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Lieb-Thirring inequalities for radial magnetic bottles in the disk

Françoise Truc*

April 22, 2011

Abstract

We consider a Schrödinger operator H with a magnetic field and Dirichlet boundary conditions on the unit disk. We assume growth conditions on the magnetic field near the boundary which guarantee compactness of the resolvent of this operator. Under some assumptions on an additional potential V the operator $H + V$ has a discrete negative spectrum and we prove a Lieb-Thirring inequality on these negative eigenvalues if moreover the magnetic field and the potential are assumed to be radially symmetric.

1 Introduction

Let us consider a particle in a domain Ω in \mathbb{R}^2 in the presence of a *magnetic field* B . We define the 2-dimensional magnetic Laplacian associated to this particle as follows:

Let A a *magnetic potential* associated to B ; it means that A is a smooth real one-form on $\Omega \subset \mathbb{R}^2$, given by $A = \sum_{j=1}^2 a_j dx_j$, and that the *magnetic field* B is the two-form $B = dA$. We have $B(x) = \mathbf{b}(x) dx_1 \wedge dx_2$ with $\mathbf{b}(x) = \partial_1 a_2(x) - \partial_2 a_1(x)$. The magnetic connection $\nabla = (\nabla_j)$ is the differential operator defined by

$$\nabla_j = \frac{\partial}{\partial x_j} - ia_j .$$

The 2-dimensional magnetic Schrödinger operator H_A is defined by

$$H_A = - \sum_{j=1}^2 \nabla_j^2 .$$

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The magnetic Dirichlet integral $h_A = \langle H_A \cdot | \cdot \rangle$ is given, for $u \in C_0^\infty(\Omega)$, by

$$h_A(u) = \int_{\Omega} \sum_{j=1}^2 |\nabla_j u|^2 |dx| . \quad (1.1)$$

From the previous definitions and the fact that the formal adjoint of ∇_j is $-\nabla_j$, it is clear that the operator H_A is symmetric on $C_0^\infty(\Omega)$.

In [4] we discuss the essential self-adjointness of this operator. The result in dimension 2 is the following

Theorem 1.1 *Assume that $\partial\Omega$ is compact and that $B(x)$ satisfies near $\partial\Omega$*

$$\mathbf{b}(x) \geq (D(x))^{-2} , \quad (1.2)$$

then the Schrödinger operator H_A is essentially self-adjoint. ($D(x)$ denotes the distance to the boundary) . This still holds true for any gauge A' such that $dA' = dA = B$.

We have, using Cauchy-Schwarz inequality,

$$|\langle \mathbf{b}(x)u, u \rangle| = |\langle [\nabla_1, \nabla_2]u, u \rangle| \leq \|\nabla_1 u\|^2 + \|\nabla_2 u\|^2 \quad u \in C_0^\infty(\Omega).$$

This gives the well-known lower bound

$$\forall u \in C_0^\infty(\Omega), \quad h_A(u) \geq \left| \int_{\Omega} \mathbf{b}(x)|u|^2 |dx| \right| . \quad (1.3)$$

In this paper, we do not use the conditions (1.2) but we assume nevertheless that $\mathbf{b}(\mathbf{x})$ grows to infinity as x approaches the boundary. The operator H_A^D defined by Friedrich's extension of the quadratic form h_A has a compact resolvent. We call such an operator a magnetic bottle, by similarity with magnetic bottles in the whole space ([1], [3], [20]). We add a suitable negative potential in order to have a discrete negative spectrum and we address the question of the existence of Lieb-Thirring inequalities ([16], [14]) in the radially symmetric case.

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2 Inequalities: the main results

Consider a magnetic field $B = \mathbf{b}(x)dx_1 \wedge dx_2$ on the unit disk $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = r^2 < 1\}$ so that

- (H_1) $\mathbf{b}(x) > 0 \quad \forall x \in \Omega$ and $\mathbf{b}(x) \rightarrow +\infty$ as $D(x) \rightarrow 0$
(i.e as x approaches the boundary.)

Then we deduce from inequality (1.3) that for any gauge A associated to B , the operator H_A^D has a compact resolvent. Consider a potential V so that

- (H_2) $V \in L^1(\Omega)$, V radial and non negative, and such that the negative spectrum of $H_A^D - V$ is discrete (for example V bounded from above),
where $H_A^D - V$ denotes the operator defined by Friedrich's extension of the quadratic form $h_A - V$. We are interested in the number $N(A, V)$ of negative eigenvalues of the operator $H_A^D - V$. Then, with the additional assumption that
- (H_3) B is radially symmetric,

we prove the following

Theorem 2.1 *If assumptions $(H_1)(H_2)(H_3)$ are verified and if moreover*

$$\mathbf{b}(x) \leq (D(x))^{-\beta}, \quad \beta < \frac{3}{2} \quad (2.1)$$

then

$$N(A, V) \leq \frac{1}{2\sqrt{1-\alpha}} \int_{[0,1[} \left[\left(\frac{1}{\alpha} - 1 \right) A^2(r) + V(r) \right] r dr$$

for any $\alpha \in]0, 1[$ and any radial gauge A associated to B .

Noticing that for any $\lambda > 0$ the constant potential $V(x) = \lambda$ is in $L^1(\Omega)$, and that $N(A, \lambda)$ denotes the number of eigenvalues of the operator H_A^D less than λ , we get the

Corollary 2.2 *If assumptions (H_1) and (H_3) are verified and if moreover*

$$\mathbf{b}(x) \leq (D(x))^{-\beta}, \quad \beta < \frac{3}{2} \quad (2.2)$$

then the number of eigenvalues of the operator H_A^D less than λ satisfies the following inequality

$$N(A, \lambda) \leq \frac{\lambda}{4\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{2\alpha} \int_{[0,1[} r A^2(r) dr \quad (2.3)$$

for any $\alpha \in]0, 1[$ and any radial gauge A associated to B .

Remark 2.3 *The minimum of the righthandside is obtained by choosing the radial gauge A_β so that $\int_0^1 A_\beta(r)rdr = 0$, and then by taking $\alpha_\lambda = \frac{-6I + \sqrt{I^2 + 4I\lambda}}{\lambda - 2I}$ with $I := \int_{[0,1[} rA_\beta^2(r)dr$.*

Remark 2.4 *The inequality of Theorem 2.1 is a "magnetic" version of the Cwikel-Lieb-Rosenblum inequality [5] [15] [17]. CLR inequalities apply to Schrödinger operators in \mathbb{R}^d for $d \geq 3$ and $A \equiv 0$ and are a particular case of Lieb-Thirring inequalities . In the case of dim 2 (and $A \neq 0$), analogues of CLR inequalities can be found in [2] and [13] (for a Aharanov-Bohm magnetic field) and more recently in [11] (for a large class of magnetic fields, in a weighted version). Our result can not be derived from [11]. Lieb-Thirring inequalities for constant magnetic fields in dim 2 and 3 are proved in [9]. In [6] and [7] magnetic Lieb-Thirring inequalities are proved for Pauli operators. For links between magnetic and non magnetic Lieb-Thirring inequalities see [8].*

Example 2.5 *Assume $b(r) \equiv 1$ in (3.1) and $\beta \neq 1$. Then the optimal gauge is $A = A_\beta(r)d\theta$ with*

$$A_\beta = \frac{1}{1-\beta} \left[\frac{1}{(1-r)^{\beta-1}} - \frac{2}{(2-\beta)(3-\beta)} \right], \quad (2.4)$$

and the corresponding minimal value of $I = \int_0^1 A^2(r)rdr$ is

$$I_\beta = \frac{P(\beta)}{2(1-\beta)^2(3-2\beta)(2-\beta)^2(3-\beta)^2}, \quad (2.5)$$

where

$$P(\beta) = 6 - 13\beta + 8\beta^2 - \beta^3 \quad (2.6)$$

is a polynomial function positive for the values of β considered in Corollary 2.2.

3 Proof of Theorem 2.1

Proof.–

Let us introduce the polar coordinates $x = (r, \theta)$, $r \in \mathbb{R}^+$, $\theta \in [0, 2\pi[$. Due to assumption (2.1) the magnetic field we have to consider is of the type

$$B(r) = \frac{b(r)}{(1-r)^\beta} dr \wedge d\theta, \quad \text{with} \quad \max_{[0,1[} b(r) \leq M \quad \text{and} \quad \beta < \frac{3}{2}. \quad (3.1)$$

We first prove the following

Lemma 3.1 *If A is a radial magnetic potential associated to B , then there exists a constant K such that A writes*

$$\text{if } \beta \neq 1 \quad A = A(r)d\theta = \frac{a(r)}{(1-r)^{\beta-1}}d\theta$$

with

$$a(r) = K(1-r)^{\beta-1} + \tilde{a}, \quad \max_{[0,1[} \tilde{a}(r) \leq C.$$

$$\text{if } \beta = 1 \quad A = A(r)d\theta = a(r) \ln(1-r)d\theta$$

with

$$a(r) = \frac{K}{\ln(1-r)} + \tilde{a}, \quad \max_{[0,1[} \tilde{a}(r) \leq C.$$

In particular

$$\int_{[0,1[} rA^2(r)dr < \infty .$$

Proof.–

Let us explain the case $\beta \neq 1$. The method for the case $\beta = 1$ is the same.

a satisfies the equation

$$(\beta - 1)a(r) - (1 - r)a'(r) = b(r) .$$

This implies that

$$a(r) = k(r)(1-r)^{\beta-1}, \quad \text{with } k(r) = \int_r^1 b(t)(1-t)^{-\beta}dt + K . \quad (3.2)$$

From (3.1) we get

$$|\int_r^1 b(t)(1-t)^{-\beta}dt| \leq M \int_r^1 (1-t)^{-\beta}dt \leq M \frac{(1-r)^{-\beta+1}}{1-\beta}$$

and the result follows. □

We come now to the proof of Theorem 2.1, following the method of [12].

The quadratic form associated to $H_A^D - V$ can be rewritten as

$$h_{A,V}(u) = \int_0^1 \int_0^{2\pi} \left[\left| \frac{\partial u}{\partial r} \right|^2 - V(r)|u|^2 + r^{-2} \left[\left[\frac{\partial}{\partial \theta} - iA(r) \right] u \right]^2 \right] r dr d\theta \quad (3.3)$$

for

$$u \in C_0^\infty([0, 1[\times [0, 2\pi[).$$

Changing variables $r = e^t$ and denoting $w(t, \theta) = u(e^t, \theta)$ for $t \in] - \infty, 0[$ and $\theta \in [0, 2\pi[$ we transfer the form $h_{A,V}(u)$ to

$$\tilde{h}_{A,V}(w) = \int_{-\infty}^0 \int_0^{2\pi} \left[\left| \frac{\partial w}{\partial t} \right|^2 - \tilde{V}(t)|w|^2 + \left[\left[\frac{\partial}{\partial \theta} - if(t) \right] w \right]^2 \right] dt d\theta \quad (3.4)$$

with

$$\tilde{V}(t) = e^{2t}V(e^t), \quad f(t) = e^tA(e^t).$$

By expanding a given function $w \in C_0^\infty([-\infty, 0[\times [0, 2\pi[)$ into a Fourier series we obtain that $\tilde{h}_{A,V}(w) = \oplus h_\ell(w_\ell)$ with

$$h_\ell(v) = \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 + \left[(\ell - f(t))^2 - \tilde{V}(t) \right] |v|^2 dt,$$

and $w_\ell = \Pi_\ell(w)$ where Π_ℓ is the projector acting as

$$\Pi_\ell(w)(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\ell(\theta - \theta')} w(r, \theta') d\theta'.$$

We have, for any $\alpha \in]0, 1[$

$$h_\ell(v) \geq \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 + \left[\left(1 - \frac{1}{\alpha}\right) f^2(t) - \tilde{V}(t) + (1 - \alpha)\ell^2 \right] |v|^2 dt.$$

Let us denote by L_α the operator associated via Friedrich's extension to the quadratic form

$$q_\alpha(v) = \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 D_t^2 + \left[\left(1 - \frac{1}{\alpha}\right) f^2(t) - \tilde{V}(t) \right] |v|^2 dt.$$

Since

$$h_\ell \geq q_\alpha + (1 - \alpha)\ell^2,$$

the number of negative eigenvalues of h_ℓ is less than the number of negative eigenvalues of $L_\alpha + (1 - \alpha)\ell^2$. So denoting by $\{-\mu_k^\alpha\}$ the negative eigenvalues of L_α and by I_ℓ the set $\{k \in \mathbb{N}; -\mu_k^\alpha + (1 - \alpha)\ell^2 < 0\}$ we get

$$N(A, V) \leq \sum_{\ell} \sum_{k \in I_\ell} 1.$$

Noticing that the sum in the righthandside is taken over the (ℓ, k) so that $\ell \leq \frac{1}{\sqrt{1-\alpha}} \sqrt{\mu_k^\alpha}$ we write

$$N(A, V) \leq \frac{1}{\sqrt{1-\alpha}} \sum_{k \in \mathbb{N}} \sqrt{\mu_k^\alpha}. \quad (3.5)$$

Let us extend the functions f and \tilde{V} to \mathbb{R} by zero and denote respectively by f_1 and \tilde{V}_1 these extensions.

Since $C_0^\infty([-\infty, 0]) \subset C_0^\infty(\mathbb{R})$, the negative eigenvalues $\{-\nu_k^\alpha\}$ of the operator L_1^α associated via Friedrich's extension to the quadratic form

$$q_1^\alpha(v) = \int_{-\infty}^0 \left| \frac{\partial v}{\partial t} \right|^2 D_t^2 + \left[\left(1 - \frac{1}{\alpha}\right) f_1^2(t) - \tilde{V}_1(t) \right] |v|^2 dt$$

verify

$$\sum_{k \in \mathbb{N}} \sqrt{\mu_k^\alpha} \leq \sum_{k \in \mathbb{N}} \sqrt{\nu_k^\alpha}. \quad (3.6)$$

Applying the sharp inequality of Hundertmarkt-Lieb-Thomas [10] (see Appendix) to the operator L_1^α we get

$$\begin{aligned} \sum_{k \in \mathbb{N}} \sqrt{\nu_k^\alpha} &\leq \frac{1}{2} \int_{-\infty}^{+\infty} \left[\left(\frac{1}{\alpha} - 1\right) f_1^2(t) + \tilde{V}_1(t) \right] dt \\ &\leq \frac{1}{2} \int_{-\infty}^0 \left[\left(\frac{1}{\alpha} - 1\right) f^2(t) + \tilde{V}(t) \right] dt \\ &\leq \frac{1}{2} \int_0^1 \left[\left(\frac{1}{\alpha} - 1\right) A^2(r) + V(r) \right] r dr \end{aligned}$$

In view of (3.5) and (3.6) we get the result. □

Proof of Remark 2.3

The choice of A is obtained by taking the minimum over the constants c of the function $F(c) = \int_0^1 (A + c)^2(r) r dr$. To get the minimum over the values of α we study the sign of the expression, for any $\alpha \in]0, 1[$, of $g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I$. A direct computation shows that the value α_λ which realizes the minimum of $g_\lambda(\alpha)$ is the positive solution of

$$\alpha^2(\lambda - 2I) + 6\alpha I - 4I = 0. \quad (3.7)$$

4 An eigenvalue asymptotic upperbound

From Corollary 2.2 we get easily an asymptotic estimate for the righthandside of (2.3) when λ tends to ∞ :

Corollary 4.1 *If assumptions (H_1) and (H_3) are verified and if moreover*

$$\mathbf{b}(x) \leq (D(x))^{-\beta}, \quad \beta < \frac{3}{2} \quad (4.1)$$

then, for any radial gauge A associated to B , the number of eigenvalues of the operator H_A^D less than λ satisfies, as λ tends to ∞

$$N(A, \lambda) \leq \frac{\lambda}{4} + \frac{\sqrt{\lambda}\sqrt{I}}{2} + O(1) , \quad (4.2)$$

where

$$I = \int_0^1 A^2(r)rdr .$$

Remark: the minimum of the righthandside is obtained by choosing the gauge so that $\int_0^1 A(r)rdr = 0$.

Example 4.2 Assume $b(r) \equiv 1$ in (3.1) and $\beta \neq 1$. Then the minimum is obtained for $I = I_\beta = \int_0^1 A_\beta^2(r)rdr$, where A_β is defined as in (2.4), so that

$$I_\beta = \frac{P(\beta)}{2(1-\beta)^2(3-2\beta)(2-\beta)^2(3-\beta)^2} , \quad (4.3)$$

where $P(\beta)$ is defined by (2.6).

Proof.–

We define as previously, for any $\alpha \in]0, 1[$,

$$g_\lambda(\alpha) := \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha} I$$

and we want to determine the asymptotic behavior as λ tends to ∞ of $g_\lambda(\alpha_\lambda)$, where α_λ is the minimum of $g_\lambda(\alpha)$.

From (3.7) we compute the following asymptotics

$$\alpha_\lambda = \frac{2\sqrt{I}}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)$$

$$\sqrt{1-\alpha_\lambda} = 1 - \frac{\sqrt{I}}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) ,$$

and this gives the result.

The minimal value is obtained as previously by taking the minimum over the constants c of the function $F(c) = \int_0^1 (A+c)^2(r)rdr$.

□

Remark 4.3 The leading term in the estimate (4.2) is also the leading term in the Weyl formula for the Dirichlet Laplacian (corresponding to the case $A = 0$) in the unit disk.

5 Appendix

We recall the sharp inequality of Hundertmark-Lieb-Thomas

Theorem 5.1 *Let*

$$Lv(t) = -v''(t) - W(t)v(t), \quad W \geq 0 \quad W \in L^1(\mathbb{R})$$

be defined in the sense of quadratic forms on \mathbb{R} , and assume that the negative spectrum of $Lv(t)$ is discrete. Denote by $\{-\nu_k, k \in \mathbb{N}\}$ the negative eigenvalues of L . Then

$$\sum_{k \in \mathbb{N}} \sqrt{\nu_k} \leq \frac{1}{2} \int_{-\infty}^{+\infty} W(t) dt .$$

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