Eigenvalue bounds for radial magnetic bottles on the disk

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Abstract

We consider a Schrödinger operator H_A^D with a non-vanishing radial magnetic field $B = dA$ and Dirichlet boundary conditions on the unit disk. We assume growth conditions on B near the boundary which guarantee in particular the compactness of the resolvent of this operator. Under some assumptions on an additional radial potential V the operator $H_A^D - V$ has a discrete negative spectrum and we obtain an upper bound of the number of negative eigenvalues. As a consequence we get an upper bound of the number of eigenvalues of H_A^D smaller than any positive value λ, which involves the minimum of B and the square of the L^2-norm of $A(r)/r$, where $A(r)$ is the specific magnetic potential defined as the flux of the magnetic field through the disk of radius r centered in the origin.

1 Introduction

Let us consider a particle in a domain Ω in $\mathbb{R}^2$ in the presence of a magnetic field B. We define the 2-dimensional magnetic Laplacian associated to this particle as follows: Let A be a magnetic potential associated to B ; it means that A is a smooth real one-form on $\Omega \subset \mathbb{R}^2$, given by $A = \sum_{j=1}^2 a_j dx_j$, and that the magnetic field B is the two-form $B = dA$. We have $B(x) = \mathbf{b}(x) dx_1 \wedge dx_2$ with $\mathbf{b}(x) = \partial_1 a_2(x) - \partial_2 a_1(x)$. The magnetic connection $\nabla = (\nabla_j)$ is the differential operator defined by

$$\nabla_j = \frac{\partial}{\partial x_j} - i a_j \ .$$

The 2-dimensional magnetic Schrödinger operator H_A is defined by

$$H_A = -\sum_{j=1}^{2} \nabla_j^2 \ .$$

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The magnetic Dirichlet integral $h_A = \langle H_A.|.\rangle$ is given, for $u \in C_0^\infty(\Omega)$, by

$$h_A(u) = \int_\Omega \sum_{j=1}^{2} |\nabla_j u|^2 |dx| \, . \tag{1.1}$$

From the previous definitions and the fact that the formal adjoint of ∇_j is $-\nabla_j$, it is clear that the operator H_A is symmetric on $C_0^\infty(\Omega)$.
In [5] we discuss the essential self-adjointness of this operator. The result in dimension 2 is the following

Theorem 1.1 *Assume that $\partial\Omega$ is compact and that $B(x)$ satisfies near $\partial\Omega$*

$$\mathbf{b}(x) \geq (D(x))^{-2} \, , \tag{1.2}$$

then the Schrödinger operator H_A is essentially self-adjoint. ($D(x)$ denotes the distance to the boundary). This still holds true for any gauge A' such that $dA' = dA = B$.

We have, using Cauchy-Schwarz inequality,

$$|\langle \mathbf{b}(x)u, u\rangle| = |\langle [\nabla_1, \nabla_2]u, u\rangle| \leq \|\nabla_1 u\|^2 + \|\nabla_2 u\|^2 \quad u \in C_0^\infty(\Omega).$$

This gives the well-known lower bound

$$\forall u \in C_0^\infty(\Omega), \; h_A(u) \geq \left| \int_\Omega \mathbf{b}(x)|u|^2 |dx| \right| \, . \tag{1.3}$$

In this paper, we do not use the conditions (1.2) but we assume nevertheless that $\mathbf{b}(\mathbf{x})$ grows to infinity as x approaches the boundary. The operator H_A^D defined by Friedrichs extension of the quadratic form h_A has a compact resolvent. By analogy with magnetic bottles on the whole space (see [1, 4, 19]), such an operator is called a magnetic bottle on the disk.

We will deal with spectral estimates for the operator H_A^D, using a perturbative method: introducing an additional non-negative bounded and radial potential V, we obtain an upper bound of the number $N(A, V)$ of negative eigenvalues of the operator $H_A^D - V$ (Theorem 2.1) and deduce, for any $\lambda > 0$, an upper bound of the number $N(H_A^D, \lambda)$ of eigenvalues of the operator H_A^D smaller than λ (Theorem 2.2). Theorem 2.1 can be seen as a magnetic version of the Cwikel-Lieb-Rosenblum inequality (see [6, 16, 18]). The CLR inequality provides a bound on the number of negative eigenvalues of Schrödinger operators in $\mathbb{R}^d$ for $d \geq 3$ (without magnetic field) and is a particular case of Lieb-Thirring inequalities (see [15, 17]).

Eigenvalue bounds were recently studied for magnetic Hamiltonians on $\mathbb{R}^2$, for constant magnetic fields (see [10]), for Aharonov-Bohm magnetic fields (see [3, 14]) and for a large class of magnetic fields (see [12]). However, in [12], the total magnetic flux $\phi = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{b}(x)dx$ has to be finite and the dependence on the magnetic field is not explicit even in the radial case. In our result, the total flux is not necessarily finite

(see example 2.4) and the upper bound involves explicitly the square of the magnetic potential.
Magnetic Lieb-Thirring inequalities were also obtained for Pauli operators (see [7, 8]), and links between magnetic and non-magnetic Lieb-Thirring inequalities were discussed in [9].

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2 Main results

We consider a smooth magnetic field $B = \mathbf{b}(x)dx_1 \wedge dx_2$ and a scalar potential V on the unit disk $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 |\ x_1^2 + x_2^2 = r^2 < 1\}$ so that

- (H_1) $\quad K = \inf_{x\in\Omega} \mathbf{b}(x) > 0$ and $\mathbf{b}(x) \to +\infty$ as $D(x) \to 0$
  (i.e as x approaches the boundary.)
- (H_2) $\quad B$ is radially symmetric (consequently we write $\mathbf{b}(r)$ instead of $\mathbf{b}(x)$)
- (H_3) $\quad V \in L^1(\Omega)$, V radial and non-negative, V bounded from above .

From assumption (H_1) and from inequality (1.3) we deduce that for any gauge A associated to B, the operator H_A^D has a compact resolvent, and assumption (H_3) entails that the negative spectrum of $H_A^D - V$ is discrete, where $H_A^D - V$ denotes the operator defined by Friedrichs extension of the quadratic form $h_A - V$.
Using assumption (H_2) we introduce polar coordinates $(r, \theta), (r \in \mathbb{R}^+, \theta \in [0, 2\pi[)$ and consider the following magnetic potential :

$$A = -a(r)\sin\theta dx_1 + a(r)\cos\theta dx_2, \quad a(r) = \frac{1}{r}\int_0^r \mathbf{b}(t)tdt\ . \tag{2.1}$$

We have $dA = B$ and

$$A = A(r)d\theta \quad \text{with} \quad A(r) = ra(r) = \int_0^r \mathbf{b}(t)tdt\ . \tag{2.2}$$

$A(r)$ is the flux of the magnetic field through the disk of radius r centered in the origin. The function $a(r) = A(r)/r$ is well-defined (and smooth) at the origin and it is the amplitude of the magnetic potential A in cartesian coordinates .
The first theorem provides an upper bound of the number $N(A, V)$ of negative eigenvalues of the operator $H_A^D - V$ where A is the magnetic potential defined by (2.2).

From now on, A denotes this specific potential.
Noticing that we have $N(A', V) = N(A, V)$ for any gauge A' so that $dA' = dA = B$, we will prove the following

Theorem 2.1 *If assumptions $(H_1)(H_2)(H_3)$ are verified and if moreover*

$$\mathbf{b}(x) \leq M(D(x))^{-\beta}\,, \quad 0 < \beta < \frac{3}{2} \tag{2.3}$$

for some $M > 0$, then

$$N(A,V) \leq \frac{1}{\sqrt{1-\alpha}} \int_0^1 [(\frac{1}{\alpha}-1)\frac{A^2(r)}{r^2}+V(r)]rdr+2\int_0^1 \left[1+|\log[r\sqrt{K}]|\right] V(r)rdr$$

for any $\alpha \in]0,1[$.
This inequality still holds when we replace in the left-hand side $N(A,V)$ by $N(A',V)$, where A' is any gauge verifying $dA' = dA = B$.

The second theorem is a consequence of the first one and provides an explicit upper bound of the number $N(H_A^D, \lambda)$ of the eigenvalues of H_A^D smaller than any positive value λ :

Theorem 2.2 *If assumptions (H_1) and (H_2) are verified and if moreover*

$$\mathbf{b}(x) \leq M(D(x))^{-\beta}\,, \quad 0 < \beta < \frac{3}{2}$$

for some $M > 0$, then the number of eigenvalues of the operator H_A^D smaller than λ satisfies, for any $\alpha \in]0,1[$, the following inequality

$$N(H_A^D, \lambda) \leq c_K\lambda + \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha}\int_0^1 \left(\frac{A(r)}{r}\right)^2 rdr \tag{2.4}$$

with

- $c_K = \dfrac{3-\log K}{2}$ *if* $0 < K \leq 1$
- $c_K = \left[\dfrac{1+\log K}{2} + \dfrac{1}{K}\right]$ *if* $K > 1$,

Inequality (2.4) still holds when we replace in the left-hand side $N(H_A^D, \lambda)$ by $N(H_{A'}^D, \lambda)$, where A' is any gauge verifying $dA' = dA = B$.

Remark 2.3 *The minimum of the right-hand side is obtained by taking*

$$\alpha_\lambda = \frac{-3I + \sqrt{I^2 + 4I\lambda}}{\lambda - 2I}$$

with $I := \displaystyle\int_0^1 \left(\frac{A(r)}{r}\right)^2 rdr$.

Example 2.4 *Consider a magnetic field B as in the definition (3.2) below, and assume $b(r) \equiv 1$ and $\beta = 1$. Then $c_K = \frac{3}{2}$, the chosen gauge is $A(r) = \int_0^r \mathbf{b}(t)tdt = -\ln(1-r) - r$ and the corresponding value of I is*

$$I = \int_0^1 \frac{[\ln(1-r)+r]^2}{r} dr = 2\zeta(3) - \frac{3}{2}\,. \tag{2.5}$$

3 Proofs

3.1 Proof of Theorem 2.1

Let us introduce the polar coordinates $x = (r,\theta), r \in \mathbb{R}^+, \theta \in [0, 2\pi[$. We have denoted by A the following vector potential :

$$A = A(r)d\theta \quad \text{with} \quad A(r) = ra(r) = \int_0^r \mathbf{b}(t)tdt\,. \tag{3.1}$$

Due to assumption (2.3) the magnetic field we consider is of the type

$$\mathbf{b}(r) = \frac{b(r)}{(1-r)^\beta}\,, \text{ with} \quad \max_{[0,1[} b(r) \le M \text{ and} \quad \beta < \frac{3}{2}. \tag{3.2}$$

We first prove the following

Lemma 3.1 *If B satisfies (3.2), then we can find some constant C so that A writes $A = A(r)d\theta = ra(r)d\theta$ where*

- *if* $\beta < 1 \quad \max_{[0,1[} a(r) \le C.$
- *if* $\beta = 1 \quad a(r) = \tilde{a}(r)\ln(1-r), \quad$ *with* $\quad \max_{[0,1[} \tilde{a}(r) \le C\,.$
- *if* $\beta > 1 \quad a(r) = \dfrac{\tilde{a}(r)}{(1-r)^{\beta-1}}, \quad$ *with* $\quad \max_{[0,1[} \tilde{a}(r) \le C.$

In particular $\displaystyle\int_0^1 \left(\frac{A(r)}{r}\right)^2 rdr < \infty\,.$

Proof.–

Let us explain the case $\beta > 1$. The method for the case $\beta = 1$ is the same.
From (3.2) we get
$0 \le \dfrac{1}{r}\displaystyle\int_0^r \mathbf{b}(t)tdt \le \frac{1}{r}\int_0^r b(t)t(1-t)^{-\beta}dt \le M\int_0^r (1-t)^{-\beta}dt \le M\dfrac{(1-r)^{-\beta+1}}{\beta-1}$ and the result follows.
The case $\beta < 1$ is straightforward.

□

We come now to the proof of Theorem 2.1, following the method of [13]. The quadratic form associated to $H_A^D - V$ can be rewritten as

$$h_{A,V}(u) = \int_0^1 \int_0^{2\pi} \left[|\frac{\partial u}{\partial r}|^2 - V(r)|u^2| + r^{-2} \left[[\frac{\partial}{\partial \theta} - iA(r)]u \right]^2 \right] r dr d\theta \qquad (3.3)$$

for any $u \in C_0^\infty([0,1[\times[0,2\pi[)$. Changing variables $r = e^t$ and denoting $w(t,\theta) = u(e^t,\theta)$ for $t \in]-\infty, 0[$ and $\theta \in [0,2\pi[$ we transfer the form $h_{A,V}(u)$ to

$$\tilde{h}_{A,V}(w) = \int_{-\infty}^0 \int_0^{2\pi} \left[|\frac{\partial w}{\partial t}|^2 - \tilde{V}(t)|w^2| + \left[[\frac{\partial}{\partial \theta} - if(t)]w \right]^2 \right] dt d\theta \qquad (3.4)$$

with

$$\tilde{V}(t) = e^{2t} V(e^t), \quad f(t) = A(e^t) \, .$$

By expanding a given function $w \in C_0^\infty([-\infty,0[\times[0,2\pi[)$ into a Fourier series we obtain that $\tilde{h}_{A,V}(w) = \oplus_{l\in\mathbb{Z}} h^\ell_{A,V}(w_\ell)$ with

$$h^\ell_{A,V}(v) = \int_{-\infty}^0 |\frac{\partial v}{\partial t}|^2 + \left[(\ell - f(t))^2 - \tilde{V}(t) \right] |v^2| \, dt \, ,$$

and $w_\ell = \Pi_\ell(w)$ where Π_ℓ is the projector acting as

$$\Pi_\ell(w)(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{il(\theta-\theta')} w(r,\theta') d\theta' \, .$$

We write, for any $\alpha \in]0,1[$ and any $\ell \in \mathbb{Z}^*$

$$h^\ell_{A,V}(v) \geq \int_{-\infty}^0 |\frac{\partial v}{\partial t}|^2 + \left[(1 - \frac{1}{\alpha}) f^2(t) - \tilde{V}(t) + (1-\alpha)\ell^2 \right] |v^2| \, dt \, .$$

Let us denote by L_α the operator associated via Friedrichs extension to the quadratic form

$$q_\alpha(v) = \int_{-\infty}^0 |\frac{\partial v}{\partial t}|^2 + \left[(1 - \frac{1}{\alpha}) f^2(t) - \tilde{V}(t) \right] |v^2| \, dt \, .$$

L_α and q_α depend on V and A but we skip the reference to V and A in notations for the sake of simplicity. Since

$$h^\ell_{A,V} \geq q_\alpha + (1-\alpha)\ell^2 \, ,$$

the number $N(h^\ell_{A,V})$ of negative eigenvalues of $h^\ell_{A,V}$ is smaller than the number of negative eigenvalues of $L_\alpha + (1-\alpha)\ell^2$. Hence denoting by $\{-\mu_k^\alpha\}$ the negative eigenvalues of L_α and by I_ℓ the set $\{k \in \mathbb{N}; -\mu_k^\alpha + (1-\alpha)\ell^2 < 0\}$ for any $\ell \in \mathbb{Z}^*$, we get

$$N(A,V) \leq \sum_{\ell\in\mathbb{Z}^*} \sum_{k\in I_\ell} 1 + N(h^0_{A,V}) \, .$$

Noticing that the sum in the right-hand side is taken over the (ℓ, k) so that $0 < |\ell| \leq \frac{1}{\sqrt{1-\alpha}}\sqrt{\mu_k^\alpha}$ we write

$$N(A,V) \leq \frac{2}{\sqrt{1-\alpha}} \sum_{k\in\mathbb{N}} \sqrt{\mu_k^\alpha} + N(h^0_{A,V}) \, . \tag{3.5}$$

Let us extend the functions f and $\tilde{V}$ to $\mathbb{R}$ by zero and denote respectively by f_1 and $\tilde{V}_1$ these extensions.
Since $C_0^\infty([-\infty, 0[) \subset C_0^\infty(\mathbb{R})$, the negative eigenvalues $\{-\nu_k^\alpha\}$ of the operator L_1^α associated via Friedrichs extension to the quadratic form

$$q_1^\alpha(v) = \int_{-\infty}^{+\infty} |\frac{\partial v}{\partial t}|^2 + \left[(1-\frac{1}{\alpha})f_1^2(t) - \tilde{V}_1(t)\right] |v^2| \, dt$$

verify

$$\sum_{k\in\mathbb{N}} \sqrt{\mu_k^\alpha} \leq \sum_{k\in\mathbb{N}} \sqrt{\nu_k^\alpha}. \tag{3.6}$$

Applying the sharp inequality of Hundertmarkt-Lieb-Thomas [11] (see Appendix) to the operator L_1^α we get

$$\begin{aligned}
\sum_{k\in\mathbb{N}} \sqrt{\nu_k^\alpha} &\leq \frac{1}{2} \int_{-\infty}^{+\infty} \left[(\frac{1}{\alpha} - 1) f_1^2(t) + \tilde{V}_1(t)\right] dt \\
&\leq \frac{1}{2} \int_{-\infty}^{0} \left[(\frac{1}{\alpha} - 1) f^2(t) + \tilde{V}(t)\right] dt \\
&\leq \frac{1}{2} \int_0^1 \left[(\frac{1}{\alpha} - 1) \frac{A^2(r)}{r^2} + V(r)\right] r dr \, .
\end{aligned} \tag{3.7}$$

To conclude we need the following

Lemma 3.2 *Assume that* $K = \inf_{x\in\Omega} \mathbf{b}(x) > 0$.*Then for any* $\varepsilon \in]0,1[$

$$N(h^0_{A,V}) = N(h^0_{A,0} - V) \leq \frac{1}{\varepsilon} \int_0^1 \left[1 + |\log(\sqrt{\frac{(1-\varepsilon)K}{\varepsilon}} r)|\right] V(r) r dr \, . \tag{3.8}$$

In particular

$$N(h^0_{A,V}) \leq 2 \int_0^1 \left[1 + |\log(\sqrt{K} r)|\right] V(r) r dr \, . \tag{3.9}$$

Proof.–

- Step 1 : From (1.3) we get that $h_A(u) \geq K \int_\Omega |u|^2 |dx| \quad \forall u \in C_0^\infty(\Omega)$, which implies for $h^0_{A,0}$ (returning to the variable r and considering $V \equiv 0$),

$$h^0_{A,0}(w) = \int_0^1 \left[|\frac{\partial w}{\partial r}|^2 + r^{-2} A^2(r) |w^2| \right] r dr$$

$$\geq K \int_0^1 |w|^2 r dr \quad \forall w \in C_0^\infty([0,1]) \, .$$

We write for any $\varepsilon \in]0,1[$

$$N(h^0_{A,0} - V) \leq N(\varepsilon h^0_{A,0} + (1-\varepsilon)K - V) \leq N\left(h^0_{A,0} + \frac{(1-\varepsilon)K}{\varepsilon} - \frac{V}{\varepsilon} \right) , \tag{3.10}$$

where we have used the fact that multiplying an operator by a positive constant does not change the number of its negative eigenvalues.

- Step 2 : We establish the following upper bound :

$$N(h^0_{A,0} + 1 - V) = N(h^0_{A,V} + 1) \leq \int_0^1 [1 + |\log r|] \, V(r) r dr \, . \tag{3.11}$$

We have

$$h^0_{A,V}(w) = \int_0^1 \left[|\frac{\partial w}{\partial r}|^2 + \left[r^{-2} A^2(r) - V(r) \right] |w^2| \right] r dr$$

$$\geq \int_0^1 \left[|\frac{\partial w}{\partial r}|^2 - V(r) |w^2| \right] r dr \quad \forall w \in C_0^\infty([0,1]) \, .$$

By the variational principle,

$$N(h^0_{A,V} + 1) \leq N(P_0 + 1 - V), \tag{3.12}$$

where P_0 is the operator generated by the closure, in $L^2([0,1], r dr)$ of the quadratic form

$$\int_0^1 |\frac{\partial w}{\partial r}|^2 r dr, \quad w \in C_0^\infty([0,1]) \, .$$

Considering the mapping $U : L^2([0,1], r dr) \to L^2([0,1], dr)$ defined by $(Uf)(r) = r^{1/2} f(r)$ we get that

$$N(P_0 + 1 - V) \leq N(T_0 + 1 - V) \tag{3.13}$$

where the operator $T_0 = U P_0 U^{-1}$ is the Sturm-Liouville operator on $L^2([0,1], dr)$ acting on its domain by

$$(T_0 u)(r) = -u"(r) - \frac{u(r)}{4r^2}, \quad u(0) = u(1) = 0 \, . \tag{3.14}$$

The upper bound (3.11) will follow from the properties of $G(r,r,1)$, the diagonal element of the integral kernel of $(T_0+1)^{-1}$. Precisely we have

$$G(r,r,1) \leq r(1+|\log r|), \quad r \in [0,1[\,. \tag{3.15}$$

The proof of (3.15) is given in Appendix . The Birman-Schwinger principle then yields

$$N(T_0+1-V) \leq \int_0^1 G(r,r,1)V(r)dr \leq \int_0^1 [1+|\log r|]\,V(r)rdr\,. \tag{3.16}$$

This ends the proof of (3.11), together with the inequalities (3.12) and (3.13).

- Step 3 : We mimick the previous method to get, for any strictly positive number k

$$N(h^0_{A,0}+k^2-V) \leq \int_0^1 [1+|\log(kr)|]\,V(r)rdr\,. \tag{3.17}$$

Due to the Birman-Schwinger principle it suffices to prove that, for any strictly positive number k

$$G(r,r,k^2) \leq r(1+|\log(kr)|), \quad r \in [0,1[\,. \tag{3.18}$$

This is done in Appendix .

- Step 4 : Returning to (3.10) and applying (3.17) with $k^2 = \frac{(1-\varepsilon)K}{\varepsilon}$ and $\frac{V}{\varepsilon}$ instead of V we get, for any $\varepsilon \in]0,1[$

$$N(h^0_{A,0}-V) \leq N\left(h^0_{A,0}+\frac{(1-\varepsilon)K}{\varepsilon}-\frac{V}{\varepsilon}\right) \tag{3.19}$$

$$\leq \frac{1}{\varepsilon}\int_0^1 \left[1+|\log(\sqrt{\frac{(1-\varepsilon)K}{\varepsilon}}r)|\right] V(r)rdr\,, \tag{3.20}$$

and taking $\varepsilon = \frac{1}{2}$ we obtain Lemma 3.2.

□

Theorem 2.1 follows from Lemma 3.2 together with inequalities (3.5), (3.6), and (3.7).

3.2 Proof of Theorem 2.2

Noticing that for any $\lambda > 0$ the constant potential $V(x) \equiv \lambda$ is in $L^1(\Omega)$, and that $N(A,\lambda)$ denotes the number of eigenvalues of the operator H^D_A less than λ, we apply

Theorem 2.1 to $V(x) \equiv \lambda$. To get the result it suffices to compute $\int_0^1 [1 + |\log(kr)|]\, rdr$. We get after computation that

$$\int_0^1 [1 + |\log(kr)|]\, rdr \ = \gamma_k, \tag{3.21}$$

with

- $\gamma_k = \dfrac{3 - 2\log k}{4}$ if $k \leq 1$
- $\gamma_k = \dfrac{1 + 2\log k}{4} + \dfrac{1}{2k^2}$ if $k > 1$.

3.3 Proof of Remark 2.3

To get the minimum over the values of α we study the sign of the expression, for any $\alpha \in]0, 1[$, of

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I \ .$$

A direct computation shows that the value α_λ which realizes the minimum of $g_\lambda(\alpha)$ is the positive solution of

$$\alpha^2(\lambda - 2I) + 6\alpha I - 4I = 0 \ . \tag{3.22}$$

4 An asymptotic eigenvalue upper bound

From Theorem 2.2 we get easily an asymptotic estimate for the right-hand side of (2.4) when λ tends to ∞ :

Corollary 4.1 *If assumptions (H_1) and (H_2) are satisfied and if moreover*

$$\mathbf{b}(x) \leq M(D(x))^{-\beta} \ , \quad 0 < \beta < \frac{3}{2}$$

for some $M > 0$, then the number of eigenvalues of the operator H_A^D smaller than λ satisfies, as $\lambda \to \infty$

$$N(H_A^D, \lambda) \leq (\frac{1}{2} + c_K)\lambda + \sqrt{\lambda}\sqrt{I} + O(1) \ , \tag{4.1}$$

where

$$I = \int_0^1 \left(\frac{A(r)}{r}\right)^2 rdr \ ,$$

and

- $c_K = \dfrac{3 - \log K}{2}$ *if* $0 < K \leq 1$

- $c_K = \left[\frac{1+\log K}{2} + \frac{1}{K}\right]$ *if* $K > 1$.

Inequality (4.1) still holds when we replace in the left-hand side $N(H_A^D, \lambda)$ *by* $N(H_{A'}^D, \lambda)$*, where* A' *is any gauge verifying* $dA' = dA = B$*.*

Proof.–

We define as previously, for any $\alpha \in]0,1[$,

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I$$

and we want to determine the asymptotic behavior as λ tends to ∞ of $g_\lambda(\alpha_\lambda)$, where α_λ is the minimum of $g_\lambda(\alpha)$.
From (3.22) we compute the following asymptotics

$$\alpha_\lambda = \frac{2\sqrt{I}}{\sqrt{\lambda}} + O(\frac{1}{\lambda})$$

$$\sqrt{1-\alpha_\lambda} = 1 - \frac{\sqrt{I}}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \, ,$$

and this gives the result.

□

Remark 4.2 *The leading term in the estimate (4.1) is of the same order than the leading term in the Weyl formula for the Dirichlet Laplacian (corresponding to the case* $A \equiv 0$*) in the unit disk.*

5 Appendix

5.1 The inequality of Hundertmarkt-Lieb-Thomas

We recall the sharp inequality of Hundertmarkt-Lieb-Thomas [11]

Theorem 5.1 *Let*

$$Lv(t) = -v"(t) - W(t)v(t), \quad W \geq 0 \quad W \in L^1(\mathbb{R})$$

be defined in the sense of quadratic forms on $\mathbb{R}$*, and assume that the negative spectrum of* L *is discrete. Denote by* $\{-\nu_k, k \in \mathbb{N}\}$ *the negative eigenvalues of* L*. Then*

$$\sum_{k\in\mathbb{N}} \sqrt{\nu_k} \leq \frac{1}{2} \int_{-\infty}^{+\infty} W(t)dt \, .$$

5.2 The Green function $G(r, r', 1)$ of the operator T_0.

Let us compute the diagonal element for the Green function $G(r, r', 1)$ of the operator T_0 defined by (3.14). $G(r, r', 1)$ is the solution of

$$((T_0 + 1)u)(r) = \delta_{r'}(r), \quad u(0) = u(1) = 0 \,. \tag{5.1}$$

We have

$G(r, r', 1) = A_1 u_1(r) + A_2 u_2(r) \quad r \leq r'$

$G(r, r', 1) = B_1 u_1(r) + B_2 u_2(r) \quad r > r' \,,$

where $u_1(r) = \sqrt{r} I_0(r)$ and $u_2(r) = \sqrt{r} K_0(r)$ are independent solutions of the related homogeneous equation, (I_0 and K_0 are the modified Bessel functions).
The coefficients depend of r' but we omit the indices for the sake of clarity. Due to the boundary conditions and to the fact that the derivative (with respect to r) of $G(r, r', 1)$ has the discontinuity in r' of a Heaviside function, they satisfy :

$$A_1 u_1(0) + A_2 u_2(0) = 0 \qquad B_1 u_1(1) + B_2 u_2(1) = 0$$

$$B_1 - A_1 = \frac{-u_2(r')}{W(r')} \qquad B_2 - A_2 = \frac{u_1(r')}{W(r')}$$

where $W(r')$ is the value of the Wronskian of u_1 and u_2 taken at the point r'.
The first equation is always satisfied since $u_1(0) = u_2(0) = 0$. Let us set $A_2 = 0$. We have $W(r') = u_1'(r')u_2(r') - u_1(r')u_2'(r') = r'\hat{W}(r')$ where $\hat{W}(r')$ is the Wronskian of the modified Bessel functions I_0 and K_0. As $r'\hat{W}(r') = 1$ (see [2]), we get after solving the above system, and doing $r = r'$:

$$G(r, r, 1) = u_1(r) \left[-u_1(r) \frac{u_2(1)}{u_1(1)} + u_2(r) \right]$$

$$= r I_0(r) \left[-I_0(r) \frac{K_0(1)}{I_0(1)} + K_0(r) \right] \,.$$

Using again the properties of the modified Bessel functions (see [2]) we can write

$$G(r, r, 1) \leq r I_0(r) K_0(r) \,.$$

The function

$$g(r) = \frac{I_0(r) K_0(r)}{1 + |\log r|}$$

has a limit at $r = 0$ equal to 1 (see [2]), so

$$c_0 = \max_{[0,1[} \frac{I_0(r) K_0(r)}{1 + |\log r|} \tag{5.2}$$

exists and

$$G(r, r, 1) \leq c_0 r (1 + |\log r|), \quad r \in [0, 1[\,.$$

Numerics suggest that g is decreasing on $[0, 1]$, so that one should have $c_0 = 1$. In next subsection, we give the proof of this result, which can not be found to our knowledge in the literature, and has been communicated to the author by J.P. Truc [20] :

Proposition 5.2 $\forall r \in]0,1] : \dfrac{I_0(r)K_0(r)}{1-\log r} \leq 1.$

5.3 Proof of Proposition 5.2

The modified Bessel function I_0 can be written as

$$I_0(r) = \sum_{k=0}^{+\infty} \frac{(\frac{r^2}{4})^k}{k!^2} = 1 + \frac{r^2}{4} + ... \tag{5.3}$$

Therefore we have

$$1 \leq I_0(r) \leq \sum_{k=0}^{+\infty} \frac{(\frac{r^2}{4})^k}{k!} = e^{\frac{r^2}{4}}$$

and

$$\forall r \in]0,1] \ : \ 1 \leq I_0(r) \leq e^{\frac{1}{4}} \ . \tag{5.4}$$

According to the expression of the modified Bessel function K_0

$$K_0(r) = -\Big(\log(r/2) + \gamma\Big) I_0(r) + \sum_{k=1}^{+\infty} \Big(\sum_{j=1}^{k} \frac{1}{j}\Big) \frac{(\frac{r^2}{4})^k}{k!^2} \tag{5.5}$$

where γ denotes the Euler constant, we compute that

$$K_0(r)I_0(r) - (1 - \log r) = \delta(r) - 1 \ , \tag{5.6}$$

where $\delta(r)$ denotes the following function :

$$\delta(r) = (1 - I_0(r)^2)\log r - \Big(-\log 2 + \gamma\Big) I_0(r)^2 + I_0(r) \sum_{k=1}^{+\infty} \Big(\sum_{j=1}^{k} \frac{1}{j}\Big) \frac{(\frac{r^2}{4})^k}{k!^2}. \tag{5.7}$$

Proposition 5.2 is then a straightforward consequence of the following Lemma

Lemma 5.3

$$\forall r \in]0,1] \ : \ \delta(r) \leq 1.$$

Proof.–

The function $\delta(r)$ splits into 3 positive parts, which we study separately .

- An upper bound for $(1 - I_0(r)^2)\log r$.
  From (5.4) we deduce $1 - I_0(r)^2 \geq 1 - e^{\frac{r^2}{2}}$, and :

  $$\forall r \in]0,1] \ : \ 0 \leq (1 - I_0(r)^2)\log r \leq \Big(e^{\frac{r^2}{2}} - 1\Big)(-\log r) \leq 0,11.$$

- An upper bound for $\Big(-\log 2 + \gamma \Big) I_0(r)^2$.
  A straightforward computation gives $-\gamma + \log 2 \leq 0.12$ so using that $I_0(r) \leq e^{\frac{1}{4}}$ we get

$$\Big(-\log 2 + \gamma \Big) I_0(r)^2 \leq 0.16.$$

- An upper bound for $I_0(r) \sum_{k=1}^{+\infty} \Big(\sum_{j=1}^{k} \frac{1}{j} \Big) \frac{(\frac{r^2}{4})^k}{k!^2}$.

  For $k \in \mathbb{N}^*$, we set $s_k = \sum_{j=1}^{k} \frac{1}{j}$. We have $s_1 = 1$. For $k \geq 2$, according to the inequality

$$\frac{1}{k} \leq \int_{k-1}^{k} \frac{dt}{t} = \log k - \log(k-1).$$

  we get that:

$$\sum_{j=2}^{k} \frac{1}{j} \leq \log k$$

  and for any integer k , $s_k \leq 1 + \log k$. Thus

$$\sum_{k=1}^{+\infty} \Big(\sum_{j=1}^{k} \frac{1}{j} \Big) \frac{(\frac{r^2}{4})^k}{k!^2} \leq \sum_{k=1}^{+\infty} \Big(\frac{1 + \log k}{k!} \Big) \frac{(\frac{r^2}{4})^k}{k!}.$$

  Noticing that, for any integer $k \geq 1$

$$0 \leq \frac{1 + \log k}{k!} \leq \frac{1 + \log k}{k} \leq 1,$$

  we can write, $\forall r \in]0, 1]$:

$$\sum_{k=1}^{+\infty} \Big(\sum_{j=1}^{k} \frac{1}{j} \Big) \frac{(\frac{r^2}{4})^k}{k!^2} \leq \sum_{k=1}^{+\infty} \frac{(\frac{r^2}{4})^k}{k!} = e^{\frac{r^2}{4}} - 1 \leq e^{\frac{1}{4}} - 1.$$

  Finally we have, for any $r \in]0, 1]$

$$I_0(r) \sum_{k=1}^{+\infty} \Big(\sum_{j=1}^{k} \frac{1}{j} \Big) \frac{(\frac{r^2}{4})^k}{k!^2} \leq e^{\frac{1}{4}} \Big(e^{\frac{1}{4}} - 1 \Big) \simeq 0.364 \, .$$

Summing the 3 previous estimates one gets : $\forall r \in]0, 1] \ : \ \delta(r) \leq 0.11 + 0.16 + 0.37 \leq 1$.

□

The optimality of the value $c_0 = 1$ is due to the fact that

$$\lim_{r \to 0+} \frac{K_0(r) I_0(r)}{1 - \ln r} = 1.$$

5.4 The Green function $G(r, r', k^2)$ of the operator T_0

We now compute the diagonal element for the Green function $G(r, r', k^2)$ of the operator T_0 defined by (3.14). $G(r, r', k^2)$ is the solution of

$$\left((T_0 + k^2)u\right)(r) = \delta_{r'}(r), \quad u(0) = u(1) = 0\,. \tag{5.8}$$

We have, as previously

$$G(r, r, k^2) = u_1(r)\left[-u_1(r)\frac{u_2(1)}{u_1(1)} + u_2(r)\right]$$

where $u_1(r) = \sqrt{r}I_0(kr)$ and $u_2(r) = \sqrt{r}K_0(kr)$ are independent solutions of the related homogeneous equation. This leads to

$$G(r, r, k^2) = rI_0(kr)\left[-I_0(kr)\frac{K_0(k)}{I_0(k)} + K_0(kr)\right] \leq rI_0(kr)K_0(kr) \leq r(1+|\log(kr)|)\,.$$

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