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**INTERACTING PARTICLES MODELS AND THE PIERI-TYPE
FORMULAS :
THE SYMPLECTIC CASE WITH NON EQUAL WEIGHTS**

MANON DEFOSSEUX

ABSTRACT. We have introduced in [4] a particles model with blocking and pushing interactions which is related to a Pieri type formula for the orthogonal group. This model has a symplectic version presented here. It leads in particular to the particles model with a wall defined in [7].

1. INTRODUCTION

In [4] an interacting particles model have been introduced which comes to be related to a Pieri-type formula for the orthogonal group. Particles of the model can move to the left or to the right and are submitted to blocking and pushing interactions. They are constrained to stay in the right half plane. Such a model is said to be with a wall. One can find for instance in [1] and [7] examples of models with a wall. In these two last references models differ only by the behavior of particles near the wall. Actually the first one is strongly related to representations of the orthogonal group whereas the second one involves representations of the symplectic compact group. Following the study of these models in the same way as in [4] we construct here an interacting particles with a wall depending on parameters. Two particular values of the parameters lead on one hand to the model studied in [7] on the other hand to a random matrix model of [2].

The paper is organized as follows. In the second section, definitions of the symplectic Gelfand-Tsetlin patterns and Schur functions are recalled. In the third one we describe the interacting particles model studied in the paper. In the fourth section we recall some properties about tensor product of particular representations of the symplectic compact group, which leads naturally to some Markov kernels involved in the interacting particles model. These Markov kernels are defined in the fifth section. Section six is devoted to a random matrix model related to the particles model. Results of the paper are stated at section seven. We sketch the proofs in the last section.

2. SYMPLECTIC GELFAND-TSETLIN PATTERNS, SYMPLECTIC SCHUR FUNCTIONS

For $n \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^n$ such that $x_n \leq \dots \leq x_1$ and $y_n \leq \dots \leq y_1$, we write $x \preceq y$ if x and y are interlaced, i.e.

$$x_n \leq y_n \leq x_{n-1} \leq \dots \leq x_1 \leq y_1.$$

When $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n+1}$ we add the relation $y_{n+1} \leq x_n$. We denote by $|x|$ the sum of the coordinates $\sum_{i=1}^n x_i$.

Definition 2.1. *Let k be a positive integer.*

(1) We denote by GT_k the set of Gelfand-Tsetlin patterns defined by

$$GT_k = \{(x^1, \dots, x^k) : x^i \in \mathbb{N}^{\lfloor \frac{i+1}{2} \rfloor} \text{ and } x^{i-1} \preceq x^i, 1 \leq i \leq k\}.$$

(2) If $x = (x^1, \dots, x^k)$ is a Gelfand-Tsetlin pattern, x^i is called the i^{th} line of x for $i \in \{1, \dots, k\}$.

(3) For $\lambda \in \mathbb{N}^{\lfloor \frac{k+1}{2} \rfloor}$ the subset of Gelfand-Tsetlin patterns having a k^{th} line equal to λ is denoted by $GT_k(\lambda)$.

Usually, a Gelfand-Tsetlin pattern is represented by a triangular array as indicated at figure 1 for $k = 2r$. For x a Gelfand-Tsetlin pattern of GT_k and $(q_i)_{i \geq 1}$ a sequence of positive real numbers one defines $w_x^k(q_1, \dots, q_{\lfloor \frac{k+1}{2} \rfloor})$ by recursion letting

$$w_x^1(q_1) = q_1^{|x^1|},$$

and

$$\begin{aligned} w_x^{2i}(q_1, \dots, q_i) &= w_x^{2i-1}(q_1, \dots, q_i) q_i^{|x^{2i-1}| - |x^{2i}|}, \\ w_x^{2i+1}(q_1, \dots, q_i, q_{i+1}) &= w_x^{2i}(q_1, \dots, q_i) q_{i+1}^{|x^{2i+1}| - |x^{2i}|}, \end{aligned}$$

for $i \in \mathbb{N}^*$.

Definition 2.2. For $\lambda \in \mathbb{N}^{\lfloor \frac{k+1}{2} \rfloor}$ such that $\lambda_1 \geq \dots \geq \lambda_{\lfloor \frac{k+1}{2} \rfloor}$, we denote by s_λ^k the symplectic Schur function defined by

$$s_\lambda^k(q) = \sum_{x \in GT_k(\lambda)} w_x^k(q),$$

for $q = (q_1, \dots, q_{\lfloor \frac{k+1}{2} \rfloor}) \in \mathbb{R}_+^{\lfloor \frac{k+1}{2} \rfloor}$.

Notice that the cardinal of $GT_k(\lambda)$ is equal to $s_\lambda^k(\mathbf{1})$, with $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{\lfloor \frac{k+1}{2} \rfloor}$.

$$\begin{array}{cccccccc} & & & & & & & 0 \\ & & & & & & & -x_1^1 & x_1^1 \\ & & & & & & & -x_1^2 & 0 & x_1^2 \\ & & & & & & & -x_1^3 & -x_2^3 & x_2^3 & x_1^3 \\ & & & & & & -x_1^4 & -x_2^4 & 0 & x_2^4 & x_1^4 \\ & & & & & & & \dots & & \dots & \\ & & & & & & -x_1^{2r-1} & \dots & -x_r^{2r-1} & x_r^{2r-1} & \dots & x_1^{2r-1} \\ -x_1^{2r} & \dots & & & -x_r^{2r} & 0 & x_r^{2r} & \dots & \dots & & & x_1^{2r} \end{array}$$

FIGURE 1. A Gelfand-Tsetlin pattern of GT_{2r}

3. INTERACTING PARTICLES MODELS

In this section we construct two processes evolving on the set GT_k of Gelfand-Tsetlin patterns. These processes can be viewed as interacting particles models. For this, we associate to a Gelfand-Tsetlin pattern $x = (x^1, \dots, x^k)$, a configuration of particles on the integer lattice \mathbb{Z}^2 putting one particle labeled by (i, j) at point $(x_j^i, k-i)$ of \mathbb{Z}^2 for $i \in \{1, \dots, k\}$, $j \in \{1, \dots, \lfloor \frac{i+1}{2} \rfloor\}$. Several particles can be located at the same point. In the sequel we identify each particles with its corresponding component.

3.1. Geometric jumps. Let $q = (q_1, \dots, q_r) \in \mathbb{R}_+^r$, $\alpha \in \mathbb{R}_+$ such that $\alpha q_i \in]0, 1[$ and $\alpha q_i^{-1} \in]0, 1[$ for $i = 1, \dots, r$, with $r = \lfloor \frac{k+1}{2} \rfloor$. Consider two independent families

$$\left(\xi_j^i \left(n + \frac{1}{2} \right) \right)_{i=1, \dots, k, j=1, \dots, \lfloor \frac{i+1}{2} \rfloor; n \geq 0}, \quad \text{and} \quad \left(\xi_j^i(n) \right)_{i=1, \dots, k, j=1, \dots, \lfloor \frac{i+1}{2} \rfloor; n \geq 1},$$

of independent geometric random variables such that

$$\mathbb{P}(\xi_j^{2i-1} \left(n + \frac{1}{2} \right) = x) = \mathbb{P}(\xi_j^{2i}(n) = x) = (\alpha q_i^{-1})^x (1 - \alpha q_i^{-1}), \quad x \in \mathbb{N},$$

and

$$\mathbb{P}(\xi_j^{2i-1}(n) = x) = \mathbb{P}(\xi_j^{2i} \left(n + \frac{1}{2} \right) = x) = (\alpha q_i)^x (1 - \alpha q_i), \quad x \in \mathbb{N}.$$

The evolution of the particles is given by a process $(X(t))_{t \geq 0}$ on the set GT_k of Gelfand-Tsetlin patterns. At each time $t \geq 0$, a particle labeled by (i, j) is at point $(X_j^i(t), k - i)$ of \mathbb{Z}^2 . Particles evolve as follows. At time 0 all particles are at zero, i.e. $X(0) = 0$. All particles try to jump to the left at times $n + \frac{1}{2}$ and to the right at times n , $n \in \mathbb{N}$. Suppose that at time n , after all particles have jumped, there is one particle at point $(X_j^i(n), k - i)$ of \mathbb{Z}^2 , for $i = 1, \dots, k$, $j = 1, \dots, \lfloor \frac{i+1}{2} \rfloor$. Positions of particles are updated recursively as follows.

At time $n + 1/2$: All particles try to jump to the left one after another in the lexicographic order pushing the other particles in order to stay in the set of Gelfand-Tsetlin patterns and being blocked by the initial configuration $X(n)$ of the particles:

- Particle $X_1^1(n)$ tries to move to the left being blocked by 0, i.e.

$$X_1^1 \left(n + \frac{1}{2} \right) = \max(X_1^1(n) - \xi_1^1 \left(n + \frac{1}{2} \right), 0).$$

- Particle $X_1^2(n)$ tries to jump to the left. It is blocked by $X_1^1(n)$. If it is necessary it pushes $X_2^3(n)$ to an intermediate position denoted by $\tilde{X}_2^3(n)$, i.e.

$$\begin{aligned} X_1^2 \left(n + \frac{1}{2} \right) &= \max(X_1^1(n), X_1^2(n) - \xi_1^2 \left(n + \frac{1}{2} \right)) \\ \tilde{X}_2^3(n) &= \min(X_2^3(n), X_1^2 \left(n + \frac{1}{2} \right)) \end{aligned}$$

- Particle $X_1^3(n)$ tries to move to the left being blocked by $X_1^2(n)$:

$$X_1^3 \left(n + \frac{1}{2} \right) = \max(X_1^2(n), X_1^3(n) - \xi_1^3 \left(n + \frac{1}{2} \right)).$$

Particle $\tilde{X}_2^3(n)$ tries to move to the left being blocked by 0, i.e

$$X_2^3 \left(n + \frac{1}{2} \right) = \max(\tilde{X}_2^3(n) - \xi_2^3 \left(n + \frac{1}{2} \right), 0).$$

Suppose now that rows 1 through $l - 1$ have been updated for some $l > 1$. Particles $X_2^l(n), \dots, X_{\lfloor \frac{l+1}{2} \rfloor}^l(n)$ of line l are pushed to intermediate positions

$$\tilde{X}_i^l(n) = \min(X_i^l(n), X_{i-1}^{l-1} \left(n + \frac{1}{2} \right)), \quad i \in \{2, \dots, \lfloor \frac{l+1}{2} \rfloor\}.$$

Then particles $X_1^l(n), \tilde{X}_2^l(n), \dots, \tilde{X}_{\lfloor \frac{l+1}{2} \rfloor}^l(n)$ try to jump to the left being blocked as follows by the initial position $X(n)$ of the particles. For $i = 1, \dots, \lfloor \frac{l+1}{2} \rfloor$,

$$X_i^l(n + \frac{1}{2}) = \max(X_i^{l-1}(n), \tilde{X}_i^l(n) - \xi_i^l(n + \frac{1}{2})),$$

with the convention that $X_{\frac{l+1}{2}}^{l-1}(n) = 0$ when l is odd.

At time $n+1$: All particles try to jump to the right one after another in the lexicographic order pushing particles in order to stay in the set of Gelfand-Tsetlin patterns and being blocked by the initial configuration $X(n + \frac{1}{2})$ of the particles. The first three lines are updated as follows.

- Particle $X_1^1(n + \frac{1}{2})$ moves to the right pushing $X_1^2(n + \frac{1}{2})$ to an intermediate position $\tilde{X}_1^2(n + \frac{1}{2})$:

$$X_1^1(n+1) = X_1^1(n + \frac{1}{2}) + \xi_1^1(n+1)$$

$$\tilde{X}_1^2(n + \frac{1}{2}) = \max(X_1^2(n + \frac{1}{2}), X_1^1(n+1))$$

- Particle $\tilde{X}_1^2(n + \frac{1}{2})$ jumps to the right pushing $X_1^3(n + \frac{1}{2})$ to an intermediate position $\tilde{X}_1^3(n + \frac{1}{2})$, i.e.

$$X_1^2(n+1) = \tilde{X}_1^2(n + \frac{1}{2}) + \xi_1^2(n+1)$$

$$\tilde{X}_1^3(n + \frac{1}{2}) = \max(X_1^3(n + \frac{1}{2}), X_1^2(n+1))$$

- Particle $X_2^3(n + \frac{1}{2})$ tries to move to the right being blocked by $X_1^2(n + \frac{1}{2})$. Particle $\tilde{X}_1^3(n + \frac{1}{2})$ moves to the right. That is

$$X_2^3(n+1) = \max(X_2^3(n + \frac{1}{2}) + \xi_2^3(n+1), X_1^2(n + \frac{1}{2}))$$

$$X_1^3(n+1) = \tilde{X}_1^3(n + \frac{1}{2}) + \xi_1^3(n+1)$$

Suppose rows 1 through $l-1$ have been updated for some $l > 1$. Then particles of line l are pushed to intermediate positions

$$\tilde{X}_i^l(n + \frac{1}{2}) = \max(X_i^{l-1}(n+1), X_i^l(n + \frac{1}{2})), i \in \{1, \dots, \lfloor \frac{l+1}{2} \rfloor\},$$

with the convention $X_{\frac{l+1}{2}}^{l-1}(n+1) = 0$ when l is odd. Then particles $\tilde{X}_1^l(n + \frac{1}{2}), \dots, \tilde{X}_{\lfloor \frac{l+1}{2} \rfloor}^l(n + \frac{1}{2})$ try to jump to the right being blocked by the initial position of the particles as follows. For $i = 1, \dots, \lfloor \frac{l+1}{2} \rfloor$,

$$X_i^l(n+1) = \min(X_{i-1}^{l-1}(n + \frac{1}{2}), \tilde{X}_i^l(n + \frac{1}{2}) + \xi_i^l(n+1)).$$

3.2. Exponential waiting times. The interacting particles model described now has been introduced in [7]. In this model particles evolve on \mathbb{Z}^2 and jump on their own volition by one rightwards or leftwards after exponentially distributed waiting times. The evolution of the particles is described by a random process $(Y(t))_{t \geq 0}$ on GT_k . As in the previous model, at time $t \geq 0$ there is one particle labeled by (i, j) at point $(Y_j^i(t), k-i)$ of the integer lattice, for $i = 1, \dots, k, j = 1, \dots, \lfloor \frac{i+1}{2} \rfloor$. Particle labeled by $(2i, j)$ tries to jump to the left by one after exponentially distributed

waiting time with mean q_i and to the right by one after exponentially distributed waiting time with mean q_i^{-1} . Particle labeled by $(2i-1, j)$ tries to jump to the left by one after exponentially distributed waiting time with mean q_i^{-1} and to the right by one after exponentially distributed waiting time with mean q_i . Waiting times are all independent. When a particle tries to jump, all particles are pushed and blocked according to the same rules as previously. That is when particle labeled by (i, j) wants to jump to the right at time $t \geq 0$ then

- (1) if $i, j \geq 2$ and $Y_j^i(t^-) = Y_{j-1}^{i-1}(t^-)$ then particles don't move and $Y(t) = Y(t^-)$.
- (2) else particles $(i, j), (i+1, j), \dots, (i+l, j)$ jump to the right by one for l the largest integer such that $Y_j^{i+l}(t^-) = Y_j^i(t^-)$ i.e.

$$Y_j^i(t) = Y_j^i(t^-) + 1, \dots, Y_j^{i+l}(t) = Y_j^{i+l}(t^-) + 1.$$

When particle labeled by (i, j) wants to jump to the left at time $t \geq 0$ then

- (1) if i is odd, $j = (i+1)/2$ and $Y_j^i(t^-) = 0$ then particle labeled by (i, j) doesn't move.
- (2) if i is odd, $j = (i+1)/2$ and $Y_j^i(t^-) \geq 1$ then $Y_j^i(t) = Y_j^i(t^-) - 1$.
- (3) if i is even or $j \neq (i+1)/2$, and $Y_j^i(t^-) = Y_j^{i-1}(t^-)$ then particles don't move.
- (4) if i is even or $j \neq (i+1)/2$, and $Y_j^i(t^-) > Y_j^{i-1}(t^-)$ then particles $(i, j), (i+1, j+1), \dots, (i+l, j+l)$ jump to the left by one for l the largest integer such that $Y_{j+l}^{i+l}(t^-) = Y_j^i(t^-)$. Thus

$$Y_j^i(t) = Y_j^i(t^-) - 1, \dots, Y_{j+l}^{i+l}(t) = Y_{j+l}^{i+l}(t^-) - 1.$$

Actually, process $(Y(t), t \geq 0)$ is obtained letting α goes to zero in the previous model. More precisely we get the following proposition.

Proposition 3.1. *The process $(X([\alpha^{-1}t]), t \geq 0)$ converges in distribution towards the process $(Y(t), t \geq 0)$ as α goes to zero.*

Proof. Proposition is obtained replacing q by α in lemma 8.9 of [4]. □

4. A PIERI TYPE FORMULA FOR THE SYMPLECTIC GROUP

Let r be a positive integer. One recalls some usual properties of the finite dimensional representations of the compact symplectic group Sp_{2r} (see for instance [5] for more details). The set of finite dimensional representations of Sp_{2r} is indexed by the set

$$\mathcal{W}_{2r} = \{\lambda \in \mathbb{R}^r : \lambda_r \in \mathbb{N}, \lambda_i - \lambda_{i+1} \in \mathbb{N}, i = 1, \dots, r-1\}.$$

For $\lambda \in \mathcal{W}_{2r}$, using standard notations, we denote by V_λ the so called irreducible representation with highest weight λ of Sp_{2r} .

Let m be an integer and λ an element of \mathcal{W}_{2r} . Consider the irreducible representations V_λ and V_{γ_m} of Sp_{2r} , with $\gamma_m = (m, 0, \dots, 0)$. The decomposition of the tensor product $V_\lambda \otimes V_{\gamma_m}$ into irreducible components is given by a Pieri-type formula for the symplectic group. It has been recalled in [2]. We have

$$(1) \quad V_\lambda \otimes V_{\gamma_m} = \bigoplus_{\beta} M_{\lambda, \gamma_m}(\beta) V_\beta,$$

where the direct sum is over all $\beta \in \mathcal{W}_{2r}$ such that there exists $c \in \mathcal{W}_{2r}$ which satisfies

$$\begin{cases} c \preceq \lambda, & c \preceq \beta \\ \sum_{i=1}^r (\lambda_i - c_i + \beta_i - c_i) = m. \end{cases}$$

In addition, the multiplicity $M_{\lambda, \gamma_m}(\beta)$ of the irreducible representation with highest weight β is the number of $c \in \mathcal{W}_{2r}$ satisfying these relations.

5. MARKOV KERNELS

Since for $\lambda \in \mathcal{W}_{2r}$ the Schur function s_λ^{2r} is the character of the irreducible representation V_λ , decomposition (1) implies

$$s_\lambda^{2r}(q) s_{\gamma_m}^{2r}(q) = \sum_{\beta \in \mathcal{W}_{2r}} M_{\lambda, \gamma_m}(\beta) s_\beta^{2r}(q).$$

Thus one defines a family $(\mu_m)_{m \geq 0}$ of Markov kernels on \mathcal{W}_{2r} letting

$$\mu_m(\lambda, \beta) = \frac{s_\beta^{2r}(q)}{s_\lambda^{2r}(q) s_{\gamma_m}^{2r}(q)} M_{\lambda, \gamma_m}(\beta),$$

for $\lambda, \beta \in \mathcal{W}_{2r}$ and $m \geq 0$. Let ξ_1, \dots, ξ_d be independent geometric random variables with respective parameters $\alpha q_1, \dots, \alpha q_{2r}$. Consider a random variable T on \mathbb{N} defined by

$$T = \sum_{i=1}^d \xi_i.$$

Lemma 5.1. *The law of T is a measure ν on \mathbb{N} defined by*

$$\nu(m) = \alpha^m a(q) s_{\gamma_m}^d(q), \quad m \in \mathbb{N},$$

where

$$a(q) = \prod_{i=1}^r (1 - \alpha q_i)(1 - \alpha q_i^{-1}).$$

Proof. Lemma follows from straightforward computations. \square

Lemma 5.1 implies in particular that the measure ν is a probability measure. Thus one defines a Markov kernel P_{2r} on \mathcal{W}_{2r} letting

$$P_{2r}(\lambda, \beta) = \sum_{m=0}^{+\infty} \mu_m(\lambda, \beta) \nu(m),$$

for $\lambda, \beta \in \mathcal{W}_{2r}$.

Proposition 5.2. *For $\lambda, \beta \in \mathcal{W}_{2r}$,*

$$(2) \quad P_{2r}(\lambda, \beta) = \sum_{c \in \mathcal{W}_r : c \preceq \lambda, \beta} a(q) \frac{s_\beta^d(q)}{s_\lambda^d(q)} \alpha^{\sum_{i=1}^r (\lambda_i + \beta_i - 2c_i)}$$

Proof. Proposition follows immediately from the tensor product rules recalled for the decomposition (1). \square

We let $\mathcal{W}_{2r-1} = \mathcal{W}_{2r}$. For $c_0, \lambda, c, \beta \in \mathcal{W}_k$, we let

$$(3) \quad S_k((c_0, \lambda), (c, \beta)) = a(q) \frac{s_\beta^k(q)}{s_\lambda^k(q)} \alpha^{\sum_{i=1}^r (\lambda_i + \beta_i - 2c_i)} \mathbf{1}_{c \preceq \beta},$$

when $k = 2r$ and

(4)

$$S_k((c_0, \lambda), (c, \beta)) = \tilde{a}(q) \frac{s_\beta^k(q)}{s_\lambda^k(q)} \alpha^{\sum_{i=1}^r (\lambda_i + \beta_i - 2c_i)} ((1 - \alpha q_r^{-1}) \mathbf{1}_{c_r > 0} + \mathbf{1}_{c_r = 0}) \mathbf{1}_{z' \preceq y'},$$

when $k = 2r - 1$, with

$$\tilde{a}(q) = (1 - \alpha q_r) \prod_{i=1}^{r-1} (1 - \alpha q_i)(1 - \alpha q_i^{-1}).$$

Notice that proposition 5.2 ensures that S_{2r} defines a Markov kernel on $\mathcal{W}_{2r} \times \mathcal{W}_{2r}$. There isn't such an argue for S_{2r-1} . Nevertheless straightforward computations show that it is also a Markov kernel. Anyway as Λ_k and Q_k defined in section 8 are Markov kernels, proposition 8.1 ensures that S_k is a Markov kernel in both the odd and the even cases. Thus one defines also a Markov kernel P_{2r-1} on \mathcal{W}_{2r-1} letting

(5)

$$P_{2r-1}(\lambda, \beta) = \sum_{c \in \mathcal{W}_r : c \preceq \lambda, \beta} \tilde{a}(q) \frac{s_\beta^d(q)}{s_\lambda^d(q)} \alpha^{\sum_{i=1}^r (\lambda_i + \beta_i - 2c_i)} ((1 - \alpha q_r^{-1}) \mathbf{1}_{c_r > 0} + \mathbf{1}_{c_r = 0}).$$

6. RANDOM MATRICES

We denote by \mathbb{H} the set of quaternions. For us, a quaternion is just a 2×2 matrix Z with complex entries which can be written as

$$Z = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$. Its conjugate Z^* is the usual adjoint of the complex matrix Z . Let us denote by $\mathcal{M}_{r,m}$ the real vector space of $r \times m$ matrices with entries in \mathbb{H} and by \mathcal{P}_r the set of $r \times r$ Hermitian matrices with entries in $i\mathbb{H}$. Since a matrix in \mathcal{P}_r is a $2r \times 2r$ Hermitian complex matrix, it has real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2r}$. Moreover $\lambda_{2r-i+1} = -\lambda_i$, for $i = 1, \dots, 2r$. We put on $\mathcal{M}_{r,m}$ the Euclidean structure defined by the scalar product,

$$\langle M, N \rangle = \text{tr}(MN^*), \quad M, N \in \mathcal{M}_{r,m}.$$

Let \mathcal{C}_r be the subset of \mathbb{R}^r defined by

$$\mathcal{C}_r = \{x \in \mathbb{R}^r : x_1 > \dots > x_r > 0\}.$$

Theorem 4.5 and proposition 4.8 of [2] imply the following proposition.

Proposition 6.1. *Let r be a positive integer and $(M(n), n \geq 0)$, be a discrete process on \mathcal{P}_r defined by*

$$M(n) = \sum_{l=1}^n Y_l \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y_l^*,$$

where the Y_l 's are independent standard Gaussian variables in $\mathcal{M}_{r,1}$. For $n \in \mathbb{N}$, let $\Lambda_1(n), \dots, \Lambda_r(n)$ be the r largest eigenvalue of $M(n)$ such that

$$\Lambda_1(n) \geq \dots \geq \Lambda_r(n).$$

Then the process $(\Lambda(n), n \geq 0)$, is a Markovian process with a transition densities p_r defined by

$$p_r(x, y) = \frac{d_r(y)}{d_r(x)} I(x, y), \quad x, y \in \mathcal{C}_r,$$

where

$$I(x, y) = \int_{\mathbb{R}_+^r} 1_{\{x, y \succ z\}} e^{-\sum_{i=1}^r (x_i + y_i - 2z_i)} dz,$$

and

$$d_r(x) = \prod_{1 \leq i < j \leq r} (x_i^2 - x_j^2) \prod_{i=1}^r x_i, \quad x, y \in \mathcal{C}_r.$$

7. RESULTS

Theorem 7.1. *The process $(Z^k(n), n \geq 0)$ with $Z^k(n) = (X^k(n - \frac{1}{2}), X^k(n))$, $n \in \mathbb{N}$, is a Markov process on $\mathcal{W}_k \times \mathcal{W}_k$ with transition kernel S_k .*

If P_k is the Markov kernel defined in (2) and (5) then theorem 7.1 implies immediately the following theorem which is our main result.

Theorem 7.2. *The process $(X^k(n))_{n \geq 0}$ is a Markov process on \mathcal{W}_k with transition kernel P_k .*

Let us notice that convergence stated in proposition 3.1, theorem 7.1 and lemma of [4] lead to theorem 2.3 of [7]. If $(\Lambda(n), n \geq 0)$ is the process of eigenvalues considered at proposition 6.1 then the following corollary holds.

Corollary 7.3. *Letting $q_i = 1$ for $i = 1, \dots, r$, the process $((1 - \alpha)X^{2r}(n), n \geq 1)$ converges in distribution towards the process of eigenvalues $(\Lambda(n), n \geq 1)$ as α goes to one.*

Proof. The Weyl dimension formula (see Knapp [5], Thm V.5.84) for the symplectic groups gives

$$s_\lambda^{2r}(\mathbf{1}) = \prod_{1 \leq i < j \leq r} \frac{(\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 2n + 2 - j - i)}{(j - i)(2n + 2 - j - i)} \prod_{i=1}^r \frac{\lambda_i + n + 1 - i}{n + 1 - i}.$$

Thus corollary follows immediately from theorem 7.2 and proposition 6.1. \square

8. PROOFS

Proof of theorem 7.1 rests on the same ingredients as the proof of proposition 8.8 of [4]. It follows from an intertwining property stated at proposition 8.1 and an application of a Pitman and Rogers criterion established in [6]. Computations are also quite similar and left to the reader. For $\lambda \in \mathcal{W}_k$ we consider the measure M_λ on $GT_k(\lambda)$ defined by

$$M_\lambda = \sum_{x \in GT_k(\lambda)} \frac{w_k(x)}{s_\lambda^k(q)} \delta_x,$$

and the measure m_λ defined as the image of the measure M_λ by the map $x \in GT_k(\lambda) \mapsto x^{k-1} \in \mathcal{W}_{k-1}$, i.e

$$m_\lambda = \sum_{\beta \in \mathcal{W}_{k-1}; \beta \preceq \lambda} q_r^{|\beta| - |\lambda|} \frac{s_\beta^{k-1}(q)}{s_\lambda^k(q)} \delta_\beta,$$

when $k = 2r$, and

$$m_\lambda = \sum_{\beta \in \mathcal{W}_{k-1}: \beta \preceq \lambda} q_r^{|\lambda| - |\beta|} \frac{s_\beta^{k-1}(\tilde{q})}{s_\lambda^k(q)} \delta_\beta,$$

when $k = 2r - 1$, with $\tilde{q} = (q_1, \dots, q_{r-1})$. One defines a kernel Λ_k from $\mathcal{W}_k \times \mathcal{W}_k$ to $\mathcal{W}_{k-1} \times \mathcal{W}_k \times \mathcal{W}_k$ letting

$$\Lambda_k((c, \lambda), (\beta, c', \lambda')) = m_\lambda(\beta) 1_{c \preceq \lambda} 1_{c=c', \lambda=\lambda'},$$

for $c, \lambda, c', \lambda' \in \mathcal{W}_k$, $\beta \in \mathcal{W}_{k-1}$. Theorem 7.1 is proved by recursion on k . Suppose that it is true for an integer $k - 1$. Let us denote by Q_k the transition kernel of the Markovian process

$$(X^{k-1}(n), Z^k(n), n \geq 0).$$

These Markov kernels satisfy the following intertwining property, which implies, using the Rogers and Pitman criterion that theorem is true for the integer k (see section 8 of [4] for more details).

Proposition 8.1.

$$\Lambda_k Q_k = S_k \Lambda_k$$

Proof. Proposition follows from straightforward computations using identities of lemma 8.3 of [4]. \square

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