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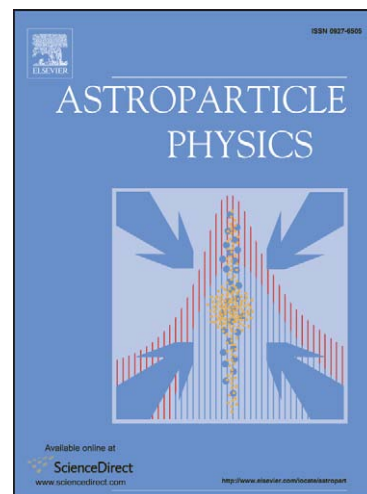
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An extension of the cosmological standard model with a bounded Hubble expansion rate

J.L. Cortés* and J. Induráin*

Departamento de Física Teórica, Universidad de Zaragoza, Zaragoza 50009, Spain

The possibility of having an extension of the cosmological standard model with a Hubble expansion rate H constrained to a finite interval is considered. Two periods of accelerated expansion arise naturally when the Hubble expansion rate approaches to the two limiting values. The new description of the history of the universe is confronted with cosmological data and with several theoretical ideas going beyond the standard cosmological model.

INTRODUCTION

According to General Relativity, if the universe is filled with the particles of the Standard Model of particle physics, gravity should lead to a deceleration of the expansion of the universe. However, in 1998 two independent evidences of present accelerated expansion were presented [1, 2] and later confirmed by different observations [3–5]. On the other hand, measurements of large scale structure [6] and CMB anisotropy [3] also indicate that the universe evolved through a period of early accelerated expansion (inflation).

There is no compelling explanation for any of these cosmic accelerations, but many intriguing ideas are being explored. In the case of inflation, the origin of the accelerated expansion can be either a modification of gravity at small scales [7] or a coupling of the expansion of the universe to the progress of phase transitions [8]. In the case of the present accelerated expansion these ideas can be classified into three main groups: new exotic sources of the gravitational field with large negative pressure [9] (Dark Energy), modifications of gravity at large scales [10] and rejection of the spatial homogeneity as a good approximation in the description of the present universe [11].

Different models (none of them compelling) of the source responsible for each of the two periods of accelerated expansion have been considered. Einstein equations admit a cosmological constant Λ , which can be realized as the stress-energy tensor of empty space. This Λ , together with Cold Dark Matter, Standard Model particles and General Relativity, form the current cosmological model (Λ CDM). However, quantum field theory predicts a value of Λ which is 120 orders of magnitude higher than observed. Supersymmetry can lower this value 60 orders of magnitude, which is still ridiculous [12]. In order to solve this paradox, dynamical Dark Energy models have been proposed.

This has also lead to explore the possibility that cosmic acceleration arises from new gravitational physics. Also here several alternative modifications of the Einstein-Hilbert action at large and small curvatures [13–19], or even higher dimensional models [20, 21], producing an accelerated expansion have been identified. All these analy-

sis include an *ad hoc* restriction to actions involving simple functions of the scalar curvature and/or the Gauss-Bonnet tensor. This discussion is sufficient to establish the point that cosmic acceleration can be made compatible with a standard source for the gravitational field, but it is convenient to consider a more general framework in order to make a systematic analysis of the cosmological effects of a modification of general relativity.

In this paper we parametrize the evolution of the universe (considered isotropic and homogeneous) with the Hubble parameter H . One finds that it is possible to restrict the domain of H to a bounded interval. This restriction naturally produces an accelerated expansion when the Hubble expansion rate approaches any of the two edges of the interval. Therefore, we find a new way to incorporate two periods of cosmic acceleration produced by a modification of general relativity. But the dependence on the two limiting values of H can be chosen independently. One could even consider a Hubble expansion rate constrained to a semi-infinite interval with a unique period of accelerated expansion. In this sense the aim to have a unified explanation of both periods of accelerated expansion is only partially achieved. This simple phenomenological approach to the problem of accelerated expansion in cosmology proves to be equivalent at the homogeneous level to other descriptions based on modifications of the Einstein-Hilbert action or the introduction of exotic components in the matter Lagrangian.

In the next section we will review how a general modification of the gravitational action leads to a generalized first Friedman equation at the homogeneous level. In the third section we will present a specific model (ACM) based on the simplest way to implement a bounded interval of H . In the fourth section we will contrast the predictions of ACM for the present acceleration with astrophysical observations. In the fifth section we will show that it is always possible to find modified gravitational actions which lead to a given generalized first Friedman equation, and present some simple examples. In the sixth section we will show that it is also always possible to find Dark Energy models which are equivalent to a given generalized first Friedman equation, and present some simple examples. The last section is devoted to summary and conclusions.

ACTION OF THE COSMOLOGICAL STANDARD MODEL EXTENSION

The spatial homogeneity and isotropy allow to reduce the gravitational system to a mechanical system with two variables $a(t)$, $N(t)$ which parametrize the Robertson-Walker geometry

$$ds^2 = N^2(t)dt^2 - a^2(t)\Delta_{ij}dx^i dx^j \quad (1)$$

$$\Delta_{ij} = \delta_{ij} + \frac{kx_i x_j}{1 - k\mathbf{x}^2}. \quad (2)$$

Invariance under parameterizations of the time variable imply that the invariant time differential $N(t)dt$ must be used. Also a rescaling of the spatial variables $x^i \rightarrow \lambda x^i$ together with $a(t) \rightarrow \lambda^{-1}a(t)$ and $k \rightarrow \lambda^{-2}k$ is a symmetry that must be kept in the Lagrangian. The action of the reduced homogeneous gravitational system can be then written as

$$I_g = \int dt N L \left(\frac{k}{a^2}, H, \frac{1}{N} \frac{dH}{dt}, \frac{1}{N} \frac{d}{dt} \left(\frac{1}{N} \frac{dH}{dt} \right), \dots \right), \quad (3)$$

with $H = \frac{1}{aN} \frac{da}{dt}$.

If we keep the standard definition of the gravitational coupling and the density (ρ) and pressure (p) of a cosmological homogeneous and isotropic fluid as a source of the gravitational field, we have the equations of the reduced system

$$\left(\frac{8\pi G_N}{3} \right) \rho = -\frac{1}{a^3} \frac{\delta I_g}{\delta N(t)} \quad (4)$$

$$-(8\pi G_N) p = -\frac{1}{Na^2} \frac{\delta I_g}{\delta a(t)}. \quad (5)$$

It is possible to choose a new time coordinate t' such that

$$\frac{dt'}{dt} = N(t). \quad (6)$$

This is equivalent to set $N = 1$ in the action (3) of the gravitational system and the evolution equations of the cosmological model reduce to a set of equations for the scale factor $a(t)$. If we introduce the notation

$$H^{(i)} = \left(\frac{d}{dt} \right)^i H \quad (7)$$

$$(\delta_i L)^{(j)} = \left(\frac{d}{dt} \right)^j [a^3 \partial_i L], \quad (8)$$

with $\partial_i L$ denoting the partial derivatives of the Lagrangian as a function of the variables $(k/a^2, H, H^{(1)}, H^{(2)}, \dots)$ we have

$$\left(\frac{8\pi G_N}{3} \right) \rho = -L + \sum_{i=2}^{\infty} \sum_{j=0}^{i-2} \frac{(-1)^{i-j}}{a^3} H^{(j)} (\delta_i L)^{(i-j-2)} \quad (9)$$

$$-(8\pi G_N) p = -3L + \frac{2k}{a^2} \partial_1 L + \sum_{i=2}^{\infty} \frac{(-1)^i}{a^3} (\delta_i L)^{(i-1)}. \quad (10)$$

In the homogeneous and isotropic approximation, the vanishing of the covariant divergence of the energy-momentum tensor leads to the continuity equation

$$\frac{d}{dt} (\rho a^3) = -p \frac{d}{dt} a^3. \quad (11)$$

In the radiation dominated era one has

$$\rho = 3p = \frac{\sigma}{a^4}, \quad (12)$$

where σ is a constant parameterizing the general solution of the continuity equation. In a period dominated by matter one has a pressureless fluid and then

$$p = 0, \quad \rho = \frac{\eta}{a^3}, \quad (13)$$

with constant η . When these expressions for the energy density and pressure are plugged in (9-10), one ends up with two compatible differential equations for the scale factor $a(t)$ which describe the evolution of the universe. From now on we will use the more common notation $\frac{dH}{dt} \equiv \dot{H}, \dots$

THE ASYMPTOTIC COSMOLOGICAL MODEL

Let us assume that the Lagrangian L of the gravitational system is such that the evolution equations (9-10) admit a solution such that $a(t) > 0$ (absence of singularities), $\dot{a} > 0$ (perpetual expansion) and $\dot{H} < 0$. In that case, one has a different value of the scale factor a and the Hubble rate H at each time and then one has a one to one correspondence between the scale factor and the Hubble rate. Since the continuity equation (together with the equation of state) gives a relation between the scale factor and the density, one can describe a solution of the evolution equations of the generalized cosmological model through a relation between the energy density and the Hubble rate, i.e. through a generalized first Friedman equation

$$\left(\frac{8\pi G_N}{3} \right) \rho = g(H), \quad (14)$$

with g a smooth function which parametrizes the different algebraic relations corresponding to different solutions of different cosmological models. Each choice for the function $g(H)$ defines a phenomenological description of a cosmological model. Then one can take it as a starting point trying to translate any observation into a partial information on the function $g(H)$ which parametrizes the cosmological model.

Eq. (14) is all one needs in order to reconstruct the evolution of the universe at the homogeneous level. The

generalized second Friedman equation is obtained by using the continuity equation (11) and the expression for ρ as a function of H . One has

$$-(8\pi G_N)p = 3g(H) + \frac{g'(H)}{H}\dot{H}. \quad (15)$$

In the matter dominated era one has

$$\frac{\dot{H}}{H^2} = -3\frac{g(H)}{Hg'(H)} \quad (16)$$

and then the assumed properties of the solution for the evolution equations require the consistency conditions

$$g(H) > 0, \quad g'(H) > 0. \quad (17)$$

In the period dominated by radiation one has

$$\frac{\dot{H}}{H^2} = -4\frac{g(H)}{Hg'(H)} \quad (18)$$

instead of (16) and the same consistency conditions (17) for the function $g(H)$ which defines the generalized first Friedman equation.

We introduce now a phenomenological cosmological model defined by the condition that the Hubble rate has an upper bound H_+ and a lower bound H_- . This can be implemented through a function $g(H)$ going to infinity when H approaches H_+ and going to zero when H approaches H_- . We will also assume that there is an interval of H in which the behavior of the energy density with the Hubble parameter is, to a good approximation, scale-free i.e. $g(H) \propto H^2$. The source of the gravitational field will be a homogeneous and isotropic fluid composed of relativistic and non-relativistic particles. The Cosmological Standard Model without curvature is recovered in the limit $H_-/H \rightarrow 0$ and $H_+/H \rightarrow \infty$ which is a good approximation for the period of decelerated expansion.

Notice that this interpretation is independent of the underlying theory of gravitation. The Hubble parameter can be used to parametrize the history of universe as long as $\dot{H} \neq 0 \forall t$. The total density can be thus expressed as a function of H . If H is bounded, then $\rho(H)$ will have a pole at $H = H_+$ and a zero at $H = H_-$. Far from these scales, the behavior of $\rho(H)$ can be assumed to be approximately scale-free.

Under these conditions, we can parametrize the dependence of the cosmological model on the lower bound H_- by

$$g(H) = H^2 h_- \left(\frac{H_-^2}{H^2} \right) \quad (19)$$

and similarly for the dependence on the upper bound H_+

$$g(H) = \frac{H^2}{h_+ \left(\frac{H^2}{H_+^2} \right)}, \quad (20)$$

where the two functions h_{\pm} satisfy the conditions

$$\lim_{x \rightarrow 0} h_{\pm}(x) = \beta^{\pm 1}, \quad \lim_{x \rightarrow 1} h_{\pm}(x) = 0. \quad (21)$$

β is a constant allowed in principle by dimensional arguments. If $\beta \neq 1$ then it can be moved to the lhs of the Friedman equation, turning $G_N \rightarrow \beta G_N$, and can be interpreted as the ratio between an effective cosmological value of the gravitational coupling and the value measured with local tests. But $\beta \neq 1$ would be in conflict with Nucleosynthesis, through the relic abundances of ${}^4\text{He}$ and other heavy elements (for 3 neutrino species) [22], so we set $\beta = 1$. Therefore

$$\lim_{x \rightarrow 0} h_{\pm}(x) = 1, \quad \lim_{x \rightarrow 1} h_{\pm}(x) = 0. \quad (22)$$

The consistency conditions (17) result in

$$h_{\pm}(x) > 0, \quad h_{\pm}(x) > x h'_{\pm}(x) \quad (23)$$

for the two functions h_{\pm} defined in the interval $0 < x < 1$. Thus, we can divide the cosmic evolution history into three periods. In the earliest, relativistic particles dominate the energy density of the universe and the generalized first Friedman equation shows a dependence on the upper bound H_+ . There is also a transition period in which the effect of the bounds can be neglected and the rhs of the first Friedman equation is scale-free; this period includes the transition from a radiation dominated universe to a matter dominated universe. In the third present period, non-relativistic particles dominate the energy density of the universe but the dependence on the lower bound H_- must be accounted for in the generalized first Friedman equation.

From the definition of the Hubble parameter, one has

$$\frac{\dot{H}}{H^2} = -1 + \frac{a\ddot{a}}{\dot{a}^2}. \quad (24)$$

Then, in order to see if there is an accelerated or decelerated expansion, one has to determine whether \dot{H}/H^2 is greater or smaller than -1 .

In the period dominated by radiation one has

$$\frac{\dot{H}}{H^2} = -2 \frac{1}{1 - \frac{x h'_+(x)}{h_+(x)}}, \quad (25)$$

with $x = H^2/H_+^2$, where we have used (20) assuming that only the dependence on the upper bound (H_+) of the Hubble parameter is relevant. A very simple choice for this dependence is given by

$$g(H) = \frac{H^2}{\left(1 - \frac{H^2}{H_+^2}\right)^{\alpha_+}}, \quad (26)$$

with α_+ a (positive) exponent which parametrizes the departure from the standard cosmological model when

the Hubble rate approaches its upper bound. With this simple choice one has a transition from an accelerated expansion for $H^2 > H_+^2/(1 + \alpha_+)$ into a decelerated expansion when $H^2 < H_+^2/(1 + \alpha_+)$, which includes the domain of validity of the standard cosmological model ($H \ll H_+$).

In the period dominated by matter (which corresponds to lower values of the Hubble rate) we assume that only the dependence on the lower bound (H_-) of the Hubble rate is relevant. Then one has

$$\frac{\dot{H}}{H^2} = -\frac{3}{2} \frac{1}{1 - \frac{xh'(x)}{h_-(x)}}, \quad (27)$$

with $x = H_-^2/H^2$. We can also consider a dependence on H_- parametrized simply by an exponent α_-

$$g(H) = H^2 \left(1 - \frac{H_-^2}{H^2}\right)^{\alpha_-}. \quad (28)$$

With this choice one has a transition from a decelerated expansion for $H^2 > H_-^2(1 + \alpha_-/2)$ leaving the domain of validity of the standard cosmological model and entering into an accelerated expansion when the Hubble rate approaches its lower bound for $H^2 < H_-^2(1 + \alpha_-/2)$.

The possibility of describing ρ as a function of H is independent of the existence of spatial curvature k . However, in the kinematics of observables in the expanding universe we do need to specify the value of k . In the rest of the paper we will assume that the universe is flat ($k = 0$), although the same analysis could be done for arbitrary k .

The properties of the expansion obtained in this simple example (a period of decelerated expansion separating two periods of accelerated expansion) are general to the class of phenomenological models with a generalized first Friedman equation (14) with $g(H)$ satisfying the consistency conditions (17) and a Hubble rate constrained to a finite interval. The specific part of the example defined by (26,28) is the simple dependence on the Hubble rate bounds and the values of the Hubble rate at the transitions between the three periods of expansion. From now on we will name this description the Asymptotic Cosmological Model (ACM). With respect to the late accelerated expansion, ACM can be seen as a generalization of Λ CDM, which can be recovered by setting $\alpha_- = 1$. It also includes an early period of exponential expansion which can be seen as a phenomenological description of the evolution of the universe at inflation in the homogeneous approximation.

Horizon problem

One can see that the horizon of a radiation-dominated universe can be made arbitrarily large as a consequence of

an upper bound on the Hubble parameter and in this way one can understand the observed isotropy of the cosmic microwave background at large angular scales.

Let us consider the effect of the modification of the cosmological model on the calculation of the distance $d_h(t_f, t_i)$ of a source of a light signal emitted at time t_i and observed at time t_f

$$d_h(t_f, t_i) = c a(t_f) \int_{t_i}^{t_f} \frac{dt}{a(t)}. \quad (29)$$

We have

$$\frac{dt}{a} = \frac{da}{a^2 \dot{H}} = \left(\frac{8\pi G_N \sigma}{3}\right)^{-1/4} \frac{g'(H) dH}{4Hg(H)^{3/4}}, \quad (30)$$

where in the first step we have used the definition of the Hubble expansion rate H and in the second step we have used the relation between the scale factor a and H as given by (12-14). We are considering both times t_i and t_f in the radiation dominated period.

The distance d_h is then given by

$$d_h(H_f, H_i) = \frac{1}{4g(H_f)^{1/4}} \int_{H_f}^{H_i} \frac{dHg'(H)}{Hg(H)^{3/4}}. \quad (31)$$

If H_i is very close to H_+ (i.e. if we choose the time t_i when the light signal is emitted well inside the period of accelerated expansion) then the integral is dominated by the region around H_i which is very close to H_+ . Then one can approximate in the integrand

$$g(H) \approx \frac{H_+^2}{\left(1 - \frac{H^2}{H_+^2}\right)^{\alpha_+}}. \quad (32)$$

On the other hand if the observation is made at a time t_f within the domain of validity of the cosmological standard model ($H_-^2 \ll H_f^2 \ll H_+^2$) then the factor $g(H_f)^{1/2}$ in front of the integral can be approximated by H_f and then one has

$$d_h(H_f, H_i) \approx \frac{\alpha_+}{4\sqrt{H_f H_i}} B_{\frac{H_i^2}{H_+^2}}(1/2, -\alpha_+/4), \quad (33)$$

where $B_z(m, n)$ is the incomplete Beta function, and it can be made arbitrarily large by choosing H_i sufficiently close to H_+ . In this way we see that a cosmological model with a finite interval of variation for H solves the horizon problem.

CONSTRAINTS OF ACM BY OBSERVATIONS

In this section we will carry out a more technical analysis about how the astrophysical observations constrain the parameter space of ACM in the matter dominance period. This analysis is based on the use of (assumed)

standard candles, basically Type Ia Supernovae [23] and CMB [25]. These observations constrain the parameter space to confidence regions in which the combination $\Omega_m \equiv (1 - H_-^2/H_0^2)^{\alpha_-}$ is constrained to be around one quarter. The consideration of both Type Ia SNe and CMB together favor $\alpha_- > 1.5$. The results of this analysis can be seen in figures (FIG. 1-3). A reader not interested in technical details might well skip this section.

We center our discussion of the experimental tests of the Asymptotic Cosmological Model in the late accelerated expansion produced when H approaches its lower bound H_- . The vast amount of supernovae data collected by [23] and [24], the data from the SDSS Baryon Acoustic Oscillation [26], the mismatch between total energy density and total matter energy density seen at CMB anisotropies [25] and the measurements of present local mass density by 2dF and SDSS [5, 27] compared with the measurements of H_0 from the HST Cepheids [28] show that the universe undergoes a surprising accelerated expansion at the present time.

We will firstly confront the model with the Supernovae Ia data from Riess *et al.* and SNLS collaboration. The usefulness of the Supernovae data as a test of Dark Energy models relies on the assumption that Type Ia SNe behave as standard candles, i.e., they have a well defined environment-independent luminosity \mathcal{L} and spectrum. Therefore, we can use measured bolometric flux $\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2}$ and frequency to determine luminosity distance d_L and redshift z . The luminosity distance is given now by

$$d_L(z) = c(1+z) \int_0^z \frac{dz'}{H(z')} \quad (34)$$

$$= \frac{c(1+z)}{3} \int_{H_0}^{H(z)} \frac{g'(H)dH}{Hg^{1/3}(H_0)g^{2/3}(H)}.$$

The computed value must be compared with the one obtained experimentally from the measured extinction-corrected distance moduli ($\mu_0 = 5\log_{10}(\frac{d_L}{1\text{Mpc}}) + 25$) for each SN. The SNe data have been compiled in references [23], and we have limited the lowest redshift at $cz < 7000\text{km/s}$ in order to avoid a possible ‘‘Hubble Bubble’’ [29, 30]. Therefore our sample consists on 182 SNe. We will determine the likelihood of the parameters from a χ^2 statistic,

$$\chi^2(H_0, H_-, \alpha_-) = \sum_i \frac{(\mu_{p,i}(z_i; H_0, H_-, \alpha_-) - \mu_{0,i})^2}{\sigma_{\mu_{0,i}}^2 + \sigma_v^2}, \quad (35)$$

where σ_v is the dispersion in supernova redshift due to peculiar velocities (we adopt $\|\mathbf{v}_p\| = 400\text{km/s}$ in units of distance moduli), $\sigma_{\mu_{0,i}}$ is the uncertainty in the individual measured distance moduli $\mu_{0,i}$, and $\mu_{p,i}$ is the value of μ_0 at z_i computed with a certain value of the set of parameters H_0, H_-, α_- . This χ^2 has been marginalized over the nuisance parameter H_0 using the adaptive method in reference [31]. The resulting likelihood distribution function $e^{-\chi^2/2}$ has been explored using Monte

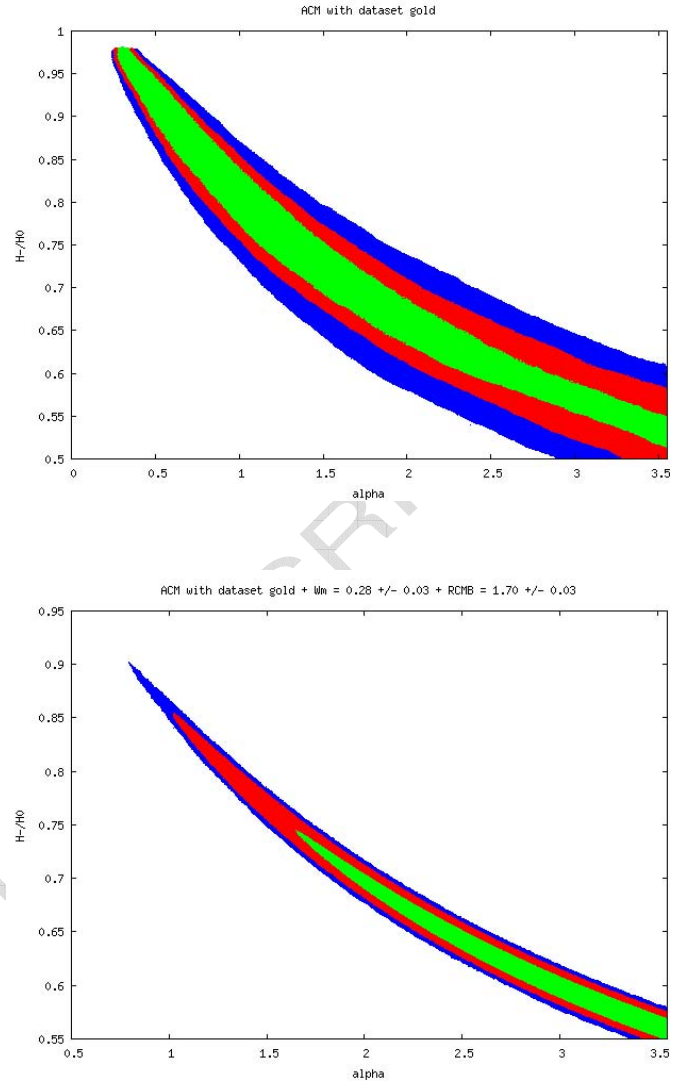


FIG. 1: Confidence regions in parameter space of the Asymptotic Cosmological Model (ACM) with (bottom) and without (top) priors from CMB and estimations of present matter energy density at 1σ (green), 2σ (red) and 3σ (blue). The Λ CDM is inside the 1σ region if we consider no priors, but is moved outside the 2σ region when we confront the SNe data with CMB and estimations from present matter energy density. In this case a value $\alpha_- > 1.5$ is favored.

Carlo Markov Chains. We get a best fit of ACM at $\alpha_- = 0.36$ and $\frac{H_-}{H_0} = 0.95$, for which $\chi^2 = 157.7$. In contrast, fixing $\alpha_- = 1$, we get $\Omega_\Lambda \equiv \frac{H_-^2}{H_0^2} = 0.66$ and $\chi^2 = 159.1$ for the best fit Λ CDM. The confidence regions are shown in Fig. 1 (top).

We can add new constraints for the model coming from measurements of the present local matter energy density from the combination of 2dF and SDSS with HST Cepheids, rendering $\Omega_m = 0.28 \pm 0.03$; and the distance to the last scattering surface from WMAP, which leads

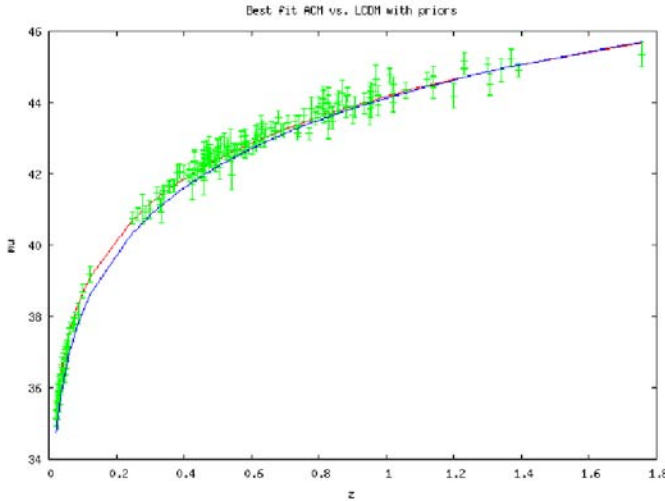


FIG. 2: Distance Moduli vs. Redshift comparison between the best fits of ACM ($H_0 = 66 \text{ Km/sMpc}$, red) and ΛCDM ($H_0 = 65 \text{ Km/sMpc}$, blue) to the SNe data with priors. ACM fits clearly better the medium redshift SNe.

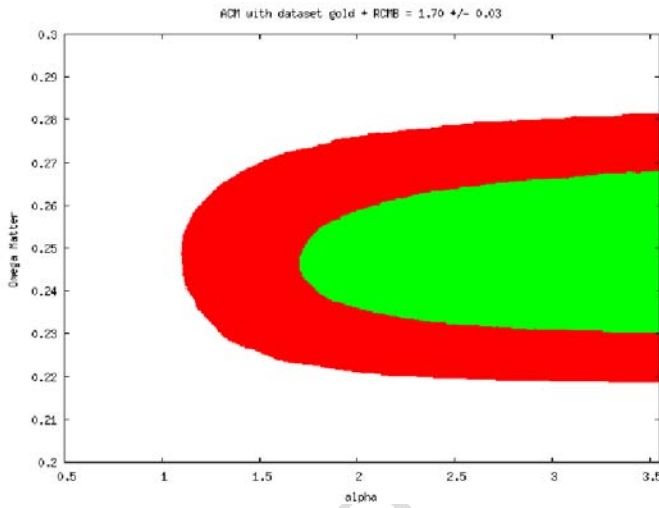


FIG. 3: Ω_m vs. α confidence regions with the CMB prior only. The confidence regions show $\Omega_m = 0.25 \pm 0.03$ and $\alpha > 1.15$ at 2σ level.

to $r_{CMB} \equiv \sqrt{\Omega_m} \int_0^{1089} \frac{H_0 dz'}{H(z')} = 1.70 \pm 0.03$ [23]. Including these priors we get a best fit of ACM at $\alpha_- = 3.54$ and $\frac{H_-}{H_0} = 0.56$ (our simulation explored the region with $\alpha_- < 3.6$), for which $\chi^2 = 164.1$. In contrast, we get $\Omega_\Lambda = 0.73$ and $\chi^2 = 169.5$ for the best fit ΛCDM , which is outside the 2σ confidence region shown in Fig. 1 (bottom). The fits of the best fit ΛCDM and ACM taking into account the priors to the SNe data are compared in Fig. 2. The information which can be extracted from the data is limited. This can be seen in Fig. 3, in which it is explicit that the data constrain mainly the value of the present matter energy density, $\Omega_m \equiv \frac{\rho_m}{\rho_C} = (1 - \frac{H^2}{H_0^2})^{\alpha_-}$.

GENERALIZED FIRST FRIEDMAN EQUATION AND $f(R)$ GRAVITY

The extension of the cosmological model considered here could be compared with recent works on a modification of gravity at large or very short distances [13–19]. It has been shown that, by considering a correction to the Einstein Hilbert action including positive and negative powers of the scalar curvature, it is possible to reproduce an accelerated expansion at large and small values of the curvature in the cosmological model. Some difficulties to make these modifications of the gravitational action compatible with the solar system tests of general relativity have lead to consider a more general gravitational action, including the possible scalars that one can construct with the Riemann curvature tensor [32], although the Gauss-Bonnet scalar is the only combination which is free from ghosts and other pathologies. In fact, there is no clear reason to restrict the extension of general relativity in this way. Once one goes beyond the derivative expansion, one should consider scalars that can be constructed with more than two derivatives of the metric and then one does not have a good justification to restrict in this way the modification of the gravitational theory. It does not seem difficult to find an appropriate function of the scalar curvature or the Gauss-Bonnet scalar, which leads to a cosmology with a bounded Hubble expansion rate.

One may ask if a set of metric $f(R)$ theories which include ACM as an homogeneous and isotropic solution exists. The answer is that a bi-parametric family of $f(R)$ actions which lead to an ACM solution exists. In the following section we will derive them and we will discuss some examples. The derivation follows the same steps of modified $f(R)$ -gravity reconstruction from any FRW cosmology [33].

We start from the action

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} [f(R) + 16\pi G_N \rho], \quad (36)$$

which leads to the “generalized first Friedman equation”

$$-18(H\ddot{H} + 4H^2\dot{H})f''(R) - 3(\dot{H} + H^2)f'(R) - \frac{1}{2}f(R) = (8\pi G_N) \rho \quad (37)$$

$$R = -6(\dot{H} + 2H^2). \quad (38)$$

If the energy density of the universe is mainly due to matter (as in the late accelerated expansion), we can use (16) and (14) to express ρ , R , \dot{H} and \ddot{H} as functions of H . Thus we get

$$[H^3 g g'^2 - 3H^2 g^2 g' + 3H^3 g^2 g''] f''(R) + \frac{[3H g g'^2 - H^2 g'^3]}{18} f'(R) - \frac{g'^3}{108} f(R) = \frac{g g'^3}{18} \quad (39)$$

$$R(H) = 18H g(H)/g'(H) - 12H^2. \quad (40)$$

$H(R)$ can be obtained from (40) and set into (39); then we get an inhomogeneous second order linear differential

equation with non-constant coefficients. Therefore, there will always be a bi-parametric family of $f(R)$ actions which present ACM as their homogeneous and isotropic solution. The difference between these actions will appear in the behavior of perturbations, which is not fixed by (14). Some of these actions are particularly easy to solve. If $g(H) = H^2$ then $R = -3H^2$ and the differential equation becomes

$$6R^2 f''(R) - Rf'(R) - f(R) + 2R = 0. \quad (41)$$

Its general solution is

$$f(R) = R + c_1 R^{\frac{1}{12}(7-\sqrt{73})} + c_2 R^{\frac{1}{12}(7+\sqrt{73})}, \quad (42)$$

which will give (14) as First Friedman equation as long as the radiation energy density can be neglected. In general the differential equation (39) will not be solvable analytically, and only approximate solutions can be found as power series around a certain singular point R_0 (a value of R such that $H(R)$ cancels out the coefficient of $f''(R)$ in (39)). These solutions will be of the form $f(R) = f_p(R) + c_1 f_1(R) + c_2 f_2(R)$ with

$$\frac{f_i(R)}{R_0} = \sum_{m=0}^{\infty} a_m^{(i)} \left(\frac{R - R_0}{R_0} \right)^{s_i+m}, \quad (43)$$

where $i = p, 1, 2$, $a_0^{(1)} = a_0^{(2)} = 1$ and the series will converge inside a certain radius of convergence. An interesting choice of R_0 is $R_0 = -12H_-^2 \equiv R_-$, which is the value of R at H_- . One can also find the approximate solution of the differential equation for $|R| \gg |R_-|$, which will be of the form

$$\frac{f_i(R)}{R} = \sum_{m=0}^{\infty} a_m^{(i)} \left(\frac{R}{R} \right)^{s_i+m}. \quad (44)$$

One can in principle assume that the action (44) could be considered as valid also in the region in which radiation begins to dominate, but this action reproduces (14) only if matter dominates. However, some of the new terms appearing in (44) will be negligible against R when radiation begins to dominate. The others can be canceled out by setting to zero the appropriate integration constant and therefore the Cosmological Standard Model will be recovered as a good approximation when radiation begins to dominate.

$f(R)$ -theories are not the only modified gravity theories studied in the literature; $f(G)$ -theories [34] are also a popular field of research. In these theories, the Einstein-Hilbert action is supplemented by a function of the Gauss-Bonnet scalar $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (not to be confused with the gravitational coupling G_N). The same approach can be used to answer what $f(G)$ actions are able to reproduce ACM homogeneous evolution, once more in analogy with the modified $f(G)$ -gravity reconstruction of a FRW cosmology [35]. The

form of the action will be

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} [R + f(G) + 16\pi G_N L_m], \quad (45)$$

from which we derive the modified Friedman equation

$$(8\pi G_N) \rho = 3H^2 - \frac{1}{2} G f'(G) + \frac{1}{2} f(G) + 12 f''(G) \dot{G} H^3. \quad (46)$$

Following the same procedure as in the case of $f(R)$ -theories we arrive to the differential equation

$$3g(H) = 3H^2 - \frac{1}{2} G f'(G) + \frac{1}{2} f(G) + \frac{864H^6 g}{g'^3} [9gg' - Hg'^2 - 3Hgg''] f''(G) \quad (47)$$

$$G = 24H^2(H^2 - 3Hg/g'). \quad (48)$$

For a given $g(H)$, one can use (48) to get $H(G)$, then set it into (47) and solve the second order linear differential equation. As in the previous case, there will be a bi-parametric family of solutions for $f(G)$ which will have (14) as their homogeneous isotropic solution as long as matter dominates. The parameters will need to be fixed in order to make the contribution of undesired terms in the Friedman equation to be negligible when radiation dominates. Again $g(H) = H^2$ is an example which can be solved analytically. The differential equation turns to be

$$12G^2 f''(G) - Gf'(G) + f(G) = 0, \quad (49)$$

which has the solution

$$f(G) = c_1 G + c_2 G^{1/12}. \quad (50)$$

In most cases this procedure will not admit an analytical solution and an approximate solution will need to be found numerically.

A similar discussion can be made to explain the early accelerated expansion (inflation) as a result of an $f(R)$ or $f(G)$ action, taking into account that in this case the dominant contribution to the energy density is radiation instead of matter. The analogue to (44) will be now a solution of the form

$$\frac{f_i(R)}{R} = \sum_{n=0}^{\infty} a_n^{(i)} \left(\frac{R}{R_+} \right)^{r_i+n}, \quad (51)$$

for $|R| \ll |R_+|$ with $R_+ = -12H_+^2$. Terms in the action which dominate over R when R becomes small enough should be eliminated in order to recover the Standard Cosmological model before matter starts to dominate.

Both accelerated expansions can be described together in the homogeneous limit by an $f(R)$ action with terms $(\frac{R}{R_-})^{s_i+m}$ with $s_i > 0$ coming from (44) and terms $(\frac{R}{R_+})^{r_i+n}$ with $r_i > 0$ coming from (51). The action will contain a term, the Einstein-Hilbert action R , which will be dominant for $H_- \ll H \ll H_+$ including the period when matter and radiation have comparable energy densities.

ALTERNATIVE DESCRIPTIONS OF THE ACM

In general, by virtue of the gravitational field equations, it is always possible to convert a modification in the gravitational term of the action to a modification in the matter content of the universe. In particular, at the homogeneous level, it is possible to convert a generalized first Friedman equation of the type (14) to an equation in which, apart from the usual matter term, there is a dark energy component with an unusual equation of state $p = p(\rho)$ and in which General Relativity is not modified.

The trivial procedure is the following. Assume that the source of the gravitational field in the modified gravity theory behaves as $p_d = \omega\rho_d$ (matter or radiation). This is the case when the modification of gravity is relevant. We can then define a dark energy or effective gravitational energy density as

$$\rho_g = \frac{3}{8\pi G_N} (H^2 - g(H)). \quad (52)$$

The continuity equation (11) allows to define a pressure for this dark energy component as $p_g = -\rho_g - \dot{\rho}_g/3H$. The equation of state of the dominant content of the universe leads to

$$\dot{H} = -3(1 + \omega) \frac{Hg(H)}{g'(H)}, \quad (53)$$

which can be used to express $\dot{\rho}_g$ as a function of H , and one gets the final expression for p_g ,

$$p_g = \frac{-3}{8\pi G_N} \left(H^2 - 2(1 + \omega)H \frac{g(H)}{g'(H)} + \omega g(H) \right). \quad (54)$$

Then, for any given $g(H)$ we can use (52) to get $H = H(\rho_g)$ and substitute in (54) to get an expression of $p_g(\rho_g; H_{\pm}, \omega)$ which can be interpreted as the equation of state of a dark energy component. In the simple case of a matter dominated universe with ACM, $\alpha_- = 1$ gives obviously a dark energy component verifying $p_g = -\rho_g$. Another simple example is $\alpha_- = 2$, for which

$$\left(\frac{8\pi G_N}{3H_-^2} \right) p_g = \frac{2}{\frac{8\pi G_N}{3H_-^2} \rho_g - 3}. \quad (55)$$

The inverse procedure is also straightforward. Suppose we have a universe filled with a standard component $p_d = \omega\rho_d$ and a dark energy fluid $p_g = p_g(\rho_g)$. Using the continuity equation of both fluids one can express their energy densities as a function of the scale factor a and then use this relation to express ρ_g as a function of ρ_d . Then the Friedman equation reads

$$H^2 = \frac{8\pi G_N}{3} (\rho_d + \rho_g(\rho_d)), \quad (56)$$

which, solving for ρ_d , is trivially equivalent to (14). This argument could be applied to reformulate any cosmological model based on a modification of the equation of state of the dark energy component [36] as a generalized first Friedman equation (14).

Until now we have considered descriptions in which there is an exotic constituent of the universe besides a standard component (pressureless matter or radiation). In these cases, pressureless matter includes both baryons and Dark Matter. However, there are also descriptions in which Dark Matter is unified with Dark Energy in a single constituent of the universe. One of these examples is the Chaplygin gas $p_g = -1/\rho_g$ [37, 38]. The use of the continuity equation leads to

$$\rho_g = \sqrt{A + Ba^{-6}}, \quad (57)$$

where A and B are integration constants. The previous method can be used to find the generalized first Friedman equation for the baryon density in this model,

$$\frac{8\pi G_N}{3} \rho_b = \frac{H^2}{k} \left(\sqrt{1 + k(1 - \frac{H^4}{H_-^4})} - 1 \right), \quad (58)$$

where $H_- = \sqrt{\frac{8\pi G}{3}} A^{1/4}$, $k = B\rho_{b0}^{-2} - 1$, and ρ_{b0} is the present value of ρ_b . In the $H \gg H_-$ limit this model can be interpreted as a universe filled with baryons and dark matter or as a universe filled with baryons and with a higher effective value of G_N . Equation (58) does not fulfill (22) because it describes the behavior of just baryon density. In the $H \gtrsim H_-$ limit, the model can be interpreted as a universe filled with baryons and a cosmological constant or as an ACM model with $\alpha_- = 1$ and filled only with baryons.

A similar example in the early period of accelerated expansion would be a universe filled with a fluid which behaves as a fluid of ultra-relativistic particles if the energy density is low enough but whose density has an upper bound

$$\rho_X = \frac{\sigma}{a^4 + C}. \quad (59)$$

This dependence for the energy density in the scale factor follows from the equation of state

$$p_X = \frac{1}{3}\rho_X - \frac{4C}{3\sigma}\rho_X^2. \quad (60)$$

This model turns out to be equivalent to a universe filled with radiation following eq. (26) with $\alpha_+ = 1$ and $H_+^2 = (\frac{8\pi G_N}{3}) \frac{\sigma}{C}$.

Another equivalent description would be to consider that the universe is also filled with some self interacting scalar field φ which accounts for the discrepancy between the standard energy-momentum and GR Einstein tensors. Given an arbitrary modified Friedman equation

(14) a potential $V(\varphi)$ can be found such that the cosmologies described by both models are the same. The procedure is similar to the one used to reconstruct a potential from a given cosmology [39]. From the point of view of the scalar field, the cosmology is defined by a set of three coupled differential equations

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \quad (61)$$

$$\frac{8\pi G_N}{3}(\rho_d + \frac{\dot{\varphi}^2}{2} + V(\varphi)) = H^2 \quad (62)$$

$$-3(1 + \omega)H\rho_d = \dot{\rho}_d. \quad (63)$$

The solution of these equations for a certain $V(\varphi)$ will give $H(t)$ and $\rho_d(t)$, and therefore $\rho_d(H)$ which is $g(H)$ up to a factor $\frac{8\pi G_N}{3}$. In this way one finds the generalized first Friedman equation associated with the introduction of a self interacting scalar field. Alternatively, given a function $g(H)$ in a generalized first Friedman equation (14) for the density ρ_d , one can find a scalar field theory leading to the same cosmology in the homogeneous limit. By considering the time derivative of (62) and using (61) and (63) one gets

$$\dot{\varphi}^2 = -(1 + \omega)\rho_d - \frac{\dot{H}}{4\pi G_N}, \quad (64)$$

where we can use (53) and (14) to get $\dot{\varphi}^2$ as a function of H . Setting this on (62) we get $V(\varphi)$ as a function of H . On the other hand, $\varphi'(H) = \dot{\varphi}/\dot{H}$, so

$$\frac{8\pi G_N}{3}(\varphi'(H))^2 = \frac{(2H - g'(H))g'(H)}{9(1 + \omega)H^2g(H)} \quad (65)$$

and

$$\frac{8\pi G_N}{3}V(\varphi(H)) = H^2 - g(H) + \frac{(1 + \omega)(2H - g'(H))g(H)}{2g'(H)}. \quad (66)$$

If $g(H)$ is such that the rhs of (65) is positive definite, it can be solved and the solution $\varphi(H)$ inverted and substituted into (66) in order to get $V(\varphi)$. This is the case of a function of the type (28) with $\alpha_- > 1$. For the particular case of an ACM expansion with $\alpha_- = 1$, the solution is a flat potential $V = V_0$ and a constant value of $\varphi = \varphi_0$.

If $g(H)$ is such that the rhs of (65) is negative definite, as it happens in (28) with $\alpha_- < 1$ or in (26), the problem can be solved by changing the sign of the kinetic term in (62). The result is a phantom quintessence in the case of (28) with $\alpha_- < 1$. The case (26) is more complicated because the energy density of the associated inflaton turns out to be negative. Moreover, it is of the same order as the energy density of ultra-relativistic particles during the whole period of accelerated expansion. Therefore, it seems that this model is inequivalent to other inflation scenarios previously studied.

In summary, there are many equivalent ways to describe the discrepancy between observed matter content

of the universe and Einstein's General Relativity. At the homogeneous level, it is trivial to find relations among them. The possibility to establish the equivalence of different descriptions is not a peculiarity of the description of this discrepancy in terms of a generalized first Friedman equation (14). The same relations can be found among generalized equations of state, scalar-tensor theories and f(R) modified gravity [40].

SUMMARY AND DISCUSSION

It may be interesting to go beyond Λ CDM in the description of the history of the universe in order to identify the origin of the two periods of accelerated expansion. We have proposed to use the expression of the energy density as a function of the Hubble parameter as the best candidate to describe the history of the universe. In this context the late time period of accelerated expansion and the early time inflation period can be easily parametrized.

We have considered a simple modification of the cosmological equations characterized by the appearance of an upper and a lower bound on the Hubble expansion rate. A better fit of the experimental data can be obtained with this extended cosmological model as compared with the Λ CDM fit. Once more precise data are available, it will be possible to identify the behavior of the energy density as a function of the Hubble parameter and then look for a theoretical derivation of such behavior.

We plan to continue with a systematic analysis of different alternatives incorporating the main features of the example considered in this work. Although the details of the departures from the standard cosmology can change, we expect a general pattern of the effects induced by the presence of the two bounds on H . We also plan to go further, considering the evolution of inhomogeneities looking for new consequences of the bounds on H .

The discussion presented in this work, which is based on a new description of the periods of accelerated expansion of the universe, can open a new way to explore either modifications of the theory of gravity or new components in the universe homogeneous fluid. Lacking theoretical criteria to select among the possible ways to go beyond Λ CDM, we think that the phenomenological approach proposed in this work is justified.

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- * Electronic address: `cortes,indurain@unizar.es`
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