non-linear Self-similar Variables

We shall use a rescaling (seen for the first time for the granular equation in the article [12] and then for example in [16]), in order to prevent the blow up of f by “zooming” on the distribution and studying it in self-similar variables.

We will assume that the cross section follows the hypotheses $\mathbf{H}_1$ and $\mathbf{H}_2$ with $a \geq 0$. The granular gases equation (1.1) now reads

$$(2.1) \quad \frac{\partial f}{\partial t} = \mathcal{E}(f)^{-a} Q_e(f, f),$$

where the collision operator is given for test functions ψ by

$$\langle Q_e(f, g), \psi \rangle = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |u| f_* g (\psi' + \psi'_* - \psi - \psi_*) b_1(\hat{u} \cdot \omega) d\omega dv dv_*.$$

The rescaling of the distribution f will be written according to [12] as

$$(2.2) \quad \begin{cases} f(t, v) = V(t)^d g(T(t), V(t)v), \\ V(0) = 1, \quad T(0) = 0, \\ \lim_{t \rightarrow T_C} T(t) = \lim_{t \rightarrow T_C} V(t) = +\infty. \end{cases}$$

We shall determine the functions T and V (the *self-similar variables*) and derive the equation followed by the distribution $g = g_s(w)$ with $s = T(t)$ and $w = V(t)v$. The term $V(t)^d$ in front of g is simply given by mass conservation. Moreover, by making a change of variables in the expression of the collision operator and thanks to the hard sphere collision kernel, one gets that

$$Q_{e(\cdot)}(g(\lambda \cdot), g(\lambda \cdot))(v) = \lambda^{-d-1} Q_{e(\frac{\cdot}{\lambda})}(g, g)(v).$$

Especially, the bilinearity of the operator Q_e together with (2.2) yields

$$(2.3) \quad Q_e(f, f)(t, \cdot) = (V(t))^{d-1} Q_e(g, g),$$

One can note at this point that our approach of the problem cannot be immediately extended to the case of variable restitution coefficient. Indeed a simple parametrization of the restitution coefficient is given by

$$1 - e^2(|u \cdot \omega|) = 2e_0|u \cdot \omega|^p,$$

for non-negative p (*e.g.* [1] for more details). This is the so-called *generalized visco-elastic hard spheres* model. We can compute the dissipation rate $D(|u|)$ in this case using the microscopic energy dissipation (1.3), and find that the upper bound of the Haff's Law is given by

$$\frac{d}{dt} \mathcal{E}(f_t) \leq -C \mathcal{E}(f_t)^{-a + \frac{3+p}{2}}, \text{ for } t < T_c.$$

for a non-negative constant C depending on p and the dimension. This partial result agrees with the formal cooling law found by T. Schwager and T. Pöschel in [19] (and rigorously proved in [1], the difficult part being the lower bound), namely that a gas of visco-elastic hard spheres ($a = 0$, $p = 1/5$) cools down slowly than a gas of hard spheres with constant coefficient of restitution:

$$0 \leq \mathcal{E}(f_t) \leq \frac{M_1}{(1 + \mu_0 t)^{5/3}}, \quad \forall t \geq 0.$$

This model arises a new difficulty: introducing the self-similar variables (2.2) in the relation (2.3) gives

$$Q_e(f, f)(t, \cdot) = (V(t))^2 Q_{\tilde{e}(t, \cdot)}(g, g),$$

that is the collision operator becomes time dependent because of $\tilde{e}$. Finally, the new collision equation is not “autonomous” any more, and this prevent us to use some previous results on this equation as we will do in the following (*e.g.* the Haff's Law). This is one of the reasons why we assumed that the restitution coefficient e is constant.

Now, if f follows the rescaling (2.2), its time derivative becomes

$$(2.4) \quad \partial_t f = (V(t))^{d-1} (T'(t)V(t)\partial_s g + V'(t)\nabla_w \cdot (wg)).$$

Thanks to the relations (2.3) and (2.4), if f is a solution to the homogeneous equation (2.1), the distribution g is a solution to

$$(2.5) \quad T'(t)V(t)\partial_s g + V'(t)\nabla_w \cdot (wg) = \mathcal{E}(f)^{-a}Q_e(g, g).$$

We shall get rid of the term in (2.5) involving a negative power of the energy to obtain a simpler equation, the classical homogeneous Boltzmann equation with an anti-drift term.

To this end, let us assume the rescaling to be “non-linear” by asking it to depend on the energy of the solution itself:

$$\begin{cases} V'(t) = \tau \mathcal{E}(f_t)^{-a}, \\ T'(t)V(t) = V'(t)/\tau, \end{cases}$$

where τ is a non-negative parameter. The functions V and T are not explicit but one can see that they are well defined and agree to (2.2). Indeed, the map $t \mapsto \mathcal{E}(f_t)^{-a}$ is increasing on $[0, T_c)$ thanks to the asymptotic behaviour (1.7), is 1 when $t = 0$ given that the energy of the initial distribution f_{in} has been normalized to this value and tends to infinity when $t \rightarrow T_c$. Moreover, one has $T(t) = \log(V(t))/\tau$.

With such an expression for V and T plugged in (2.5), the distribution g is a solution to the following equation:

$$(2.6) \quad \partial_s g + \tau \nabla_w \cdot (wg) = Q_e(g, g).$$

Actually, this equation is the homogeneous granular equation for inelastic hard spheres with constant restitution coefficient, complemented with an *anti-drift* term. This last term will act like an input of energy and will prevent the blow up of g_s to a Dirac mass when $s \rightarrow \infty$. This equation has been thoroughly studied in the articles [16, 17]. Especially, one has the following result.

Theorem 2.1 ([16]). *Let $g_{in} \in \mathcal{C}_{1,0}$ be an initial datum for (2.6) with a constant restitution coefficient e . The energy of a solution g to the Cauchy problem is such that*

$$(2.7) \quad 0 < c_0 \leq \mathcal{E}(g_s) \leq c_1 < \infty, \forall s \geq 0.$$

Moreover, there exists a self-similar profile $0 \leq G \in L^1_2 \cap \mathcal{C}_{1,0}$:

$$\tau \nabla_w \cdot (wG) - Q_e(G, G) = 0.$$

Remark 3. The above Theorem will be shown in a new proof in Section 4 because the sub-critical case $0 \leq a < 1/2$ includes the granular equation with drift (2.6).

Summarizing, the distribution f defined by

$$\begin{cases} f(t, v) = V(t)^d g(T(t), V(t)v), \\ V'(t) = \tau \mathcal{E}(f_t)^{-a}, \\ T(t) = \log(V(t))/\tau, \end{cases}$$

is solution to the homogeneous granular equation (2.1) for all times as soon as the function g is solution to the drift/collision equation (2.6). Conversely, if f is a solution to (2.1), one can associate a solution g to the rescaled equation (2.6) by setting

$$g(s, w) = e^{-d\tau s} f(V^{-1}(e^{\tau s}), e^{-\tau s} w).$$

Note that the inverse V^{-1} of V is well defined, by the discussion above.

Moreover, using the change of variables $w = V(t)v$, the following relation holds between the energies of f and g :

$$\mathcal{E}(g_{T(t)}) = (V(t))^2 \mathcal{E}(f_t).$$

Then the function V can be written for all $0 \leq t < T_c$ as

$$V(t) = \left(\frac{\mathcal{E}(g_{T(t)})}{\mathcal{E}(f_t)} \right)^{1/2}.$$

Using this expression, the bounds (2.7) of the energy of g and the raw cooling process (1.7), one has another proof that $V(t) \rightarrow \infty$ when $t \rightarrow T_c$. Given that $T(t) = \log(V(t))$, then $T(t) \rightarrow \infty$ when $t \rightarrow T_c$, which are the properties we are looking for.

Finally, if G is a self-similar profile, that is a stationary solution to the equation (2.6) (such a profile exists thanks to Theorem 2.1), we may associate a self-similar solution F (also known as a *blow-up profile*) to the original equation (2.1) by setting

$$F(t, v) = V(t)^d G(V(t)v), \quad \forall (t, v) \in (0, T_c) \times \mathbb{R}^d,$$

which prove the last assertion of Theorem 1.2.

2.2 Two Useful Results

We will show in this Subsection two asymptotic results concerning solutions to the homogeneous granular equation with and without drift term, that we will need in order to prove the cooling process.

Let $h = h_t(v)$ for non-negative t and $v \in \mathbb{R}^d$ be a solution to the homogeneous inelastic Boltzmann equation

$$\partial_t h = Q(h, h),$$

that is the collision equation (2.1) with $a = 0$. One can use the works [1] or [16] to give the cooling process of h , namely the (sub-critical) Haff's Law:

$$(2.8) \quad \frac{m}{(1 + \mu_0 t)^2} \leq \mathcal{E}(h_t) \leq \frac{M}{(1 + \mu_0 t)^2}, \quad \forall t > 0.$$

An upper control of the third order moment of h can be deduced from this law. Let m_l denotes the $2l$ -th order moment of a distribution h , that is

$$m_l = m_l(h) = \int_{\mathbb{R}^d} h(v) |v|^{2l} dv.$$

Lemma 2.1. *Let h be a solution to the equation (2.1) for $a = 0$, with an initial condition $h_{in} \in \mathcal{C}_{1,0}$. There exists a non-negative constant κ such that if $m_{3/2}(h_{in}) \leq \kappa$, then for all $t > 0$,*

$$(2.9) \quad m_{3/2}(h_t) \leq \frac{\kappa}{(1 + \mu_0 t)^3}.$$

Proof. Multiplying the equation (2.1) with $a = 0$ by $|v|^3$, one gets after integration in the velocity space

$$\frac{d}{dt}m_{3/2}(h_t) = \int_{\mathbb{R}^d} Q_e(h_t, h_t)|v|^3 dv.$$

Some more informations are then needed regarding the third order moment of the collision operator. A.V. Bobylev, I.M. Gamba and V.A. Panferov had shown in [4] (Lemma 3) the following estimate:

$$\int_{\mathbb{R}^d} Q_e(h, h)|v|^3 dv \leq -(1 - \gamma)m_2(h) + \gamma S_{3/2}(h),$$

where $0 < \gamma < 1$ and

$$\begin{aligned} S_{3/2}(h) &= \sum_{k=1}^{\lfloor \frac{5}{4} \rfloor} \binom{3/2}{k} (m_{k+1/2}m_{p-k} + m_k m_{p-k+1/2}), \\ &= \frac{3}{2}(m_{3/2}m_{1/2} + m_1^2), \\ &= \frac{3}{2}\mathcal{E}(h)^2, \end{aligned}$$

because $m_1 = \mathcal{E}(h)$ and $m_{1/2} = 0$ by momentum conservation. Besides, by convexity, $m_2 \geq m_{3/2}^{4/3}$. Thus, $m_{3/2}(h_t)$ verifies thanks to the sub-critical Haff's Law (2.8):

$$\begin{aligned} \frac{d}{dt}m_{3/2}(h_t) &\leq -(1 - \gamma)m_{3/2}(h_t)^{4/3} + \frac{3}{2}\gamma\mathcal{E}(h_t)^2, \\ &\leq -(1 - \gamma)m_{3/2}(h_t)^{4/3} + \frac{3}{2}\frac{\gamma M^2}{(1 + \mu_0 t)^4}. \end{aligned}$$

Then, one has

$$\frac{d}{dt} \left(m_{3/2}(h_t) - \frac{\kappa}{(1 + \mu_0 t)^3} \right) \leq -(1 - \gamma)m_{3/2}(h_t)^{4/3} + \frac{3}{2}\frac{\gamma M^2}{(1 + \mu_0 t)^4} + \frac{3\mu_0 \kappa}{(1 + \mu_0 t)^4},$$

where κ is a non-negative constant to be determined thereafter.

If κ is assumed such that $m_{3/2}(h_{in}) \leq \kappa$, then the inequality (2.9) is fulfilled for $t = 0$ and by continuity, the lemma is proved for $t < t_*$ with a positive time t_* . Else, one gets $m_{3/2}(h_{t_*}) = \kappa(1 + \mu_0 t_*)^{-3}$, and the last differential inequality reads

$$\begin{aligned} \frac{d}{dt} \left(m_{3/2}(h_{t_*}) - \frac{\kappa}{(1 + \mu_0 t_*)^3} \right) &\leq -(1 - \gamma) \left(\frac{\kappa}{(1 + \mu_0 t_*)^3} \right)^{4/3} + \frac{3}{2}\frac{\gamma M^2}{(1 + \mu_0 t_*)^4} + \frac{3\mu_0 \kappa}{(1 + \mu_0 t_*)^4}, \\ &= \left(-(1 - \gamma)\kappa^{4/3} + 3\mu_0 \kappa + \frac{3}{2}\gamma M^2 \right) \frac{1}{(1 + \mu_0 t_*)^4}. \end{aligned}$$

If the constant κ is chosen large enough, the right hand side of this inequality is negative, which concludes the proof. $\square$

Thanks to this estimate, one is now able to compute the limit of the time derivative of the energy of a distribution $g = g_s(w)$ solution to (2.6):

Proposition 2.1. *If g is a solution to the homogeneous inelastic collision equation with drift term (2.6), then*

$$\lim_{s \rightarrow +\infty} \frac{d}{ds} \mathcal{E}(g_s) = 0.$$

Proof. It is possible to define a new distribution h by setting as in [18]

$$h(t, v) := V_0(t)^d g(T_0(t), V_0(t)v),$$

in which

$$\begin{cases} V_0(t) = 1 + \mu_0 t, \\ T_0(t) = \log(1 + \mu_0 t). \end{cases}$$

This is also the rescaling (4.2) for the sub-critical case $a = 0$ and h is a solution to the homogeneous inelastic Boltzmann equation

$$\partial_t h = Q(h, h).$$

Thanks to the expression of this rescaling,

$$\mathcal{E}(g_{T_0(t)}) = V_0(t)^2 \mathcal{E}(h_t)$$

Differentiating the relation with respect to time, one gets

$$\frac{d}{dt} \mathcal{E}(g_{T_0(t)}) = A(t) + B(t),$$

where we have set

$$A(t) = 2\mu_0(1 + \mu_0 t) \mathcal{E}(h_t), \quad B(t) = (1 + \mu_0 t)^2 \mathcal{E}'(h_t).$$

Therefore, using once again the sub-critical Haff's Law (2.8), if $t > 0$,

$$\frac{2\mu_0 m}{1 + \mu_0 t} \leq A(t) \leq \frac{2\mu_0 M}{1 + \mu_0 t}$$

and then $A(t) \rightarrow 0$ if $t \rightarrow \infty$. Moreover, by the weak expression of $Q(h, h)$, the equation (2.1) with $a = 0$ and the hypothesis **H**₂ about the bounds of the cross section,

$$\begin{aligned} B(t) &= (1 + \mu_0 t)^2 \int_{\mathbb{R}^d} Q(h, h) |v|^2 dv, \\ &\leq -\beta_e (1 + \mu_0 t)^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} h h_* |v - v_*|^3 dv dv_*. \end{aligned}$$

Especially, $B(t) \leq 0$. On the other hand, one has by Fubini Theorem and mass conservation that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} h h_* |v - v_*|^3 dv dv_* \leq 8m_{3/2}(h).$$

Then, using the inequality (2.9) of Lemma 2.1 and the weak expression of $Q(h, h)$, B is such that:

$$B(t) \geq -\frac{\kappa \beta'_e}{1 + \mu_0 t},$$

and then $B(t) \rightarrow 0$ if $t \rightarrow \infty$. The conclusion of the Proposition is finally given by the fact that the map T_0 is one-to-one. $\square$

We are now ready to prove Theorem 1.2.

2.3 Anomalous Cooling Process

We will prove in this Subsection the cooling process of an anomalous gas for non-negative a , that is the points (i), (ii) and (iii) of Theorem 1.2. Both upper and lower bounds for the energy of f will be obtained together, thanks to the usual Haff's Law for classical granular gases, applied on the equation (2.6). The limit of the time derivative of the energy of a solution to the rescaled Boltzmann equation will be computed to this end.

For the sake of simplicity, let us denote for $t < T_c$

$$E(t) := \mathcal{E}(f_t) \text{ and } \bar{E}(t) := \mathcal{E}(g_{T(t)}),$$

and set $\tau = 1$. We have already seen that $V(t) = \bar{E}(t)^{1/2} E(t)^{-1/2}$ and that

$$\begin{aligned} E(t)^{-a} &= V'(t), \\ &= \frac{\bar{E}'(t)}{2(E(t)\bar{E}(t))^{1/2}} - \frac{\bar{E}(t)^{1/2} E'(t)}{2E(t)^{3/2}}. \end{aligned}$$

Thanks to the expression of $\bar{E}$ and a chain rule, one has on the one hand

$$\begin{aligned} \bar{E}'(t) &= \frac{d}{dt} \left(\int_{\mathbb{R}^d} g_{T(t)}(w) |w|^2 dw \right), \\ &= T'(t) \chi(t), \text{ with } \chi(t) := \left(\frac{d}{ds} \mathcal{E}(g_s) \right) \Big|_{s=T(t)}. \end{aligned}$$

On the other hand, the time derivative of T is given by

$$T'(t) = \frac{V'(t)}{V(t)} = \frac{E(t)^{-a+1/2}}{\bar{E}(t)^{1/2}}.$$

Finally, gathering the last three equalities and dividing by $E(t)^{-a}$, it comes that

$$(2.10) \quad \frac{\chi(t)}{2\bar{E}(t)} - \frac{1}{2} \bar{E}(t)^{1/2} E'(t) E(t)^{a-3/2} = 1.$$

Following the result of Proposition 2.1, one has $\chi(t) \rightarrow 0$ when $t \rightarrow T_c$, because $T(t) \rightarrow \infty$ when $t \rightarrow T_c$. Thus, thanks to the uniform bounds (2.7) on $\bar{E}$,

$$\lim_{t \rightarrow T_c} \frac{\chi(t)}{2\bar{E}(t)} = 0.$$

Therefore, using the equality (2.10) and the bounds of $\bar{E}$ once more, there exists a time $t_0 > 0$ and two constant D_1 and D_2 such that if $t_0 < t < T_c$,

$$-\infty < -D_1 \leq E'(t) E(t)^{a-3/2} \leq -D_2 < 0.$$

Integrating the two sides of this inequality with successively $a < 1/2$, $a = 1/2$ and $a > 1/2$, one has shown respectively the points (i), (ii) and (iii) of Theorem 1.2 concerning cooling process. Let us now show the result concerning self-similar profiles of equation (1.1).

3 Proof of Theorem 1.3

We will study in this Section the uniqueness of the self-similar profiles of the granular Boltzmann equation with drift (2.6), together with their attractiveness. Using the cooling process found in the previous Subsection, this will allow us to state a Theorem concerning the convergence of the solutions to (1.1) towards the self-similar solutions. Let us fix $a \geq 0$ and denote the constant restitution coefficient by e .

We will use a Theorem concerning the convergence toward self-similar profiles for small inelasticity in the equation (2.6), which has been shown in [17]. For this, we have to set

$$\tau = \tau_e := 1 - e,$$

in order to balance the dissipation of kinetic energy by the drift. Thanks to this scaling, one has:

Theorem 3.1 ([17], Theorem 1.1.iv). *There exists a constructive $e_* \in (0, 1)$ such that for all $e \in [e_*, 1)$, the self-similar profile G_e from Theorem 2.1 is unique and globally attractive on bounded subsets of L_3^1 : for any $M > 0$, there exists $\eta \in (0, 1)$ such that if*

$$g_{in} \in L_3^1 \cap \mathcal{C}_{1,0}, \quad \|g_{in}\|_{L_3^1} \leq M_0,$$

the solution g to the granular equation with drift (2.6) satisfies

$$(3.1) \quad \|g_t - G_e\|_{L_2^1} \leq e^{-(1-\eta)\nu_e t},$$

where $\nu_e = \tau_e + \mathcal{O}(\tau_e^2)$.

Let us show thanks to this Theorem the trend to the self-similar solution of our problem. We have already seen in Subsection 2.1 that if g is a solution to (2.6) then f is a solution to (1.1) with $f(t, v) = V(t)^d g(T(t), V(t)v)$, where

$$(3.2) \quad \begin{cases} V(t) = \left(\frac{\mathcal{E}(g_{T(t)})}{\mathcal{E}(f_t)} \right)^{1/2}, \\ T(t) = \frac{\log(V(t))}{\tau_e}. \end{cases}$$

Thus, if G_e is the unique self-similar profile of the equation (2.6), one can find a self-similar solution to the equation (1.1) by setting $F_e(t, v) = V(t)^d G_e(V(t)v)$. The uniqueness of this solution up to a translation of the time can be shown as in [17] to prove the first point of Theorem 1.3.

Moreover, the transformation $w \rightarrow V(t)v$ and the rate of convergence (3.1) give

$$\begin{aligned} \|f(t, \cdot) - F_e(t, \cdot)\|_{L^1} &= V(t)^d \int_{\mathbb{R}^d} |g(T(t), V(t)v) - G_e(V(t)v)| dv, \\ &\leq \|g_{T(t)} - G_e\|_{L_2^1}, \\ &\leq e^{-(1-\eta)\nu_e T(t)}. \end{aligned}$$

Besides, thanks to the expression (3.2) of the self-similar variables and the choice of $\tau_e = 1 - e$,

$$\nu_e T(t) = \log(V(t)) + \mathcal{O}(1 - e).$$

The positive lower bound of $\mathcal{E}(g_{T(t)})$ of Theorem 2.1 together with the last inequality yields

$$\|f(t, \cdot) - F_e(t, \cdot)\|_{L^1} \leq C\mathcal{E}(f_t)^{\mu_e/2},$$

where $\mu_e = 1 - \eta + \mathcal{O}(1 - e)$ and C is a non-negative constant. Finally, using the cooling process of Theorem 1.2 that we have shown in the last Subsection, we can conclude the proof of Theorem 1.3, that is the attractiveness of the self-similar solution to the equation (1.1), together with the rate of convergence depending on the (weak) inelasticity $1 - e$ and a .

4 Alternative Proof in the Sub-critical Case: the Classical Self-similar Variables

In this section, we are going to work with another set of self-similar variables in order to give a partial (sub-critical case) but alternative and self-contained proof of Theorems 1.2 and 1.3. We shall derive the cooling process (or Haff's Law) and existence of self-similar profiles, with constructive constants. The cooling part is partly inspired of the article [1] and the core of the proof is based on a new estimate involving the third order moment of the rescaled distribution g , given in Lemma 4.1. We were not able to extend this Lemma to the critical and super-critical cases, and then to give their cooling process with this proof. The part concerning existence of self-similar profile is partly based on the article [16].

We will present in the following Subsection the classical rescaling, allowing to prevent the blow up for $0 \leq a \leq 1/2$.

4.1 Classical Self-similar Variables

If $a \neq 1/2$, the differential inequality (1.6) gives the following control of the energy of f :

$$(4.1) \quad 0 \leq \mathcal{E}(f_t) \leq \left(\mathcal{E}(f_{in})^{a-\frac{1}{2}} - \frac{\beta_e}{2\alpha} t \right)^{2\alpha}, \quad \forall t < T_c,$$

where $\alpha = \frac{1}{2a-1}$.

The study of the cooling process will be conditioned by the sign of α . We shall focus on the rescaling of f for negative α . Indeed, if one has $0 \leq a < 1/2$, α is negative and we know from [18] that $T_c = \infty$: the gas is sub-critical and the inequality (4.1) gives one half of the Haff's Law, namely the upper bound which is its easy part. One needs to rescale the equation (2.1) so as to get some new informations on $\mathcal{E}(f)$ to obtain the lower bound.

The rescaling of the distribution f will still be written as

$$(4.2) \quad \begin{cases} f(t, v) = V_a(t)^d g(T_a(t), V_a(t)v), \\ V_a(0) = 1, \quad T_a(0) = 0. \end{cases}$$

We shall determine as in Subsection 2.1 the functions T_a and V_a and derive the equation followed by the distribution $g = g_s(w)$ with $s = T_a(t)$ and $w = V_a(t)v$. We have seen

thanks to (2.3) and (2.4) that if f is a solution to the homogeneous equation (2.1), the function g is a solution to

$$(4.3) \quad T'_a(t)V_a(t)\partial_s g + V'_a(t)\nabla_w \cdot (wg) = (V_a(t))^{2a}\mathcal{E}(g)^{-a}Q_e(g, g).$$

We also used for deriving this equation the relation between the energy of f and the one of g , namely that $\mathcal{E}(g_{T_a(t)}) = (V_a(t))^2\mathcal{E}(f_t)$.

We want the rescaling to be consistent with the steady-state solutions (the self-similar profiles) to the homogeneous granular equation. If $G = G(v)$ is such a profile, then $\mathcal{E}(G)$ is a constant, and $V'_a(t) = C(V_a(t))^{2a}$ for a non-negative constant C . This last expression exactly integrates as

$$V_a(t) = (1 + \mu_a t)^{-\alpha},$$

for a non-negative constant μ_a depending on the initial condition $g_{in} = f_{in}$. Moreover, for the sake of simplicity one wants the equation on g to be autonomous. The easier way to do this is to assume that

$$T'_a(t)\frac{V_a(t)}{V'_a(t)} = 1,$$

that is $T_a(t) = -\alpha \log(1 + \mu_a t)$. Moreover, using the expressions of α and V_a ,

$$\frac{(V_a(t))^{2a}}{V'_a(t)} = -\frac{1}{\alpha\mu_a}.$$

Summarizing this and using equation (4.3), for any *self-similar profile* G solution to the stationary equation

$$(4.4) \quad \nabla_v \cdot (vG) - K_a\mathcal{E}(G)^{-a}Q_e(G, G) = 0,$$

where $K_a = -\frac{1}{\alpha\mu_a} > 0$, we may associate a *self-similar solution* F to the original equation (2.1) by setting

$$F(t, v) = (1 + \mu_a t)^{-d\alpha}G(-(1 + \mu_a t)^{-\alpha}v).$$

Besides, G is a *stationary solution* to the following rescaled equation

$$(4.5) \quad \partial_s g + \nabla_w \cdot (wg) = K_a\mathcal{E}(g)^{-a}Q_e(g, g).$$

More generally, the distribution

$$f(t, v) := (1 + \mu_a t)^{-d\alpha}g(-\alpha \log(1 + \mu_a t), (1 + \mu_a t)^{-\alpha}v)$$

is solution to the equation (2.1) for all times as soon as the function g is solution to (4.5). Similarly, if f is a solution to (2.1), one can associate a solution g to the rescaled equation by defining g as

$$g(s, w) = e^{-ds}f(\mu_a^{-1}(e^{-s/\alpha} - 1), e^{-s}w).$$

According to the expression of the rescaling (4.2), we have shown the following result:

Proposition 4.1. *The distribution f is a solution to (2.1) as soon as one can write*

$$\begin{cases} f(t, v) = V_a(t)^d g(T_a(t), V_a(t)v), \\ V_a(t) = (1 + \mu_a t)^{-\alpha}, \\ T_a(t) = \log V(t), \end{cases}$$

where g is a solution to (4.5). The converse also holds true. Moreover, the following relation holds for any non-negative integer k

$$\forall t \geq 0, \begin{cases} \|f_t\|_{L^1(|\cdot|^k)} = (1 + \mu_a t)^{k\alpha} \|g_{-\alpha \log(1+\mu_a t)}\|_{L^1(|\cdot|^k)}, \\ \|g_s\|_{L^1(|\cdot|^k)} = e^{ks} \|f_{\mu_a^{-1}(\exp(-s/\alpha)-1)}\|_{L^1(|\cdot|^k)}. \end{cases}$$

Thus, the knowledge of the behaviour of the 2^{nd} order moment of g (its energy), which seems easier to obtain because of the expected lack of explosion, will give us the cooling law for f by using the relation

$$(4.6) \quad \mathcal{E}(f_t) = \frac{\mathcal{E}(g_{-\alpha \log(1+\mu_a t)})}{(1 + \mu_a t)^{-2\alpha}},$$

where $\alpha = 1/(2a - 1)$ is negative.

Remark 4. The non-linear rescaling of Section 2 can give us a similar relation as (4.6), but the lack of explicit representation of the self-similar variables in this case prevents us to find the cooling law just by using the bounds on the energy of the g .

Now that the classical self-similar variables have been introduced, we shall be able to prove the part of Theorems 1.2 and 1.3 concerning the sub-critical case, using estimates on the moments in self-similar variables.

4.2 Sub-critical Cooling Process

In this subsection, we are going to study the energy of a distribution g in classical self-similar variables for the sub-critical case, solution for a non-negative constant $a < 1/2$ to

$$(4.7) \quad \partial_s g + \nabla_w \cdot (wg) = \mathcal{E}(g)^{-a} Q_e(g, g).$$

The constant K_a has been taken equal to one in this last equation, which is made possible by assuming for example that it appears in the definition of the collisional cross section b . According to the Proposition 4.1 we shall give some positive upper and lower bounds for $\mathcal{E}(g)$, in order to yield the cooling law of f .

Upper Bound Let $g \in \mathcal{C}(0, +\infty; L_2^1)$ be a solution to equation (4.7), whose initial condition $g_{in} \in \mathcal{C}_{1,0}$. Using the weak form of the collision operator, it is easy to show that these quantities are conserved with respect to time. Multiplying the rescaled equation by $|v|^2$, integrating in velocity and applying successively the flux-divergence formula, Jensen's and Hölder's inequalities, it comes as in Subsection 1.2 that:

$$\begin{aligned} \frac{d}{ds} \mathcal{E}(g_s) &\leq -\beta_e \mathcal{E}(g_s)^{-a} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g_s(w) g_s(w_*) |w - w_*|^3 dw dw_* + \int_{\mathbb{R}^d} g_s(w) w \cdot \nabla |w|^2 dw, \\ &\leq -\beta_e \mathcal{E}(g_s)^{-a} \int_{\mathbb{R}^d} g_s(w) |w|^3 dw + 2 \mathcal{E}(g_s), \\ &\leq \beta_e \mathcal{E}(g_s) \left(\frac{2}{\beta_e} - \mathcal{E}(g_s)^{-a+\frac{1}{2}} \right). \end{aligned}$$

From this later differential inequality, and according to the fact that $-a + 1/2 = -1/2\alpha$ is positive, if the energy is such that

$$\mathcal{E}(g_s) \geq \left(\frac{2}{\beta_e}\right)^{-2\alpha},$$

then $\mathcal{E}(g_s) \leq \mathcal{E}(g_{in})$. One has therefore shown a maximum principle which allows to write the uniform upper bound for $\mathcal{E}(g_s)$:

$$(4.8) \quad \sup_{s \geq 0} \mathcal{E}(g_s) \leq \max \left\{ \left(\frac{2}{\beta_e}\right)^{-2\alpha}, \mathcal{E}(g_{in}) \right\} =: M.$$

Lower Bound Some estimates on the third order moment of g will be needed in this section. Therefore, g_{in} will be taken in $L^1_3 \cap \mathcal{C}_{1,0}$. For a sake of clarity, we will drop the s -dependency on g when it is not absolutely necessary. Thanks to the Fubini Theorem for non-negative functions and to conservation of mass, one has

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} gg_* |w - w_*|^3 dw dw_* &\leq 2^3 \int_{\mathbb{R}^d} g(w_*) dw_* \int_{\mathbb{R}^d} g(w) |w|^3 dw \\ &= 8m_{3/2}, \end{aligned}$$

where the quantity m_k for non-negative k denotes the $2k$ -th order moment of g . Multiplying the rescaled equation by $|v|^2$ and integrating in velocity again, a new differential inequality involving the energy and the third order moment of g is found, namely

$$(4.9) \quad \frac{d}{ds} \mathcal{E}(g) \geq 2\mathcal{E}(g) - \beta'_e m_{3/2} \mathcal{E}(g)^{-a},$$

where $\beta'_e = 2\beta_2(1 - e^2) |\mathbb{S}^{d-2}|$ thanks to Hypothesis **H₂** on the cross section. A control of $m_{3/2}$ by a certain power (at least greater than $1 + a$) of $\mathcal{E}(g)$ will then be needed in order to give a lower bound to the energy.

Lemma 4.1. *Let g be a solution to the equation (4.7) for $0 \leq a < 1/2$, with an initial condition $g_{in} \in L^1_3 \cap \mathcal{C}_{1,0}$. There exists two non-negative constants κ_a and K such that if $\mathcal{E}(g_{in}) \leq K$ and*

$$m_{3/2}(g_{in}) \leq \kappa_a \mathcal{E}(g_{in})^{1+a+\eta},$$

then the following inequality holds for all η in the range $(-a + \frac{1}{2}, 1 - 2a)$:

$$(4.10) \quad m_{3/2}(g_s) \leq \kappa_a \mathcal{E}(g_s)^{1+a+\eta}, \quad \forall s \geq 0.$$

Proof. Multiplying the rescaled equation (4.7) by $|w|^3$, one gets after integration in velocity space and by the flux-divergence formula

$$\frac{d}{ds} m_{3/2}(g_s) = \mathcal{E}(g)^{-a} \int_{\mathbb{R}^d} Q_e(g, g) |w|^3 dw + 3m_{3/2}.$$

Using again the estimates of [4], the third order moment of $Q_e(g, g)$ is such that

$$\int_{\mathbb{R}^d} Q_e(g, g) |w|^3 dw \leq -(1 - \gamma)m_2 + \gamma S_{3/2},$$

where $0 < \gamma < 1$ and

$$S_{3/2} = \frac{3}{2} \mathcal{E}(g)^2,$$

because $m_1 = \mathcal{E}(g)$ and $m_{1/2} = 0$ by momentum conservation. Moreover, applying the Jensen's inequality to the function $x \rightarrow x^{4/3}$ and to the probability measure $g(w)dw$, the fourth order moment of g is such that

$$m_2 = \int_{\mathbb{R}^d} (|w|^3)^{4/3} g(w) dw \geq m_{3/2}^{4/3}.$$

Thus, one can have a differential inequality for $m_{3/2}$, namely

$$\frac{d}{ds} m_{3/2}(g_s) \leq \mathcal{E}(g)^{-a} \left(-(1-\gamma)m_{3/2}^{4/3} + \frac{3}{2}\gamma\mathcal{E}(g)^2 \right) + 3m_{3/2}.$$

Gathering this latter inequality with (4.9) and differentiating in time, it comes that

$$(4.11) \quad \begin{aligned} \frac{d}{ds} U(s) \leq & -(1-\gamma)m_{3/2}^{4/3}\mathcal{E}(g)^{-a} + \frac{3}{2}\gamma\mathcal{E}(g)^{2-a} - 2\kappa_a(1+a+\eta)\mathcal{E}(g)^{1+a+\eta} \\ & + m_{3/2}(3 + \beta'_e \kappa_a(1+a+\eta)\mathcal{E}(g)^\eta), \end{aligned}$$

where κ_a is a non-negative constant to be determined thereafter and

$$U(s) := m_{3/2}(g_s) - \kappa_a \mathcal{E}(g_s)^{1+a+\eta}.$$

If $0 < \eta < 1-2a$, then $1+a+\eta < 2-a$ (because $0 \leq a < 1/2$ yields $2-a > 3/2 > 1+a$). Thus, if the energy of g is such that

$$\mathcal{E}(g) < \left(\frac{4\kappa_a(1+a+\eta)}{3\gamma N_0} \right)^{\frac{3}{1-2a-\eta}},$$

for a constant $N_0 > 1$, then using monotonicity argument,

$$\frac{3}{2}\gamma\mathcal{E}(g)^{2-a} - 2\kappa_a(1+a+\eta)\mathcal{E}(g)^{1+a+\eta} \leq -2\kappa_a(1+a+\eta)\frac{N_0-1}{N_0}\mathcal{E}(g)^{1+a+\eta}.$$

Besides, thanks to the Jensen's inequality and the conservation of mass,

$$m_{3/2} \geq \mathcal{E}(g)^{3/2}.$$

Then, the inequality (4.11) becomes

$$\begin{aligned} \frac{d}{ds} U(s) \leq & -(1-\gamma)\mathcal{E}(g)^{2-a} - 2\kappa_a(1+a+\eta)\frac{N_0-1}{N_0}\mathcal{E}(g)^{1+a+\eta} \\ & + m_{3/2}(3 + \beta'_e \kappa_a(1+a+\eta)\mathcal{E}(g)^\eta). \end{aligned}$$

Let us suppose that the positive constant κ_a is such that $m_{3/2}(g_{in}) \leq \kappa_a \mathcal{E}(g_{in})^{1+a+\eta}$. If this inequality is strict, then for $s < s_*$ with a positive time s_* , the lemma is proved. Else, one gets $m_{3/2}(g_{s_*}) = \kappa_a \mathcal{E}(g_{s_*})^{1+a+\eta}$, and the last differential inequality now reads

$$\begin{aligned} \frac{d}{ds} U(s_*) \leq & -(1-\gamma)\mathcal{E}(g_{s_*})^{2-a} - 2\kappa_a(1+a+\eta)\frac{N_0-1}{N_0}\mathcal{E}(g_{s_*})^{1+a+\eta} \\ & + \kappa_a \mathcal{E}(g_{s_*})^{1+a+\eta} (3 + \beta'_e \kappa_a(1+a+\eta)\mathcal{E}(g_{s_*})^\eta), \\ \leq & -(1-\gamma)\mathcal{E}(g_{s_*})^{2-a} + \kappa_a \mathcal{E}(g_{s_*})^{1+a+\eta} \left(3 + (1+a+\eta) \left(-2 + N_1 + \frac{2}{N_0} \right) \right), \end{aligned}$$

if the energy at time s_* is such that

$$\mathcal{E}(g_{s_*}) < \min \left\{ \left(\frac{4\kappa_a(1+a+\eta)}{3\gamma N_0} \right)^{\frac{3}{1-2a-\eta}}, \left(\frac{N_1}{\beta'_e \kappa_a} \right)^{1/\eta} \right\} =: K(N_0, N_1),$$

for a positive constant N_1 .

Provided that $a < 1/2$, then $1+a+\eta > 3/2$ if $\eta > -a+1/2$. The conditions that η has to fulfil are now sharpened:

$$-a+1/2 < \eta < 1-2a.$$

Thus $3-2(1+a+\eta) = -\Lambda$ for a non-negative constant Λ and the differential inequality on $U(s_*)$ is now

$$\begin{aligned} \frac{d}{ds} U(s_*) &\leq - (1-\gamma) \mathcal{E}(g_{s_*})^{2-a} + \kappa_a \mathcal{E}(g_{s_*})^{1+a+\eta} \left(-\Lambda + (1+a+\eta) \left(N_1 + \frac{2}{N_0} \right) \right), \\ &< 0, \end{aligned}$$

where N_0 and N_1 are respectively large and small enough to have

$$N_1 + \frac{2}{N_0} < \frac{\Lambda}{(1+a+\eta)}.$$

One has shown that for the times $s > s_*$, the quantity $U(s) = m_{3/2}(g_s) - \kappa_a \mathcal{E}(g_s)^{1+a+\eta}$ is non-increasing (and is equal to 0 in s_*). Taking $K = K(N_0, N_1)$ concludes the proof. $\square$

Using the result (4.10) of the Lemma 4.1 in the inequality (4.9), one finds that if $0 \leq a < 1/2$

$$\frac{d}{ds} \mathcal{E}(g_s) \geq \nu_a \mathcal{E}(g_s) \left(\frac{2}{\nu_a} - \mathcal{E}(g_s)^\eta \right),$$

in which $\nu_a := \beta'_e \kappa_a$. The lower bound comes again using a maximum principle which yields:

$$(4.12) \quad \sup_{s \geq 0} \mathcal{E}(g_s) \geq \min \left\{ \left(\frac{2}{\nu_a} \right)^{1/\eta}, \mathcal{E}(g_{in}) \right\} =: m > 0,$$

for all $\eta \in (-a + \frac{1}{2}, 1-2a)$.

Gathering the uniform bounds (4.8) and (4.12) on the energy of a solution g to the rescaled equation (4.7), and the relation (4.6) between the energy of a solution f to the classical homogeneous granular equation (2.1) and the one of g , one has shown the point (i) of Theorem 1.2, that is the one concerning cooling in infinite time.

We are now going to use these uniform bounds to prove the existence of self-similar profiles in classical self-similar variables, that is the existence of a distribution G solution to (4.4) with $K_a = 0$ and $0 \leq a < 1/2$:

$$\nabla_w \cdot (wG) - \mathcal{E}(G)^{-a} Q_e(G, G) = 0.$$

4.3 Stability Estimate

Let us assume that $0 \leq a < 1/2$ and that $\mathbf{H}_1$ and $\mathbf{H}_2$ hold.

Lemma 4.2. *Let $0 \leq g_{in}, h_{in} \in L^1_3$ be the initial condition of the Cauchy problem for the rescaled equation (4.7), and $g, h \in C(0, T; L^1_2) \cap L^\infty(0, T; L^1_3)$ the two associated solutions. Then, there exists a constant C_{stab} depending on a, g_{in} and h_{in} such that*

$$\|g_s - h_s\|_{L^1_2} \leq \|g_{in} - h_{in}\| e^{C_{stab}s}.$$

Proof. Multiplying the equation satisfied by $g - h$ by $\varphi(s, w) := \text{sgn}(g_s(w) - h_s(w))(1 + |w|^2)$ and integrating in velocity, one gets using the chain rule

$$\frac{d}{ds} \|g_s - h_s\|_{L^1_2} = - \int_{\mathbb{R}^d} \nabla_w \cdot (w(g_s(w) - h_s(w))) \varphi(s, w) dw + I(s),$$

where the quantity I is given by

$$\begin{aligned} I &:= \int_{\mathbb{R}^d} (\mathcal{E}(g)^{-a} Q_e(g, g) - \mathcal{E}(h)^{-a} Q_e(h, h)) \varphi(w) dw, \\ &\leq \int_{\mathbb{R}^d} (m^{-a} Q_e(g, g) - M^{-a} Q_e(h, h)) \varphi(w) dw, \end{aligned}$$

thanks to the uniform bounds of the energy obtained in Subsection 4.2. Therefore, using the weak form (1.4) of the collision operator and the bounds of the cross section, we have on the one hand by symmetry and thanks to the change of variable $(w, w_*) \rightarrow (w_*, w)$,

$$\begin{aligned} I &\leq C_1 \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |w - w_*| ((g - h)h_* + g(g_* - h_*)) (\varphi' + \varphi'_* - \varphi - \varphi_*) dw dw dw_*, \\ &\leq C_1 \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} |w - w_*| (g - h)(g_* + h_*) (\varphi' + \varphi'_* - \varphi - \varphi_*) dw dw dw_*, \\ &\leq C_2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} |w - w_*| |g - h| (g_* + h_*) (1 + |w_*|^2) dw dw_*. \end{aligned}$$

On the other hand,

$$(4.13) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} |w - w_*| \psi_1(w) \psi_2(w_*) (1 + |w_*|^2) dw dw_* \leq \|\psi_1\|_{L^1_2} \|\psi_2\|_{L^1_3},$$

which is equivalent to the following inequality:

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|w - w_*|}{(1 + |w|^2)(1 + |w_*|^2)^{1/2}} d\mu(w, w_*) \leq 1,$$

where μ is the probability distribution defined by

$$\mu(dw, dw_*) := \frac{1}{\|\psi_1\|_{L^1_2} \|\psi_2\|_{L^1_3}} \psi_1(w) \psi_2(w_*) (1 + |w|^2) (1 + |w_*|^2)^{3/2} dw dw_*.$$

To prove (4.13), it then remains to use that

$$|w - w_*| \leq (1 + |w|^2) (1 + |w_*|^2)^{1/2},$$

which can be shown by taking the difference of the square of its two members. Finally, setting $\psi_1 = g - h$ and $\psi_2 = g + h$ in (4.13) yields

$$I(s) \leq C_2 \|g_s - h_s\|_{L_2^1} \|g_s + h_s\|_{L_3^1}.$$

Coming back to the drift term, one has thanks to the divergence formula,

$$\begin{aligned} - \int_{\mathbb{R}^d} \nabla_w \cdot (w(g(w) - h(w))) \varphi(w) dv &= -d \|g - h\|_{L_2^1} + \int_{\mathbb{R}^d} |g(w) - h(w)| \nabla_w \cdot w(1 + |w|^2) dw, \\ &= (d + 2) \|g - h\|_{L_2^1}. \end{aligned}$$

Gathering these results, one gets

$$\frac{d}{ds} \|g_s - h_s\|_{L_2^1} \leq C_{stab} \|g_s - h_s\|_{L_2^1},$$

with $C_{stab} := C_2 \sup_{s \geq 0} \|g_s + h_s\|_{L_3^1} + d + 2$ and the conclusion comes thanks to the Gronwall Lemma. $\square$

We are now ready to prove the existence of self-similar profiles in the sub-critical case.

4.4 Existence of Self-similar Profiles in the Sub-critical Case

We will prove the existence of a self-similar profile for the equation (4.5) thanks to the following abstract Theorem, seen in the paper [13] of M. Escobedo, S. Mischler, R. Ricard or in another form in the one [14] of I. Gamba, V. Panferov and C. Villani. This is a simple consequence of the Schauder-Tychonoff fixed point Theorem (*e.g.* the textbook [11] of Edwards, pp. 161-163).

Theorem 4.1. *Let $\mathcal{Y}$ be a Banach space, and $(T_s)_{s \geq 0}$ a continuous semi-group on $\mathcal{Y}$ such that:*

- (i) *there exists a non-empty convex subset $\mathcal{Z}$ of $\mathcal{Y}$ weakly (sequentially) compact, invariant under the action of T_s , for all $s \geq 0$;*
- (ii) *T_s is weakly (sequentially) continuous on $\mathcal{Z}$, for all $s \geq 0$.*

Then, there exists $z_0 \in \mathcal{Z}$ such that $T_s z_0 = z_0$, for all $s \geq 0$.

We shall apply Theorem 4.1 to the evolution semi-group T_s of the equation (2.1) with $a < 1/2$, on the Banach space $\mathcal{Y} := L_2^1$. We have to check some properties of this semi-group.

First of all, $(T_s)_{s \geq 0}$ is continuous on $\mathcal{Y}$ thanks to the study of the Cauchy problem (Theorem 1.1). Moreover, the stability estimate of the Lemma 4.2 means that T_s is (strongly) continuous on each bounded subset of $\mathcal{Z}$.

Besides, if one sets

$$\mathcal{Z} := \left\{ 0 \leq f \in \mathcal{Y} \cap \mathcal{C}_{1,0}; \|f\|_{L_3^1} + \|f\|_{L^p} \leq M_0 \right\}$$

for $p > 1$ and $M_0 > 0$, then thanks again to Theorem 1.1 and to the uniform propagation of L^p norms, $T_s \mathcal{Z} \subset \mathcal{Z}$, for all $s \geq 0$ (if one sets M sufficiently large). The assertion

concerning L^p norms can be shown by doing the same proof than the one of Subsection 3.3 of [16], thanks to the energy bounds of g found in Subsection 4.2.

Then, being bounded for the L^p norm, and then uniformly integrable, the set $\mathcal{Z}$ is weakly compact in $\mathcal{Y}$, thanks to the Dunford-Pettis compactness criterion (*e.g.* the textbook [10] of C. Dellacherie and Meyer, p. 27).

Finally, one can apply Theorem 4.1 to prove the existence of a non-negative distribution $G \in L^1_3 \cap \mathcal{C}_{1,0} \cap L^p$ solution to the stationary equation (4.4), that is a self-similar profile for the granular equation. This ends the alternative proof of Theorem 1.3.

Acknowledgement

The author would like to thanks Francis Filbet and Clément Mouhot for the fruitful discussions and comments on this article.

References

- [1] R. ALONSO AND B. LODS, *Free cooling and high-energy tails of granular gases with variable restitution coefficient*, SIAM J. Math. Anal., 42 (2010), pp. 2499–2538.
- [2] A. BOBYLEV, J. CARRILLO, AND I. GAMBA, *On some properties of kinetic and hydrodynamic equations for inelastic interactions*, J. Stat. Phys., 98 (2000), pp. 743–773.
- [3] ———, *Erratum on: “On some properties of kinetic and hydrodynamic equations for inelastic interactions”*, J. Stat. Phys., 103 (2001), pp. 1137–1138.
- [4] A. BOBYLEV, I. GAMBA, AND V. PANFEROV, *Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions*, J. Stat. Phys., 116 (2004), pp. 1651–1682.
- [5] A. V. BOBYLEV AND C. CERCIGNANI, *Moment equations for a granular material in a thermal bath*, J. Stat. Phys., 106 (2002), pp. 547–567.
- [6] A. V. BOBYLEV, C. CERCIGNANI, AND I. M. GAMBA, *Generalized kinetic Maxwell type models of granular gases*, in Mathematical models of granular matter, vol. 1937 of Lecture Notes in Math., Springer, Berlin, 2008, pp. 23–57.
- [7] A. V. BOBYLEV, C. CERCIGNANI, AND G. TOSCANI, *Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials*, J. Stat. Phys., 111 (2003), pp. 403–417.
- [8] N. BRILLIANTOV AND T. PÖSCHEL, *Kinetic Theory of Granular Gases*, Oxford University Press, USA, 2004.
- [9] C. CERCIGNANI, R. ILLNER, AND C. STOICA, *On diffusive equilibria in generalized kinetic theory*, J. Stat. Phys., 105 (2001), pp. 337–352.
- [10] C. DELLACHERIE AND P. MEYER, *Probabilities and Potential*, North-Holland, 1978.
- [11] R. E. EDWARDS, *Functional Analysis. Theory and Applications*, Holt, Rinehart and Winston, New York, 1965.

- [12] M. H. ERNST AND R. BRITO, *Scaling solutions of inelastic Boltzmann equations with over-populated high energy tails*, J. Stat. Phys., 109 (2002), pp. 407–432. Special issue dedicated to J. Robert Dorfman on the occasion of his sixty-fifth birthday.
- [13] M. ESCOBEDO, S. MISCHLER, AND M. RODRIGUEZ RICARD, *On self-similarity and stationary problem for fragmentation and coagulation models*, in Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, vol. 22, Elsevier, 2005, pp. 99–125.
- [14] I. GAMBA, V. PANFEROV, AND C. VILLANI, *On the Boltzmann equation for diffusively excited granular media*, Commun. Math. Phys., 246 (2004), pp. 503–541.
- [15] P. HAFF, *Grain flow as a fluid-mechanical phenomenon*, J. Fluid Mech., 134 (1983), pp. 401–30.
- [16] S. MISCHLER AND C. MOUHOT, *Cooling process for inelastic Boltzmann equations for hard spheres, Part II: Self-similar solutions and tail behavior*, J. Stat. Phys., 124 (2006), pp. 703–746.
- [17] ———, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*, Commun. Math. Phys., 288 (2009), pp. 431–502.
- [18] S. MISCHLER, C. MOUHOT, AND M. RICARD, *Cooling process for inelastic Boltzmann equations for hard spheres, Part I: The Cauchy problem*, J. Stat. Phys., 124 (2006), pp. 655–702.
- [19] T. SCHWAGER AND T. PÖSCHEL, *Coefficient of normal restitution of viscous particles and cooling rate of granular gases*, Phys. Rev. E, 57 (1998), pp. 650–654.
- [20] K. SHIDA AND T. KAWAI, *Cluster formation by inelastically colliding particles in one-dimensional space*, Phys. A., 162 (1989), pp. 145 – 160.
- [21] C. VILLANI, *Mathematics of Granular Materials*, J. Stat. Phys., 124 (2006), pp. 781–822.