Lieb-Thirring inequalities for radial magnetic bottles in the disk

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Abstract

We consider a Schrödinger operator H_A with a non vanishing radial magnetic field $B = dA$ and Dirichlet boundary conditions on the unit disk. We assume growth conditions on B near the boundary which guarantee in particular the compactness of the resolvent of this operator. Under some assumptions on an additional radial potential V the operator $H_B + V$ has a discrete negative spectrum and we prove a Lieb-Thirring inequality on these negative eigenvalues. As a consequence we get an explicit upperbound of the number $N(H_A, \lambda)$ of eigenvalues of H_A less than any positive value λ, which depends on the minimum of B and on the integral of the square of any gauge associated to B.

1 Introduction

Let us consider a particle in a domain Ω in $\mathbb{R}^2$ in the presence of a *magnetic field* B. We define the 2-dimensional magnetic Laplacian associated to this particle as follows:

Let A a *magnetic potential* associated to B ; it means that A is a smooth real one-form on $\Omega \subset \mathbb{R}^2$, given by $A = \sum_{j=1}^2 a_j dx_j$, and that the *magnetic field* B is the two-form $B = dA$. We have $B(x) = \mathbf{b}(x) dx_1 \wedge dx_2$ with $\mathbf{b}(x) = \partial_1 a_2(x) - \partial_2 a_1(x)$. The magnetic connection $\nabla = (\nabla_j)$ is the differential operator defined by

$$\nabla_j = \frac{\partial}{\partial x_j} - i a_j \ .$$

The 2-dimensional magnetic Schrödinger operator H_A is defined by

$$H_A = -\sum_{j=1}^{2} \nabla_j^2 \ .$$

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The magnetic Dirichlet integral $h_A = \langle H_A.|.\rangle$ is given, for $u \in C_0^\infty(\Omega)$, by

$$h_A(u) = \int_\Omega \sum_{j=1}^{2} |\nabla_j u|^2 |dx| \, . \tag{1.1}$$

From the previous definitions and the fact that the formal adjoint of ∇_j is $-\nabla_j$, it is clear that the operator H_A is symmetric on $C_0^\infty(\Omega)$.
In [5] we discuss the essential self-adjointness of this operator. The result in dimension 2 is the following

Theorem 1.1 *Assume that $\partial\Omega$ is compact and that $B(x)$ satisfies near $\partial\Omega$*

$$\mathbf{b}(x) \geq (D(x))^{-2} \, , \tag{1.2}$$

then the Schrödinger operator H_A is essentially self-adjoint. ($D(x)$ denotes the distance to the boundary). This still holds true for any gauge A' such that $dA' = dA = B$.

We have, using Cauchy-Schwarz inequality,

$$|\langle \mathbf{b}(x)u, u\rangle| = |\langle [\nabla_1, \nabla_2]u, u\rangle| \leq \|\nabla_1 u\|^2 + \|\nabla_2 u\|^2 \quad u \in C_o^\infty(\Omega).$$

This gives the well-known lower bound

$$\forall u \in C_o^\infty(\Omega), \; h_A(u) \geq \left| \int_\Omega \mathbf{b}(x)|u|^2 |dx| \right| \, . \tag{1.3}$$

In this paper, we do not use the conditions (1.2) but we assume nevertheless that $\mathbf{b}(\mathbf{x})$ grows to infinity as x approaches the boundary. The operator H_A^D defined by Friedrich's extension of the quadratic form h_A has a compact resolvent. We call such an operator a magnetic bottle, by similarity with magnetic bottles in the whole space ([1], [4], [21]). We add a suitable negative potential in order to have a discrete negative spectrum and we address the question of the existence of Lieb-Thirring inequalities ([17], [15]) in the radially symmetric case.

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2 Inequalities: the main results

We consider a magnetic field $B = \mathbf{b}(x)dx_1 \wedge dx_2$ and a scalar potential V on the unit disk $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 |\ x_1^2 + x_2^2 = r^2 < 1\}$ so that

- (H_1) $K = \inf_{x\in\Omega} \mathbf{b}(x) > 0$ and $\mathbf{b}(x) \to +\infty$ as $D(x) \to 0$
  (i.e as x approaches the boundary.)
- (H_2) B is radially symmetric,
- (H_3) $V \in L^1(\Omega)$, V radial and non negative, V bounded from above .

From assumption (H_1) and from inequality (1.3) we deduce that for any gauge A associated to B, the operator H_A has a compact resolvent, and assumption (H_3) entails that the negative spectrum of $H_A - V$ is discrete, where $H_A - V$ denotes the operator defined by Friedrich's extension of the quadratic form $h_A - V$. Using assumption (H_2) we can write any vector potential as $A = A(r)d\theta$.
The first theorem deals with the number $N(A, V)$ of negative eigenvalues of the operator $H_A - V$. Noticing that we have $N(A, V) = N(A', V)$ for any gauge A' so that $dA' = dA = B$, we will prove that

Theorem 2.1 *If assumptions $(H_1)(H_2)(H_3)$ are verified and if moreover*

$$\mathbf{b}(x) \leq (D(x))^{-\beta}\,, \quad \beta < \frac{3}{2} \tag{2.1}$$

then

$$N(A,V) \leq \frac{1}{\sqrt{1-\alpha}} \int_{[0,1[} [(\frac{1}{\alpha}-1)A'^2(r)+V(r)]rdr+2\int_0^1 \left[1+|\log[r\sqrt{K}]|\right] V(r)rdr$$

for any $\alpha \in]0,1[$ and any radial gauge A' such that $dA' = dA = B$.

The second theorem is a consequence of the first one and provides an explicit upper-bound of the number $N(H_A, \lambda)$ of the eigenvalues of H_A less than any positive value λ :

Theorem 2.2 *If assumptions (H_1) and (H_2) are verified and if moreover*

$$\mathbf{b}(x) \leq (D(x))^{-\beta}\,, \quad \beta < \frac{3}{2}$$

then the number of eigenvalues of the operator H_A less than λ satisfies the following inequality

$$N(A,\lambda) \leq c_K\lambda + \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha}\int_{[0,1[} rA'^2(r)dr \tag{2.2}$$

with

- $c_K = \dfrac{3 - \log K}{2}$ *if* $K \leq 1$

  and

- $c_K = \dfrac{1 + \log K}{2} + \dfrac{1}{K}$ *if* $K > 1$,

for any $\alpha \in]0,1[$ *and any radial gauge* A' *such that* $dA' = dA = B$.

Remark 2.3 *The minimum of the righthandside is obtained by choosing the radial gauge* A' *so that* $\int_0^1 A'(r) r dr = 0$ *, and then by taking* $\alpha_\lambda = \dfrac{-6I + \sqrt{I^2 + 4I\lambda}}{\lambda - 2I}$ *with* $I := \int_{[0,1[} r A'^2(r) dr$.

Remark 2.4 *The inequality of Theorem 2.1 is a "magnetic" version of the Cwikel-Lieb-Rosenblum inequality [6] [16] [18]. CLR inequalities apply to Schrödinger operators in* $\mathbb{R}^d$ *for* $d \geq 3$ *and* $A \equiv 0$ *and are a particular case of Lieb-Thirring inequalities. In the case of dim 2 (and* $A \neq 0$*), analogues of CLR inequalities can be found in [3] and [14] (for a Aharanov-Bohm magnetic field) and more recently in [12] (for a large class of magnetic fields, in a weighted version). Let us emphasize that the bounds in [12] in the radial case do not depend on the magnetic field and are obtained only for bounded magnetic potentials* A*, assumption which we do not need (example 2.5, with* $1 < \beta < 3/2$*). Moreover the constants in our results are explicit. This implies that our theorems can not be derived from [12].*
Concerning general magnetic Lieb-Thirring inequalities we refer to [10] for Lieb-Thirring inequalities for constant magnetic fields in dim 2 and 3 which depend on the field strength, to [7] and [8] for magnetic Lieb-Thirring inequalities related to Pauli operators, and to [9] for links between magnetic and non magnetic Lieb-Thirring inequalities.

Example 2.5 *Consider a magnetic field* B *as in the definition (3.1) below, and assume* $b(r) \equiv 1$ *and* $\beta \neq 1$ *. Then* $c_K = \frac{3}{2}$*, the optimal gauge is* $A' = A_\beta(r) d\theta$ *with*

$$A_\beta = \frac{1}{1-\beta} \left[\frac{1}{(1-r)^{\beta-1}} - \frac{1}{(2-\beta)(3-\beta)} \right] , \tag{2.3}$$

and the corresponding minimal value of I *is*

$$I_\beta = \int_0^1 A_\beta^2(r) r dr = \frac{1}{(1-\beta)^2 (3-2\beta)(4-2\beta)} . \tag{2.4}$$

3 Proofs

3.1 Proof of Theorem 2.1

Let us introduce the polar coordinates $x = (r, \theta), r \in \mathbb{R}^+, \theta \in [0, 2\pi[$. Due to assumption (2.1) the magnetic field we have to consider is of the type

$$B(r) = \frac{b(r)}{(1-r)^\beta} dr \wedge d\theta \ , \text{ with } \quad \max_{[0,1[} b(r) \leq M \text{ and } \quad \beta < \frac{3}{2}. \tag{3.1}$$

We first prove the following

Lemma 3.1 *If B satisfies (3.1), then, for any radial magnetic potential A associated to B, there exists a constant K such that A writes*

- *if* $\beta \neq 1 \quad A = A(r)d\theta = \dfrac{a(r)}{(1-r)^{\beta-1}} d\theta$
  with $a(r) = K(1-r)^{\beta-1} + \tilde{a}, \quad \max\limits_{[0,1[} \tilde{a}(r) \leq C.$
- *if* $\beta = 1 \quad A = A(r)d\theta = a(r)\ln(1-r)d\theta$
  with $a(r) = \dfrac{K}{\ln(1-r)} + \tilde{a}, \quad \max\limits_{[0,1[} \tilde{a}(r) \leq C.$

In particular $\displaystyle\int_{[0,1[} rA^2(r)dr < \infty$.

Proof.–

Let us explain the case $\beta \neq 1$. The method for the case $\beta = 1$ is the same. The function $a(r)$ satisfies the equation

$$(\beta - 1)a(r) - (1-r)a'(r) = b(r) \ .$$

This implies that

$$a(r) = k(r)(1-r)^{\beta-1}, \text{ with } \quad k(r) = \int_r^1 b(t)(1-t)^{-\beta} dt + K \ . \tag{3.2}$$

From (3.1) we get
$|\int_r^1 b(t)(1-t)^{-\beta}dt| \leq M \int_r^1 (1-t)^{-\beta}dt \leq M\frac{(1-r)^{-\beta+1}}{1-\beta}$ and the result follows.

□

We come now to the proof of Theorem 2.1, following the method of [13]. The quadratic form associated to $H_A^D - V$ can be rewritten as

$$h_{A,V}(u) = \int_0^1 \int_0^{2\pi} \left[|\frac{\partial u}{\partial r}|^2 - V(r)|u^2| + r^{-2} \left[[\frac{\partial}{\partial \theta} - iA(r)]u \right]^2 \right] r dr d\theta \tag{3.3}$$

for any $u \in C_0^\infty([0,1[\times[0,2\pi[)$. Changing variables $r = e^t$ and denoting $w(t,\theta) = u(e^t,\theta)$ for $t \in]-\infty,0[$ and $\theta \in [0,2\pi[$ we transfer the form $h_{A,V}(u)$ to

$$\tilde{h}_{A,V}(w) = \int_{-\infty}^{0}\int_{0}^{2\pi}\left[|\frac{\partial w}{\partial t}|^2 - \tilde{V}(t)|w^2| + \left[[\frac{\partial}{\partial\theta} - if(t)]w\right]^2\right]dtd\theta \qquad (3.4)$$

with

$$\tilde{V}(t) = e^{2t}V(e^t), \quad f(t) = e^t A(e^t)\ .$$

By expanding a given function $w \in C_0^\infty([-\infty,0[\times[0,2\pi[)$ into a Fourier series we obtain that $\tilde{h}_{A,V}(w) = \oplus_{l\in\mathbb{Z}} h_{\ell,V}(w_\ell)$ with

$$h_{\ell,V}(v) = \int_{-\infty}^{0}|\frac{\partial v}{\partial t}|^2 + \left[(\ell - f(t))^2 - \tilde{V}(t)\right]|v^2|\ dt\ ,$$

and $w_\ell = \Pi_\ell(w)$ where Π_ℓ is the projector acting as

$$\Pi_\ell(w)(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} e^{il(\theta-\theta')}w(r,\theta')d\theta'\ .$$

We write, for any $\alpha \in]0,1[$ and any $\ell \in \mathbb{Z}^*$

$$h_{\ell,V}(v) \geq \int_{-\infty}^{0}|\frac{\partial v}{\partial t}|^2 + \left[(1-\frac{1}{\alpha})f^2(t) - \tilde{V}(t) + (1-\alpha)\ell^2\right]|v^2|\ dt\ .$$

Let us denote by L_α the operator associated via Friedrich's extension to the quadratic form

$$q_\alpha(v) = \int_{-\infty}^{0}|\frac{\partial v}{\partial t}|^2 D_t^2 + \left[(1-\frac{1}{\alpha})f^2(t) - \tilde{V}(t)\right]|v^2|\ dt\ .$$

L_α and q_α depend on V but we skip the reference to V in notations for the sake of simplicity. Since

$$h_{\ell,V} \geq q_\alpha + (1-\alpha)\ell^2\ ,$$

the number $N(h_{\ell,V})$ of negative eigenvalues of $h_{\ell,V}$ is less than the number of negative eigenvalues of $L_\alpha + (1-\alpha)\ell^2$. So denoting by $\{-\mu_k^\alpha\}$ the negative eigenvalues of L_α and by I_ℓ the set $\{k \in \mathbb{N}; -\mu_k^\alpha + (1-\alpha)\ell^2 < 0\}$ for any $\ell \in \mathbb{Z}^*$, we get

$$N(A,V) \leq \sum_{\ell\in\mathbb{Z}^*}\sum_{k\in I_\ell} 1 + N(h_{0,V})\ .$$

Noticing that the sum in the righthandside is taken over the (ℓ,k) so that $0 < |\ell| \leq \frac{1}{\sqrt{1-\alpha}}\sqrt{\mu_k^\alpha}$ we write

$$N(A,V) \leq \frac{2}{\sqrt{1-\alpha}}\sum_{k\in\mathbb{N}}\sqrt{\mu_k^\alpha} + N(h_{0,V})\ . \qquad (3.5)$$

Let us extend the functions f and $\tilde{V}$ to $\mathbb{R}$ by zero and denote respectively by f_1 and $\tilde{V}_1$ these extensions.
Since $C_0^\infty([-\infty,0[) \subset C_0^\infty(\mathbb{R})$, the negative eigenvalues $\{-\nu_k^\alpha\}$ of the operator L_1^α associated via Friedrich's extension to the quadratic form

$$q_1^\alpha(v) = \int_{-\infty}^0 |\frac{\partial v}{\partial t}|^2 D_t^2 + \left[(1-\frac{1}{\alpha})f_1^2(t) - \tilde{V}_1(t)\right]|v^2|\, dt$$

verify

$$\sum_{k\in\mathbb{N}} \sqrt{\mu_k^\alpha} \leq \sum_{k\in\mathbb{N}} \sqrt{\nu_k^\alpha}. \tag{3.6}$$

Applying the sharp inequality of Hundertmarkt-Lieb-Thomas [11] (see Appendix) to the operator L_1^α we get

$$\begin{aligned}\sum_{k\in\mathbb{N}} \sqrt{\nu_k^\alpha} &\leq \frac{1}{2}\int_{-\infty}^{+\infty}\left[(\frac{1}{\alpha}-1)f_1^2(t) + \tilde{V}_1(t)\right]dt \\ &\leq \frac{1}{2}\int_{-\infty}^{0}\left[(\frac{1}{\alpha}-1)f^2(t) + \tilde{V}(t)\right]dt \\ &\leq \frac{1}{2}\int_0^1\left[(\frac{1}{\alpha}-1)A^2(r) + V(r)\right]r dr\,. \end{aligned} \tag{3.7}$$

To conclude we need the following

Lemma 3.2 *Assume that $K = \inf_{x\in\Omega} \mathbf{b}(x) > 0$. Then for any $\varepsilon \in]0,1[$*

$$N(h_{0,V}) = N(h_{0,0} - V) \leq \frac{1}{\varepsilon}\int_0^1 \left[1 + |\log(\sqrt{\frac{(1-\varepsilon)K}{\varepsilon}}r)|\right] V(r) r dr\,, \tag{3.8}$$

In particular

$$N(h_{0,V}) \leq 2\int_0^1 \left[1 + |\log(\sqrt{K}r)|\right] V(r) r dr\,. \tag{3.9}$$

Proof.–

Step 1 :From (1.3) we get that $h_A(u) \geq K\int_\Omega |u|^2|dx| \quad \forall u \in C_0^\infty(\Omega)$, which implies for $h_{0,0}$ (returning to the variable r and considering $V \equiv 0$),

$$\begin{aligned} h_{0,0}(w) &= \int_0^1 \left[|\frac{\partial w}{\partial r}|^2 + r^{-2}A^2(r)|w^2|\right] r dr \\ &\geq K\int_0^1 |w|^2 r dr \quad \forall w \in C_o^\infty([0,1])\,. \end{aligned}$$

We write for any $\varepsilon \in]0,1[$

$$N(h_{0,0}-V) \leq N(\varepsilon h_{0,0}+(1-\varepsilon)K-V) \leq N\left(h_{0,0}+\frac{(1-\varepsilon)K}{\varepsilon}-\frac{V}{\varepsilon}\right), \tag{3.10}$$

where we have used the fact that multiplying an operator by a positive constant does not change the number of its negative eigenvalues.

Step 2 : We establish the following upperbound :

$$N(h_{0,0}+1-V) = N(h_{0,V}+1) \leq \int_0^1 [1+|\log r|]\, V(r) r dr\,. \tag{3.11}$$

We have

$$h_{0,V}(w) = \int_0^1 \left[|\frac{\partial w}{\partial r}|^2 + \left[r^{-2}A^2(r) - V(r)\right] |w^2| \right] r dr$$

$$\geq \int_0^1 \left[|\frac{\partial w}{\partial r}|^2 - V(r)|w^2| \right] r dr \quad \forall w \in C_o^\infty([0,1])\,.$$

By the variational principle,

$$N(h_{0,V}+1) \leq N(P_0+1-V), \tag{3.12}$$

where P_0 is the operator generated by the closure, in $L^2([0,1], rdr)$ of the quadratic form

$$\int_0^1 |\frac{\partial w}{\partial r}|^2 r dr, \quad w \in C_o^\infty([0,1])\,.$$

Considering the mapping $U : L^2([0,1], rdr) \to L^2([0,1], dr)$ defined by $(Uf)(r) = r^{1/2} f(r)$ we get that

$$N(P_0+1-V) \leq N(T_0+1-V) \tag{3.13}$$

where the operator $T_0 = UP_0U^{-1}$ is the Sturm Liouville operator on $L^2([0,1], dr)$ acting on its domain by

$$(T_0 u)(r) = -u"(r) - \frac{u(r)}{4r^2}, \quad u(0) = u(1) = 0\,. \tag{3.14}$$

The upperbound (3.11) will follow from the properties of $G(r,r,1)$, the diagonal element of the integral kernel of $(T_0+1)^{-1}$. Precisely we have

$$G(r,r,1) \leq r(1+|\log r|), \quad r \in [0,1[\,. \tag{3.15}$$

The proof of (3.15) is given in Appendix B. The Birman-Swinger principle then yields

$$N(T_0+1-V) \leq \int_0^1 G(r,r,1) V(r) dr \leq \int_0^1 [1+|\log r|]\, V(r) r dr\,. \tag{3.16}$$

This ends the proof of (3.11), together with the inequalities (3.12) and (3.13).

Step 3 : We mimick the previous method to get, for any strictly positive number k

$$N(h_{0,0} + k^2 - V) \leq \int_0^1 [1 + |\log(kr)|]\, V(r) r dr\ . \tag{3.17}$$

Due to the Birman-Swinger principle it suffices to prove that, for any strictly positive number k

$$G(r, r, k^2) \leq r(1 + |\log(kr)|), \quad r \in [0, 1[\ . \tag{3.18}$$

This is done in Appendix C.

Step 4 :Returning to (3.10) and applying (3.17) with $k^2 = \frac{(1-\varepsilon)K}{\varepsilon}$ and $\frac{V}{\varepsilon}$ instead of V we get, for any $\varepsilon \in]0, 1[$

$$N(h_{0,0} - V) \leq N\left(h_{0,0} + \frac{(1-\varepsilon)K}{\varepsilon} - \frac{V}{\varepsilon}\right) \tag{3.19}$$

$$\leq \frac{1}{\varepsilon} \int_0^1 \left[1 + |\log(\sqrt{\frac{(1-\varepsilon)K}{\varepsilon}} r)|\right] V(r) r dr\ , \tag{3.20}$$

and taking $\varepsilon = \frac{1}{2}$ we obtain Lemma 3.2.

□

Theorem 2.1 follows from Lemma 3.2 together with inequalities (3.5), (3.6), and (3.7).

3.2 Proof of Theorem 2.2

Noticing that for any $\lambda > 0$ the constant potential $V(x) \equiv \lambda$ is in $L^1(\Omega)$, and that $N(A, \lambda)$ denotes the number of eigenvalues of the o perator H_A^D less than λ, we apply Theorem 2.1 to $V(x) \equiv \lambda$. To get the result it suffices to compute $\int_0^1 [1 + |\log(kr)|]\, r dr$. We get after computation that

$$\int_0^1 [1 + |\log(kr)|]\, r dr\ = \gamma_k, \tag{3.21}$$

with

- $\gamma_k = \dfrac{3 - 2\log k}{4}$ if $k \leq 1$
- $\gamma_k = \dfrac{1 + 2\log k}{4} + \dfrac{1}{2k^2}$ if $k > 1$.

3.3 Proof of Remark 2.3

The choice of $A' = A + c$ is obtained by taking the minimum over the constants c of the function $F(c) = \int_0^1 (A+c)^2(r)rdr$. To get the minimum over the values of α we study the sign of the expression, for any $\alpha \in]0,1[$, of

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I \ .$$

A direct computation shows that the value α_λ which realizes the minimum of $g_\lambda(\alpha)$ is the positive solution of

$$\alpha^2(\lambda - 2I) + 6\alpha I - 4I = 0 \ . \tag{3.22}$$

4 An eigenvalue asymptotic upperbound

From Theorem 2.2 we get easily an asymptotic estimate for the righthandside of (2.2) when λ tends to ∞ :

Corollary 4.1 *If assumptions (H_1) and (H_2) are satisfied and if moreover*

$$\mathbf{b}(x) \leq (D(x))^{-\beta} \ , \quad \beta < \frac{3}{2}$$

then the number of eigenvalues of the operator H_A less than λ satisfies, as λ tends to ∞

$$N(A,\lambda) \leq (\frac{1}{2} + c_K)\lambda + \sqrt{\lambda}\sqrt{I} + O(1) \ , \tag{4.1}$$

where

$$I = \int_0^1 A'^2(r)rdr \ ,$$

and

- $c_K = \dfrac{3 - \log K}{2}$ *if* $K \leq 1$

  and

- $c_K = \dfrac{1 + \log K}{2} + \dfrac{1}{K}$ *if* $K > 1$,

This holds for any radial gauge A' associated to B, and the minimum of the righthandside is obtained by choosing A' so that $\int_0^1 A'(r)rdr = 0$.

Example 4.2 *Assume $b(r) \equiv 1$ in (3.1) and $\beta \neq 1$. Then $c_K = \frac{3}{2}$ and the minimum is obtained for $I = I_\beta = \int_0^1 A_\beta^2(r)rdr$, where A_β is defined as in (2.3), so that*

$$I_\beta = \frac{1}{(1-\beta)^2(3-2\beta)(4-2\beta)} \ . \tag{4.2}$$

Proof.–

We define as previously, for any $\alpha \in]0,1[$,

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I$$

and we want to determine the asymptotic behavior as λ tends to ∞ of $g_\lambda(\alpha_\lambda)$, where α_λ is the minimum of $g_\lambda(\alpha)$.
From (3.22) we compute the following asymptotics

$$\alpha_\lambda = \frac{2\sqrt{I}}{\sqrt{\lambda}} + O(\frac{1}{\lambda})$$

$$\sqrt{1-\alpha_\lambda} = 1 - \frac{\sqrt{I}}{\sqrt{\lambda}} + O(\frac{1}{\lambda}) \ ,$$

and this gives the result.
The minimal value is obtained as previously by taking the minimum over the constants c of the function $F(c) = \int_0^1 (A+c)^2(r) r dr$.

□

Remark 4.3 *The leading term in the estimate (4.1) is of the same order than the leading term in the Weyl formula for the Dirichlet Laplacian (corresponding to the case $A \equiv 0$) in the unit disk.*

5 Appendix A

We recall the sharp inequality of Hundertmarkt-Lieb-Thomas

Theorem 5.1 *Let*

$$Lv(t) = -v"(t) - W(t)v(t), \quad W \geq 0 \quad W \in L^1(\mathbb{R})$$

be defined in the sense of quadratic forms on $\mathbb{R}$, and assume that the negative spectrum of L is discrete. Denote by $\{-\nu_k, k \in \mathbb{N}\}$ the negative eigenvalues of L. Then

$$\sum_{k\in\mathbb{N}} \sqrt{\nu_k} \leq \frac{1}{2} \int_{-\infty}^{+\infty} W(t) dt \ .$$

6 Appendix B

Let us compute the diagonal element for the Green function $G(r, r', 1)$ of the operator T_0 defined by (3.14). $G(r, r', 1)$ is the solution of

$$((T_0 + 1)u)(r) = \delta_{r'}(r), \quad u(0) = u(1) = 0\,. \tag{6.1}$$

We have

$G(r, r', 1) = A_1 u_1(r) + A_2 u_2(r) \quad r \leq r'$

$G(r, r', 1) = B_1 u_1(r) + B_2 u_2(r) \quad r > r'$,

where $u_1(r) = \sqrt{r} I_0(r)$ and $u_2(r) = \sqrt{r} K_0(r)$ are independent solutions of the related homogeneous equation, (I_0 and K_0 are the modified Bessel functions).
The coefficients depend of r' but we omit the indices for the sake of clarity. Due to the boundary conditions and to the fact that the derivative (with respect to r) of $G(r, r', 1)$ has the discontinuity in r' of a Heaviside function, they satisfy :

$$A_1 u_1(0) + A_2 u_2(0) = 0 \qquad B_1 u_1(1) + B_2 u_2(1) = 0$$

$$B_1 - A_1 = \frac{-u_2(r')}{W(r')} \qquad B_2 - A_2 = \frac{u_1(r')}{W(r')}$$

where $W(r')$ is the value of the Wronskian of u_1 and u_2 taken at the point r'.
The first equation is always satisfied since $u_1(0) = u_2(0) = 0$. Let us set $A_2 = 0$. We have $W(r') = u_1'(r')u_2(r') - u_1(r')u_2'(r') = r'\hat{W}(r')$ where $\hat{W}(r')$ is the Wronskian of the modified Bessel functions I_0 and K_0. As $r'\hat{W}(r') = 1$ (see [2]), we get after solving the above system, and doing $r = r'$:

$$G(r, r, 1) = u_1(r)\left[-u_1(r)\frac{u_2(1)}{u_1(1)} + u_2(r)\right]$$

$$= rI_0(r)\left[-I_0(r)\frac{K_0(1)}{I_0(1)} + K_0(r)\right]\,.$$

Using again the properties of the modified Bessel functions we can write

$$G(r, r, 1) \leq rI_0(r)K_0(r)$$

and observe (see figure 1) that the function

$$g(r) = \frac{I_0(r)K_0(r)}{1 + |\log r|}$$

is decreasing on $]0, \infty[$ and has a limit at $r = 0$ equal to 1, so we get

$$G(r, r, 1) \leq r(1 + |\log r|), \quad r \in [0, 1[\,.$$

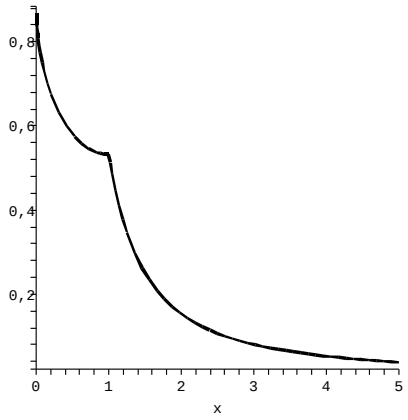

Figure 1: The function g

7 Appendix C

We now compute the diagonal element for the Green function $G(r, r', k^2)$ of the operator T_0 defined by (3.14). $G(r, r', k^2)$ is the solution of

$$\left((T_0 + k^2)u\right)(r) = \delta_{r'}(r), \quad u(0) = u(1) = 0\,. \tag{7.1}$$

We have, as previously

$$G(r, r, k^2) = u_1(r)\left[-u_1(r)\frac{u_2(1)}{u_1(1)} + u_2(r)\right]$$

where $u_1(r) = \sqrt{r}I_0(kr)$ and $u_2(r) = \sqrt{r}K_0(kr)$ are independent solutions of the related homogeneous equation. This leads to

$$G(r, r, k^2) = rI_0(kr)\left[-I_0(kr)\frac{K_0(k)}{I_0(k)} + K_0(kr)\right] \leq rI_0(kr)K_0(kr) \leq r(1+|\log(kr)|)\,.$$

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