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LANDAU-LIFSCHITZ-GILBERT EQUATION WITH APPLIED ELECTRIC CURRENT

Gaël Bonithon

University Bordeaux 1
351 cours de la Libération
33405 TALENCE cedex France

Abstract. In this paper, we are concerned with a model of electric current effect in ferromagnetic materials, that is Landau-Lifschitz equation adding a transport term. We prove classical existence theorem in the general three dimensional case, and we justify a one dimensional approximation for wich we have the explicit behavior of the magnetisation.

1. **Introduction.** Landau-Lifschitz equation describe spontaneous magnetisation behavior in ferromagnetic materials and writes

$$\partial_t u = \mathcal{H}_{eff} \wedge u + u \wedge (\mathcal{H}_{eff} \wedge u) \tag{1.1}$$

where the magnetic moment u takes its value in S^2, and the effective field $\mathcal{H}_{eff}$ is derived from micromagnetic energy (see [2]), in particular in presence of an external magnetic field. In case of electric current injection (see [3] and [11] for some physical developmnents on electric current injection in ferromagnetic materials), Thiaville and Miltat propose in [14] a first transport term addition, that is simply $(v \cdot \nabla) u$ considering Gilbert form of equation (1.1) :

$$\partial_t u = \mathcal{H}_{eff} \wedge u + u \wedge \partial_t u - (v \cdot \nabla) u \tag{1.2}$$

where v can be seen as the current (carriers) speed. But because of undesired treshold effect in simulations, the same authors are led to introduce in [13] a global transport term that competes with the two dervatives in time in the last equation, that is

$$\partial_t u = \mathcal{H}_{eff} \wedge u + \alpha u \wedge \partial_t u - (v \cdot \nabla) u + \beta u \wedge (v \cdot \nabla) u \tag{1.3}$$

where β is positive and α strictly positive (the same model has been proposed independently in [9]).
In the following, we consider equation (1.3) posed in a regular open set Ω of $\mathbb{R}^3$, with

$$\mathcal{H}_{eff} = \Delta u + \mathcal{H}(u)$$

where $\mathcal{H}$ is defined by

$$\begin{cases} \text{rot } (\mathcal{H}(u)) = 0 \\ \text{div } (\mathcal{H}(u) + \bar{u}) = 0 \end{cases}$$

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with $\bar{u}$ equals u in Ω and zero outside.
We will use functions spaces notations like L^p for applications with vectorial value or numerical value without difference in non ambigous cases, and hypothesis on v or u will be given precisely in the following, depending on the kind of solution we will consider.
The paper is organized as follow : in a first part, we give classical existence results with an overview of the proofs, and a second part is dedicated to the study of a one dimensional approximation in nanowires.

2. **General Results.** Let us now give a first existence result for the global problem, that is with $\mathcal{H}$ defined as above.

Theorem 1. *For all u_0 in $H^1(\Omega)$ with $|u_0| \equiv 1$, and for all v in $C_b^0\left(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3\right)$ there exist a unitary vector field u in $H^1(\mathbb{R}^+ \times \Omega)$ satisfying equation (1.2) and the following energy estimate for all positive t :*

$$\int_0^t \int_\Omega |\partial_t u|^2 + \mathcal{E}(u(t)) \leqslant \mathcal{E}(u_0)\left(1 + I(t)\exp\left(I(t)\right)\right)$$

with

$$I(t) = \int_0^t \|v\|^2_{L^\infty(\Omega\times\mathbb{R}^3)}(s)ds$$

and

$$\mathcal{E}(u(t)) = \int_\Omega |\nabla u(t)|^2 + 2\int_\Omega \Phi(u(t)) - \int_\Omega \mathcal{H}(u(t)) \cdot u(t)$$

Remark 1. By simplicity, theorem 1 is written for equation (1.2), but as the proof shows it clearly, the same result is true for equation (1.3).
In the estimate of the theorem, we see that $v \in L^2(\mathbb{R}^+)$ ensures a bouded energy, and as a consequence u in $H^1(\mathbb{R}^+ \times \Omega)$. But in general, the estimate gives a time dependent bound, and this is the main difference with the corresponding result for Landau-Lifschitz equation.

We now introduce two other formulations of equation (1.3), equivalent for sufficiently regular unitary vector field. The first one can be seen as derived from Landau-Lifschitz equation :

$$(1+\alpha^2)\partial_t u = \mathcal{H}_{eff} \wedge u + \alpha u \wedge (\mathcal{H}_{eff} \wedge u) - (1+\alpha\beta)\left(v \cdot \nabla\right) u - (\alpha - \beta) u \wedge \left(v \cdot \nabla\right) u \quad (2.4)$$

The second one is then deduced from (2.4) in order to construct strong solutions for equation (1.3), as made in [6] for Landau-Lifschitz equation.

$$\begin{aligned}(1+\alpha^2)\partial_t u &= \alpha\left(\Delta u + |\nabla u|^2 u\right) + \mathcal{H}_{eff} \wedge u + \alpha u \wedge (\chi(u) \wedge u) \\ &\quad -(1+\alpha\beta)\left(v \cdot \nabla\right) u - (\alpha - \beta) u \wedge \left(v \cdot \nabla\right) u\end{aligned} \quad (2.5)$$

To clarify the part of the physical constraint $|u| \equiv 1$ in the existence and uniquness of "strong solutions", we define this term only from regularity point of view.

Definition 1. A solution of equations (1.3), (2.4) or (2.5) is said strong if

- $u \in C^0\left(0, t; H^2(\Omega)\right) \cap L^2\left(0, t; H^3(\Omega)\right)$ for all $0 < t < T$

- u depends continiously on u_0 for the topology of $C^0\left(0,t;H^2\left(\Omega\right)\right)$ for all $0<t<T$

local in time if $T<\infty$ and global in time otherwise.

Theorem 2. *For v in $C_b^1\left(\mathbb{R}^+\times\Omega\times\mathbb{R}^3\right)$, and for all u_0 in $H^2\left(\Omega\right)$ sutch that $\partial_\nu u_0\equiv 0$ on $\partial\Omega$ there exist a unique strong solution u of equation (2.5). Moreover, if u_0 takes its value in S^2, then $|u|\equiv 1$ and u is the unique strong solution of formulations (2.4) and (1.3).*

Proof of Theorem 1. Proof is based on a construction of Alouges and Soyeurs for Landau-Lifschitz equation (see [1]). We first consider a relaxed problem

$$\begin{cases} \partial_t u_\lambda = & \mathcal{H}_{eff}(u_\lambda) - u_\lambda\wedge\partial_t u_\lambda - \mathcal{T}(u_\lambda)\wedge\left(v\cdot\nabla\right)u_\lambda & \text{in } \mathbb{R}^+\times\Omega \\ & -\frac{4}{\lambda}\left(|u_\lambda|^2-1\right)u_\lambda & \\ \partial_\nu u_\lambda = 0 & & \text{on } \mathbb{R}^+\times\partial\Omega \\ u_\lambda(t=0)=u_0 & & \text{in } \Omega \end{cases} \tag{2.6}$$

where $\mathcal{T}$ is identity in the unitary ball, and projection on S^2 otherwise. We prove existence of weak solutions for equation (2.6) by a Galerkine approximation, using a bound for equation (2.6) similar to the theorem's one and independent of Galerkine approximation's degree. Aubin Lemma permits us to take the limit in non linear terms of equation (2.6), and by the same arguments, we take the limit in lambda to obtain by construction $|u|\equiv 1$. To prove that u satisfy equation (1.2), or its integral form in fact, we choose a function test of the form $u_\lambda\wedge\phi$ to refind equation (1.3) by cross product properties.

□

Remark 2. As said in remark 1, v in $L^2\left(\mathbb{R}^+\right)$ gives u in $H^1\left(\mathbb{R}^+\times\Omega\right)$ but obviously similar result with u in $H^1\left(]0,T[\times\Omega\right)$ for all positive T is true for v only bouded and weakly continous in its third variable.
The truncate term permits us to obtain energy estimate of theorem 1 but makes this bound be time dependent too, and more generaly, it's not easy to integrate transport term contribution in micromagnetic energy, as it's the case for the terms of the effective field.

Proof of theorem 2. Proof of existence and uniquness of strong solutions for equation (2.5) is given in [6], because non linear transport term estimations can be seen as particular cases of the estimations made in the reference above for the term $|\nabla u|^2 u$.
Let now u_0 be with value in S^2. Multiplying equation (2.5) by $u(t)\in H^2\left(\Omega\right)\subset L^\infty\left(\Omega\right)$, we obtain (with $\alpha=\beta=1$) :

$$\partial_t\left(|u|^2\right) = \Delta u\cdot u + |\nabla u|^2|u|^2 - \frac{1}{2}\left(v\cdot\nabla\right)u\cdot u$$

but

$$\frac{1}{2}\Delta\left(|u|^2\right) = |\nabla u|^2 + u\cdot\Delta u \quad \text{et} \quad \left(v\cdot\nabla\right)u\cdot u = \nabla u.v\cdot u = v\cdot{}^t(\nabla u).u = \frac{1}{2}v\cdot\nabla\left(|u|^2\right)$$

so with $b:=|u|^2-1$,

$$\begin{cases} 2\ \partial_t b = \Delta b + 2|\nabla u|^2 b - \dfrac{1}{2} v \cdot \nabla b & \text{in } [0,T[\times\Omega \\ \partial_\nu b = 0 & \text{on } [0,T[\times\partial\Omega \\ b(t=0) = 0 & \text{in } \Omega \end{cases} \tag{2.7}$$

Now, because of the dissipative term, we can conclude directly that b is zero by classical energy estimates using Young inequality and Gronwall Lemma.
We have then directly, by cross product properties, that u is strong solution of equations (1.3) and (2.4). By equivalence of the three formulations for regular solutions with value in S^2, uniquness of strong solutions for equations (1.3) and (2.4) is equivalent to pointwise norm conservation by these equations.
Adapting equation (2.7) to equation (1.3) (or equation (2.4) with different multiplicative constants) we obtain :

$$\begin{cases} \partial_t b = -v \cdot \nabla b & \text{in } [0,T[\times\Omega \\ \partial_\nu b = 0 & \text{on } [0,T[\times\partial\Omega \\ b(t=0) = 0 & \text{in } \Omega \end{cases} \tag{2.8}$$

We are going to prove that b is in fact solution in classical sense, and by caracteristics method we will obtain that b is zero. Remark first that :

- $H^2(\Omega)$ is an algebra, so $b \in C^0\left(0,t;H^2(\Omega)\right)$
- $\frac{1}{2}\nabla b = {}^t\nabla u.u \in C^0\left(0,t;H^1(\Omega)\right) \otimes C^0\left(0,t;H^2(\Omega)\right) \hookrightarrow C^0\left(0,t;H^1(\Omega)\right)$
- by (2.8), $\partial_t b \in C^0\left(0,t;H^1(\Omega)\right)$ so $b \in C^1\left(0,t;H^1(\Omega)\right)$
- $b \in L^2\left(0,t;H^3(\Omega)\right) \otimes L^2\left(0,t;H^3(\Omega)\right) \hookrightarrow L^1\left(0,t;C^1\left(\overline{\Omega}\right)\right)$

so finally $b \in C^1\left([0,T[\times\overline{\Omega}\right)$ and equation (2.8) is quasilinear without second member so b is constant along caracteristics. Let now (t,x) be in $[0,T[\times\Omega$, if b is nonzero there exist a unique caracteristic γ passing by x at time t, and defined by :

$$\begin{cases} \gamma'(s) = v(s,\gamma(s),\lambda) \\ \gamma(t) = x \end{cases}$$

where $\lambda = b(t,\gamma(t))$. Let $]t_0,t_1[$ be the existence intervale of γ, then if $t_0 = 0$ and $\gamma(0) \in \Omega$, $b(t,x) = b(t,\gamma(t)) = b(0,\gamma(0)) = 0$, otherwise $\gamma(t_0) \in \partial\Omega$ and $b(t,x) = b(t_0,\gamma(t_0))$. So we have to show that b is zero on the boundary.
As Ω is regular, we can decompose equation (2.8) in tangencial and normal directions in some neyborwood V of $\partial\Omega$:

$$\begin{cases} \partial_t b = -v_\nu \partial_\nu b - v_\tau \cdot \nabla_\tau b & \text{in } [0,T[\times(V\cap\Omega) \\ \partial_\nu b = 0 & \text{on } [0,T[\times\partial\Omega \\ b(t=0) = 0 & \text{in } V\cap\Omega \end{cases}$$

And taking the limit in x, we obtain (as v_ν is suposed bounded on $\partial\Omega$) :

$$\begin{cases} \partial_t b = -v_\tau \cdot \nabla_\tau b & \text{on } [0,T[\times\partial\Omega \\ b(t=0) = 0 & \text{on } \partial\Omega \end{cases}$$

Now, by regularity and hypothesis on v, we can conclude using classical methods that $b \equiv 0$.

□

3. **One Dimensionnal Approximations.** In this section, we follow a work of Carbou and Labbé on Bloch Wall motion in nanowires submitted to an external magnetic field. These authors use in [4] and [5] a one dimensional approximation of Landau-Lifschitz equation, that is a one dimensional approximation of the demagnetising field $\mathcal{H}$ defined at the end of section 1, given by Sanchez in [10] for perfectly cylindrical wires. Following its method, we justify a model used by Thiaville and Miltat in [12] for Bloch wall motion in nanowires to an external magnetic field magnetic field, and also used by the same authors in [13] and [14] for wall motion caused by electric current injection.
We consider an infinite wire in direction e_1, of section $\mathcal{E}_k$ defined by

$$\mathcal{E}_k(x_0) = \left\{ X = (x_1, x_2) \in \{x = x_0\}, \left(\frac{x_1}{k}\right)^2 + x_2^2 < 1 \right\} \tag{3.9}$$

and we recall that in this case, magnetostatic equations writes

$$\begin{cases} \text{div } (\mathcal{H}(u)) = 0 & \text{dans } \Omega_k \cup^c \Omega_k \\ \text{rot } (\mathcal{H}(u) = 0) & \text{dans } \Omega_k \cup^c \Omega_k \\ [\mathcal{H}(u) \cdot \nu] = u \cdot \nu & \text{sur } \partial\Omega_k \end{cases} \tag{3.10}$$

The following theorem include the result given by Sanchez in [10].

Theorem 3. *The problem (3.10) as a unique solution in $\mathbb{R}^2$ for all $k \in \mathbb{R}_\star^+ \setminus \{1\}$, given by*

$$\begin{aligned} \mathcal{H}(u)(X) \quad = \quad & -\frac{1}{k+1} u_k - \mathbf{1}_{D(0,1)}(z_1) \frac{1}{z_1 - z_2} \left(\frac{1}{z_1} u_k - z_1 u_{-k} \right) \\ & -\mathbf{1}_{D(0,1)} z_2 \frac{1}{z_2 - z_1} \left(\frac{1}{z_2} u_k - z_2 u_{-k} \right) \end{aligned} \tag{3.11}$$

where $D(0,1) = \{z \in \mathbb{C}, |z| < 1\}$, and denoting also X the complex number associated to (x_1, x_2),

$$u_k = \begin{pmatrix} u_1 \\ k u_2 \end{pmatrix} \qquad \text{and} \qquad z_j = \frac{\overline{X} + (-1)^j \sqrt{\overline{X}^2 - (k^2 - 1)}}{k - 1}, \text{ for } j = 1, 2$$

Moreover, $\mathcal{H}(u)$ is constant in Ω_k for all $k \in \mathbb{R}^+$ and we have

$$\mathcal{H}(u)(X) = -\frac{1}{k+1} u_k$$

Proof of theorem 3. As rot $(\mathcal{H}(u)) = 0$ in $\mathbb{R}^2$, there exists a potential ϕ such that $\mathcal{H}(u) = -\nabla\phi$ and

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega_k \cup^c \Omega_k \\ \partial_\nu \phi = -u \cdot \nu & \text{on } \partial\Omega_k \end{cases}$$

this gives

$$\phi(X) = \int_{\partial\Omega_k} \frac{1}{2\pi} \ln\left(|X - Y|\right) (-u \cdot \nu)(Y) d\sigma(Y)$$

and

$$\mathcal{H}(u)(X) = \int_{\partial\Omega_k} \frac{1}{2\pi} (u \cdot \nu)(Y) \frac{X - Y}{|X - Y|^2} d\sigma(Y)$$

Using natural parametrisation of $\partial\Omega_k = \partial\mathcal{E}_k(x_0)$ and complex numbers, we have

$$\begin{aligned}\mathcal{H}(u)(X) &= \int_{|z|=1}\left(u_1\left(z+\frac{1}{z}\right)-iku_2\left(z-\frac{1}{z}\right)\right)\frac{1}{2\overline{X}-\frac{k+1}{z}-(k-1)z}\frac{dz}{iz}\\ &= -\frac{1}{2i\pi}(T_1+T_2)\end{aligned}$$

where

$$T_1=\int_{|z|=1}\frac{u_1+iku_2}{zP(z)}\qquad\text{and}\qquad T_2=\int_{|z|=1}\frac{(u_1-iku_2)z}{P(z)}$$

with $P(z)=(k-1)z^2-2\bar{X}z+(k+1)$.

We now use Residus Theorem to obtain

$$T_1=2i\pi(u_1+iku_2)\left(\frac{1}{k+1}+\mathbf{1}_{D(0,1)}(z_1)\frac{1}{z_1(z_1-z_2)}+\mathbf{1}_{D(0,1)}(z_2)\frac{1}{z_2(z_2-z_1)}\right)$$

and

$$T_2=-2i\pi(u_1-iku_2)\left(\mathbf{1}_{D(0,1)}(z_1)\frac{z_1}{z_1-z_2}+\mathbf{1}_{D(0,1)}(z_2)\frac{z_2}{z_2-z_1}\right)$$

so

$$\begin{aligned}\mathcal{H}(u)(X) &= -\frac{1}{k+1}u_k-\mathbf{1}_{D(0,1)}(z_1)\frac{1}{z_1-z_2}\left(\frac{1}{z_1}u_k-z_1u_{-k}\right)\\ &\quad -\mathbf{1}_{D(0,1)}z_2\frac{1}{z_2-z_1}\left(\frac{1}{z_2}u_k-z_2u_{-k}\right)\end{aligned}$$

By classical optimisation methods, we show that

$$\min_{\mathcal{E}_k(x_0)}|z_j|^2>1,\ j=1,2$$

that gives the result.

□

We then obtain an one dimensional approximation for $\mathcal{H}$: $\mathcal{H}(u)=\partial_x^2u+u_1e_1-u_2e_2$, where u_1 is the wire's direction and u_2 is transverse to the Bloch's Wall. For this approximation, we can extend results given in [4] and [5] for equation (1.1).We obviously have stability of the following static profile

$$M_0=\begin{pmatrix} th(x)\\ 0\\ \dfrac{1}{ch(x)}\end{pmatrix}$$

and with a magnetic field injection δe_1, we use local inversion theorem to show wall motion for all non zero applied magnetic field, and magnetic structure rotation toward the wire.

In the case of an applied electric current, equation (1.2) has a static solution for positive v, that is we have a treshold effect :

$$u(x)=R_\theta M_0(ex)$$

with

$$\theta = \arcsin\left(\sqrt{e-1}\right), \text{ and } e = \frac{3-\sqrt{9-8\left(1+v^2\right)}}{2\left(1+v^2\right)}$$

for $|v| < \dfrac{1}{2\sqrt{2}}$.

In the general case of equation (1.3), we also have an explicit solution given by

$$u(t,x) = R_\theta M_0(ex - ct)$$

with

$$c = \frac{\beta}{\alpha}ve, \text{ and } e = \frac{3-\sqrt{9-8\left(1+\left(1-\frac{\beta}{\alpha}\right)^2 v^2\right)}}{2\left(1+\left(1-\frac{\beta}{\alpha}\right)^2 v^2\right)}$$

for $|v| < \dfrac{1}{\left(1-\frac{\beta}{\alpha}\right)2\sqrt{2}}$.

In the particular case $\alpha = \beta$, we have $u(t,x) = M_0(x - vt)$, stable for all $v \in \mathbb{R}$.

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E-mail address: gael.bonithon@laposte.net