The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part II: the non-zero degree case

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Abstract

We consider minimizers of a Ginzburg-Landau energy with a discontinuous and rapidly oscillating pinning term, subject to a Dirichlet boundary condition of degree $d > 0$. The pinning term models an unbounded number of small impurities in the domain. We prove that for strongly type II superconductor with impurities, minimizers have exactly d isolated zeros (vortices). These vortices are of degree 1 and pinned by the impurities. As in the standard case studied by Bethuel, Brezis and Hélein, the macroscopic location of vortices is governed by vortex/vortex and vortex/ boundary repelling effects. In some special cases we prove that their macroscopic location tends to minimize the renormalized energy of Bethuel-Brezis-Hélein. In addition, impurities affect the microscopic location of vortices. Our technics allows us to work with impurities having different size. In this situation we prove that vortices are pinned by the largest impurities.

Contents

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1 Introduction

In this article we let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected domain and let $a_\varepsilon : \Omega \to \{b, 1\}$, $b \in (0, 1)$ be a measurable function. We associate to a_ε the pinned Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2} \left(a_\varepsilon(x)^2 - |u(x)|^2 \right)^2 \right\} \, \mathrm{d}x. \qquad (1.1)$$

Here, $u : \Omega \to \mathbb{C}$ is in the Sobolev space $H^1(\Omega, \mathbb{C})$ and $\varepsilon > 0$ is the inverse of the Ginzburg-Landau parameter.

Our goal is to consider a discontinuous and rapidly oscillating pinning term (the pinning term is $a_\varepsilon : \Omega \to \{b, 1\}$). Our pinning term is periodic with respect to a $\delta \times \delta$-grid with $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (in some cases we drop the periodic hypothesis).

We are interested in the minimization of (1.1) in $H^1(\Omega, \mathbb{C})$ subject to a Dirichlet boundary condition: we fix $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ and thus the set of the test functions is

$$H_g^1 := \{ u \in H^1(\Omega, \mathbb{C}) \,|\, \mathrm{tr}_{\partial\Omega} u = g \}.$$

The situation where $d = \deg_{\partial\Omega}(g) = 0$ was studied in detail in [12]. The non zero degree case ($d = \deg_{\partial\Omega}(g) > 0$) is the purpose of the present article. Recall that for $\Gamma \subset \mathbb{R}^2$ a Jordan curve and $g \in H^{1/2}(\Gamma, \mathbb{S}^1)$, the degree (winding number) of g is defined as

$$\deg_\Gamma(g) := \frac{1}{2\pi} \int_\Gamma g \times \partial_\tau g \, \mathrm{d}\tau.$$

Here "$\times$" stands for the vectorial product in $\mathbb{C}$, *i.e.* $z_1 \times z_2 = \mathrm{Im}(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$, τ is the direct unit tangent vector of Γ ($\tau = \nu^\perp$ where ν is the outward normal unit vector of $\mathrm{int}(\Gamma)$, the bounded open set whose boundary is Γ) and ∂_τ is the tangential derivative on Γ.

This energy is a simplification of the full Ginzburg-Landau energy (see Eq. (1.2) below) whose minimizers model the state of a Type II superconductor (the parameter ε corresponds to a material parameter, this parameter is small for Type II superconductor) [23], [20]. The pinning term allows to model a heterogenous superconductor (see [15] or Introduction of [11]).

Physical informations which can be obtained with the simplification of the full Ginzburg-Landau energy are quantization and location of zeros of minimizers. Their zeros represent the centers of small areas where the superconductivity is destroyed. These areas are called vorticity defects. Here the superconductor is a cylinder whose cross section is Ω and the vorticity defects (under some special conditions) takes the form of small wires parallel to the superconductor [23], [20].

Before going further, let us summarize two previous works in related directions [17], [1]. In these works, the role of the pinning term is identified: its points of *minimum* attract the vorticity defects.

In [17], Lassoued and Mironescu considered the case where $a_\varepsilon \equiv a$. Here, the pinning term $a = \begin{cases} b & \text{in } \omega \\ 1 & \text{in } \Omega \setminus \omega \end{cases}$, $0 < b < 1$, and ω is a smooth inner domain of Ω. These authors proved that the vorticity defects are quantified by $\deg_{\partial\Omega}(g)$, localized in ω and that their position is governed by a renormalized energy (in the spirit of [4]).

In [1], Aftalion, Sandier and Serfaty considered a smooth and ε-dependent pinning term a_ε. Their study allows to consider the case where the pinning term has fast oscillations: it is a perturbation of a fixed smooth function $\tilde{b} : \Omega \to [b, 1]$ s.t. $a_\varepsilon \geq \tilde{b}$.

In contrast with [17], [1] is dedicated to the study of a full Ginzburg-Landau energy GL_ε with the pinning term a_ε

$$GL_\varepsilon(u, A) = \frac{1}{2} \int_\Omega \left\{ |\mathrm{curl}A - h_{\mathrm{ex}}|^2 + |(\nabla - iA)u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2 \right\}. \qquad (1.2)$$

We denoted by $A \in \mathbb{R}^2$ the electromagnetic vector potential of the induced field and by $h_{\mathrm{ex}} \gg 1$ the intensity of the applied magnetic field (see [20] for more details).

They considered the following hypotheses on $a_\varepsilon, \tilde{b}$:

- $|\nabla a_\varepsilon| \leq C h_{\mathrm{ex}}$

- there is $\sigma_\varepsilon \in \mathbb{R}$ s.t. $\sigma_\varepsilon = o\left((\ln|\ln\varepsilon|)^{-1/2}\right)$ and for all $x \in \Omega$, we have

$$\min_{B(x,\sigma_\varepsilon)} \left\{ a_\varepsilon - \tilde{b} \right\} = 0.$$

In the study of the full Ginzburg-Landau functional without pinning term GL_ε^0 (GL_ε^0 is obtained from (1.2) by taking $a_\varepsilon \equiv 1$), the vorticity defects appear for large apply magnetic field. They are characterized by two facts: the presence of isolated zeros x_i of a map u with a non zero degree around small circles centered in x_i and the existence of a magnetic field inside the domain ($\mathrm{curl}(A) \simeq h_{\mathrm{ex}}$ inside small discs). The nature of the superconductivity makes that both facts appear together. Assume that the intensity of the applied field h_{ex} depends on $0 < \varepsilon < 1$ and that $h_{\mathrm{ex}}/|\ln\varepsilon| \to \Lambda \in \mathbb{R}_+^*$. For the full Ginzburg-Landau energy without pinning term GL_ε^0, it is well known (see *e.g.* [20]) that there is an inner domain ω_Λ (non decreasing w.r.t. Λ) s.t., when $\varepsilon \to 0$, the vorticity defects are "uniformly located" by ω_Λ (in this situation the number of vortices is unbounded).

In [1] (study of a full Ginzburg-Landau functional with a pinning term), the authors proved the existence of ω_Λ, an inner set of Ω, where the penetration of the magnetic field is located. In contrast with the situation without pinning term, the presence of a_ε makes that, in general, the vortices are not uniformly located in ω_Λ. Although in the proofs of the main results of [1], the minimal points of $\tilde{b}$ seem play the role of a pinning site, this fact is not proved. They expect that the most favorable pinning sites should be close to the *minima* of $\tilde{b}$: ω_Λ should be located close to the points of *minimum* of $\tilde{b}$.

One of our goals is to prove that the minimum points of a rapidly oscillating and discontinuous pinning term attract the vorticity defects.

Before going further, we construct our (periodic) pinning term a_ε.

Construction 1. The periodic pinning term
Consider

- $\delta = \delta(\varepsilon) \in (0,1)$, $\lambda = \lambda(\varepsilon) \in (0,1]$;

- $\omega \subset Y = (-1/2, 1/2)^2$ be a smooth bounded and simply connected open set s.t. $(0,0) \in \omega$ and $\overline{\omega} \subset Y$ (here Y is the unit cell).

For $k, l \in \mathbb{Z}$ we denote

$$Y_{k,l}^\delta := \delta \cdot Y + (\delta k, \delta l), \qquad \Omega_\delta^{\mathrm{incl}} = \bigcup_{Y_{k,l}^\delta \subset \Omega} \overline{Y_{k,l}^\delta}, \qquad \omega^\lambda = \lambda \cdot \omega,$$

$$\omega_{\mathrm{per}}^\lambda = \bigcup_{(k,l) \in \mathbb{Z}^2} \left\{ \omega^\lambda + (k,l) \right\} \text{ and } \omega_\varepsilon = \bigcup_{\substack{(k,l) \in \mathbb{Z}^2 \text{ s.t.} \\ Y_{k,l}^\delta \subset \Omega}} \left\{ \delta \cdot \omega^\lambda + (\delta k, \delta l) \right\}.$$

4

For $b \in (0, 1)$, we define

$$
\begin{array}{ccccccc}
a^\lambda : & \mathbb{R}^2 & \to & \{b, 1\} & \qquad a_\varepsilon : & \mathbb{R}^2 & \to & \{b, 1\} \\
& x & \mapsto & \begin{cases} b & \text{if } x \in \omega_{\text{per}}^\lambda \\ 1 & \text{otherwise} \end{cases} \text{ and} & & x & \mapsto & \begin{cases} b & \text{if } x \in \omega_\varepsilon \\ 1 & \text{otherwise} \end{cases} .
\end{array}
$$

The values of the periodic pinning term are represented Figure 1. The connected components of $\{a_\varepsilon = b\} = \omega_\varepsilon$ are called inclusions or impurities.

(a) The pining term is periodic on a $\delta \times \delta$-grid

(b) The parameter λ controls the size of an inclusion in the cell
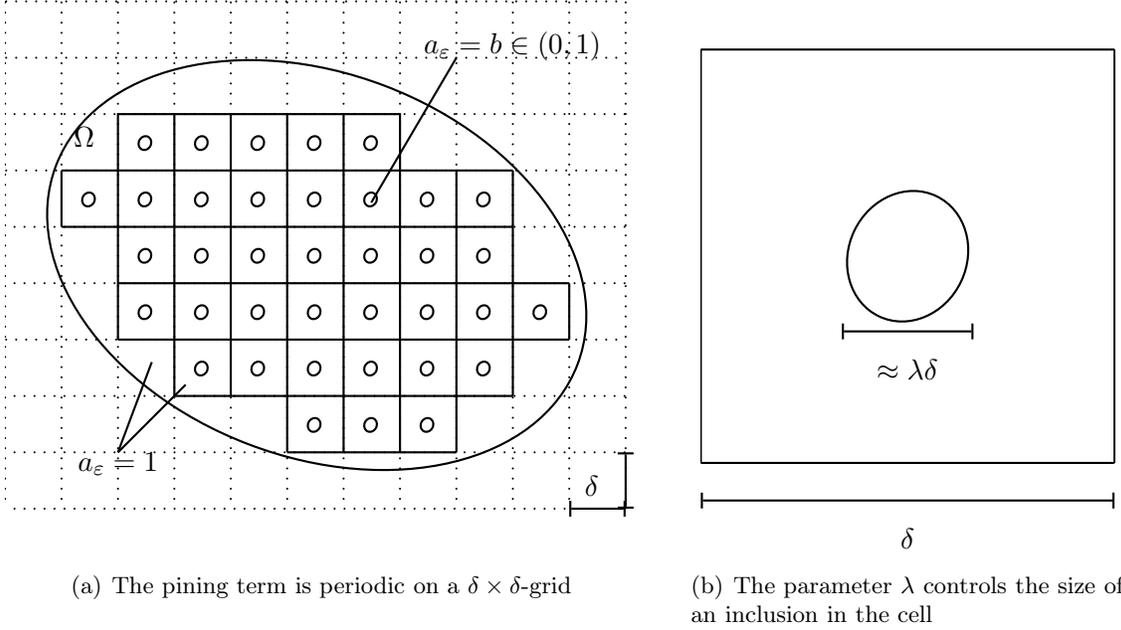
Figure 1: The periodic pinning term

In the rest of this article $\lambda = \lambda(\varepsilon)$ and $\delta = \delta(\varepsilon)$ are functions of ε. We assume that $\delta \to 0$ as $\varepsilon \to 0$. In addition, we assume that either $\lambda \equiv 1$, or $\lambda \to 0$ as $\varepsilon \to 0$. The case $\lambda \to 0$ is the *diluted case*.

We make the (technical) assumption

$$
\lim_\varepsilon \frac{|\ln(\lambda\delta)|^3}{|\ln \varepsilon|} = 0. \tag{1.3}
$$

Remark 2. • This is slightly more restrictive than asking that $\lambda\delta \gg \varepsilon^\alpha$ for all $\alpha \in (0, 1)$.

• Hypothesis (1.3) is technical, a more natural hypothesis should be $\lambda\delta \gg \varepsilon$ or $\lambda\delta \gg \varepsilon^\alpha$ for some $\alpha \in (0, 1)$.

• In [1] and in the situation where we have a bounded number of zeros (the applied magnetic field is not too large), the smooth pinning term a_ε^0 satisfies the condition $|\nabla a_\varepsilon^0| \le C|\ln \varepsilon|$. In order to compare this assumption with (1.3), we may consider a regularization of our pinning term by a mollifier $\rho_t(x) = t^{-2}\rho(x/t)$. A suitable scale t to have a complete view of the variations of a_ε is $t = \lambda\delta$. Thus, $|\nabla(\rho_{\lambda\delta} * a_\varepsilon)|$ is of order $\frac{1}{\lambda\delta}$. Consequently, the condition (1.3) allows to consider a more rapidly oscillating than

the condition in [1]. Indeed, we have $\ln|\nabla a_\varepsilon^0| \lesssim \ln|\ln\varepsilon|$ and on the other hand (1.3) is equivalent to $\ln|\nabla(\rho_{\lambda\delta} * a_\varepsilon)| \lesssim |\ln(\lambda\delta)| = o(|\ln\varepsilon|^{1/3})$.

The goal of this article is to study the minimizers of

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(a_\varepsilon^2 - |u|^2\right)^2 \right\}, \, u \in H_g^1$$

in the asymptotic $\varepsilon \to 0$. A standard method (initiated in [17]) consists in decoupling E_ε into a sum of two functionals. The key tool in this method is U_ε **the** unique global minimizer of E_ε in H_1^1 (see [17]). Clearly, U_ε satisfies

$$\begin{cases} -\Delta U_\varepsilon = \dfrac{1}{\varepsilon^2}U_\varepsilon(a_\varepsilon^2 - U_\varepsilon^2) & \text{in } \Omega \\ U_\varepsilon = 1 & \text{on } \partial\Omega \end{cases}. \tag{1.4}$$

From the uniqueness of U_ε, by construction of a test function, it is easy to get that $b \leq U_\varepsilon \leq 1$.

This special solution may be seen as a regularization of a_ε. For example, one may easily prove that U_ε is exponentially close to a_ε far away from $\partial\omega_\varepsilon$ (a more complete description of U_ε is done Appendix D.1). Namely, we have

Proposition 3. *There are $C, \alpha > 0$ independent of $\varepsilon, R > 0$ s.t.*

$$|a_\varepsilon - U_\varepsilon| \leq Ce^{-\frac{\alpha R}{\varepsilon}} \text{ in } V_R := \{x \in \Omega \,|\, \text{dist}(x, \partial\omega_\varepsilon) \geq R\}, \tag{1.5}$$

$$|\nabla U_\varepsilon| \leq \frac{Ce^{-\frac{\alpha R}{\varepsilon}}}{\varepsilon} \text{ in } W_R := \{x \in \Omega \,|\, \text{dist}(x, \partial\omega_\varepsilon), \text{dist}(x, \partial\Omega) \geq R\}. \tag{1.6}$$

A similar result was proved in [13] (Proposition 2). The above proposition yields by the same arguments.

As in [17], we define

$$F_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2|\nabla v|^2 + \frac{1}{2\varepsilon^2}U_\varepsilon^4(1 - |v|^2)^2 \right\}.$$

Then we have for all $v \in H_g^1$, (see [17])

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v).$$

Therefore, u_ε is a minimizer of E_ε if and only if $u_\varepsilon = U_\varepsilon v_\varepsilon$ where v_ε is a minimizer of F_ε in H_g^1. Consequently, the study of a minimizer $u_\varepsilon = U_\varepsilon v_\varepsilon$ of E_ε in H_g^1 (location of zeros and asymptotics) can be performed by combining the asymptotic of U_ε with one of v_ε.

Our main result is the following

Theorem 1. *Assume that λ, δ satisfy (1.3) and that $\lambda \to 0$.*

Quantization. *There are $\varepsilon_0 > 0$, $c > 0$ and $\eta_0 > 0$ s.t. for $0 < \varepsilon < \varepsilon_0$:*

1. *v_ε has exactly d zeros $x_1^\varepsilon, ..., x_d^\varepsilon$,*

2. *$B(x_i^\varepsilon, c\lambda\delta) \subset \omega_\varepsilon$,*

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3. *for $\rho = \rho(\varepsilon) \downarrow 0$ s.t. $|\ln \rho|/|\ln \varepsilon| \to 0$, there is $C > 0$ independent of ε satisfying*

$$|v_\varepsilon| \geq 1 - C\sqrt{\frac{|\ln \rho|}{|\ln \varepsilon|}} \ \ in \ \Omega \setminus \cup\overline{B(x_i^\varepsilon, \rho)},$$

4. *for $\varepsilon < \varepsilon_0$*

- *There are two repulsive effects: $|x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0$ for $i \neq j$ and $\mathrm{dist}(x_i^\varepsilon, \partial\Omega) \geq \eta_0$;*
- $\deg_{\partial B(x_i^\varepsilon, \delta)}(v_\varepsilon) = 1.$

Location.

- *The macroscopic location of the zeros tends to minimize the renormalized energy of Bethuel-Brezis-Hélein $W_g : \{\{x_1, ..., x_d\} \subset \Omega \,|\, x_i \neq x_j \text{ for } i \neq j\} \to \mathbb{R}$ (defined in [4], Chapter I Eq. (47)):*

$$\limsup W_g(x_1^\varepsilon, ..., x_d^\varepsilon) = \min_{\substack{a_1,...,a_d \in \Omega \\ a_i \neq a_j}} W_g(a_1, ..., a_d)$$

- *The microscopic location of the zeros inside ω_ε tends to depend **only** on ω and b:*

  – *since $x_i^\varepsilon \in \omega_\varepsilon$, we have $x_i^\varepsilon = (k_\varepsilon\delta, l_\varepsilon\delta) + \lambda\delta y_i^\varepsilon$ with $k_\varepsilon, l_\varepsilon \in \mathbb{Z}$ and $y_i^\varepsilon \in \omega$;*
  – *for $\varepsilon_n \downarrow 0$ s.t. $y_i^{\varepsilon_n} \to \hat{a}_i$, we have $\hat{a}_i \in \omega$ which minimizes a renormalized energy $\tilde{W}_1 : \omega \to \mathbb{R}$ (given in [13] Eq. (90)) which depends only on ω and $b \in (0, 1)$.*

Remark 4. 1. The renormalized energy defined in [4]

$$W_g : \{\{x_1, ..., x_d\} \subset \Omega \,|\, x_i \neq x_j \text{ for } i \neq j\} \to \mathbb{R}$$

governs the location of the zeros in the situation where $a_\varepsilon \equiv 1$ (homogenous case): the zeros tend to minimize W_g. In [4] (Chapter 1), the authors defined a renormalized energy in a more general setting

$$W_g^{\mathrm{BBH}} : \left\{ \{(x_1, d_1), ..., (x_N, d_N)\} \ \middle| \ \begin{array}{l} x_i \in \Omega, \ x_i \neq x_j \text{ for } i \neq j \\ d_i \in \mathbb{Z} \text{ is s.t. } \sum_{i=1}^N d_i = d \end{array} \right\} \to \mathbb{R}.$$

Here $W_g(x_1, ..., x_d) = W_g^{\mathrm{BBH}}(\{(x_1, 1), ..., (x_d, 1)\})$, *i.e.*, in this article we will consider only the renormalized energy with the degrees equal 1 and thus we do not specify the degrees in its notation.

2. From smoothness of W_g (see [4] and [10]), *Location part* of Theorem 1 implies that up to pass to a subsequence, the zeros converge to a minimizer of W_g.

3. This macroscopic location is strongly correlated with the Dirichlet boundary condition $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$.

4. The result about the macroscopic position of the periodic and diluted pinning term may be sum up as: *the macroscopic position of the zeros tends to be the same than in the homogenous case ($a_\varepsilon \equiv 1$).*

5. The microscopic location of the zeros (position inside an inclusion) is independent of the boundary condition. For example, in the situation $\omega = B(0, r_0)$, *i.e.*, the inclusions are discs, this location should be the center of the inclusion. This fact is not proved yet.

6. In Assertion 4. of *Quantization part*, $\deg_{\partial B(x_i^\varepsilon, \delta)}(v_\varepsilon) = \deg_{\partial B(x_i^\varepsilon, \delta)}(v_\varepsilon/|v_\varepsilon|).$

2 Main results

We present in this section several extensions of the above result dropping either the dilution of the inclusion ($\lambda \equiv 1$ instead of $\lambda \to 0$) or the periodic structure. The main results of this section are obtained under the condition: $\lambda\delta$ satisfies (1.3).

Our sharper results are shared into four theorems:

- The first theorem (Theorem 2) gives informations on the zeros of minimizers $u_\varepsilon, v_\varepsilon$ (quantization and location).

- The second theorem (Theorem 3) establishes the asymptotic behavior of v_ε.

- The third theorem (Theorem 4) establishes, under the additional hypothesis $\lambda \to 0$, that the microscopic position of the zeros is independent of the boundary condition g.

- The last theorem (Theorem 5) gives an expansion of $F_\varepsilon(v_\varepsilon)$.

The technics developed in this paper allows to consider either the case $\lambda \to 0$ or $\lambda \equiv 1$. The results in the diluted case are more precise. One may drop the periodic structure for the pinning term and consider impurities (the connected components of $\omega_\varepsilon = \{a_\varepsilon = b\}$) with different sizes (adding the hypothesis $\lambda \to 0$).

More precisely we may consider the pinning term defined as follow:

Construction 5. The general diluted pinning term

- Fix $P \in \mathbb{N}^*$, $j \in \{1, ..., P\}$ and $1 > \varepsilon > 0$. We consider $M_j^\varepsilon \in \mathbb{N}$ and

$$\mathcal{M}_j^\varepsilon = \begin{cases} \emptyset & \text{if } M_j^\varepsilon = 0 \\ \{1, ..., M_j^\varepsilon\} & \text{if } M_j^\varepsilon \in \mathbb{N}^* \end{cases}.$$

- The sets $\mathcal{M}_j^\varepsilon$'s are s.t. (for sufficiently small ε) one may fix $y_{i,j}^\varepsilon \in \Omega$ s.t. for $(i,j) \neq (i',j')$, $i \in \mathcal{M}_j^\varepsilon$, $i' \in \mathcal{M}_{j'}^\varepsilon$ we have

$$|y_{i,j}^\varepsilon - y_{i',j'}^\varepsilon| \geq \delta^j + \delta^{j'} \text{ and dist}(y_{i,j}^\varepsilon, \partial\Omega) \geq \delta^j. \tag{2.1}$$

We denote $\widehat{\mathcal{M}}_j^\varepsilon := \{y_{i,j}^\varepsilon \,|\, i \in \mathcal{M}_j^\varepsilon\}$.

For sake of simplicity, we assume that there is $\eta > 0$ s.t. for small ε, we have $M_1^\varepsilon \geq d = \deg_{\partial\Omega}(g)$ and

$$\min\left\{ \min_{i=1,...,d} \text{dist}(y_{i,1}^\varepsilon, \partial\Omega), \min_{\substack{i,i'=1,...,d \\ i \neq i'}} |y_{i,1}^\varepsilon - y_{i',1}^\varepsilon| \right\} \geq \eta. \tag{2.2}$$

- We now define the domain which models the impurities:

$$\omega_\varepsilon = \bigcup_{j=1}^{P} \bigcup_{i \in \mathcal{M}_j^\varepsilon} \left\{ y_{i,j}^\varepsilon + \delta^j \cdot \omega^\lambda \right\}, \ \omega^\lambda = \lambda \cdot \omega.$$
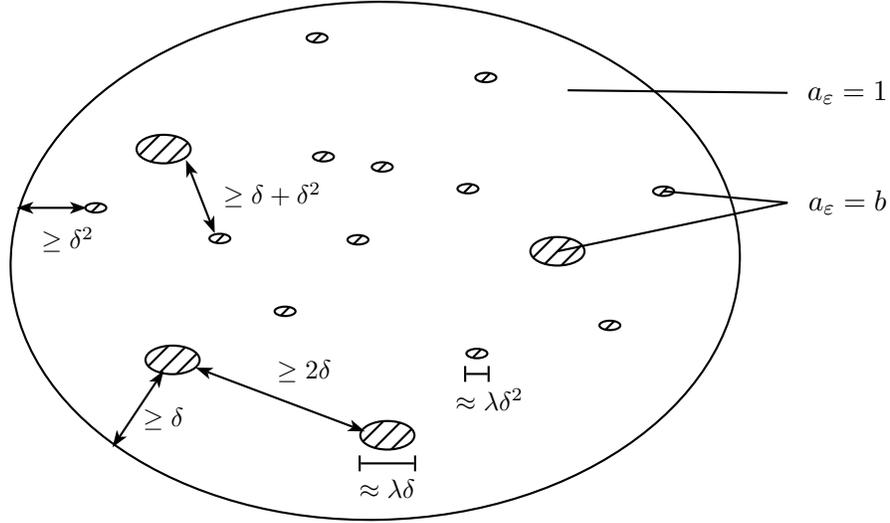
Figure 2: Representation of the general diluted pinning term with $P = 2$

The pinning term is

$$a_\varepsilon : \quad \mathbb{R}^2 \quad \to \quad \{b, 1\}$$
$$x \quad \mapsto \quad \begin{cases} 1 & \text{if } x \notin \omega_\varepsilon \\ b & \text{if } x \in \omega_\varepsilon \end{cases}$$

The values of the pinning term are represented Figure 2.

Our main results are

Theorem 2. *Assume that λ, δ satisfy (1.3) and if the pinning term is not periodic (represented Figure 2) then we assume also that $\lambda \to 0$.*
There is $\varepsilon_0 > 0$ s.t.:

1. *for $0 < \varepsilon < \varepsilon_0$, v_ε has exactly d zeros $x_1^\varepsilon, ..., x_d^\varepsilon$,*

2. *there are $c > 0$ and $\eta_0 > 0$ s.t. for $\varepsilon < \varepsilon_0$, $B(x_i^\varepsilon, c\lambda\delta) \subset \omega_\varepsilon$ and*

$$\min_i \left\{ \min_{j \neq i} |x_i^\varepsilon - x_j^\varepsilon|, \text{dist}(x_i^\varepsilon, \partial\Omega) \right\} \geq \eta_0.$$

   In particular, if the pinning term is not periodic, then the zeros are trapped by the largest inclusions (those of size $\lambda\delta$).

3. *for $\rho = \rho(\varepsilon) \downarrow 0$ s.t. $|\ln\rho|/|\ln\varepsilon| \to 0$, we have for $\varepsilon < \varepsilon_0$,*

$$|v_\varepsilon| \geq 1 - C\sqrt{\frac{|\ln\rho|}{|\ln\varepsilon|}} \quad \text{in } \Omega \setminus \cup\overline{B(x_i^\varepsilon, \rho)}.$$

   Here C is independent of ε.

4. *for $\varepsilon < \varepsilon_0$, $\deg_{\partial B(x_i^\varepsilon, \delta)}(v_\varepsilon) = 1$.*

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Remark 6. Hypothesis (2.2) is used to simplify the statements. Without this hypothesis, some of the results are subject to technical considerations on $\delta, \lambda, b...$ For example if we consider the pinning term a_ε defined in $\Omega = B(0,2)$ by

$$
\begin{aligned}
a_\varepsilon: \quad B(0,2) &\rightarrow \{b,1\} \\
x &\mapsto \begin{cases} b & \text{if } x \in B(0,\lambda\delta) \cup B(1,\lambda\delta^2) \\ 1 & \text{otherwise} \end{cases},
\end{aligned}
$$

and $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ s.t. $\deg_{\partial\Omega}(g) = 2$, then Hypothesis (2.2) is not satisfied. In this situation, we may prove that, for sufficiently small ε, v_ε has exactly two zeros and if $2(1-2b^2)|\ln\lambda| + (1-3b^2)|\ln\delta| \rightarrow +\infty$ (resp. $-\infty$), then the zeroes are in $B(0,\lambda\delta)$ (resp. there is one zero inside $B(0,\lambda\delta)$ and one zero inside $B(1,\lambda\delta^2)$).

Theorem 3. *Assume that λ, δ satisfy (1.3) and if the pinning term is not periodic (represented Figure 2) then we assume also that $\lambda \rightarrow 0$.*

Let $\varepsilon_n \downarrow 0$, up to a subsequence, we have the existence of $a_1, ..., a_d \in \Omega$, d distinct points s.t. $x_i^{\varepsilon_n} \rightarrow a_i$ and

$$
|v_{\varepsilon_n}| \rightarrow 1 \text{ and } v_{\varepsilon_n} \rightharpoonup v_* \text{ in } H^1_{\text{loc}}(\overline{\Omega} \setminus \{a_1, ..., a_d\}, \mathbb{S}^1)
$$

where v_* solves

$$
\begin{cases} -\text{div}(\mathcal{A}\nabla v_*) = (\mathcal{A}\nabla v_* \cdot \nabla v_*)v_* & \text{in } \Omega \setminus \{a_1, ..., a_d\} \\ v_* = g & \text{on } \partial\Omega \end{cases}.
$$

Here $\mathcal{A}$ is the homogenized matrix of $a^2\left(\dfrac{\cdot}{\delta}\right)\text{Id}_{\mathbb{R}^2}$ if $\lambda \equiv 1$ and $\mathcal{A} = \text{Id}_{\mathbb{R}^2}$ if $\lambda \rightarrow 0$.

In addition, for each $M > 0$, $v'_{\varepsilon,i}(\cdot) = v_\varepsilon\left(x_i^\varepsilon + \dfrac{\varepsilon}{b}\cdot\right)$ converges, up to a subsequence, in $C^1(B(0,M))$ to $f(|x|)\dfrac{x}{|x|}e^{i\theta_i}$ where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the universal function defined in [19] and $\theta_i \in \mathbb{R}$.

Theorem 4. *Assume, in addition to the hypotheses of Theorem 3, that $\lambda \rightarrow 0$.*

Let $[x] = [(x_1, x_2)] = ([x_1], [x_2]) \in \mathbb{Z}^2$ be the vectorial integer part of the point $x \in \mathbb{R}^2$. For x_i^ε a zero of v_ε, let

$$
y_i^\varepsilon = \frac{\frac{x_i^\varepsilon}{\delta} - [\frac{x_i^\varepsilon}{\delta}]}{\lambda} \in \omega.
$$

Then, as $\varepsilon \rightarrow 0$, up to pass to a subsequence, we have $y_i^\varepsilon \rightarrow \hat{\hat{a}}_i \in \omega$. Here, $\hat{\hat{a}}_i$ minimizes a renormalized energy $\tilde{W}_1 : \omega \rightarrow \mathbb{R}$ (given in [13] Eq. (90)) which depends only on ω and b. In particular, $\hat{\hat{a}}_i$ is independent of the boundary condition g.

Theorem 5. *Assume that λ, δ satisfy (1.3) and if the pinning term is not periodic (represented Figure 2) then we assume also that $\lambda \rightarrow 0$.*

We have the following expansion

$$
F_\varepsilon(v_\varepsilon) = J_{\varepsilon,\varepsilon} + db^2(\pi \ln b + \gamma) + o_\varepsilon(1)
$$

where $J_{\varepsilon,\varepsilon}$ is defined in (3.6) and $\gamma > 0$ is the universal constant defined in [4] Lemma IX.1.

This article is divided in two parts:

- In the first one (Section 3) we consider two auxiliary minimization problems for weighted Dirichlet functionals associated to $\mathbb{S}^1$-valued maps.

- The second part (Section 4) is devoted to the proofs of Theorems 1, 2, 3, 4, 5. The main tool is an η-ellipticity result (Lemma 19). This lemma reduces (under the assumption that λ, δ satisfy (1.3)) the study of F_ε to the one of the auxiliary problems considered Section 3.

3 Shrinking holes for weighted Dirichlet functionals

This section is devoted to the study of two minimization problems and it is divided in three subsections.

The first and the second subsections are related with minimizations of weighted Dirichlet functionals among $\mathbb{S}^1$-valued maps. In both subsections, the considered weights are the more general one: $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$. The third subsection deals the weight $\alpha = U_\varepsilon^2$ in the situation where U_ε is the minimizer of E_ε in H_1^1 with a_ε represented Figure 1 (the periodic case with or without dilution) or Figure 2 (the general diluted case).

Notation 7. In Section 3 we fix :

- a smooth simply connected domain $\Omega \subset \mathbb{R}^2$;

- a boundary condition $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ s.t. $d := \deg_{\partial\Omega}(g) > 0$;

- a smooth and bounded open subset $\Omega' \prec \mathbb{R}^2$ s.t. $\overline{\Omega} \subset \Omega'$;

- an extension of g which is in $C^\infty(\overline{\Omega'} \setminus \Omega, \mathbb{S}^1)$ (this extension is also denoted by g).

We will also consider (uniformly bounded) families of points/degres $\{(x_1, d_1), ..., (x_N, d_N)\} = \{\mathbf{x}, \mathbf{d}\}$ s.t.

- $x_i \in \Omega$, $x_i \neq x_{i'}$ for $i \neq i'$;

- d_i are s.t. $d_i \in \mathbb{N}^*$ and $\sum_i d_i = d$ (thus $N \leq d$).

According to the considered problems, for $0 < \rho \leq 8^{-1} \min_{i \neq i'} |x_i - x_{i'}|$ we will use the following perforated domains

- $\Omega_\rho := \Omega_\rho(\mathbf{x}) = \Omega \setminus \cup_i \overline{B(x_i, \rho)}$;

- $\Omega'_\rho := \Omega'_\rho(\mathbf{x}) = \Omega' \setminus \cup_i \overline{B(x_i, \rho)}$.

3.1 Existence results

In this subsection we prove the existence of solutions of two minimization problems whose studies will be the purpose of the rest of Section 3 (Subsections 3.2 & 3.3).

3.1.1 Existence of minimal maps defined in a perforated domain

Let $\mathbf{x} = (x_1, ..., x_N)$ be $1 \leq N \leq d$ distinct points of Ω and let $\mathbf{d} = (d_1, ..., d_N) \in (\mathbb{N}^*)^N$ s.t. $\sum_i d_i = d$.

For $0 < \rho < 8^{-1} \min_{i \neq j} |x_i - x_j|$, we denote $\Omega_\rho = \Omega \setminus \cup \overline{B(x_i, \rho)}$.

We define

$$\mathcal{I}_\rho(\mathbf{x},\mathbf{d}) = \mathcal{I}_\rho := \left\{ w \in H^1(\Omega_\rho,\mathbb{S}^1) \,|\, w = g \text{ on } \partial\Omega \text{ and } \deg_{\partial B(x_i,\rho)}(w) = d_i \right\}$$

and for $0 < \rho < 8^{-1} \min\{\min_{i \neq j} |x_i - x_j|, \min_i \operatorname{dist}(x_i,\partial\Omega)\}$

$$\mathcal{J}_\rho(\mathbf{x},\mathbf{d}) = \mathcal{J}_\rho := \left\{ w \in H^1(\Omega_\rho,\mathbb{S}^1) \,|\, w = g \text{ on } \partial\Omega \text{ and } w(x_i + \rho e^{\imath\theta}) = e^{\imath(d_i\theta + \theta_i)}, \theta_i \in \mathbb{R} \right\}.$$

From the compatibility condition $\deg_{\partial\Omega}(g) = d = \sum d_i$, we have $\mathcal{I}_\rho(\mathbf{x},\mathbf{d}), \mathcal{J}_\rho(\mathbf{x},\mathbf{d}) \neq \emptyset$ and it is clear that $\mathcal{J}_\rho(\mathbf{x},\mathbf{d}) \subset \mathcal{I}_\rho(\mathbf{x},\mathbf{d})$.

In Subsection 3.2, we compare the minimal energies corresponding to a weighted Dirichlet functional in the above sets. Here, we just state existence results.

Proposition 8. *Let* $\alpha \in L^\infty(\Omega)$ *be s.t.* $b^2 \leq \alpha \leq 1$. *Consider the minimization problems*

$$\widehat{\mathcal{I}}_{\rho,\alpha}(\mathbf{x},\mathbf{d}) = \inf_{w \in \mathcal{I}_\rho} \frac{1}{2} \int_{\Omega_\rho} \alpha |\nabla w|^2$$

and

$$\widehat{\mathcal{J}}_{\rho,\alpha}(\mathbf{x},\mathbf{d}) = \inf_{w \in \mathcal{J}_\rho} \frac{1}{2} \int_{\Omega_\rho} \alpha |\nabla w|^2.$$

In both minimization problems the infima *are attained.*

Moreover, if $\alpha \in W^{1,\infty}(\Omega)$, *then, denoting* $w_{\rho,\alpha}^{\mathrm{deg}}$ *(resp.* $w_{\rho,\alpha}^{\mathrm{Dir}}$*) a global minimizer of* $\frac{1}{2} \int_{\Omega_\rho} \alpha |\nabla \cdot|^2$ *in* $\mathcal{I}_\rho(\mathbf{x},\mathbf{d})$ *(resp. in* $\mathcal{J}_\rho(\mathbf{x},\mathbf{d})$*) we have* $w_{\rho,\alpha}^{\mathrm{deg}} \in H^2(\Omega_\rho,\mathbb{S}^1)$ *(resp.* $w_{\rho,\alpha}^{\mathrm{Dir}} \in H^2(\Omega_\rho,\mathbb{S}^1)$*) and*

$$\begin{cases} -\operatorname{div}(\alpha \nabla w_{\rho,\alpha}^{\mathrm{deg}}) = \alpha |\nabla w_{\rho,\alpha}^{\mathrm{deg}}|^2 w_{\rho,\alpha}^{\mathrm{deg}} \text{ in } \Omega_\rho \\ w_{\rho,\alpha}^{\mathrm{deg}} \in \mathcal{I}_\rho \text{ and } w_{\rho,\alpha}^{\mathrm{deg}} \times \partial_\nu w_{\rho,\alpha}^{\mathrm{deg}} = 0 \text{ on } \partial B(x_i,\rho), \, i = 1,...,N \end{cases}, \tag{3.1}$$

$$\begin{cases} -\operatorname{div}(\alpha \nabla w_{\rho,\alpha}^{\mathrm{Dir}}) = \alpha |\nabla w_{\rho,\alpha}^{\mathrm{Dir}}|^2 w_{\rho,\alpha}^{\mathrm{Dir}} \text{ in } \Omega_\rho \\ w_{\rho,\alpha}^{\mathrm{Dir}} \in \mathcal{J}_\rho \text{ and } \int_{\partial B(x_i,\rho)} \alpha w_{\rho,\alpha}^{\mathrm{Dir}} \times \partial_\nu w_{\rho,\alpha}^{\mathrm{Dir}} = 0, \, i = 1,...,N \end{cases}. \tag{3.2}$$

The proof of this standard result is postponed to Appendix A.

In the special case $\alpha = U_\varepsilon^2$, we denote

$$\widehat{\mathcal{I}}_{\rho,\varepsilon}(\mathbf{x},\mathbf{d}) = \inf_{w \in \mathcal{I}_\rho} \frac{1}{2} \int_{\Omega_\rho} U_\varepsilon^2 |\nabla w|^2 \text{ and } \widehat{\mathcal{J}}_{\rho,\varepsilon}(\mathbf{x},\mathbf{d}) = \inf_{w \in \mathcal{J}_\rho} \frac{1}{2} \int_{\Omega_\rho} U_\varepsilon^2 |\nabla w|^2.$$

3.1.2 Existence of an optimal perforated domain

For $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$ we define

$$I_{\rho,\alpha} := \inf_{\substack{x_1,...,x_N \in \Omega \\ |x_i - x_j| \geq 8\rho \\ d_1,...,d_N > 0, \sum d_i = d}} \inf_{\substack{w \in H_g^1(\Omega_\rho',\mathbb{S}^1) \\ \deg_{\partial B(x_i,\rho)}(w) = d_i}} \frac{1}{2} \int_{\Omega_\rho'} \alpha |\nabla w|^2 \tag{3.3}$$

and

$$J_{\rho,\alpha} := \inf_{\substack{x_1,...,x_d \in \Omega \\ |x_i - x_j| \geq 8\rho \\ \operatorname{dist}(x_i,\partial\Omega) \geq 8\rho}} \inf_{\substack{w \in H_g^1(\Omega_\rho,\mathbb{S}^1) \\ w(x_i + \rho e^{\imath\theta}) = e^{\imath(\theta + \theta_i)}, \theta_i \in \mathbb{R}}} \frac{1}{2} \int_{\Omega_\rho} \alpha |\nabla w|^2. \tag{3.4}$$

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Here $\Omega'_\rho = \Omega' \setminus \cup\overline{B(x_i, \rho)}$.

In the special case $\alpha = U_\varepsilon^2$, we denote

$$I_{\rho,\varepsilon} := \inf_{\substack{x_1,...,x_N \in \Omega \\ |x_i - x_j| \geq 8\rho \\ d_1,...,d_N > 0, \sum d_i = d}} \quad \inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i,\rho)}(w) = d_i}} \frac{1}{2} \int_{\Omega'_\rho} U_\varepsilon^2 |\nabla w|^2 \qquad (3.5)$$

and

$$J_{\rho,\varepsilon} := \inf_{\substack{x_1,...,x_d \in \Omega \\ |x_i - x_j| \geq 8\rho \\ \text{dist}(x_i, \partial\Omega) \geq 8\rho}} \quad \inf_{\substack{w \in H_g^1(\Omega_\rho, \mathbb{S}^1) \\ w(x_i + \rho e^{\imath\theta}) = e^{\imath(\theta+\theta_i)}, \theta_i \in \mathbb{R}}} \frac{1}{2} \int_{\Omega_\rho} U_\varepsilon^2 |\nabla w|^2. \qquad (3.6)$$

We have the following result

Proposition 9. *For $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, there are $\mathbf{x}_{\rho,\alpha}^{\deg}, \mathbf{x}_{\rho,\alpha}^{\mathrm{Dir}} \in \Omega^d$ and $\mathbf{d}_{\rho,\alpha} \in (\mathbb{N}^*)^N$ (with $\mathbf{d}_{\rho,\alpha} = (d_1,...,d_N), \sum d_i = d$) s.t. $\{\mathbf{x}_{\rho,\alpha}^{\deg}, \mathbf{d}_{\rho,\alpha}\}$ minimizes $I_{\rho,\alpha}$ and $\mathbf{x}_{\rho,\alpha}^{\mathrm{Dir}}$ minimizes $J_{\rho,\alpha}$.*

The proof of this result is in Appendix B.

3.2 Dirichlet Vs Degree Conditions in a fixed perforated domain

Let $\eta_{\mathrm{stop}} > 0$ be s.t. $\eta_{\mathrm{stop}} < 10^{-5} \cdot 9^{-d^2} \mathrm{diam}(\Omega)$ and let $N \in \{1,...,d\}$.

Consider $x_1,...,x_N \in \Omega$, N distinct points of Ω satisfying the condition $\eta_{\mathrm{stop}} < 10^{-3} \cdot 9^{-d^2} \min \mathrm{dist}(x_i, \partial\Omega)$, and let $\rho > 0$ be s.t. $\min\{\eta_{\mathrm{stop}}, \min_{i\neq j}|x_i - x_j|\} \geq 8\rho$. Roughly speaking η_{stop} controls the distance between the points and $\partial\Omega$.

The main result of this section is

Proposition 10. *There is $C_0 > 0$ depending only on $g, \Omega, \eta_{\mathrm{stop}}$ and b s.t. for $\alpha \in L^\infty(\Omega, [b^2, 1])$ we have*

$$\widehat{\mathcal{I}}_{\rho,\alpha}(\mathbf{x}, \mathbf{d}) \leq \widehat{\mathcal{J}}_{\rho,\alpha}(\mathbf{x}, \mathbf{d}) \leq \widehat{\mathcal{I}}_{\rho,\alpha}(\mathbf{x}, \mathbf{d}) + C_0.$$

Here, $\widehat{\mathcal{I}}_{\rho,\alpha}$ and $\widehat{\mathcal{J}}_{\rho,\alpha}$ are defined Proposition 8.

The rigorous proof of Proposition 10 is presented in Appendix C. Here, we simply present the main lines of the proof.

Two situations are possible:

1. $N = 1$ or the points $x_1,...,x_N$ are well separated: $\frac{1}{4}\min_{i\neq j}|x_i - x_j| > \eta_{\mathrm{stop}}$,

2. The points $x_1,...,x_N$ are not well separated: $\frac{1}{4}\min_{i\neq j}|x_i - x_j| \leq \eta_{\mathrm{stop}}$.

If the points are well separated (or $N = 1$), Proposition 10 can be easily proved: it is a direct consequence of Proposition 45 and Lemma 44 in Appendix C. These results, whose statements and proofs are postponed in Appendix C, give essentially the existence of test functions into two kinds of domains.

The domains are

- the thin domain $\Omega_{10^{-1}\eta_{\mathrm{stop}}}(\mathbf{x}) = \Omega \setminus \cup\overline{B(x_i, 10^{-1}\eta_{\mathrm{stop}})}$ obtained by perforating Ω by "large", "well separated" and "far from $\partial\Omega$" discs,

- the thick annulars $B(x_i, 10^{-1}\eta_{\mathrm{stop}}) \setminus \overline{B(x_i, \rho)}$.

The proof is made in three steps:

Step 1: Using Lemma 44, we obtain a constant C_1 (depending only on $g, \Omega, \eta_{\text{stop}}$) s.t.

$$\widehat{\mathcal{J}}_{10^{-1}\eta_{\text{stop}},\alpha}(\mathbf{x}, \mathbf{d}) \le C_1.$$

Step 2: With the help of Proposition 45, we obtain the existence of a constant C_b (depending only on b) s.t. for $\tilde{d} \in \mathbb{N}$, denoting $A_\rho^i = B(x_i, 10^{-1}\eta_{\text{stop}}) \setminus \overline{B(x_i, \rho)}$, we have

$$\inf_{\substack{w \in H^1(A_\rho^i, \mathbb{S}^1) \\ w(x_1 + 10^{-1}\eta_{\text{stop}} e^{i\theta}) = \text{Cst}_1 e^{i\tilde{d}\theta} \\ w(x_1 + \rho e^{i\theta}) = \text{Cst}_2 e^{i\tilde{d}\theta}}} \frac{1}{2} \int_{A_\rho^i} \alpha |\nabla w|^2 \le \inf_{\substack{w \in H^1(A_\rho^i, \mathbb{S}^1) \\ \deg_{\partial B(x_i, \rho)} = \tilde{d}}} \frac{1}{2} \int_{A_\rho^i} \alpha |\nabla w|^2 + C_b \tilde{d}^2.$$

Step 3: By extending a minimizer of $\widehat{\mathcal{J}}_{10^{-1}\eta_{\text{stop}},\alpha}(\mathbf{x}, \mathbf{d})$ by the ones of $\dfrac{1}{2} \displaystyle\int_{A_\rho^i} \alpha |\nabla \cdot|^2$ with Dirichlet conditions, we can construct a map which proves the result taking $C_0 = C_1 + d^3 C_b$.

3.3 Optimal perforated domains for the degree conditions

Recall that we fixed $\Omega' \supset \Omega$ a smooth bounded domain s.t. $\text{dist}(\partial\Omega', \Omega) > 0$ and a smooth $\mathbb{S}^1$-valued extension of g to $\Omega' \setminus \overline{\Omega}$ (still denoted by g).

In this section, we study the minimization problem

$$I_{\rho,\varepsilon} := \inf_{\substack{x_1, \ldots, x_N \in \Omega \\ |x_i - x_j| \ge 8\rho \\ d_1, \ldots, d_N > 0, \sum d_i = d}} \inf_{\substack{w \in H^1_g(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i, \rho)}(w) = d_i}} \frac{1}{2} \int_{\Omega'_\rho} U_\varepsilon^2 |\nabla w|^2 \tag{3.7}$$

where

$$\Omega'_\rho = \Omega' \setminus \cup \overline{B(x_i, \rho)}$$

and

$$H^1_g(\Omega'_\rho, \mathbb{S}^1) = \left\{ w \in H^1(\Omega'_\rho, \mathbb{S}^1) \,|\, w = g \text{ in } \Omega' \setminus \overline{\Omega \cup B(x_i, \rho)} \right\};$$

here, we extended U_ε with the value 1 outside Ω. We recall that we denoted by U_ε **the unique global minimizer** of E_ε in H_1^1.

In this subsection we assume that Hypothesis (1.3) holds ($|\ln(\lambda\delta)|^3/|\ln \varepsilon| \to 0$). This is not optimal for the statements but it makes the proofs simpler (this hypothesis may be relaxed, but it appears as a crucial and technical hypothesis for the methods developed Section 4).

A first purpose of this section is the study of the behavior of $I_{\rho,\varepsilon}$ when $\rho = \rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. In view of the application we have in mind we suppose that $\lambda\delta^{P+1} \gg \rho(\varepsilon) \ge \varepsilon$ but this is not crucial for our arguments (here $P = 1$ if U_ε is associated associated with the periodic pinning term) .

A second objective of our study is to exhibit the behavior of almost minimal configurations $\{(x_1^n, \ldots, x_N^n), (d_1^n, \ldots, d_N^n)\}$.

For fixed ρ, ε, the existence of a minimal configuration of points $\mathbf{x}_{\rho,\varepsilon}$ is the purpose of Proposition 9. In this section we consider only almost minimal configurations.

Notation 11. For $\varepsilon_n \downarrow 0$, we say that $\{(x_1^n, ..., x_N^n), (d_1^n, ..., d_N^n)\}$ is an almost minimal configuration for $\rho = \rho(\varepsilon_n) \downarrow 0$ when $x_1^n, ..., x_N^n \in \Omega$, $|x_i^n - x_j^n| \geq 8\rho$, $d_1^n, ..., d_N^n > 0, \sum d_i^n = d$ and there is $C > 0$ (independent of n) s.t.

$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n, \rho)}(w) = d_i^n}} \frac{1}{2} \int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho, \varepsilon_n} \leq C.$$

Roughly speaking, we establish in this section two repelling effects for the points: point/point and point/$\partial\Omega$; and an attractive effect for the points by the inclusions ω_ε.

3.3.1   The case of the periodic pinning term

The main result of this section establishes that when $\varepsilon_n, \rho \downarrow 0$, an almost minimal configuration $\{(x_1^n, ..., x_N^n), (d_1^n, ..., d_N^n)\}$ is s.t. (for sufficiently large n)

- the points x_i^n's cannot be mutually close,

- the degrees d_i^n's are necessarily all equal to 1,

- the points x_i^n's cannot approach $\partial\Omega$,

- there is $c > 0$ s.t. $B(x_i^n, c\lambda\delta) \subset \omega_\varepsilon$ for all i.

These facts are expressed in the following proposition (whose proof is postponed to Appendix D).

Proposition 12. *[The case of a periodic pinning term]*
Assume that λ, δ satisfy (1.3) and let a_ε be the periodic the pinning term (represented Figure 1).

Let $\varepsilon_n \downarrow 0$, $\rho = \rho(\varepsilon_n) \downarrow 0$, $x_1^n, ..., x_N^n \in \Omega$ be s.t. $|x_i^n - x_j^n| \geq 8\rho$, $\rho \geq \varepsilon_n$ and let $d_1^n, ..., d_N^n \in \mathbb{N}^*$ be s.t. $\sum d_i^n = d$.

1. *Assume that there is $i_0 \in \{1, ..., N\}$ s.t. $d_{i_0}^n \neq 1$ or that there are $i_0 \neq j_0$ s.t. $|x_{i_0}^n - x_{j_0}^n| \to 0$. Then*

$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n, \rho)}(w) = d_i^n}} \left\{ \frac{1}{2} \int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho, \varepsilon_n} \right\} \to \infty.$$

2. *Assume that there is $i_0 \in \{1, ..., N\}$ s.t. $\mathrm{dist}(x_{i_0}^n, \partial\Omega) \to 0$. Then*

$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n, \rho)}(w) = d_i^n}} \left\{ \frac{1}{2} \int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho, \varepsilon_n} \right\} \to \infty.$$

3. *Assume that $\dfrac{\rho}{\lambda\delta} \to 0$ and that there is $i_0 \in \{1, ..., N\}$ s.t. $x_{i_0}^n \notin \omega_\varepsilon$ or s.t. $x_{i_0}^n \in \omega_\varepsilon$ and $\dfrac{\mathrm{dist}(x_{i_0}^n, \partial\omega_\varepsilon)}{\lambda\delta} \to 0$. Then*

$$\inf_{\substack{w \in H_g^1(\Omega'_\rho, \mathbb{S}^1) \\ \deg_{\partial B(x_i^n, \rho)}(w) = d_i^n}} \left\{ \frac{1}{2} \int_{\Omega'_\rho} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho, \varepsilon_n} \right\} \to \infty.$$

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A straightforward consequence of Proposition 12 is the following

Corollary 13. *1. Consider an almost minimal configuration $\{\mathbf{x}_{\rho,\varepsilon}, \mathbf{d}_{\rho,\varepsilon}\} \in \Omega^N \times \mathbb{N}^{*N}$, i.e., assume that there is $w_{\rho,\varepsilon} \in H^1_g(\Omega' \setminus \cup \overline{B(x_i^{\rho,\varepsilon}, \rho)}, \mathbb{S}^1)$ verifying*

$$\deg_{\partial B(x_i^{\rho,\varepsilon}, \rho)}(w) = d_i^{\rho,\varepsilon} \text{ and } \frac{1}{2} \int_{\Omega' \setminus \cup \overline{B(x_i^{\rho,\varepsilon}, \rho)}} U_\varepsilon^2 |\nabla w|^2 \leq I_{\rho,\varepsilon} + C.$$

(Here, C is independent of ε.)

Then, there is some η_0 independent of ε s.t., for small ε, we have

$$|x_i^{\rho,\varepsilon} - x_j^{\rho,\varepsilon}|, \text{dist}(x_i^{\rho,\varepsilon}, \partial\Omega) \geq \eta_0 \text{ and } d_i = 1 \text{ for all } i \neq j, \ i,j \in \{1, ..., N\}.$$

In particular, we have $N = d$.

2. If, in addition, $\rho = \rho(\varepsilon)$ is s.t. $\rho \geq \varepsilon$ and $\dfrac{\rho}{\lambda\delta} \to 0$, then there is $c > 0$ (independent of ε) s.t., for small ε, we have $B(x_i^{\rho,\varepsilon}, c\lambda\delta) \subset \omega_\varepsilon$.

Proof of Corollary 13. We prove the first part. Let $C > 0$. We argue by contradiction and we assume that for all $n \in \mathbb{N}^*$ there are $0 < \varepsilon_n \leq \rho = \rho(\varepsilon_n) \leq 1/n$, $\mathbf{x}_n = \mathbf{x}_{\rho,\varepsilon_n}$, $(d_1, ..., d_N)$ and $w_n = w_{\rho,\varepsilon_n}$ satisfying the hypotheses of Corollary 13 and s.t.

$$\min\left\{|x_i^n - x_j^n|, \text{dist}(x_i^n, \partial\Omega)\right\} \to 0 \text{ or s.t. there is } i \in \{1, ..., N\} \text{ for which we have } d_i \neq 1.$$

By construction we have that $\{\mathbf{x}_{\rho,\varepsilon_n}, \mathbf{d}\}$ is an almost minimal configuration for I_{ρ,ε_n} with $\rho = \rho(\varepsilon_n) \geq \varepsilon_n$. Clearly from Proposition 12 we find a contradiction.

The proof of the second part is similar. $\square$

We end this subsection by the following direct consequence of Corollary 13

Corollary 14. *For sufficiently small ε, ρ, an almost minimal configuration $(x_1, ..., x_d)$ for $J_{\rho,\varepsilon}$ is an almost minimal configuration for $I_{\rho,\varepsilon}$.*

Moreover, there is $C_0 > 0$ s.t. $J_{\rho,\varepsilon} \leq I_{\rho,\varepsilon} + C_0$, C_0 is independent of small ε, ρ.

Proof. Let $C \geq 0$ and let $(x_1, ..., x_d), (x_1', ..., x_d') \in \Omega^d$ be s.t.

$$\hat{\mathcal{J}}_{\rho,\varepsilon}(x_1, ..., x_d) \leq J_{\rho,\varepsilon} + C$$

and

$$\hat{\mathcal{I}}_{\rho,\varepsilon}(x_1', ..., x_d') \leq I_{\rho,\varepsilon} + C.$$

From Corollary 13, there is $\eta_0 = \eta_0(C) > 0$ s.t. for $\varepsilon \leq \rho \leq \eta_0$, $\min_i \text{dist}(x_i', \partial\Omega) \geq \eta_0$. Using Proposition 10 we have the existence of C_0 s.t.

$$\begin{aligned}
\hat{\mathcal{I}}_{\rho,\varepsilon}(x_1, ..., x_d) \leq \hat{\mathcal{J}}_{\rho,\varepsilon}(x_1, ..., x_d) &\leq J_{\rho,\varepsilon} + C \leq \hat{\mathcal{J}}_{\rho,\varepsilon}(x_1', ..., x_d') + C \\
&\leq \hat{\mathcal{I}}_{\rho,\varepsilon}(x_1', ..., x_d') + C + C_0 \\
&\leq I_{\rho,\varepsilon} + 2C + C_0.
\end{aligned}$$

$\square$

3.3.2 A more precise result for the case of the periodic pinning term with dilution

In this section we focus on the periodic pinning term (represented Figure 1) with dilution: $\lambda \to 0$.

Notation 15. We define two kinds of configuration of distinct points of Ω:

- We say that for $\varepsilon_n \downarrow 0$ and $\rho = \rho(\varepsilon_n) \to 0$, d distinct points of Ω, $\mathbf{x}_n = (x_1^n, ..., x_d^n)$ form a quasi-minimizer of J_{ρ,ε_n} when $\mathcal{J}_{\rho,\varepsilon_n}(\mathbf{x}_n) - J_{\rho,\varepsilon_n} \to 0$.

- We say that for $\varepsilon_n \downarrow 0$ and $\rho = \rho(\varepsilon_n) \to 0$, d distinct points of Ω, $\mathbf{x}_n = (x_1^n, ..., x_d^n)$ form a quasi-minimizer of W_g, the *renormalized energy* of Bethuel-Brezis-Hélein (see [4]) when $W_g(\mathbf{x}_n) \to \min W_g$.

Proposition 16. *[Asymptotic location of optimal perforations]*
Assume that λ, δ satisfy (1.3) and that $\lambda \to 0$.
Let $\varepsilon_n \downarrow 0$, $\rho = \rho(\varepsilon_n) \to 0$, $\rho \geq \varepsilon_n$ and $\mathbf{x}_n = (x_1^n, ..., x_d^n)$ be d distinct points of Ω.
If the points $\mathbf{x}_n$ form a quasi-minimizer of J_{ρ,ε_n}, then $\mathbf{x}_n = (x_1^n, ..., x_d^n)$ form a quasi-minimizer of W_g.

This proposition is proved Appendix E.

3.3.3 The case of a general pinning term with variable sizes of inclusions

We assume that a_ε is the general pinning term represented Figure 2 with the hypothesis on the dilution: $\lambda \to 0$.

Proposition 17. *[The case of a non-periodic pinning term]*
Assume that λ, δ satisfy (1.3) and $\lambda \to 0$.
Let $\rho = \rho(\varepsilon)$ s.t. $\rho \geq \varepsilon$ and $\dfrac{\rho}{\lambda \delta^{3/2}} \to 0$. If $\{\mathbf{x}_{\rho,\varepsilon}, \mathbf{d}_{\rho,\varepsilon}\}$ is an almost minimal configuration for $I_{\rho,\varepsilon}$, then $N = d$ (thus $d_i = 1$ for all i) and there are $c, \eta_0 > 0$ (independent of ε) s.t. for sufficiently small ε:

1. $|x_i^{\rho,\varepsilon} - x_j^{\rho,\varepsilon}|, \operatorname{dist}(x_i^{\rho,\varepsilon}, \partial\Omega) \geq \eta_0$ for all $i \neq j$, $i, j \in \{1, ..., N\}$.

2. $B(x_i^{\rho,\varepsilon}, c\lambda\delta) \subset \omega_\varepsilon$ (the centers of the holes are included in the largest inclusions).

Moreover, there is $C_0 > 0$ s.t. $J_{\rho,\varepsilon} \leq I_{\rho,\varepsilon} + C_0$, C_0 is independent of small ε, ρ. And thus an almost minimal configuration $\mathbf{x}_{\rho,\varepsilon}$ for $J_{\rho,\varepsilon}$ is an almost minimal configuration for $I_{\rho,\varepsilon}$

This proposition is proved Appendix E (Subsection E.3).

4 The pinned Ginzburg-Landau functional

In this section, we turn to the main purpose of this article: the study of minimizers of E_ε (defined in (1.1)) in H_g^1.

The pinning term is the periodic one (represented Figure 1) or the non periodic one (represented Figure 2).

Recall that we fix $\delta = \delta(\varepsilon)$, $\delta \to 0$, $\lambda = \lambda(\varepsilon)$, $\lambda \equiv 1$ or $\lambda \to 0$ satisfying (1.3). If the pinning term is not periodic then we add the hypothesis $\lambda \to 0$.

4.1 Sharp Upper Bound, η-ellipticity and Uniform Convergence

4.1.1 Sharp Upper Bound and an η-ellipticity result

We may easily prove the following upper bound.

Lemma 18. *Assume that $\dfrac{\rho}{\lambda\delta} \to 0$ (or $\dfrac{\rho}{\lambda\delta^{3/2}} \to 0$ if the pinning term is not periodic), then we have*

$$\inf_{v \in H_g^1(\Omega,\mathbb{C})} F_\varepsilon(v) \le db^2(\pi \ln \frac{b\rho}{\varepsilon} + \gamma) + J_{\rho,\varepsilon} + o_\varepsilon(1), \tag{4.1}$$

where $\gamma > 0$ is a universal constant defined in [4], Lemma IX.1.

Proof. We construct a suitable test function $\tilde{w}_\varepsilon \in H_g^1$ (for sufficiently small ε).

From Proposition 9, one may consider $(x_1^\varepsilon, ..., x_d^\varepsilon) = \mathbf{x}^\varepsilon \in \Omega^d$, a minimal configuration for $J_{\rho,\varepsilon}$.

Note that since $\dfrac{\rho}{\lambda\delta} \to 0$ (or $\dfrac{\rho}{\lambda\delta^{3/2}} \to 0$ if the pinning term is not periodic), from Corollaries 13 & 14 (or Proposition 17 if the pinning term is not periodic), there are $\eta > 0$ and $c > 0$ s.t. for small ε we have $B(x_i^\varepsilon, c\lambda\delta) \subset \omega_\varepsilon$ and $\min_i \{\min_{i \ne j} |x_i - x_j|, \text{dist}(x_i, \partial\Omega)\} \ge \eta$.

Let w_ε be a minimal map in $\mathcal{J}_{\rho,\varepsilon}(\mathbf{x}^\varepsilon, \mathbf{1})$ (Proposition 8). We denote $\mathbf{1} := (1, ..., 1) \in \mathbb{N}^d$

Let $u_{\varepsilon/(b\rho)} \in H^1(B(0,1), \mathbb{C})$ be a global minimizer of

$$E_{\varepsilon/(b\rho)}^0(u) = \frac{1}{2} \int_{B(0,1)} \left\{ |\nabla u|^2 + \frac{b^2\rho^2}{2\varepsilon^2}(1 - |u|^2)^2 \right\}, \, u \in H_{x/|x|}^1(B(0,1), \mathbb{C}).$$

We consider the test function

$$\tilde{w}_\varepsilon(x) = \begin{cases} w_\varepsilon & \text{in } \Omega_\rho \\ \alpha_i^\varepsilon u_{\varepsilon/(b\rho)}\left(\dfrac{x - x_i^\varepsilon}{\rho}\right) & \text{in } B(x_i^\varepsilon, \rho) \end{cases}.$$

Here the constants $\alpha_i^\varepsilon \in \mathbb{S}^1$ are s.t. $w_\varepsilon(x_i^\varepsilon + \rho e^{\imath\theta}) = \alpha_i^\varepsilon e^{\imath\theta}$.

Estimate (4.1) is obtained by using the fact that $E_\varepsilon^0(u_\varepsilon) = \pi |\ln \varepsilon| + \gamma + o_\varepsilon(1)$ as $\varepsilon \to 0$ (see [4] Lemma IX.1) and Proposition 3. $\qquad\square$

Note that

$$I_{\rho,\varepsilon} \le J_{\rho,\varepsilon} \le \pi d |\ln \rho| + C. \tag{4.2}$$

We now turn to the η-ellipticity.

We denote by v_ε a global minimizer of F_ε in H_g^1. We extend $|v_\varepsilon|$ with the value 1 outside Ω.

One of the main ingredients in this work is the following result.

Lemma 19. *[η-ellipticity lemma]*

Let $0 < \alpha < 1/2$. Then the following results hold:

1. *If for $\varepsilon < \varepsilon_0$*

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^\alpha) \cap \Omega) \le \chi^2 |\ln \varepsilon| - C_1,$$

   then we have

$$|v_\varepsilon| \ge 1 - C\chi \text{ in } B(x, \varepsilon^{2\alpha}).$$

   Here, $\chi_\varepsilon \in (0,1)$ is s.t. $\chi_\varepsilon \to 0$ and $\varepsilon_0 > 0$, $C > 0$, $C_1 > 0$ depend only on $b, \alpha, \chi, \Omega, \|g\|_{C^1(\partial\Omega)}$.

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2. *If for* $\varepsilon < \varepsilon_0$

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^\alpha) \cap \Omega) \leq C|\ln \varepsilon|,$$

then we have

$$|v_\varepsilon| \geq \mu \ in \ B(x, \varepsilon^{2\alpha}).$$

Here, $\mu \in (0, 1)$ *and* $\varepsilon_0, C > 0$ *depend only on* $b, \alpha, \mu, \Omega, \|g\|_{C^1(\partial\Omega)}$.

This result is a direct consequence of Lemma 1 in [13].

4.1.2 Uniform convergence of $|v_\varepsilon|$ outside ω_ε

With the help of Lemma 19, we are in position to establish uniform convergence of $|v_\varepsilon|$ to 1 far away from $\overline{\omega_\varepsilon}$.

Proposition 20. *Let* $10^{-2} \cdot \operatorname{dist}(\omega, \partial Y) > \mu > 0$ *and* $K_\varepsilon^\mu = \{x \in \Omega \,|\, \operatorname{dist}(x, \omega_\varepsilon) \geq \mu\lambda\delta\}$. *Then, for sufficiently small* ε, *we have*

$$|v_\varepsilon| \geq 1 - C\sqrt{\frac{|\ln(\lambda\delta)|}{|\ln \varepsilon|}} \ in \ K_\varepsilon^\mu.$$

Here C *is independent of* ε *and* μ.

Furthermore, if for some small ε, *we have* $|v_\varepsilon(x)| < 1 - C\sqrt{\dfrac{|\ln(\lambda\delta)|}{|\ln \varepsilon|}}$, *then*

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) \geq \frac{2(\pi d + 1)}{b^2(1 - b^2)}|\ln(\lambda\delta)|.$$

Proof. Using Lemma 19.1 with $\alpha = 1/4$ and $\chi = \sqrt{\dfrac{2(\pi d + 1)}{b^2(1 - b^2)}\dfrac{|\ln(\lambda\delta)|}{|\ln \varepsilon|}}$, we obtain the existence of $C > 0$ s.t. for $\varepsilon > 0$ sufficiently small:

if $F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) < \dfrac{2(\pi d + 1)}{b^2(1 - b^2)}|\ln(\lambda\delta)|$, then we have $|v_\varepsilon| \geq 1 - C\chi$ in $B(x, \varepsilon^{1/2})$.

In order to prove Proposition 20, we argue by contradiction. There are $\varepsilon_n \downarrow 0$, $\mu > 0$ and $x_n \in K_{\varepsilon_n}^\mu$ s.t.

$$|v_{\varepsilon_n}(x_n)| < 1 - C\chi.$$

From (1.5), we find

$$|U_{\varepsilon_n} - 1| \leq C_0 \mathrm{e}^{-\frac{\alpha\mu}{2\xi}} \ in \ K_{\varepsilon_n}^{\mu/2}, \ \xi = \frac{\varepsilon_n}{\lambda\delta}. \tag{4.3}$$

Consequently, Lemma 19, the definition of C and (4.3) imply that for large n,

$$\frac{1}{2}\int_{B(x_n, \varepsilon_n^{1/4})} \left\{|\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2\right\} \geq \frac{2(\pi d + 1)}{b^2(1 - b^2)}|\ln(\lambda\delta)| + o_\varepsilon(1). \tag{4.4}$$

We extend v_ε to $\Omega' := \Omega + B(0, 1)$ with the help of a fixed smooth $\mathbb{S}^1$-valued map v s.t. $v = g$ on $\partial\Omega$. We also extend U_ε and a_ε with the value 1 outside Ω.

For n sufficiently large, we have

$$\frac{1}{2}\int_{\Omega'} \left\{|\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2\right\} \leq C|\ln \varepsilon_n|.$$

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Theorem 4.1 in [20] applied with $r = 10^{-2} \cdot \lambda \delta \mu$ and for large n, implies the existence of $\mathcal{B}^n = \{B_j^n\}$ a finite disjoint covering by balls of

$$\left\{ x \in \Omega' \,\middle|\, \text{dist}(x, \partial\Omega') > \frac{\varepsilon_n}{b} \text{ and } 1 - |v_{\varepsilon_n}(x)| \geq \left(\frac{\varepsilon_n}{b}\right)^{1/8} \right\}$$

s.t.

$$\text{rad}\,(\mathcal{B}^n) \leq 10^{-2} \cdot \lambda \delta \mu$$

satisfying

$$\frac{1}{2} \int_{\cup B_j^n} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} \geq \pi \sum_j d_j^n (|\ln \varepsilon_n| - |\ln(\lambda\delta)|) - C$$

$$= \pi \sum_j d_j^n |\ln \xi| - C.$$

Here, $\text{rad}\,(\mathcal{B}^n) = \sum_j \text{rad}(B_j^n)$, $\text{rad}(B)$ stands for the radius of the ball B, $\xi = \varepsilon_n/(\lambda\delta)$ and the integers d_j^n are defined by

$$d_j^n = \begin{cases} |\deg_{\partial B_j^n}(v_{\varepsilon_n})| & \text{if } B_j^n \subset \{x \in \Omega' \,|\, \text{dist}(x, \partial\Omega') > \frac{\varepsilon_n}{b}\} \\ 0 & \text{otherwise} \end{cases}.$$

Since $B_j \subset \Omega + B_{1/2} \subset \{x \in \Omega' \,|\, \text{dist}(x, \partial\Omega') > \frac{\varepsilon_n}{b}\}$, we obtain

$$\frac{1}{2} \int_{\cup B_j^n} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} \geq \pi d |\ln \xi| - C. \tag{4.5}$$

From (4.3) and (1.3) we have

$$F_\xi(v_{\varepsilon_n}, \cup_j B_j \cup B(x_n, \varepsilon_n^{1/4})) \geq \frac{b^2(1-b^2)}{2} \int_{B(x_n, \varepsilon_n^{1/4})} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} +$$

$$+ \frac{b^2}{2} \int_{\cup_j B_j} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} + o_n(1). \tag{4.6}$$

By combining (4.1) (with $\rho = \lambda^2\delta^2$), (4.2), (4.4), (4.5) and (4.6), we find that

$$\begin{aligned} \pi d b^2 \ln[(\lambda\delta)/\xi] + \pi d |\ln[(\lambda\delta)^2]| &\geq F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega') - \mathcal{O}_n(1) \\ &\geq F_{\varepsilon_n}(v_{\varepsilon_n}, \cup_j B_j \cup B(x_n, \varepsilon_n^{1/4})) - \mathcal{O}_n(1) \\ &\geq \pi d b^2 |\ln \xi| + 2(\pi d + 1)|\ln(\lambda\delta)| - \mathcal{O}_n(1) \end{aligned}$$

which is a contradiction. This completes the proof of Proposition 20. $\qquad\square$

4.2 Bad discs

4.2.1 Construction and first properties of bad discs

A fundamental tool in this article is the use of *ad-hoc* coverings of $\{|v_\varepsilon| \leq 7/8\}$ by small discs. The best radius for a covering of $\{|v_\varepsilon| \leq 7/8\}$ should be of the order ε. But the construction of such covering need some preliminary results.

Roughly speaking, the way to get a "sharp" covering is to consider a trivial covering and to "clean" it by dropping some discs with the help an "energetic test" (η-ellipticity result).

Here, we used two kinds of energetic tests: Lemma 19 and Theorem III.3 in [4]. Theorem III.3 in [4] gives the most precise results (it allows to deal with discs with radius $\mathcal{O}(\varepsilon)$) but it needs a bound on the potential part $\varepsilon^{-2}\int_\Omega(1-|v_\varepsilon|^2)^2$ which is the purpose of Proposition 30. In order to prove this bound (Proposition 30), we first use larger discs (discs with radius ρ, $\varepsilon \ll \rho \ll \lambda\delta^{P+1}$). The construction of intermediate coverings is done via Lemma 19.

We first consider

Notation 21. A trivial covering of Ω by discs

For $\varepsilon > 0$, we fix a family of discs $\left(B(x_i, \varepsilon^{1/4})\right)_{i\in I}$ s.t

$$x_i \in \Omega, \ \forall\, i \in I,$$

$$B(x_i, \varepsilon^{1/4}/4) \cap B(x_i, \varepsilon^{1/4}/4) = \emptyset \text{ if } i \neq j,$$

$$\cup_{i\in I} B(x_i, \varepsilon^{1/4}) \supset \Omega.$$

Then we select discs (using Lemma 19) and we define

Notation 22. The initial good/bad discs

- Let $C_0 = C_0(1/4, 7/8)$, $\varepsilon_0 = \varepsilon_0(1/4, 7/8)$ be defined by Lemma 19.2. For $\varepsilon < \varepsilon_0$, we say that $B(x_i, \varepsilon^{1/4})$ is an *initial good disc* if

$$F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq C_0 |\ln \varepsilon|$$

  and $B(x_i, \varepsilon^{1/4})$ is an *initial bad disc* if

$$F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) > C_0 |\ln \varepsilon|. \tag{4.7}$$

- We let $J = J(\varepsilon) := \{i \in I \,|\, B(x_i, \varepsilon^{1/4}) \text{ is an initial bad disc}\}$.

  An easy consequence of Lemma 18 is

Lemma 23. *The number of initial bad discs is bounded*
  There is an integer N which depends only on g and Ω s.t.

$$\operatorname{Card} J \leq N.$$

Proof. Since each point of Ω is covered by at most $C > 0$ (universal constant) discs $B(x_i, \varepsilon^{1/4})$, we have

$$\sum_{i\in J} F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq CF_\varepsilon(v_\varepsilon, \Omega).$$

The previous assertion implies that $\operatorname{Card} J \leq \dfrac{C\pi d}{C_0} + 1$. $\qquad\square$

Let $\rho(\varepsilon) = \rho \downarrow 0$ be s.t.

$$\frac{\rho}{\lambda\delta^{P+1}} \to 0 \text{ and } \frac{|\ln \rho|^3}{|\ln \varepsilon|} \to 0. \tag{4.8}$$

Note that from Assumption (1.3), such a ρ exists, *e.g.*, $\rho = (\lambda\delta)^{P+2}$. (Recall that if the pinning term is periodic then $P = 1$)

The following result is a straightforward variant of Theorem IV.1 in [4].

Lemma 24. *Separation of the initial bad discs*

Let $\varepsilon_n \downarrow 0$. Then (possibly after passing to a subsequence and relabeling the indices), we may choose $J' \subset J$ and a constant κ independent of n s.t.

$$J' = \{1, ..., N'\}, \ N' = \text{Cst},$$

$$|x_i - x_j| \geq 16\kappa\rho \ for \ i, j \in J', \ i \neq j,$$

and

$$\cup_{i \in J} B(x_i, \varepsilon_n^{1/4}) \subset \cup_{i \in J'} B(x_i, \kappa\rho).$$

Notation 25. The ρ-bad disc

For $i \in J'$, we say that $B(x_i, 2\kappa\rho)$ is a ρ-bad disc.

Proposition 26. *We have*

1. $\dfrac{\rho}{\text{dist}(B(x_i, 2\kappa\rho), \partial\Omega)} \to 0,$

2. $\deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n}) > 0,$

3. $F_{\varepsilon_n}(v_{\varepsilon_n}, B(x_i, 2\kappa\rho)) \geq \pi b^2 \deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n}) \ln \dfrac{\rho}{\varepsilon_n} - \mathcal{O}(1),$

4. $|v_{\varepsilon_n}| \geq 1 - C\sqrt{\dfrac{|\ln \rho|}{|\ln \varepsilon_n|}} \ in \ \Omega \setminus \cup_{i \in J'} \overline{B(x_i, 2\kappa\rho)}.$

Proof. We prove Assertions 1., 2. and 3.. Set

$$J_0' := \{i \in J' \,|\, \deg_{\partial(B(x_i, 2\kappa\rho)\cap\Omega)}(v_{\varepsilon_n}) > 0\}.$$

Since $|v_{\varepsilon_n}| \geq \frac{7}{8}$ in $\Omega \setminus \cup_{i \in J'} \overline{B(x_i, 2\kappa\rho)}$, we have

$$0 < d = \sum_{I \in J'} \deg_{\partial(B(x_i, 2\kappa\rho)\cap\Omega)}(v_{\varepsilon_n}) \leq \sum_{I \in J_0'} \deg_{\partial(B(x_i, 2\kappa\rho)\cap\Omega)}(v_{\varepsilon_n}). \tag{4.9}$$

Consequently $J_0' \neq \emptyset$.

Up to a subsequence, we may assume that J_0' is independent of n.

From Proposition 20, for all $i \in J_0'$, we have $\text{dist}(B(x_i, \varepsilon^{1/4}), \partial\Omega) \gtrsim \delta$ (or δ^P if the pinning term is not periodic). Consequently, for $i \in J_0'$ we find

$$\frac{\text{dist}(B(x_i, 2\kappa\rho), \partial\Omega)}{\rho} \to +\infty \tag{4.10}$$

since $\dfrac{\rho}{\lambda\delta^{P+1}} \to 0$.

Assertions 1., 2. and 3. will follow from the estimate

$$F_{\varepsilon_n}(v_{\varepsilon_n}, B(x_i, 2\kappa\rho)) \geq b^2\pi\deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n}) \ln \frac{\rho}{\varepsilon_n} - \mathcal{O}(1), \tag{4.11}$$

valid for $i \in J_0'$. Indeed, assume for the moment that (4.11) holds for $i \in J_0'$.

Then, by combining (4.1), (4.2), (4.7), (4.8), (4.9) and (4.11), we find that $J_0' = J'$, *i.e.*, 2. holds. Consequently, by combining Assertion 2. with (4.10), Assertion 1. yields and from Assertion 2. and (4.11), Assertion 3. holds.

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We now turn to the proof of (4.11), which relies on Proposition 4.1 in [20]. We apply this proposition in the domain $B = B(0, 2\kappa)$, to the function $v'(x) = v_{\varepsilon_n}[\rho(x - x_i)]$ and with the rescaled parameter $\xi_{\text{meso}} = \dfrac{\varepsilon}{\rho}$.

Note that, from (4.8), $\varepsilon \ll \xi_{\text{meso}} \ll \rho \ll \lambda \delta^{P+1}$ and $|\ln \varepsilon| \sim |\ln \xi_{\text{meso}}| \gg |\ln(\lambda \delta)|$. Clearly, v' satisfies

$$\int_B \left\{ |\nabla v'|^2 + \frac{1}{\xi_{\text{meso}}^2}(1 - |v'|^2)^2 \right\} = \int_{B(x_i, 2\kappa\rho)} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{\varepsilon^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\}$$
$$= \mathcal{O}(|\ln \varepsilon|) = \mathcal{O}(|\ln \xi_{\text{meso}}|).$$

Hence, one may apply the following result of Sandier and Serfaty: there is $(B_j)_{j \in I}$, a finite covering of
$$\{x \in B(0, 2\kappa - \xi_{\text{meso}}/b) \,|\, |v'(x)| \leq 1 - (\xi_{\text{meso}}/b)^{1/8}\}$$
with disjoint balls B_j of radius $r_j < 10^{-3}$ s.t.

$$\frac{1}{2} \int_{B \cap \cup B_j} \left\{ |\nabla v'|^2 + \frac{b^2}{\xi_{\text{meso}}^2}(1 - |v'|^2)^2 \right\} \geq \pi \sum_j d_j |\ln \xi_{\text{meso}}| - \mathcal{O}(1);$$

here $d_j = \begin{cases} |\deg_{\partial B_j}(v')| & \text{if } B_j \subset B(0, 2\kappa - \xi_{\text{meso}}/b) \\ 0 & \text{otherwise} \end{cases}$.

Note that from construction, $\{|v_{\varepsilon_n}| \leq 7/8\} \subset \cup_J B(x_i, \varepsilon_n^{1/4}) \subset \cup_{J'} B(x_i, \kappa\rho)$. Consequently:

$$\text{if } \deg_{\partial(B_j \cap B(0, 2\kappa - \xi_{\text{meso}}/b))}(v') \neq 0, \text{ then we have } B_j \subset B(0, \frac{3}{2}\kappa).$$

Therefore, $\sum d_j = \deg_{\partial B(0, 2\kappa)}(v') = \deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n})$ and

$$\frac{1}{2} \int_{B(x_i, 2\kappa\rho)} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} \geq \pi \deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n})|\ln \xi_{\text{meso}}| - \mathcal{O}(1)$$
$$= \pi \deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n}) \ln \frac{\rho}{\varepsilon} - \mathcal{O}(1).$$

Thus (4.11) holds.

The last assertion is obtained using Lemmas 18 & 19. Indeed, note that the proof of (4.11) gives a more precise result

$$F_{\varepsilon_n}(v_{\varepsilon_n}, B(x_i, \frac{3}{2}\kappa\rho)) \geq b^2 \pi \deg_{\partial B(x_i, 2\kappa\rho)}(v_{\varepsilon_n}) \ln \frac{\rho}{\varepsilon_n} - \mathcal{O}(1).$$

Let $x \in \Omega \setminus \cup_{J'} B(x_i, 2\kappa\rho)$ then $B(x, \varepsilon_n^{1/4}) \cap B(x_i, \frac{3}{2}\kappa\rho) = \emptyset$. Consequently, using Lemma 18 and the previous lower bound, we obtain:

$$F_{\varepsilon_n}(v_{\varepsilon_n}, B(x, \varepsilon_n^{1/4})) \leq I_{2\kappa\rho, \varepsilon_n} + C_0 \leq \pi d |\ln \rho| + C_0.$$

Therefore, from Lemma 19, there is $C > 0$, independent of x s.t. $|v_{\varepsilon_n}(x)| \geq 1 - C\sqrt{\dfrac{|\ln \rho|}{|\ln \varepsilon_n|}}$.

$\square$

4.2.2 Location and degree of bad discs

Let $w_n = \dfrac{v_{\varepsilon_n}}{|v_{\varepsilon_n}|} \in H^1(\Omega \setminus \cup_{J'} \overline{B(x_i, 2\kappa\rho)}, \mathbb{S}^1)$.

Proposition 27. *The map w_n is an almost minimal function for $I_{2\kappa\rho,\varepsilon_n}$.*

Proof. Indeed, denote $K_n = \dfrac{1}{2}\displaystyle\int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} U_{\varepsilon_n}^2 |\nabla w_n|^2$, then we have

$$
\begin{aligned}
K_n \quad &\leq \quad F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega \setminus \cup_{J'}\overline{B(x_i,2\kappa\rho)}) + \int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} U_{\varepsilon_n}^2(1-|v_{\varepsilon_n}|^2)|\nabla w_n|^2 \\
&= \quad F_{\varepsilon_n}(v_{\varepsilon_n},\Omega) - F_{\varepsilon_n}(v_{\varepsilon_n}, \cup_{J'}B(x_i,2\kappa\rho)) + \int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} U_{\varepsilon_n}^2(1-|v_{\varepsilon_n}|^2)|\nabla w_n|^2
\end{aligned}
$$

$$
\leq (4.1), \text{Prop } 26 \leq \quad I_{2\kappa\rho,\varepsilon_n} + C\sqrt{\frac{|\ln\rho|}{|\ln\varepsilon_n|}}\int_{\Omega\setminus\cup_{J'}\overline{B(x_i,2\kappa\rho)}} U_{\varepsilon_n}^2|\nabla w_n|^2 + \mathcal{O}(1)
$$

$$
\leq (4.1), \text{Prop } 26 \leq \quad I_{2\kappa\rho,\varepsilon_n} + C\sqrt{\frac{|\ln\rho|}{|\ln\varepsilon_n|}}F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega \setminus \cup_{J'}\overline{B(x_i,2\kappa\rho)}) + \mathcal{O}(1)
$$

$$
\leq \begin{matrix}(4.1),(4.2)\\ \text{Prop } 26\end{matrix} \leq \quad I_{2\kappa\rho,\varepsilon_n} + C\sqrt{\frac{|\ln\rho|^3}{|\ln\varepsilon_n|}} + \mathcal{O}(1)
$$

$$
\leq (4.8) \leq \quad I_{2\kappa\rho,\varepsilon_n} + \mathcal{O}(1).
$$

$\square$

Remark 28. Note that the penultimate line in the proof of Proposition 27 is the main use of (1.3) (which is express in (4.8)).

By combining Proposition 12 with Proposition 27 in the periodic case or Proposition 17 if the pinning term is not periodic, we obtain the following

Corollary 29. *The configuration $\{(x_1,...,x_{N'}), (\deg_{\partial B(x_1,2\kappa\rho)}(v_{\varepsilon_n}),...,\deg_{\partial B(x_{N'},2\kappa\rho)}(v_{\varepsilon_n}))\}$ is an almost minimal configuration of $I_{2\kappa\rho,\varepsilon_n}$ and consequently, $N' = d$, $\deg_{\partial B(x_i,2\kappa\rho)}(v_{\varepsilon_n}) = 1$ for all i and there is $\eta_0 > 0$ independent of large n s.t.*

$$
\min\left\{\min_{i\neq j}|x_i - x_j|, \min_i \operatorname{dist}(x_i, \partial\Omega)\right\} > 2\eta_0,
$$

$$
B(x_i, 2\eta_0\lambda\delta) \subset \omega_\varepsilon.
$$

4.3 H^1_{loc}-weak convergence

In order to keep notations simple, we replace from now on, $2\kappa\rho$ by $\rho/2$.

Using Corollary 29, there is $\{a_1,...,a_d\} \subset \Omega$ s.t. possibly after passing to a subsequence, we have $x_i^n = x_i \to a_i$.

Let $\rho_0 > 0$ be defined as

$$
\rho_0 = 10^{-2} \cdot \min_{k\neq l}\left\{\operatorname{dist}(a_k, \partial\Omega), |a_k - a_l|\right\}.
$$

4.3.1 The contribution of the modulus is bounded in the whole domain

We are now in position to bound the potential part of $F_\varepsilon(v_\varepsilon)$. More precisely we have

Proposition 30. *We have* $\int_\Omega \left\{ |\nabla|v_{\varepsilon_n}||^2 + \frac{1}{\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} = \mathcal{O}(1).$

Proof. From (4.1), Proposition 26 (Assertion 1., 2. and 3.) and Proposition 27, we infer that

$$\int_{\Omega \setminus \cup_i \overline{B(x_i, \rho/2)}} \left\{ |\nabla|v_{\varepsilon_n}||^2 + \frac{1}{\varepsilon_n^2}(1 - |v_{\varepsilon_n}|^2)^2 \right\} = \mathcal{O}(1).$$

Consequently it suffices to obtain a similar estimate in $B(x_i, \rho/2)$. Note that $B(x_i, \rho) \subset \omega_\varepsilon$. Thus, if we set

$$u'(x) = \frac{u_{\varepsilon_n}(x_i + \rho x)}{b} : B(0, 1) \to \mathbb{C},$$

then u' solves

$$-\Delta u' = \frac{1}{[\varepsilon_n/(b\rho)]^2} u'(1 - |u'|^2) \text{ in } B(0, 1).$$

From [5], we obtain

$$\frac{1}{2} \int_{B(0, 1/2)} \left\{ |\nabla|u'||^2 + \frac{b^2 \rho^2}{2\varepsilon_n^2}(1 - |u'|^2)^2 \right\} = \mathcal{O}(1).$$

This estimate is the subject of Theorem 1 for the potential part and Proposition 1 in [5] for the gradient of the modulus (see also Corollary 1 in [5]).

Set $K_n = \frac{1}{2} \int_{B(0, 1/2)} \left\{ |\nabla|u'||^2 + \frac{b^2 \rho^2}{2\varepsilon_n^2}(1 - |u'|^2)^2 \right\}.$ Using Proposition 3, we obtain

$$
\begin{aligned}
K_n = \mathcal{O}(1) &= \frac{1}{2b^2} \int_{B(x_i, \rho/2)} \left\{ |\nabla|U_{\varepsilon_n} v_{\varepsilon_n}||^2 + \frac{b^4}{2\varepsilon_n^2}\left(1 - \frac{|U_{\varepsilon_n} v_{\varepsilon_n}|^2}{b^2}\right)^2 \right\} \\
&= \frac{1}{2} \int_{B(x_i, \rho/2)} \left\{ |\nabla|v_{\varepsilon_n}||^2 + \frac{b^2}{2\varepsilon_n^2}\left(1 - |v_{\varepsilon_n}|^2\right)^2 \right\} + o_n(1).
\end{aligned}
$$

Consequently, Proposition 30 holds. $\qquad\square$

4.3.2 We bound the energy in a fixed perforated domain

Proposition 31. *For $0 < \eta \le \rho_0$, there is $C(\eta) > 0$ independent of n s.t. we have*

$$\frac{1}{2} \int_{\Omega \setminus \overline{B(a_i, \eta)}} |\nabla v_{\varepsilon_n}|^2 \le C(\eta). \tag{4.12}$$

Proof. We argue by contradiction and we assume that there is $\eta > 0$ s.t., up to pass to a subsequence, we have $\int_{\Omega \setminus \overline{B(a_i, \eta)}} |\nabla v_{\varepsilon_n}|^2 \to \infty$.

Because $\int_{\Omega \setminus \overline{B(a_i, \eta)}} |\nabla v_{\varepsilon_n}|^2 = \int_{\Omega \setminus \overline{B(a_i, \eta)}} |v_{\varepsilon_n}|^2 |\nabla w_n|^2 + |\nabla(|v_{\varepsilon_n}|)|^2$, from Propositions 26 & 30 we get $\int_{\Omega \setminus \overline{B(a_i, \eta)}} |\nabla w_n|^2 \to \infty$. Therefore, we have $\int_{\Omega \setminus \overline{B(x_i, 10^{-1}\eta)}} |\nabla w_n|^2 \to \infty$.

It is clear that we may get a map $\tilde{w}_n \in \mathcal{J}_{10^{-1}\eta, \varepsilon_n}(\mathbf{x}_{\varepsilon_n}, \mathbf{1})$ s.t. $\int_{\Omega \setminus \overline{B(x_i, 10^{-1}\eta)}} |\nabla \tilde{w}_n|^2 \le C(\eta).$

For $i = 1, ..., d$, using Proposition (45) (Appendix C, Section C.3, Page 44), we get the existence of a map $\tilde{w}_{i,n} \in H^1(B(x_i, 10^{-1}\eta) \setminus \overline{B(x_i, \rho/2)}, \mathbb{S}^1)$ s.t. $\tilde{w}_{i,n}(x_i + 10^{-1}\eta e^{i\theta}) = \tilde{w}_n(x_i + 10^{-1}\eta e^{i\theta})$ and

$$\int_{B(x_i, 10^{-1}\eta) \setminus \overline{B(x_i, \rho/2)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_{i,n}|^2 \le \int_{B(x_i, 10^{-1}\eta) \setminus \overline{B(x_i, \rho/2)}} U_{\varepsilon_n}^2 |\nabla w_n|^2 + \mathcal{O}(1)$$

Therefore by extending $\tilde{w}_n$ with $\tilde{w}_{i,n}$ in $B(x_i, 10^{-1}\eta) \setminus \overline{B(x_i, \rho/2)}$ we get a map still denoted $\tilde{w}_n \in H_g^1(\Omega \setminus \cup \overline{B(x_i, \rho/2)}, \mathbb{S}^1)$ s.t.

$$\frac{1}{2} \int_{\Omega \setminus \cup \overline{B(x_i, \rho/2)}} |\nabla w_n|^2 - \frac{1}{2} \int_{\Omega \setminus \cup \overline{B(x_i, \rho/2)}} |\nabla \tilde{w}_n|^2 \to \infty$$

which is in contradiction with Proposition 27.

$\square$

Consequently, there is $v_* \in H_{\mathrm{loc}}^1(\overline{\Omega} \setminus \{a_1, ..., a_d\}, \mathbb{S}^1)$ s.t., up to pass to a subsequence, $v_{\varepsilon_n} \rightharpoonup v_*$ in $H_{\mathrm{loc}}^1(\overline{\Omega} \setminus \{a_1, ..., a_d\})$. Next section is dedicate to the limiting equation of v_*.

4.3.3 We establish the limiting equation

In order to obtain the expression of the homogenized problem, we use the *unfolding operator* (see [8], definition 2.1).

The use of the unfolding operator needs a slightly modification of the cell period. More precisely, instead of considering the $\delta \times \delta$-grid whose vertices-grid are the points $\{\delta(k,l) + (1/2, 1/2) \,|\, k, l \in \mathbb{Z}\}$, we consider the one whose vertices-grid are $\{\delta(k,l) \,|\, k, l \in \mathbb{Z}\}$.

Thus instead of having cells which contain one inclusion at their center we have cells with quarters of inclusion at their vertices. (See Figure 3)

More specifically, we define, for $\Omega_0 \subset \mathbb{R}^2$ an open set, $p \in (1, \infty)$ and $\delta > 0$,

$$\mathcal{T}_\delta: \quad L^p(\Omega_0) \quad \to \quad L^p(\Omega_0 \times \tilde{Y})$$
$$\phi \quad \mapsto \quad \mathcal{T}_\delta(\phi)(x,y) = \begin{cases} \phi\left(\delta\left[\dfrac{x}{\delta}\right] + \delta y\right) & \text{for } (x,y) \in \tilde{\Omega}_\delta^{\mathrm{incl}} \times \tilde{Y} \\ 0 & \text{for } (x,y) \in \Lambda_\delta \times \tilde{Y} \end{cases}.$$

Here, $\tilde{Y} = (0,1)^2$, $[s]$ is the integer part of $s \in \mathbb{R}$ and

$$\tilde{\Omega}_\delta^{\mathrm{incl}} := \bigcup_{\substack{\tilde{Y}_\delta^K \subset \Omega_0, K \in \mathbb{Z}^2 \\ \tilde{Y}_\delta^K = \delta \cdot (K + \tilde{Y})}} \overline{\tilde{Y}_\delta^K}, \quad \Lambda_\delta := \Omega_0 \setminus \tilde{\Omega}_\delta^{\mathrm{incl}} \quad \text{and} \quad \left[\frac{x}{\delta}\right] := \left(\left[\frac{x_1}{\delta}\right], \left[\frac{x_2}{\delta}\right]\right).$$

An adaptation of a result of Sauvageot ([21], Theorem 4) gives the following

Proposition 32. *Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth bounded open set. Let $v_n \in H^2(\Omega_0, \mathbb{C})$ be s.t.*

1. *$|v_n| \le 1$ and $\displaystyle\int_{\Omega_0} (1 - |v_n|^2)^2 \to 0$,*

2. *$v_n \rightharpoonup v_*$ in $H^1(\Omega_0)$ for some $v_* \in H^1(\Omega_0, \mathbb{S}^1)$,*

3. *there are $H_n \in W^{1,\infty}(\Omega_0, [b^2, 1])$ and $\delta = \delta_n \downarrow 0$ s.t. $\mathcal{T}_\delta(H_n)(x,y) \to H_0(y)$ in $L^2(\Omega_0 \times \tilde{Y})$,*

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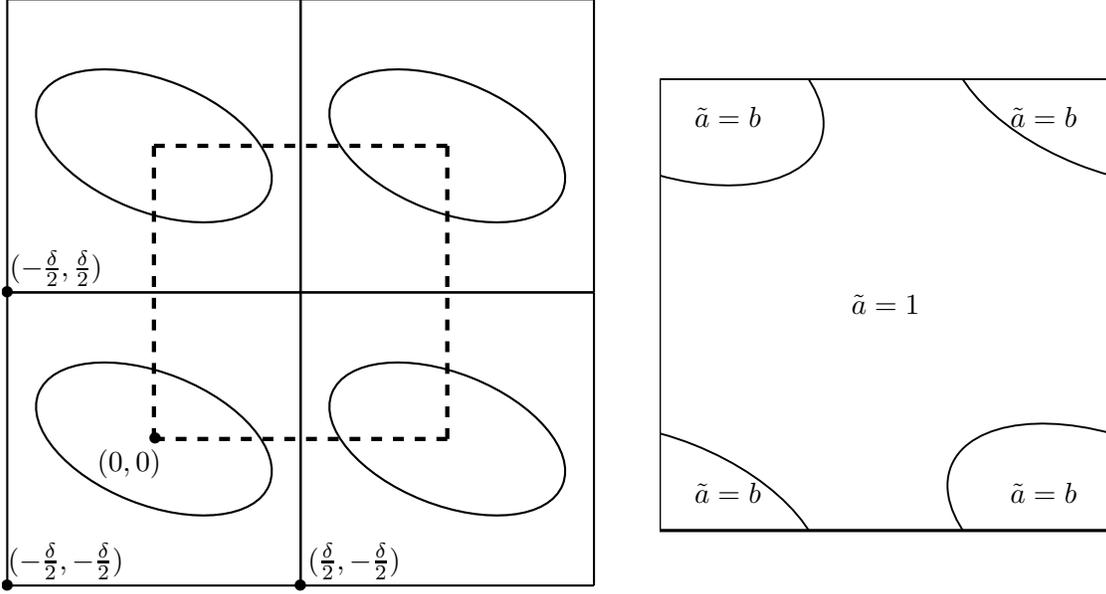
(a) Four period-cells which are obtained from $Y = (-1/2, 1/2)^2$ and the new period-cell in dash obtained from $\tilde{Y} = (0, 1)^2$

(b) The new unit-cell $\tilde{Y}$ with four quarters of an inclusion and the values of $\tilde{a} = a_{|\tilde{Y}}^\lambda$

Figure 3: The modification of the reference cell

4. $-\mathrm{div}(H_n \nabla v_n) = v_n f_n(x)$, $f_n \in L^\infty(\Omega_0, \mathbb{R})$.

Then v_* is a solution of

$$- \mathrm{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_*$$

where $\mathcal{A}$ is the homogenized matrix of $H_0(\frac{\cdot}{\delta})\mathrm{Id}_{\mathbb{R}^2}$. (See Appendix F to have more details about $\mathcal{A}$)

The proof of Proposition 32 is postponed to Appendix F.

We apply the above proposition with $\Omega_0 = \Omega \setminus \cup \overline{B(a_i, \eta)}$, $\delta = \delta_n \downarrow 0$ the sequence which defines a_{ε_n} and $H_n = U_{\varepsilon_n}^2$. By application of Proposition 3, we obtain

$$\mathcal{T}_\delta(U_{\varepsilon_n}^2)(x, y) \overset{L^2(\Omega_0 \times \tilde{Y})}{\to} \begin{cases} \tilde{a}^2(y) & \text{if } \lambda \equiv 1 \\ 1 & \text{if } \lambda \to 0 \end{cases}.$$

Note that the $\tilde{Y}$-periodic extension of $\tilde{a}$ in $\mathbb{R}^2$ is equal to the Y-periodic extension of $1 - (1 - b^2)\mathbb{1}_{\omega^\lambda}$ which is a^λ (defined Construction 1).

We find that v_* solves

$$\begin{array}{ll} -\mathrm{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_* & \text{if } \lambda \equiv 1 \\ -\Delta v_* = |\nabla v_*|^2 v_* & \text{if } \lambda \to 0 \end{array}.$$

Here $\mathcal{A}$ is the homogenized matrix of $[a^\lambda(\frac{\cdot}{\delta})]^2 \mathrm{Id}_{\mathbb{R}^2}$.

4.4 The small bad discs

4.4.1 Definition

With the help the bound on the potential part of the minimizers $\frac{1}{\varepsilon^2} \int_\Omega (1 - |v_\varepsilon|^2)^2 \le C$ (Proposition 30), in the spirit of [4] (Theorem III.3), we may detect the vorticity defects

(the connected components of $\{|v_\varepsilon| \le 7/8\}$) by smaller discs (discs with radius of order ε) than the ρ-bad discs (Notation 25).

Notation 33. The small bad discs

The construction is done as follows:

- We consider a covering of Ω as in Notation 21 (page 21). We fix $\rho = \rho(\varepsilon) \downarrow 0$ s.t. Assumption (4.8) holds. For sufficiently small ε, we denote

$$S_\rho(\varepsilon) = \{B(x, R/2) \,|\, B(x, R) \text{ given by Notation 25, page 22}\}.$$

- Following [4] (Theorem III.3), for $l \ge 2$, there are $\kappa_l, \mu_l > 0$ (depending only on Ω, g and l) s.t. for $x \in \Omega$, if

$$\frac{1}{\varepsilon^2} \int_{B(x, 2\kappa_l\varepsilon)} (1 - |v_\varepsilon|^2)^2 \le \mu_l$$

then

$$|v_\varepsilon| \ge 1 - \frac{1}{l^2} \text{ in } B(x, \kappa_l\varepsilon).$$

We fix $l \ge 2$ and we drop the subscript l. We now consider a covering of $\cup_{B \in S_\rho(\varepsilon)} B$ by discs $(B(x_i^\varepsilon, \kappa\varepsilon))_{i \in I}$ s.t

$$x_i^\varepsilon \in \cup_{B \in S_\rho(\varepsilon)} B, \, \forall i \in I,$$
$$B(x_i^\varepsilon, \kappa\varepsilon/4) \cap B(x_j^\varepsilon, \kappa\varepsilon/4) = \emptyset \text{ if } i \ne j,$$
$$\cup_{i \in I} B(x_i^\varepsilon, \kappa\varepsilon) \supset \cup_{B \in S_\rho(\varepsilon)} B.$$

We say that $B(x_i^\varepsilon, \kappa\varepsilon)$ is a *small good disc* if

$$\frac{1}{\varepsilon^2} \int_{B(x_i^\varepsilon, 2\kappa\varepsilon)} (1 - |v_\varepsilon|^2)^2 < \mu.$$

- If $B(x_i^\varepsilon, \kappa\varepsilon)$ is not a small good disc, then we call it a *small bad disc*. We denote $J \subset I$ the set of indices of small bad discs.

Following [4], using Proposition 30, there is $N_l = N > 0$ (depending only on Ω, g and l) s.t. $\text{Card}(J) \le N$.

Using a standard separation process (Lemma 42), for $\varepsilon_n \downarrow 0$, possibly after passing to a subsequence and relabeling the discs, there are $J' \subset J$ and $\kappa' \in \{\kappa, ..., 9^{N-1}\kappa\}$ s.t.

$$\{|v_{\varepsilon_n}| < 1 - 1/l^2\} \subset \cup_{i \in J} B(x_i^{\varepsilon_n}, \kappa\varepsilon_n) \subset \cup_{i \in J'} B(x_i^{\varepsilon_n}, \kappa'\varepsilon_n)$$

and

$$\frac{|x_i^{\varepsilon_n} - x_j^{\varepsilon_n}|}{\varepsilon_n} \ge 8\kappa' \text{ if } i, j \in J', i \ne j.$$

By a standard iterative procedure, we may assume that the small bad discs are mutually far away in the ε-scale.

Proposition 34. *Possibly after passing to a subsequence, we have, for large R and $J'' \subset J'$,*

$$\{|v_{\varepsilon_n}| < 1 - 1/l^2\} \subset \cup_{i \in J''} B(x_i^{\varepsilon_n}, R\varepsilon_n),$$

where, for $i \ne j$,

$$\frac{|x_i^{\varepsilon_n} - x_j^{\varepsilon_n}|}{\varepsilon_n} \to \infty \text{ as } n \to \infty.$$

Notation 35. The small and separated bad discs

The discs $\{B(x_i^{\varepsilon_n}, R\varepsilon_n) \,|\, i \in J''\}$ obtained in Proposition 34 are the small and separated bad discs.

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4.4.2 Each ρ-bad disc contains exactly one small bad disc

By construction, we know that the small and separated bad discs (defined Notation 35) are covered by the ρ-bad discs defined Notation 25 (page 22). We next prove that there are exactly d small bad discs and consequently, there is exactly one small bad discs per ρ-bad discs.

Proposition 36. *For large n and for all $i \in J''$, we have*

$$\deg_{\partial B(x_i^{\varepsilon n}, R\varepsilon_n)}(v_{\varepsilon_n}) = 1.$$

Proof. First we prove that, for large n and for all i, we have

$$\deg_{\partial B(x_i^{\varepsilon n}, R\varepsilon_n)}(v_{\varepsilon_n}) \neq 0.$$

We argue by contradiction and we assume that, up to a subsequence, there is i s.t. $\deg_{\partial B(x_i^{\varepsilon n}, R\varepsilon_n)}(v_{\varepsilon_n}) = 0$.

Set

$$M_n = \min\left\{ b\min_{i \neq j} \frac{|x_i^{\varepsilon n} - x_j^{\varepsilon n}|}{8R\varepsilon_n}, \delta^{-1} \right\} \tag{4.13}$$

and set

$$
\begin{array}{rccc}
u_n' : & B(0, M_n) & \to & \mathbb{C} \\
& x & \mapsto & \dfrac{u_{\varepsilon_n}\left(\dfrac{\varepsilon_n}{b}x + x_i^{\varepsilon n}\right)}{b}
\end{array}.
$$

Note that, $B(x_i^{\varepsilon n}, M_n\varepsilon_n) \subset \omega_{\varepsilon_n}$ and by Proposition 34, we have $M_n \to \infty$.

It is easy to check that u_n' solves $-\Delta u_n' = u_n'(1 - |u_n'|^2)$. Following [7], up to a subsequence,

$$u_n' \to u_0 \text{ in } C^2_{\text{loc}}(\mathbb{R}^2); \tag{4.14}$$

here $u_0 : \mathbb{R}^2 \to \mathbb{C}$ solves $-\Delta u_0 = u_0(1 - |u_0|^2)$ in $\mathbb{R}^2$.

Then two cases occur: $\displaystyle\int_{\mathbb{R}^2}(1 - |u_0|^2)^2 < \infty$ or $\displaystyle\int_{\mathbb{R}^2}(1 - |u_0|^2)^2 = \infty$.

Assume first that $\displaystyle\int_{\mathbb{R}^2}(1 - |u_0|^2)^2 < \infty$. From [7], noting that the degree of u_0 on large circles centered in 0 is 0, we obtain that $u_0 = \text{Cst} \in \mathbb{S}^1$ and consequently $\displaystyle\int_{\mathbb{R}^2}(1 - |u_0|^2)^2 = 0$.

Since $u_n' \to u_0$ in $L^4(B(0, 2bR))$ $(R \geq \kappa)$, we find that

$$
\begin{aligned}
\int_{B(0, 2bR)}(1 - |u_n'|^2)^2 &= \frac{b^2}{\varepsilon_n^2}\int_{B(x_i^{\varepsilon n}, 2R\varepsilon_n)}(1 - |u_n/b|^2)^2 \\
&= \frac{b^2}{\varepsilon_n^2}\int_{B(x_i^{\varepsilon n}, 2R\varepsilon_n)}(1 - |v_{\varepsilon_n}|^2)^2 + o_n(1) \to 0.
\end{aligned}
$$

Noting that $B(x_i^{\varepsilon n}, \kappa\varepsilon_n)$ is a small bad disc and that $B(x_i^{\varepsilon n}, 2\kappa\varepsilon_n) \subset B(x_i^{\varepsilon n}, 2R\varepsilon_n)$, we have a contradiction.

Therefore $\displaystyle\int_{\mathbb{R}^2}(1 - |u_0|^2)^2 = \infty$. Consequently, there is $M_0 > 0$ s.t.

$$\int_{B(0, bM_0)}(1 - |u_0|^2)^2 \geq \sup_n\left\{ \frac{4b^2}{\varepsilon_n^2}\int_\Omega(1 - |v_{\varepsilon_n}|^2)^2 \right\}.$$

Thus, for large n we have

$$
\begin{aligned}
\int_{B(0,bM_n)} (1 - |u'_n|^2)^2 &= \frac{b^2}{\varepsilon_n^2} \int_{B(x_i^{\varepsilon_n}, M_n \varepsilon_n)} (1 - |u_{\varepsilon_n}/b|^2)^2 \\
&= \frac{b^2}{\varepsilon_n^2} \int_{B(x_i^{\varepsilon_n}, M_n \varepsilon_n)} (1 - |v_{\varepsilon_n}|^2)^2 + o_n(1) \\
&\geq \sup_n \left\{ \frac{2b^2}{\varepsilon_n^2} \int_\Omega (1 - |v_{\varepsilon_n}|^2)^2 \right\},
\end{aligned}
$$

which is a contradiction with $B(x_i^{\varepsilon_n}, M_n \varepsilon_n) \subset \Omega$.

Consequently we obtain that for large n, $\deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \neq 0$.

Now we prove that

$$
\deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) = 1 \text{ for all } i \text{ and large } n. \tag{4.15}
$$

Note that each small bad disc contains at least a zero of v_{ε_n}. Consequently, for ρ satisfying (4.8), all small bad discs are included in a ρ-bad disc $B(y, \rho)$ defined Notation 25 (page 22). (For sake of simplicity we wrote $B(y, \rho)$ instead of $B(y, 2\kappa\rho)$).

If $B(y, \rho)$ is a ρ-bad disc, we denote $\Lambda_y = \{i \in J'' \mid x_i^{\varepsilon_n} \in B(y, \rho)\}$. Clearly, if $\mathrm{Card}(\Lambda_y) = 1$, then (4.15) holds.

We define

$$
\mathbf{æ}_n^y := \begin{cases} 10^{-2} \min_{i,j \in \Lambda_y, i \neq j} |x_i^{\varepsilon_n} - x_j^{\varepsilon_n}| & \text{if } \mathrm{Card}(\Lambda_y) > 1 \\ R\varepsilon_n & \text{otherwise} \end{cases}.
$$

From Proposition 34, if $\mathrm{Card}(\Lambda_y) > 1$ then $\mathbf{æ}_n / \varepsilon_n \to \infty$.

For simplicity, we assume that $y = 0$ and we let

$$
\tilde{B} = B(0, 8) \setminus \cup_{i \in \Lambda_0} \overline{B\left(\frac{x_i^{\varepsilon_n}}{\rho}, \frac{\mathbf{æ}_n^0}{\rho} \right)}.
$$

Remark 37. Note that from Corollary 29 we have $B(y, 16\rho) \subset \omega_\varepsilon$.

Clearly, we are in position to apply Theorem 2 in [14] in the perforated domain $\tilde{B}$. After scaling, we find that

$$
\frac{1}{2} \int_{B(y, 8\rho) \setminus \overline{\cup B(x_i^{\varepsilon_n}, \mathbf{æ}_n^y)}} |\nabla v_{\varepsilon_n}|^2 \geq \pi \left| \sum_{i \in \Lambda_y} \deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \right| \ln \frac{\rho}{\mathbf{æ}_n^y} - C = \pi \ln \frac{\rho}{\mathbf{æ}_n^y} - C.
$$

In order to prove (4.15), we observe the case where there is y s.t. $\mathrm{Card}(\Lambda_y) > 1$. Recall that if for all y centers of ρ-bad discs we have $\mathrm{Card}(\Lambda_y) = 1$, then (4.15) holds. Since $\deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \neq 0$, if $\mathrm{Card}(\Lambda_y) > 1$, then we have

$$
\sum_{i \in \Lambda_y} |\deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n})| > 1.
$$

We obtain easily the following lower bound for $i \in \Lambda_y$:

$$
\frac{1}{2} \int_{B(x_i^{\varepsilon_n}, \mathbf{æ}_n^y) \setminus \overline{B(x_i^{\varepsilon_n}, R\varepsilon_n)}} |\nabla v_{\varepsilon_n}|^2 \geq \pi \left| \deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \right| \ln \frac{\mathbf{æ}_n^y}{R\varepsilon_n} - C.
$$

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Summing for $i \in \Lambda_y$, we obtain that

$$\sum_{i \in \Lambda_y} \frac{1}{2} \int_{B(x_i^{\varepsilon n}, \ae_n^y) \setminus \overline{B(x_i^{\varepsilon_n}, R\varepsilon_n)}} |\nabla v_{\varepsilon_n}|^2 \geq 2\pi \ln \frac{\ae_n^y}{R\varepsilon_n} - C.$$

Consequently, we deduce that

$$\sum_y \frac{1}{2} \int_{B(y, 8\rho) \setminus \overline{\cup B(x_i^{\varepsilon n}, R\varepsilon_n)}} |\nabla v_{\varepsilon_n}|^2 \geq \pi d \ln \frac{\rho}{R\varepsilon_n} + \pi \sum_{y \text{ s.t. } \mathrm{Card}(\Lambda_y) > 1} \ln \frac{\ae_n^y}{R\varepsilon_n} - \mathcal{O}_n(1).$$

From Lemma 18 and Propositions 26 & 27, we deduce easily

$$\frac{1}{2} \int_{\cup B(y, 8\rho) \setminus \overline{\cup B(x_i^{\varepsilon n}, R\varepsilon_n)}} U_{\varepsilon_n}^2 |\nabla v_{\varepsilon_n}|^2 = \pi d b^2 \ln \frac{\rho}{\varepsilon_n} + \mathcal{O}_n(1).$$

Combining the previous estimates, we obtain that

$$\{y \text{ center of } \rho\text{-bad discs} \,|\, \mathrm{Card}(\Lambda_y) > 1\} = \emptyset,$$

and thus $\deg_{\partial B(x_i^{\varepsilon n}, R\varepsilon_n)}(v_{\varepsilon_n}) = 1$ for large n. $\qquad\square$

Corollary 38. *For large n, there is a unique zero inside each small and separated bad discs defined Notation 35 (page 28).*

Proof. From Proposition 36, one may assume that $v_{\varepsilon_n}(x_i^{\varepsilon n}) = 0$.

Let $i \in \{1, ..., d\}$. In view of (4.14), if we denote

$$
\begin{array}{rccc}
u_n' : & B(0, M_n) & \to & \mathbb{C} \\
& x & \mapsto & \dfrac{u_{\varepsilon_n}(\frac{\varepsilon_n}{b}x + x_i^{\varepsilon n})}{b} ,
\end{array}
\tag{4.16}
$$

then, up to a subsequence,

$$u_n' \to u_0 \text{ in } C^1(\overline{B(0, bR)}). \tag{4.17}$$

Here M_n is defined in (4.13).

Using the main result of [19], we have the existence of a universal function $f : \mathbb{R}^+ \to [0, 1]$ s.t.

$$
\begin{cases}
u_0(x) = f(|x|)e^{i(\theta + \theta_i)} \text{ where } x = |x|e^{i\theta}, \ \theta_i \in \mathbb{R} \\
\quad \text{and } f : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is increasing.}
\end{cases}
\tag{4.18}
$$

Therefore, we may apply Theorem 2.3 in [2] in order to obtain that, for large n, u_n' has a unique zero in $B(0, bR)$. Consequently, for large n, v_{ε_n} has a unique zero in $B(x_i^{\varepsilon n}, R\varepsilon_n)$. $\qquad\square$

Corollary 39. *One may consider that R depends only on l (R is independent of the extraction we consider), i.e, for $l \geq 2$ there is $R_l > 0$ s.t. for small ε, denoting $\{x_i^\varepsilon \,|\, i \in \{1, ..., d\}\}$ the set of zeros of a minimizer v_ε, we have*

$$\{|v_\varepsilon| < 1 - 1/l^2\} \subset \cup_i B(x_i^\varepsilon, R_l \varepsilon).$$

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Proof. From Corollary 38, one may assume that $v_{\varepsilon_n}(x_i^{\varepsilon_n}) = 0$.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as in (4.18) and u_n' as in (4.16). For $l \geq 2$, consider $R_l > 0$ s.t.

$$l \mapsto R_l \text{ is increasing and } f(bR_l) \geq 1 - \frac{1}{2l^2}.$$

Note that from [22], one may consider $R_l \simeq \sqrt{2}l/b$.

By uniqueness of f, the full sequence $|u_n'|$ converges to f in $L^\infty\,[B(0, b\max\{R, R_l\})]$. Consequently, for n sufficiently large, since f is not decreasing,

$$\{|v_{\varepsilon_n}| < 1 - 1/l^2\} \subset \cup_i B(x_i^{\varepsilon_n}, R_l\varepsilon_n).$$

$\square$

4.5 Asymptotic expansion of $F_\varepsilon(v_\varepsilon)$

This section is essentially devoted to proof Theorem 5. The key argument in this proof is Proposition 40.

4.5.1 Statement of the main result and a corollary

We state a technical and fundamental result and a direct corollary.

Proposition 40. *For all $\varepsilon_n \downarrow 0$, up to a subsequence, there is $\rho = \rho(\varepsilon_n)$ s.t. $\varepsilon_n \ll \rho \ll \lambda\delta^{3/2}$ and s.t. when $n \to \infty$ the following holds*

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\rho,\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1), \tag{4.19}$$

where $J_{\rho,\varepsilon}$ is defined in (3.6) and γ is the universal constant defined in [4], Lemma IX.1.

Corollary 41. *Let $\varepsilon_n \downarrow 0, \rho$ be as in Proposition 40. Then we have*

$$J_{\varepsilon_n,\varepsilon_n} - J_{\rho,\varepsilon_n} = \pi db^2 \ln \frac{\rho}{\varepsilon_n} + o_n(1).$$

Proof of Corollary 41. Using Proposition 9, we may consider $\mathbf{x}_n = (x_1^n, ..., x_d^n) \in \Omega^d$ a minimal configuration of points for J_{ρ,ε_n}, *i.e.* s.t.

$$\hat{\mathcal{J}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1}) = J_{\rho,\varepsilon_n}.$$

Combining Corollaries 13 & 14 (or Proposition 17 if the pinning term is not periodic), we have the existence of $c > 0$ s.t. $B(x_i^n, c\lambda\delta) \subset \omega_\varepsilon$.

Therefore, for a minimal map w_n of $\hat{\mathcal{J}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1})$, we may easily construct a map $\tilde{w}_n \in H^1(\Omega \setminus \cup_i \overline{B(x_i, \varepsilon_n)}, \mathbb{S}^1)$ s.t. $\tilde{w}_n \in \mathcal{J}_{\varepsilon_n}(\mathbf{x}_n, \mathbf{1})$ and

$$
\begin{aligned}
J_{\varepsilon_n,\varepsilon_n} &\leq \frac{1}{2} \int_{\Omega \setminus \overline{B(x_i, \varepsilon_n)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 \\
&= \frac{1}{2} \int_{\Omega \setminus \overline{B(x_i, \rho)}} U_{\varepsilon_n}^2 |\nabla w_n|^2 + \frac{1}{2} \int_{\cup B(x_i, \rho) \setminus \overline{B(x_i, \varepsilon_n)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 \\
&= J_{\rho,\varepsilon_n} + db^2 \pi \ln \frac{\rho}{\varepsilon_n} + o_n(1).
\end{aligned}
\tag{4.20}
$$

On the other hand, Lemma 18 combined with Proposition 40 yield

$$J_{\rho,\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1) \leq F_{\varepsilon_n}(v_{\varepsilon_n}) \leq J_{\varepsilon_n,\varepsilon_n} + db^2(\pi \ln b + \gamma). \tag{4.21}$$

We conclude with the help of (4.20) and (4.21). $\square$

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4.5.2 Proof of Theorem 5

We are now in position to prove Theorem 5, *i.e.*, we are going to prove that

$$F_\varepsilon(v_\varepsilon) = J_{\varepsilon,\varepsilon} + db^2(\pi \ln b + \gamma) + o_\varepsilon(1).$$

Indeed, using Lemma 18, it suffices to prove that

$$F_\varepsilon(v_\varepsilon) \geq J_{\varepsilon,\varepsilon} + db^2(\pi \ln b + \gamma) + o_\varepsilon(1).$$

This estimate is equivalent to:

for all $\varepsilon_n \downarrow 0$, up to subsequence, we have $F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\varepsilon_n,\varepsilon_n} + db^2(\pi \ln b + \gamma) + o_n(1)$.

Let $\varepsilon_n \downarrow 0$. Then, up to a subsequence, there is $\rho = \rho_n$ given by Proposition 40 s.t.

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\rho,\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1).$$

We deduce from Corollary 41 that

$$
\begin{aligned}
F_{\varepsilon_n}(v_{\varepsilon_n}) &\geq J_{\varepsilon_n,\varepsilon_n} - db^2 \ln \frac{\rho}{\varepsilon_n} + db^2(\pi \ln \frac{b\rho}{\varepsilon_n} + \gamma) + o_n(1) \\
&= J_{\varepsilon_n,\varepsilon_n} + db^2(\pi \ln b + \gamma) + o_n(1),
\end{aligned}
$$

which ends the proof of Theorem 5.

4.5.3 Proof of Proposition 40

In order to construct ρ, we first define a suitable extraction.
For $l \in \mathbb{N} \setminus \{0,1\}$, consider R_l given by Corollary 39.
Using Proposition 36 and Corollary 38, for sufficiently large n, v_{ε_n} has exactly d zeros $x_1^n = x_1, ..., x_d^n = x_d$.
Clearly, these zeros are well separated and far from $\partial\Omega$ (independently of n).
Fix $i \in \{1, ..., d\}$ and consider

$$
\begin{aligned}
u_n' : \quad B(0, \lambda^2\delta^2/\varepsilon_n) &\to \quad \mathbb{C} \\
x &\mapsto \quad \frac{u_{\varepsilon_n}(\frac{\varepsilon_n}{b}x + x_i)}{b}.
\end{aligned}
$$

For simplicity, assume $x_i = 0$.
Up to a subsequence, one has, as in (4.18),

$$u_n' \to u_0 \text{ in } C^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}), \ u_0(x) = f(|x|)e^{i(\theta+\theta_i)}$$

where $x = |x|e^{i\theta}$, $\theta_i \in \mathbb{R}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing.
Consequently, for $l \in \mathbb{N} \setminus \{0,1\}$, one may construct an extraction $(n_l)_{l \geq 2}$ s.t., denoting $u_{n_l}' = u_l' = |u_l'|e^{i(\theta+\phi_l')}$ and $v_{\varepsilon_{n_l}} = v_l$, we have

$$\{|v_l| < 1 - 1/l^2\} \subset \cup_i B(x_i, R_l\varepsilon_{n_l}), \tag{4.22}$$

$$\rho_l := R_l\varepsilon_{n_l} \leq \frac{\lambda^2\delta^2}{l},$$

$$\left| \int_{B(0,bR_l)} |\nabla u_l'|^2 + \frac{1}{2}\left(1 - |u_l'|^2\right)^2 - \int_{B(0,bR_l)} |\nabla u_0|^2 + \frac{1}{2}\left(1 - |u_0|^2\right)^2 \right| \leq \frac{1}{l}, \qquad (4.23)$$

and

$$\|\phi_l' - \theta_i\|_{C^1(\overline{B(0,bR_l)})} \leq \frac{1}{l}. \qquad (4.24)$$

Here $R_l \simeq \sqrt{2}l/b$ and is defined in Corollary 39.

Following the proof of Proposition 1, Step 2 in [7], one has

$$\int_{B(0,\frac{\lambda^2\delta^2}{\varepsilon_{n_l}})\setminus\overline{B(0,R_l)}} |\nabla\phi_l'|^2 \leq C \text{ independently of } l. \qquad (4.25)$$

In $B(0, \lambda^2\delta^2) \setminus \overline{B(0, \varepsilon_{n_l})}$, we denote $v_{n_l} = v_l = |v_l|e^{\imath(\theta+\phi_l)}$ ($e^{\imath\theta} = x/|x|$). By conformal invariance, (4.24) implies that

$$\|\phi_l - \theta_i\|_{L^\infty(\partial B(0,\rho_l))} + |\phi_l|_{H^{1/2}(\partial B(0,\rho_l))} \leq \frac{C}{l}. \qquad (4.26)$$

Denote $W_l = B(0, 2\rho_l) \setminus \overline{B(0, \rho_l)}$ and consider $\psi_i^l \in H^{1/2}(\partial W_l, \mathbb{R})$ s.t.

$$\psi_i^l = \begin{cases} \phi_l - \theta_i & \text{on } \partial B(0, \rho_l) \\ 0 & \text{on } \partial B(0, 2\rho_l) \end{cases}.$$

Using (4.26), it is clear that $\|\psi_i^l\|_{L^\infty(\partial W_l)} + |\psi_i^l|_{H^{1/2}(\partial W_l)} = \mathcal{O}(1/l)$. From this, it is straightforward that there exists a constant $C_0 > 0$ (independent of l) and $\Psi_i^l \in H^1(W_l, \mathbb{R})$ s.t.

$$\mathrm{tr}_{\partial W_l}\Psi_i^l = \psi_i^l \text{ and } \frac{1}{2}\int_{W_l} |\nabla\Psi_i^l|^2 \leq \frac{C_0}{l^2}.$$

Finally we define $\Psi_l \in H^1(\Omega \setminus \cup\overline{B(x_i, \rho_l)}, \mathbb{R})$ by

$$\Psi_l = \begin{cases} \Psi_i^l(\cdot - x_i) & \text{in } x_i + W_l \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{w}_l = \frac{v_l}{|v_l|}e^{-\imath\Psi_l} \in \mathcal{J}_{\rho_l}(\mathbf{x}, \mathbf{1}) \text{ with } \mathbf{x} = (x_1, ..., x_d).$$

Therefore, denoting $w_l = \frac{v_l}{|v_l|} = e^{\imath(\theta+\phi_l)}$, $U_l = U_{\varepsilon_{n_l}}$ and $\Omega_{\rho_l} = \Omega \setminus \overline{B(x_i, \rho_l)}$, we have

$$\hat{\mathcal{J}}_{\rho_l,\varepsilon_{n_l}}(\mathbf{x}, \mathbf{1}) \leq \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla\tilde{w}_l|^2 = \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla w_l|^2 + 2U_l^2\nabla(\theta+\phi_l)\cdot\nabla\Psi_l + o_l(1).$$

From (4.25), we obtain easily that

$$\left| \int_{\Omega_{\rho_l}} \nabla(\theta+\phi_l)\cdot\nabla\Psi_l \right| = \sum_i \left| \int_{x_i+W_l} \nabla(\theta+\phi_l)\cdot\nabla\Psi_i^l(\cdot - x_i) \right| = o_l(1)$$

and consequently

$$\hat{\mathcal{J}}_{\rho_l,\varepsilon_{n_l}}(\mathbf{x}, \mathbf{1}) \leq \frac{1}{2}\int_{\Omega_{\rho_l}} U_l^2|\nabla w_l|^2 + o_l(1). \qquad (4.27)$$

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On the other hand, from direct computations, one has

$$\frac{1}{2} \int_{\Omega_{\rho_l}} U_l^2 |\nabla v_l|^2 \geq \frac{1}{2} \int_{\Omega_{\rho_l}} U_l^2 |\nabla w_l|^2 + \frac{1}{2} \int_{\Omega_{\rho_l}} U_l^2 (|v_l|^2 - 1) |\nabla(\theta + \phi_l)|^2.$$

Using the same argument as Mironescu in [19], one may obtain that

$$\frac{1}{2} \int_{\Omega_{\rho_l}} (1 - |v_l|^2)^{1/2} |\nabla(\theta + \phi_l)|^2 \leq C \text{ with } C \text{ independent of } l. \tag{4.28}$$

From (4.28) and (4.22), we obtain

$$\frac{1}{2} \int_{\Omega_{\rho_l}} U_l^2 |\nabla v_l|^2 \geq \frac{1}{2} \int_{\Omega_{\rho_l}} U_l^2 |\nabla w_l|^2 - o_l(1).$$

Therefore, with (4.27),

$$F_{\varepsilon_{n_l}}(v_l, \Omega_{\rho_l}) + o_l(1) \geq \frac{1}{2} \int_{\Omega_{\rho_l}} U_l^2 |\nabla v_l|^2 + o_l(1) \geq \hat{\mathcal{J}}_{\rho_l, \varepsilon_{n_l}}(\mathbf{x}, \mathbf{1}). \tag{4.29}$$

In order to complete the proof of (4.19), it suffices to estimate the contribution of the discs $B(x_i, \rho_l)$.

One has (using (4.23))

$$
\begin{aligned}
F_{\varepsilon_{n_l}}(v_l, B(x_i, \rho_l)) &= \frac{b^2}{2} \int_{B(0,\rho_l)} \left| \nabla\left(\frac{u_l}{b}\right) \right|^2 + \frac{b^2}{2\varepsilon_{n_l}^2} \left(1 - \left|\frac{u_l}{b}\right|^2\right)^2 + o_l(1) \\
&= \frac{b^2}{2} \int_{B(0,bR_l)} |\nabla u_l'|^2 + \frac{1}{2}\left(1 - |u_l'|^2\right)^2 + o_l(1) \\
&= \frac{b^2}{2} \int_{B(0,bR_l)} |\nabla u_0|^2 + \frac{1}{2}\left(1 - |u_0|^2\right)^2 + o_l(1).
\end{aligned}
$$

From Proposition 3.11 in [20], one has

$$\frac{1}{2} \int_{B(0,bR_l)} |\nabla u_0|^2 + \frac{1}{2}\left(1 - |u_0|^2\right)^2 = \pi \ln(bR_l) + \gamma + o_l(1),$$

hence

$$F_{\varepsilon_{n_l}}(v_l, B(x_i, \rho_l)) = b^2 [\pi \ln(bR_l) + \gamma] + o_l(1). \tag{4.30}$$

By combining (4.29) with (4.30), we obtain (4.19) with $\rho_l = R_l \varepsilon_{n_l}$.

4.6 Proof of Theorems 1, 2, 3 and 4

We prove Quantization part of Theorem 1 and Theorem 2:

- The existence of exactly d zeros is a direct consequence of Corollary 38.

- The facts that they are well included in ω_ε, well separated and that v_ε has a degree equal to 1 on small circles around the zeros are obtained by Proposition 36 and Corollary 38.

- The lower bound for $|v_\varepsilon|$ is given by Proposition 26.4.

We prove Macroscopic location part of Theorem 1:

- The macroscopic location part of Theorem 1 is a direct consequence of Theorem 5 (proved Subsection 4.5.2), Proposition 26, (4.30) and Proposition 16.

  Indeed, from Theorem 5, Proposition 26.4 and (4.30), we get that the zeros form a quasi-minimizer of $J_{\rho,\varepsilon}$ (defined Notation 15, page 17). By using Proposition 16 we deduce that they are a quasi-minimizer of the renormalized energy W_g (defined Notation 15). Thus, by smoothness of W_g, the zeros tend to a minimal configuration of W_g.

We prove Microscopic location part of Theorem 1 & 4:

- In the case where $\lambda \to 0$, the fact that we may localize the zeros inside the inclusions (microscopic location part of Theorem 1 and Theorem 4) is obtained via Theorem 4 in [13].

  Indeed we take $f_n(x) = \mathrm{tr}_{\partial B((k_n,l_n),\delta/2)} v_{\varepsilon_n} ((k_n,l_n) + \delta x)$ with (k_n, l_n) a center of a cell containing a zero of v_{ε_n}. Using the main result of [18], one may easily prove that f_n satisfies the conditions (A1) and (A2) in [13]. Thus we can apply Theorem 4 in [13] and infer that the location of the zero inside the inclusion is governed by a renormalized energy which is independent of the boundary condition.

Theorem 3 is obtained by combining:

- The weak H^1-convergence of v_{ε_n} to v_* is a direct consequence of Proposition 31. The limiting equation for v_* is a direct consequence of Proposition 32 (this is explained right after Proposition 32).

- The behavior in an ε-neighborhood of the zeros of v_{ε_n} is given by (4.16), (4.17) and (4.18) (noting that in (4.17) we have $R = R_l \to +\infty$ as $l \to \infty$).

A   Proof of Proposition 8

We prove the existence of minimal map in $\mathcal{I}_\rho$ and in $\mathcal{J}_\rho$. The main ingredient is the fact that these sets are closed under H^1-weak convergence (see [16] or below). Thus, considering a minimizing sequence for $\dfrac{1}{2} \displaystyle\int_{\Omega_\rho} \alpha |\nabla \cdot|^2$ in above sets, we obtained the result.

We consider

- $\theta_i : \Omega_\rho \to \mathbb{R}$ the main argument of $x - x_i$, *i.e.* $\mathrm{e}^{\imath\theta_i} = \frac{x-x_i}{|x-x_i|}$. Note that the θ_i are multivalued function with smooth gradient.

- For $d_i \in \mathbb{N}^*$ (given by the definition of $\mathcal{I}_\rho$ or $\mathcal{J}_\rho$) we let $\theta_0 = \sum d_i \theta_i$ and thus $\mathrm{e}^{\imath\theta_0} = \Pi_i \left(\frac{x-x_i}{|x-x_i|} \right)^{d_i}$.

From Lemma 11 in [6], there is $\phi_0 \in C^\infty(\partial\Omega, \mathbb{R})$ s.t. $g\mathrm{e}^{-\imath\theta_0} = \mathrm{e}^{\imath\phi_0}$.

Note that

$$w \in \mathcal{I}_\rho \Longleftrightarrow w = \mathrm{e}^{\imath(\theta_0+\phi)} \text{ with } \phi \in H^1(\Omega_\rho, \mathbb{R}) \text{ and } \mathrm{tr}_{\partial\Omega}\phi = \phi_0, \qquad (A.1)$$

$$w \in \mathcal{J}_\rho \Longleftrightarrow \begin{cases} w = \mathrm{e}^{\imath(\theta_0+\phi)} \text{ with } \phi \in H^1(\Omega_\rho, \mathbb{R}), \\ \displaystyle\sum_{j\neq i} d_j\theta_j + \phi = \mathrm{Cst}_i \text{ on } \partial B(x_i, \rho) \text{ and } \mathrm{tr}_{\partial\Omega}\phi = \phi_0 \end{cases} . \qquad (A.2)$$

Clearly, from (A.1) and (A.2), $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ are H^1-weakly closed.

We now prove the second part of Proposition 8.

One may easily obtain that for some $\lambda : \Omega_\rho \to \mathbb{R}$, denoting $w = e^{\iota(\theta_0 + \phi)}$, $\phi \in H^1(\Omega_\rho, \mathbb{R})$ (and thus $w \in \mathcal{I}_\rho$), we have

$$-\operatorname{div}(\alpha \nabla w) = \lambda w \Longleftrightarrow \left\{ -\operatorname{div}\left[\alpha \nabla(\theta_0 + \phi)\right] = 0 \text{ and } \lambda = \alpha |\nabla w|^2 \right\}. \qquad (A.3)$$

This observation is a direct consequence of the following identity

$$-\operatorname{div}\left[\alpha \nabla e^{\iota(\theta_0 + \phi)}\right] = -\operatorname{div}\left[\alpha \nabla(\theta_0 + \phi)\right] \iota e^{\iota(\theta_0 + \phi)} + \alpha |\nabla(\theta_0 + \phi)|^2 e^{\iota(\theta_0 + \phi)}.$$

Note that under these notations one has $|\nabla w| = |\nabla(\theta_0 + \phi)|$. Thus w is a minimizer in $\mathcal{I}_\rho$ or $\mathcal{J}_\rho$ if and only if $\theta_0 + \phi$ minimizes the weighted Dirichlet functional under the condition fixed by the RHS of (A.1) or (A.2).

Consequently, we find that $\theta_0 + \phi$ minimizes the weighted Dirichlet functional under its Dirichlet boundary condition.

Therefore, we obtain easily that $-\operatorname{div}\left[\alpha \nabla(\theta_0 + \phi)\right] = 0$. The identity $\nabla(\theta_0 + \phi) = w \times \nabla w$ yields $-\operatorname{div}(\alpha \nabla w) = \lambda w$.

Hence, the Euler-Lagrange equations in (3.1) and (3.2) are direct consequences of (A.3).

The condition on the boundary of the holes for $w_{\rho,\alpha}^{\deg}$ (resp. $w_{\rho,\alpha}^{\mathrm{Dir}}$) follows from multiplying the equation satisfied by $\theta_0 + \phi_{\rho,\alpha}^{\deg}$, $w_{\rho,\alpha}^{\deg} = e^{\iota(\theta_0 + \phi_{\rho,\alpha}^{\deg})}$ (resp. $\theta_0 + \phi_{\rho,\alpha}^{\mathrm{Dir}}$, $w_{\rho,\alpha}^{\mathrm{Dir}} = e^{\iota(\theta_0 + \phi_{\rho,\alpha}^{\mathrm{Dir}})}$) by $\psi \in \mathcal{D}(\Omega, \mathbb{R})$ (resp. $\psi \in \mathcal{D}(\Omega, \mathbb{R})$ s.t $\psi \equiv \mathrm{Cst}_i$ in $B(x_i, \rho)$).

Since α is sufficiently smooth, we can rewrite the Euler-Lagrange equation as

$$-\Delta \phi = \frac{\nabla \alpha \cdot \nabla(\phi + \theta_0)}{\alpha} \text{ with } \frac{\nabla \alpha \cdot \nabla(\phi + \theta_0)}{\alpha} \in L^2(\Omega_\rho).$$

So, by elliptic regularity $\phi_{\rho,\alpha}^{\deg}, \phi_{\rho,\alpha}^{\mathrm{Dir}} \in H^2(\Omega_\rho, \mathbb{R})$, and consequently $w_{\rho,\alpha}^{\deg}, w_{\rho,\alpha}^{\mathrm{Dir}} \in H^2(\Omega_\rho, \mathbb{S}^1)$.

B   Proof of Proposition 9

We prove the existence of a minimal configuration $\{\mathbf{x}, \mathbf{d}\} = \{(x_1, ..., x_N), (d_1, ..., d_n)\}$ for $I_{\rho,\alpha}$.

Let $(\{\mathbf{x}_n, \mathbf{d}_n\})_n$ be a minimizing sequence of configuration of $I_{\rho,\alpha}$, i.e.,

$$\inf_{\substack{w \in H^1(\Omega_\rho^n, \mathbb{S}^1) \text{ s.t.} \\ w = g \text{ in } \Omega' \setminus \overline{\Omega \bigcup \cup B(x_i^n, \rho)} \\ \deg_{\partial B(x_i^n, \rho)}(w) = d_i^n \text{ for all } i}} \frac{1}{2} \int_{\Omega_\rho^n} \alpha |\nabla w|^2 \to I_{\rho,\alpha};$$

here $\Omega_\rho^n = \Omega' \setminus \cup \overline{B(x_i^n, \rho)}$.

Up to a subsequence, we have $N_n = N = \mathrm{Cst}$, $\mathbf{d}_n = \mathbf{d} = \mathrm{Cst}$ and $\mathbf{x}_n \to \mathbf{x}$ with $\mathbf{x} = (x_1, ..., x_N)$ s.t. $\min_{i \neq j} |x_i - x_j| \geq 8\rho$.

Consider $w_n \in \mathcal{I}_\rho(\mathbf{x}_n, \mathbf{d})$ a minimal map. Since w_n is bounded independently of n in $H^1(\Omega_\rho^n)$, up to a subsequence, we have $w_n \rightharpoonup w_0$ in $H^1_{\mathrm{loc}}(\Omega_\rho^0)$, $\Omega_\rho^0 = \Omega' \setminus \cup \overline{B(x_i, \rho)}$.

Clearly the following properties hold:

- $w_0 \in H^1_{\mathrm{loc}}(\Omega_\rho^0, \mathbb{S}^1)$ and $w_0 = g$ in $\Omega_\rho^0 \setminus \overline{\Omega}$.

- For all compact $K \subset \Omega_\rho^0$ we have $\dfrac{1}{2} \displaystyle\int_K \alpha |\nabla w_0|^2 \leq \liminf \dfrac{1}{2} \displaystyle\int_K \alpha |\nabla w_n|^2 \leq I_{\rho,\alpha}$.

Thus $w_0 \in H_g^1(\Omega_\rho^0, \mathbb{S}^1)$ and $\int_{\Omega_\rho^0} \alpha |\nabla w_0|^2 \leq I_{\rho,\alpha}$.

Now, it suffices to check that $\deg_{\partial B(x_i, \rho)}(w_0) \in \mathbb{N}^*$ for all i. Since w_0 is $\mathbb{S}^1$-valued, this fact is equivalent to $\deg_{\partial B(x_i, \rho')}(w_0) \in \mathbb{N}^*$ for all i and for all $\rho' \in (\rho, 2\rho)$.

In view of the facts:

- for $\rho' \in (\rho, 2\rho)$ we have $w_n' = w_{n|\Omega' \setminus \cup \overline{B(x_i, \rho')}} \rightharpoonup w_0' = w_{0|\Omega' \setminus \cup \overline{B(x_i^n, \rho')}}$

- the set $\mathcal{I}' := \{ w' \in H^1(\Omega' \setminus \cup \overline{B(x_i, \rho')}, \mathbb{S}^1) \, | \, \deg_{\partial B(x_i, \rho')}(w') = d_i \text{ for all } i \in \{1, ..., N\} \}$ is closed under the H^1-weak convergence (see Appendix A or [16]),

since $w_n' \in \mathcal{I}'$, we obtain that $w_0' \in \mathcal{I}'$. Therefore $\{\mathbf{x}, \mathbf{d}\} = \{(x_1, ..., x_N), (d_1, ..., d_n)\}$ is a minimal configuration for $I_{\rho,\alpha}$.

Now we prove the existence of a minimal configuration for $J_{\rho,\alpha}$.

Let $(\mathbf{x}_n)_n$ be a minimizing sequence of configuration for $J_{\rho,\alpha}$, i.e.,

$$\hat{\mathcal{J}}_{\rho,\alpha}(\mathbf{x}_n, \mathbf{1}) \to J_{\rho,\alpha}.$$

Up to a subsequence, one may assume that there is $\mathbf{x} = (x_1, ..., x_d) \in \Omega^d$ s.t. $x_i^n \to x_i$, $|x_i - x_j| \geq 8\rho$ and $\text{dist}(x_i, \partial \Omega) \geq 8\rho$.

Let $\eta_n = 8 \max |x_i^n - x_i|$. There is a smooth diffeomorphism $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$\begin{cases} \phi_n \equiv \text{Id}_{\mathbb{R}^2} & \text{in } \mathbb{R}^2 \setminus \cup \overline{B(x_i^n, \rho + \eta_n^{1/2})} \\ \phi_n \left[x_i + (1 + \eta_n)x\right] = x_i^n + x & \text{for } x \in B(0, \rho) \\ \|\phi_n - \text{Id}_{\mathbb{R}^2}\|_{C^1(\mathbb{R}^2)} = o_n(1) \end{cases}.$$

For example we can consider $\phi_n = \text{Id}_{\mathbb{R}^2} + H_n$ with

$$\begin{cases} H_n \equiv 0 & \text{in } \mathbb{R}^2 \setminus \cup \overline{B(x_i^n, \rho + \eta_n^{1/2})} \\ H_n \left[x_i + (1 + \eta_n)x\right] = \left[1 - \psi_n(|x|)\right](x_i^n - x_i - \eta_n x) & \text{for } x \in B(0, \dfrac{\rho + \eta_n^{1/2}}{1 + \eta_n}) \end{cases}.$$

Here $\psi_n : \mathbb{R}^+ \to [0, 1]$ is a smooth function satisfying

$$\psi_n(r) = \begin{cases} 0 & \text{if } r \leq \rho \\ 1 & \text{if } r \geq \rho + \eta_n^{1/2}/2 \end{cases} \text{ and } |\psi_n'| = \mathcal{O}(\eta_n^{-1/2}).$$

For $w_n \in \mathcal{J}_\rho(\mathbf{x}_n, \mathbf{1})$ a minimal map, we consider

$$\tilde{w}_n : \begin{array}{ccc} \Omega \setminus \cup_i \overline{B(x_i, (1 + \eta_n)\rho)} & \to & \mathbb{S}^1 \\ x & \mapsto & w_n\left[\phi_n(x)\right] \end{array}.$$

Clearly $\tilde{w}_n$ is well defined and we have

$$\int_{\Omega \setminus \cup_i \overline{B(x_i, (1+\eta_n)\rho)}} \alpha |\nabla \tilde{w}_n|^2 = \int_{\Omega \setminus \cup_i \overline{B(x_i^n, \rho)}} \alpha |\nabla w_n|^2 + o_n(1),$$

$$\tilde{w}_n \left[x_i + (1 + \eta_n)\rho e^{i\theta}\right] = w_n\left[\phi(x_i + (1 + \eta_n)\rho e^{i\theta})\right] = w_n\left[x_i^n + \rho e^{i\theta}\right] = e^{i(\theta + \theta_i)}.$$

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We can extend $\tilde{w}_n$ in $\cup_i B(x_i, (1 + \eta_n)\rho) \setminus \overline{B(x_i, \rho)}$ by $\tilde{w}_n(x_i + re^{i\theta}) = e^{i(\theta + \theta_i)}$, $\rho < r < (1 + \eta_n)\rho$.

Clearly, we have $\tilde{w}_n \in \mathcal{J}_{\rho,\alpha}(\mathbf{x}, \mathbf{1})$ and $\dfrac{1}{2} \displaystyle\int_{\Omega \setminus \cup_i \overline{B(x_i, \rho)}} \alpha |\nabla \tilde{w}_n|^2 = J_{\rho,\alpha} + o_n(1)$.

Thus considering $w \in \mathcal{J}_{\rho,\alpha}(\mathbf{x}, \mathbf{1})$ a minimizer of $\dfrac{1}{2} \displaystyle\int_{\Omega \setminus \cup_i \overline{B(x_i, \rho)}} \alpha |\nabla \cdot |^2$, we obtain

$$\frac{1}{2} \int_{\Omega \setminus \cup_i \overline{B(x_i, \rho)}} \alpha |\nabla w|^2 \leq \frac{1}{2} \int_{\Omega \setminus \cup_i \overline{B(x_i, \rho)}} \alpha |\nabla \tilde{w}_n|^2 = J_{\rho,\alpha} + o_n(1).$$

Letting $n \to \infty$ we deduce that the configuration $\mathbf{x} = (x_1, ..., x_d)$ is minimal.

C Proof of Proposition 10

As explained Section 3.2, Proposition 10 is easily established when either $N = 1$ or when the points are well separated. It remains to consider the case where $N \geq 2$ and there are $i \neq j$ s.t. $|x_i - x_j| \leq 4\eta_{\text{stop}}$.

C.1 The separation process

We assume that $N \geq 2$ and that the points are not well separated. Our purpose is to compare the energy of $\hat{\mathcal{J}}_{\rho,\alpha}$ to the energy of $\hat{\mathcal{I}}_{\rho,\alpha}$. To this purpose, we decompose Ω_ρ into several regions and we compare energies in each regions. These regions are constructed recursively using the following version of Theorem IV.1 in [4].

Lemma 42. *Let $N \geq 2$, $x_1, ..., x_N \in \mathbb{R}^2$ and $\eta > 0$. There are $\kappa \in \{9^0, ..., 9^{N-1}\}$ and $\{y_1, ..., y_{N'}\} \subset \{x_1, ..., x_N\}$ s.t.*

$$\cup_{i=1}^{N} B(x_i, \eta) \subset \cup_{i=1}^{N'} B(y_i, \kappa\eta)$$

and

$$|y_i - y_j| \geq 8\kappa\eta \text{ for } i \neq j.$$

We let $x_1^0, ..., x_N^0$ denote the initial points $x_1, ..., x_N$ and $N_0 = N$ the initial number of points. For $k \geq 1$ (here, k is an iteration in the construction of the regions), we let N_k denote the number of points selected at Step k, and denote the points we select by $x_1^k, ..., x_{N_k}^k$.

The recursive construction is made in such a way that $N_k > N_{k+1}$ and $N_k \geq 1$ for all $k \geq 1$.

The process will stop at the end of Step k if and only if one of the following conditions yields

Rule 1: there is a unique point in the selection (*i.e.* $N_k = 1$),

Rule 2: $\min_{i \neq j} |x_i^k - x_j^k| > 4\eta_{\text{stop}}$.

Step k, $k \geq 1$: Let $\eta_k' = \frac{1}{4} \min_{i \neq j} |x_i^{k-1} - x_j^{k-1}|$.
Using Lemma 42, there are

$$\kappa_k \in \{9^1, ..., 9^{N_{k-1}-1}\} \text{ and } \{x_1^k, ..., x_{N_k}^k\} \subset \{x_1^{k-1}, ..., x_{N_{k-1}}^{k-1}\}$$

s.t.
$$\cup_{i=1}^{N_{k-1}} B(x_i^{k-1}, \eta_k') \subset \cup_{i=1}^{N_k} B(x_i^k, \kappa_k \eta_k') \text{ and } |x_i^k - x_j^k| \geq 8\kappa_k \eta_k' \text{ for } i \neq j.$$

We denote $\eta_k = 2\kappa_k \eta_k'$. We stop the construction if $N_k = 1$ (Rule 1) or if $\frac{1}{4} \min |x_i^{k-1} - x_j^{k-1}| > \eta_{\text{stop}}$ (Rule 2).

In Figure 4 & 5 both stop-conditions are presented.

(a) The initial balls

(b) The first step: a selection of two centers

(c) The process stops at the end of the first step since there are two well separated balls.
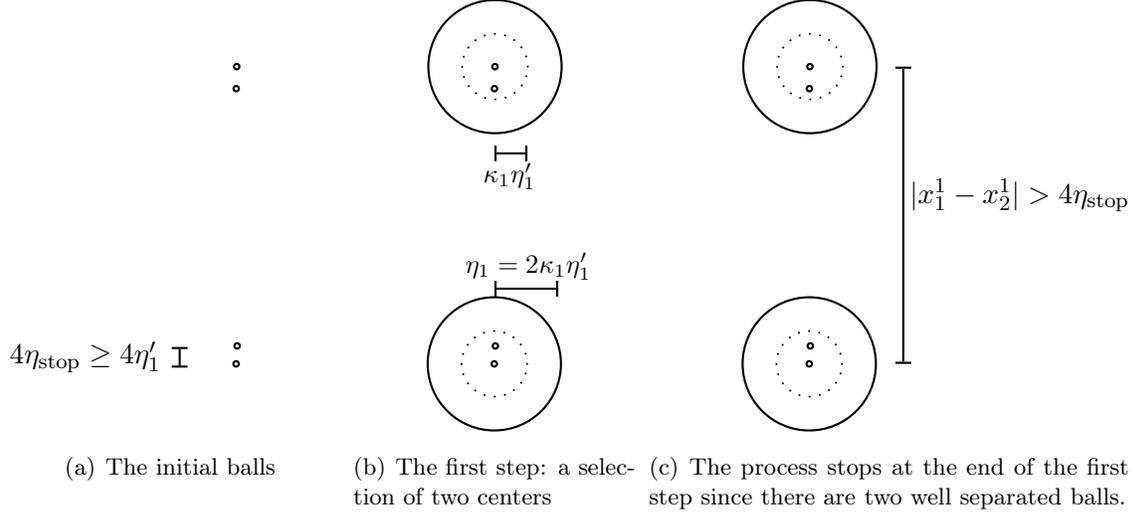
Figure 4: The process stops when we obtain well separated balls

(a) The initial balls

(b) The first step: a selection of three centers

(c) The second step: it remains a unique ball (the picture is at scale 1/2)
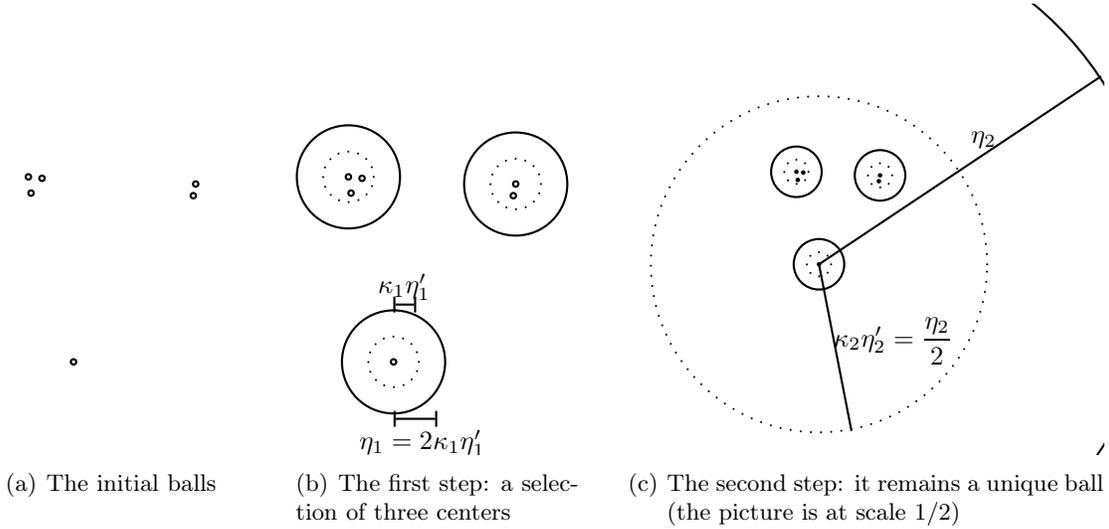
Figure 5: The process stops when we obtain a unique ball

Remark 43.   i. From the definitions of η_k' and η_k, we have $N_k < N_{k-1}$ and $\eta_{k-1} \leq \eta_k' < \eta_k$.

ii. The balls $B(x_j^k, 2\eta_k)$ are disjoint.

iii. Denoting $\Lambda_j^k \subset \{1, ..., N_{k-1}\}$ the set of indices i s.t. $x_i^{k-1} \in B(x_j^k, \kappa_k \eta_k')$, then for $i \in \Lambda_j^k$

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we have $B(x_i^{k-1}, \eta_k') \subset B(x_j^k, \kappa_k \eta_k')$. Furthermore, by construction, $|x_i^{k-1} - x_j^{k-1}| \geq 4\eta_k'$.

C.2 The separation process gives a natural partition of Ω

Let $\Omega, g, x_1, ..., x_N, \mathbf{d}$ and ρ, η_{stop} like in Section 3.2 with $N \geq 2$ and s.t. the points are not well separated.

We apply the separation process. The process stops after K steps, $1 \leq K \leq N-1$.

We denote

$\{y_1, ..., y_{N'}\} \subset \{x_1, ..., x_N\}$ the selection that we obtain, *i.e.*, $y_j = x_j^K$ and $N' = N_K$,

$$\eta = \begin{cases} 9^N \cdot \eta_{\text{stop}} & \text{if } N' = 1 \\ \min\left\{9^N \cdot \eta_{\text{stop}}, \frac{1}{4} \min |y_i - y_j|\right\} & \text{if } N' > 1 \end{cases}, \text{ so } \eta \geq \max(\eta_K, \eta_{\text{stop}}), \quad \text{(C.1)}$$

$$\Lambda_j = \{i \in \{1, ..., N\} \,|\, x_i \in B(y_j, \eta)\} \text{ and } \eta_0 = \rho.$$

We denote

$$D_{j,k} = B(x_j^k, \eta_k) \setminus \cup_{x_i^{k-1} \in B(x_j^k, \eta_k)} \overline{B(x_i^{k-1}, \eta_k')}, \, k \in \{1, ..., K\}, \, j \in \{1, ..., N_k\}, \quad \text{(C.2)}$$

$$R_{j,k} = B(x_j^k, \eta_{k+1}') \setminus \overline{B(x_j^k, \eta_k)}, \, k \in \{0, ..., K-1\}, \, j \in \{1, ..., N_k\}, \quad \text{(C.3)}$$

$$R_j = B(y_j, \eta) \setminus \overline{B(y_j, \eta_K)}, \, j \in \{1, ..., N'\} \quad \text{(C.4)}$$

and

$$D = \Omega \setminus \cup_{j \in \{1, ..., N'\}} \overline{B(y_j, \eta)}.$$

Note that by construction of η_k', η_k and x_i^k the following properties are satisfied:

$$\text{the balls } B(x_i^{k-1}, 2\eta_k') \text{ are disjoint} \quad \text{(C.5)}$$

and

$$2 \cdot 9\eta_k' \leq \eta_k \leq 9^N \eta_k'. \quad \text{(C.6)}$$

Therefore

$$\Omega_\rho = D \bigcup \cup_{j,k} \overline{D_{j,k}} \bigcup \cup_{j,k} R_{j,k} \bigcup \cup_j \overline{R_j} \text{ with disjoint unions.} \quad \text{(C.7)}$$

C.3 Construction of test functions

Construction of test functions in D and $D_{j,k}$

Lemma 44. *1. Let $\eta > 0$. There is $C_1(\eta) > 0$ (depending on Ω, g and η) s.t. if $x_1, ..., x_N \in \Omega$ satisfy $\min_{i \neq j} |x_i - x_j|, \min_i \text{dist}(x_i, \partial\Omega) > 4\eta$ and $d_1, ..., d_N \in \mathbb{N}^*$ are s.t. $\sum d_i = d$ then there is $w \in H_g^1(\Omega \setminus \cup \overline{B(x_i, \eta)}, \mathbb{S}^1)$ s.t. $w(x) = \frac{(x - x_i)^{d_i}}{\eta^{d_i}}$ on $\partial B(x_i, \eta)$ and*

$$\int_{\Omega \setminus \overline{B(x_i, \eta)}} |\nabla w|^2 \leq C_1(\eta).$$

Moreover C_1 can be considered decreasing with η.
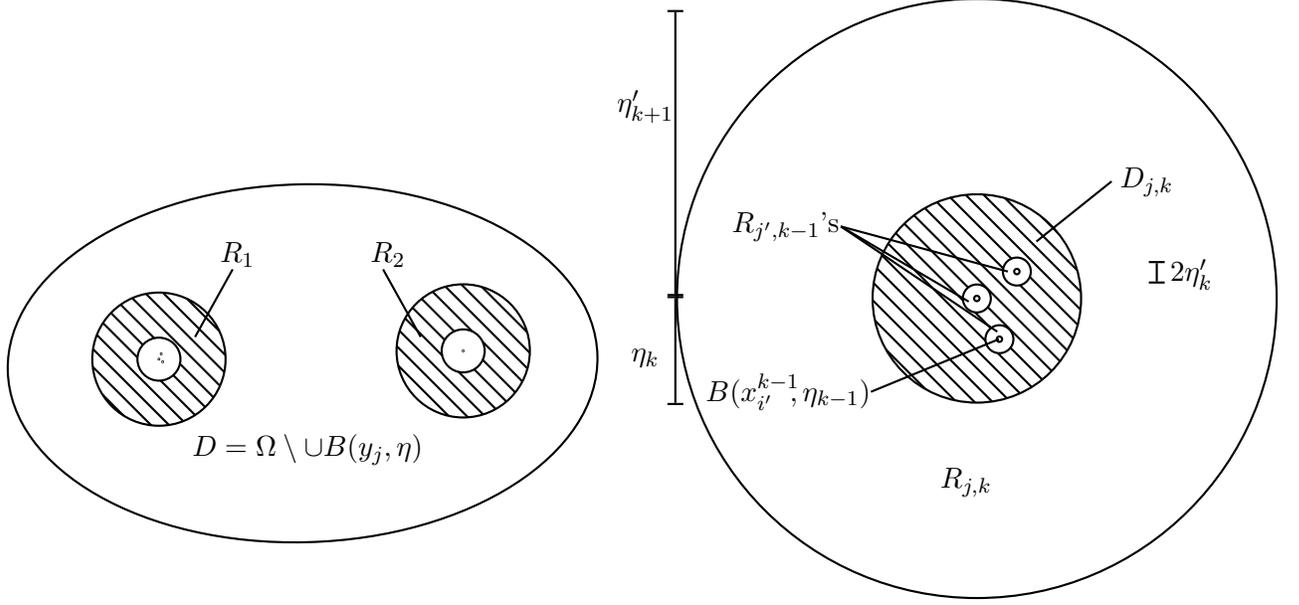
(a) The macroscopic perforated domain and the first mesoscopic rings

(b) A mesoscopic ring and a mesoscopic perforated domain

2. *Let $\eta > 0, \kappa \geq 8, d_0, d_1, ..., d_N \in \mathbb{N}^*$ be s.t. $\sum_{1 \leq i \leq N} d_i = d_0$. Then, there is $C_2(\kappa, d_0)$ s.t. for $x_1, ..., x_N \in B(0, \kappa\eta)$ satisfying $\min_{i \neq j} |x_i - x_j| \geq 4\eta$ we can associate a map $w \in H^1(B(0, 2\kappa\eta) \setminus \cup \overline{B(x_i, \eta)}, \mathbb{S}^1)$ s.t.*

$$
w(x) = \begin{cases} \dfrac{x^{d_0}}{(2\kappa\eta)^{d_0}} & \text{on } \partial B(0, 2\kappa\eta) \\ \dfrac{(x - x_i)^{d_i}}{\eta^{d_i}} & \text{on } \partial B(x_i, \eta) \end{cases}
$$

and

$$
\int_{B(0, 2\kappa\eta) \setminus \cup \overline{B(x_i, \eta)}} |\nabla w|^2 \leq C_2(\kappa, d_0).
$$

Moreover C_2 can be considered increasing with κ, d_0.

Proof. In order to prove 1., we consider, *e.g.*, the test function defined in $\Omega_\eta := \Omega \setminus \cup \overline{B(x_i, \eta)}$ by

$$
w = e^{\imath H} \Pi_i \frac{(x - x_i)^{d_i}}{|x - x_i|^{d_i}} \text{ with } H \text{ s.t. } \begin{cases} H : \Omega_\eta \to \mathbb{R} \\ H \equiv 0 \text{ in } \{\text{dist}\,[x, \partial\Omega_\eta] \geq \eta\} \\ -\Delta H = 0 \text{ in } \{\text{dist}\,[x, \partial\Omega_\eta] < \eta\} \\ w \in H^1_g(\Omega_\eta, \mathbb{S}^1) \text{ and } w(x) = \frac{(x - x_i)^{d_i}}{\eta^{d_i}} \text{ on } \partial B(x_i, \eta) \end{cases} .
$$

Assertion 2. was essentially established in [14], Section 3. We adapt here the argument

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in [14]. By conformal invariance, we may assume that $\eta = 1$. We let

$$
w(x) = \begin{cases}
\Pi_i \dfrac{\left[x + 2x_i \left(\frac{|x|}{\kappa} - 2\right)\right]^{d_i}}{\left|x + x_i \left(\frac{|x|}{\kappa} - 2\right)\right|^{d_i}} & \text{in } B(0, 2\kappa) \setminus \overline{B(0, \frac{3\kappa}{2})} \\[4mm]
\Pi_i \dfrac{(x - x_i)^{d_i}}{|x - x_i|^{d_i}} & \text{in } B(0, \frac{3\kappa}{2}) \setminus \cup \overline{B(x_i, 3/2)} \\[4mm]
\dfrac{(x - x_i)^{d_i}}{|x - x_i|^{d_i}} e^{\imath(2|x - x_i| - 2)\varphi_i} & \text{in } B(x_i, 3/2) \setminus \cup \overline{B(x_i, 1)}
\end{cases} ;
$$

here $\varphi_i \in C^\infty(B(x_i, 3/2), \mathbb{R})$ is defined by $e^{\imath\varphi_i} = \Pi_{j \neq i} \dfrac{(x - x_j)^{d_j}}{|x - x_j|^{d_j}}$ and $\varphi_i(x_i) \in [0, 2\pi)$. Clearly $\|\varphi_i\|_{H^1(B(x_i, 3/2) \setminus \overline{B(x_i, 1)})}$ is bounded by a constant which depends only on d_0. $\qquad\square$

By (C.1) and Lemma 44.1, one may find a map $w_0 \in H^1(D, \mathbb{S}^1)$ s.t.

$$
w_0 = \begin{cases}
g & \text{on } \partial\Omega \\
w_0(x) = \dfrac{(x - y_j)^{\tilde{d}_j}}{\eta^{\tilde{d}_j}} & \text{on } \partial B(y_j, \eta)
\end{cases}
\quad (\text{where } \tilde{d}_j = \sum_{x_i \in B(y_j, \eta)} d_i)
$$

satisfying in addition

$$
\int_D |\nabla w_0|^2 \leq C_1(\eta) \leq C_1(\eta_{\text{stop}}). \tag{C.8}
$$

For each $D_{j,k}$, combining (C.2), (C.5), (C.6) and using Lemma 44.2, there exists a map $w_{j,k} \in H^1(D_{j,k}, \mathbb{S}^1)$ s.t.

$$
w_{j,k}(x) = \begin{cases}
\dfrac{(x - x_j^k)^{\tilde{d}_{j,k}}}{\eta_k^{\tilde{d}_{j,k}}} & \text{for } x \in \partial B(x_j^k, \eta_k) \\[4mm]
\dfrac{(x - x_i^{k-1})^{\tilde{d}_{i,k-1}}}{\eta_k'^{\tilde{d}_{i,k-1}}} & \text{for } x \in \partial B(x_i^{k-1}, \eta_k')
\end{cases} .
$$

Here,

$$
\tilde{d}_{j,k} = \sum_{x_i \in B(x_j^k, \eta_k)} d_i
$$

and

$$
\int_{D_{j,k}} |\nabla w_{j,k}|^2 \leq C_2(2\kappa_k, d_{j,k}) \leq C_2(2 \cdot 9^{d-1}, d). \tag{C.9}
$$

Construction of test functions in R_j's and $R_{j,k}$'s

For $R > r > 0$ and $x_0 \in \mathbb{R}^2$ we denote $\mathscr{R}(x_0, R, r) := B(x_0, R) \setminus \overline{B(x_0, r)}$. For $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, we define

$$
\mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d}) = \inf_{\substack{w \in H^1(\mathscr{R}(x_0, R, r), \mathbb{S}^1) \\ \deg_{\partial B(x_0, R)}(w) = \tilde{d}}} \frac{1}{2} \int_{\mathscr{R}(x_0, R, r)} \alpha |\nabla w|^2 \tag{C.10}
$$

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and

$$\mu_\alpha^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}) = \inf_{\substack{w \in H^1(\mathscr{R}(x_0,R,r),\mathbb{S}^1) \\ w(x_0+Re^{\imath\theta})=e^{\imath\tilde{d}\theta} \\ w(x_0+re^{\imath\theta})e^{-\imath\tilde{d}\theta}=\mathrm{Cst}}} \frac{1}{2} \int_{\mathscr{R}(x_0,R,r)} \alpha|\nabla w|^2. \qquad (\mathrm{C}.11)$$

In the special case $\alpha = U_\varepsilon^2$, we denote

$$\mu_\varepsilon(\mathscr{R}(x_0, R, r), \tilde{d}) = \mu_{U_\varepsilon^2}(\mathscr{R}(x_0, R, r), \tilde{d})$$

and

$$\mu_\varepsilon^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}) = \mu_{U_\varepsilon^2}^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}).$$

Note that the minimization problems (C.10) and (C.11) admit solutions; this is obtained by adapting the proof of Proposition 8.

We present an adaptation of a result of Sauvageot, Theorem 2 in [21].

Proposition 45. *There is $C_b > 0$ depending only on $b \in (0, 1)$ s.t. for $R > r > 0$ and $\alpha \in L^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying $1 \geq \alpha \geq b^2$, we have*

$$\mu_\alpha^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}) \leq \mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d}) + \tilde{d}^2 C_b.$$

Proof. This result was obtained by Sauvageot with $\alpha \in W^{1,\infty}(\mathbb{R}^2, [b^2, 1])$. We may extend this estimate to $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$.

Indeed, let $(\rho_t)_{1>t>0}$ be a classical mollifier, namely $\rho_t(x) = t^{-2}\rho(x/t)$ with $\rho \in C^\infty(\mathbb{R}^2, [0, 1])$, $\mathrm{Supp}\,\rho \subset B(0, 1)$ and $\int_{\mathbb{R}^2} \rho = 1$.

Set $\alpha_t = \alpha * \rho_t \in W^{1,\infty}(B(x_0, R), [b^2, 1])$. We have

$$\lim_{t\to0} \mu_{\alpha_t}(\mathscr{R}(x_0, R, r), \tilde{d}) = \mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d}) \qquad (\mathrm{C}.12)$$

and

$$\lim_{t\to0} \mu_{\alpha_t}^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}) = \mu_\alpha^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}). \qquad (\mathrm{C}.13)$$

We prove (C.12), Equality (C.13) follows with the same lines.

Let w be a minimizer of $\mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d})$. By using Dominated convergence theorem, since $\alpha_t \to \alpha$ in $L^1(B(x_0, R))$, we obtain that $\alpha_t|\nabla w|^2 \to \alpha|\nabla w|^2$ in $L^1(\mathscr{R}(x_0, R, r))$ as $t \to 0$. Consequently

$$\lim_{t\to0} \mu_{\alpha_t}(\mathscr{R}(x_0, R, r), \tilde{d}) \leq \mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d}).$$

On the other hand, let w_t be a minimizer of $\mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d})$ and let $t_n \downarrow 0$. Up to a subsequence, $w_{t_n} \rightharpoonup w_0$ in $H^1(\mathscr{R}(x_0, R, r))$ as $n \to \infty$ and $\sqrt{\alpha_{t_n}}\nabla w_{t_n} \rightharpoonup \sqrt{\alpha}\nabla w_0$ in $L^2(\mathscr{R}(x_0, R, r))$.

Since the class $\mathcal{I} := \{w \in H^1(\mathscr{R}(x_0, R, r), \mathbb{S}^1) \,|\, \deg_{B(x_0,R)}(w) = \tilde{d}\}$ is closed under the H^1-weak convergence (see Appendix A or [16]), we obtain that $w_0 \in \mathcal{I}$. Consequently, we have

$$\liminf_{t\to0} \mu_{\alpha_t}(\mathscr{R}(x_0, R, r), \tilde{d}) \geq \mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d}).$$

Thus the proof of (C.12) is complete.

Therefore, without loss of generality, we may assume that α is Lipschitz.

One may easily prove that if $R \leq 4r$, then $\mu_\alpha^{\mathrm{Dir}}(\mathscr{R}(x_0, R, r), \tilde{d}) \leq 2\tilde{d}^2\pi \ln 4$. Thus we assume that $R > 4r$. Clearly, it suffices to obtain the result for $\tilde{d} = 1$ and $x_0 = 0$.

Let w be a global minimizer of $\mu_\alpha(\mathscr{R}(x_0, R/2, 2r), 1)$. As explained Section A, denoting $x/|x| = e^{\iota\theta}$, one may write $w = e^{\iota(\theta+\phi)}$ for some $\phi \in H^2(\mathscr{R}(x_0, R/2, 2r), \mathbb{R})$. Now we switch to polar coordinates.

Consider

$$I = \left\{ \rho \in [2r, R/2] \ \Big| \ \int_0^{2\pi} \alpha |\nabla(\theta+\phi)|^2 (\rho, \theta) \, d\theta \leq \frac{1}{\rho^2} \int_0^{2\pi} \alpha(\rho, \theta) \, d\theta \right\}.$$

Then I is closed (since $\phi \in H^2$). On the other hand, I is non empty, by the mean value theorem.

Let $r_1 = \min I$ and $r_2 = \max I$. We may assume that $\phi(r_2, 0) = 0$ and $\phi(r_1, 0) = \theta_0$. We construct a test function:

$$\phi'(\rho, \theta) = \begin{cases} 0 & \text{if } 2r_2 \leq \rho \leq R \\ \dfrac{2r_2 - \rho}{r_2}\phi(r_2, \theta) & \text{if } r_2 \leq \rho \leq 2r_2 \\ \phi(\rho, \theta) & \text{if } r_1 \leq \rho \leq r_2 \\ \dfrac{2\rho - r_1}{r_1}\phi(r_1, \theta) + 2\dfrac{r_1 - \rho}{r_1}\theta_0 & \text{if } r_1/2 \leq \rho \leq r_1 \\ \theta_0 & \text{if } r \leq \rho \leq r_1/2 \end{cases}.$$

As explained in [21], there is C depending only on b s.t.

$$\frac{1}{2} \int_{\mathscr{R}(0, R/2, 2r)} \alpha \left(|\nabla(\theta + \phi')|^2 - |\nabla(\theta + \phi)|^2 \right) \leq C.$$

Thus the result follows. $\qquad\square$

As a direct consequence of Proposition 45 (the two first assertions of the next proposition are direct), we have

Proposition 46. *Let $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, $R > r_1 > r > 0$, $\tilde{d} \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^2$, we have*

1. *$\mu_\alpha(\mathscr{R}(x_0, R, r), \tilde{d}) = \tilde{d}^2 \mu_\alpha(\mathscr{R}(x_0, R, r), 1)$,*

2. *$b^2\pi \ln \dfrac{R}{r} \leq \mu_\alpha(\mathscr{R}(x_0, R, r), 1) \leq \pi \ln \dfrac{R}{r}$,*

3. *$\mu_\alpha(\mathscr{R}(x_0, R, r), 1) \leq \mu_\alpha(\mathscr{R}(x_0, R, r_1), 1) + \mu_\alpha(\mathscr{R}(x_0, r_1, r), 1) + 2C_b$ where C_b is given by Proposition 45 and depends only on b.*

We turn to the construction of test functions in R_j and $R_{j,k}$.

Using Proposition 45, there is C_b depending only on $b \in (0, 1)$ s.t. for $\alpha \in L^\infty(\Omega, [b^2, 1])$ and for all $k \in \{1, ..., K-1\}$, $j \in \{1, ..., N_k\}$, there is $w_{\alpha,j,k} \in H^1(R_{j,k}, \mathbb{S}^1)$ s.t.

$$w_{\alpha,j,k}(x) = \begin{cases} \dfrac{(x - x_j^k)^{\tilde{d}_{j,k}}}{\eta'^{\tilde{d}_{j,k}}_{k+1}} & \text{for } x \in \partial B(x_j^k, \eta'_{k+1}) \\ \gamma_{\alpha,j,k}\dfrac{(x - x_j^k)^{\tilde{d}_{j,k}}}{\eta_k^{\tilde{d}_{j,k}}} & \text{for } x \in \partial B(x_j^k, \eta_k) \text{ where } \gamma_{\alpha,j,k} \in \mathbb{S}^1 \end{cases}$$

and s.t. for all $w \in H^1(R_{j,k}, \mathbb{S}^1)$ satisfying $\deg_{\partial B(x_j^k, \eta_k)}(w) = \tilde{d}_{j,k}$ one has

$$\int_{R_{j,k}} \alpha |\nabla w_{\alpha,j,k}|^2 \leq \int_{R_{j,k}} \alpha |\nabla w|^2 + C_b \tilde{d}_{j,k}^2 \leq \int_{R_{j,k}} \alpha |\nabla w|^2 + 2d^2 C_b. \tag{C.14}$$

Now we consider the rings R_j. For $j \in \{1, ..., N'\}$, we denote

$$\tilde{d}_j = \sum_{x_i \in B(y_j, \eta)} d_i.$$

Using Proposition 45, for $j \in \{1, ..., N'\}$, we obtain $w_{\alpha,j} \in H^1(R_j, \mathbb{S}^1)$ s.t.

$$w_{\alpha,j}(x) = \begin{cases} \dfrac{(x - y_j)^d}{\eta^d} & \text{for } x \in \partial B(y_j, \eta) \\[2ex] \gamma_{\alpha,j} \dfrac{(x - y_j)^d}{\eta_K^d} & \text{for } x \in \partial B(y_j, \eta_K) \text{ where } \gamma_{\alpha,j} \in \mathbb{S}^1 \end{cases}$$

and s.t. for all $w \in H^1(R_j, \mathbb{S}^1)$ satisfying $\deg_{\partial B(y_j, \eta)}(w) = \tilde{d}_j$ one has

$$\int_{R_j} \alpha |\nabla w_{\alpha,j}|^2 \leq \int_{R_j} \alpha |\nabla w|^2 + 2d^2 C_b. \tag{C.15}$$

C.4   Proof of Proposition 10

Note that there are at most d^2 regions $D_{j,k}$, at most d^2 rings $R_{j,k}$ and at most d rings R_j. Consequently, denoting

$$C_4(\eta_{\text{stop}}) = C_1(\eta_{\text{stop}}) + d^2 C_2(2 \cdot 9^{d-1}, d) + 4d^4 C_b$$

and using (C.7), (C.8), (C.9), (C.14), (C.15), one may construct a test function $w_\alpha \in \mathcal{J}_\rho$ (up to multiply by some $\mathbb{S}^1$-Constants each function previously constructed) s.t. for all $w \in \mathcal{I}_\rho$, one has

$$\int_{\Omega_\rho} \alpha |\nabla w_\alpha|^2 \leq \int_{\Omega_\rho} \alpha |\nabla w|^2 + C_4. \tag{C.16}$$

Clearly, (C.16) allows us to prove Proposition 10 with $C_0 = C_4/2$.

D   Proof of Proposition 12

D.1   Description of the special solution U_ε

From Proposition 3, we know that far away $\partial \omega_\varepsilon$, U_ε is uniformly close to a_ε. Here we prove that, in a neighborhood of $\partial \omega_\varepsilon$, U_ε is very close to *a cell regularization of* a_ε.

Let

$$a^\lambda: \quad Y = (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \quad \rightarrow \qquad \{b, 1\}$$
$$x \qquad \qquad \mapsto \quad \begin{cases} b & \text{if } x \in \omega^\lambda = \lambda \cdot \omega \\ 1 & \text{otherwise} \end{cases}.$$

Consider V_ξ **the** unique minimizer of

$$E_\xi^{a^\lambda}(V, Y) = \frac{1}{2} \int_Y |\nabla V|^2 + \frac{1}{2\xi^2}(a^{\lambda 2} - V^2)^2, \; V \in H_1^1(Y, \mathbb{R}). \tag{D.1}$$

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Lemma 47. *We have the existence of $C, \gamma > 0$ s.t. for $\varepsilon > 0$ and $x \in Y$*

$$|U_\varepsilon[y_{i,j}^\varepsilon + \delta^j x] - V_{\varepsilon/\delta^j}(x)| \leq C \mathrm{e}^{-\frac{\gamma \delta^j}{\varepsilon}}.$$

Thus in the periodic case, we have U_ε which is almost a $\delta \cdot (\mathbb{Z} \times \mathbb{Z})$-periodic function in $\Omega_\delta^{\mathrm{incl}}$ in the sense that

$$|U_\varepsilon(x) - U_\varepsilon[x + (\delta k, \delta l)]| \leq C \mathrm{e}^{-\frac{\gamma \delta}{\varepsilon}} \text{ if } x, x + (\delta k, \delta l) \in \Omega_\delta^{\mathrm{incl}} \text{ and } k, l \in \mathbb{Z}.$$

Proof. **Step 1.** We first prove that, for all $s > 0$ and for sufficiently small ε, we have $U_\varepsilon^2 \geq \dfrac{b^2 + 1}{2} - s$ in $\Omega \setminus \omega_\varepsilon$. The same argument leads to $U_\varepsilon^2 \leq \dfrac{b^2 + 1}{2} + s$ in $\overline{\omega_\varepsilon}$ and for sufficiently small ξ: $V_\xi^2 \geq \dfrac{b^2 + 1}{2} - s$ in $Y \setminus \omega^\lambda$ and $V_\xi^2 \leq \dfrac{b^2 + 1}{2} + s$ in ω^λ.

From Proposition 3, it suffices to prove that for

$$R = \alpha^{-1} \ln \frac{C}{1 - \sqrt{\frac{1+b^2}{2}}},$$

we have $U_\varepsilon^2 \geq \dfrac{b^2 + 1}{2} - s$ in $\{x \in \Omega \setminus \omega_\varepsilon \,|\, \mathrm{dist}(x, \partial \omega_\varepsilon) < R\varepsilon\}$ (for sufficiently small ε). Here $C > 1, \alpha > 0$ are given by (1.5).

We fix $0 < s < 1$ and we let $z_\varepsilon = y_{i,j}^\varepsilon + \lambda \delta^j z_\varepsilon^0 \in \partial \omega_\varepsilon$, $z_\varepsilon^0 \in \partial \omega$. For $x \in B(z_\varepsilon, \lambda \delta^{P+1})$, we write $x = z_\varepsilon + \varepsilon \tilde{x}$ with $\tilde{x} \in B(0, \lambda \delta^{P+1}/\varepsilon)$. Here $P = 1$ and $y_{i,j}^\varepsilon \in \delta \mathbb{Z} \times \delta \mathbb{Z}$ if we are in the periodic situation.

We define

$$\tilde{U}_\varepsilon(\tilde{x}) : \begin{array}{ccc} B(0, \lambda \delta^{P+1}/\varepsilon) & \to & [b, 1] \\ \tilde{x} & \mapsto & U_\varepsilon(z_\varepsilon + \varepsilon \tilde{x}) \end{array}.$$

It is easy to check that

$$\begin{cases} -\Delta \tilde{U}_\varepsilon = \tilde{U}_\varepsilon(\tilde{a}_\varepsilon^2 - \tilde{U}_\varepsilon^2) \text{ in } B(0, \lambda \delta^{P+1}/\varepsilon) \\ \tilde{U}_\varepsilon \in H^1 \cap L^\infty(B(0, \lambda \delta^{P+1}/\varepsilon), [b, 1]) \end{cases} \tag{D.2}$$

where

$$\tilde{a}_\varepsilon = \begin{cases} b & \text{in } \dfrac{\omega_\varepsilon - z_\varepsilon}{\varepsilon} \cap B(0, \lambda \delta^{P+1}/\varepsilon) \\ 1 & \text{in } \dfrac{(\mathbb{R}^2 \setminus \omega_\varepsilon) - z_\varepsilon}{\varepsilon} \cap B(0, \lambda \delta^{P+1}/\varepsilon) \end{cases}.$$

Clearly

$$\begin{aligned} \frac{\omega_\varepsilon - z_\varepsilon}{\varepsilon} \cap B(0, \lambda \delta^{P+1}/\varepsilon) &= \left[\frac{\lambda \delta^j}{\varepsilon} \cdot (\omega - z_\varepsilon^0) \right] \cap B(0, \lambda \delta^{P+1}/\varepsilon) \\ &= \frac{\lambda \delta^j}{\varepsilon} \cdot \left[(\omega - z_\varepsilon^0) \cap B(0, \delta^{P+1-j}) \right], \end{aligned}$$

and thus

$$\frac{(\mathbb{R}^2 \setminus \omega_\varepsilon) - z_\varepsilon}{\varepsilon} \cap B(0, \lambda \delta^{P+1}/\varepsilon) = \frac{\lambda \delta^j}{\varepsilon} \cdot \left[(\mathbb{R}^2 \setminus \omega) - z_\varepsilon^0) \cap B(0, \delta^{P+1-j}) \right].$$

Note that $\lambda \delta^{P+1}/\varepsilon \to \infty$ and $\delta^{P+1-j} \to 0$, thus by smoothness of ω, up to a subsequence, we have $\dfrac{\lambda \delta^j}{\varepsilon} \cdot \left\{ [(\mathbb{R}^2 \setminus \omega) - z_\varepsilon^0] \cap B(0, \delta^{P+1-j}) \right\} \to \mathcal{R}_{\theta_0}(\mathbb{R} \times \mathbb{R}^+)$. Here $\mathcal{R}_{\theta_0}$ is the vectorial rotation of angular $\theta_0 \in [0, 2\pi)$.

For sake of simplicity, we assume that $\theta_0 = 0$.

From (D.2) and standard elliptic estimates, we obtain that $\tilde{U}_\varepsilon$ is bounded in $W^{2,p}(B(0,R))$ for $p \geq 2$, $R > 0$. Thus up to consider a subsequence, we obtain that $\tilde{U}_\varepsilon \to \tilde{U}_b$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ ($\varepsilon \to 0$) where $\tilde{U}_b \in C^1(\mathbb{R}^2, [b, 1])$ is a solution of

$$\begin{cases} -\Delta \tilde{U}_b = \tilde{U}_b(1 - \tilde{U}_b^2) & \text{in } \mathbb{R} \times \mathbb{R}^+ \\ -\Delta \tilde{U}_b = \tilde{U}_b(b^2 - \tilde{U}_b^2) & \text{in } \mathbb{R} \times \mathbb{R}^- \\ \tilde{U}_b \in C^1(\mathbb{R}^2) \cap H^2_{\text{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \end{cases} \tag{D.3}$$

It is proved in [15] (Theorem 2.2), that (D.3) admits a unique positive solution. Moreover $\tilde{U}_b(x, y) = U_b(y)$ ($\tilde{U}_b$ is independent of its first variable) and U_b is **the unique** solution of

$$\begin{cases} -U_b'' = U_b(1 - U_b^2) & \text{in } \mathbb{R}^+ \\ -U_b'' = U_b(b^2 - U_b^2) & \text{in } \mathbb{R}^- \\ U_b \in C^1(\mathbb{R}, \mathbb{R}), \ U_b' > 0, & \lim_{+\infty} U_b = 1, \lim_{-\infty} U_b = b \end{cases} .$$

Note that since the limit is unique, the convergence is valid for the whole sequence.

This solution U_b may be explicitly obtained by looking for U_b under the form

$$U_b(x) = \begin{cases} \dfrac{Ae^{\sqrt{2}x} - 1}{Ae^{\sqrt{2}x} + 1} & \text{if } x \geq 0 \\ b\dfrac{Be^{-b\sqrt{2}x} - 1}{Be^{-b\sqrt{2}x} + 1} & \text{if } x < 0 \end{cases} .$$

We get $B = -\dfrac{3b^2 + 1 + 2b\sqrt{2(b^2 + 1)}}{1 - b^2}$, $A = \dfrac{B(1 + b) + 1 - b}{B(1 - b) + 1 + b}$ and

$$U_b(0) = b\frac{B - 1}{B + 1} = \frac{1 + b^2 + b\sqrt{2(b^2 + 1)}}{2b + \sqrt{2(b^2 + 1)}} = \frac{1 - b^2}{2b + \sqrt{2(b^2 + 1)}} + b = \sqrt{\frac{b^2 + 1}{2}}$$

Since $U_b(0)^2 = \dfrac{b^2 + 1}{2}$ and U_b is an increasing function, for $x \geq 0$, $U_b(x)^2 \geq \dfrac{b^2 + 1}{2}$. From the convergence $\tilde{U}_\varepsilon \to \tilde{U}_b$ in $L^\infty(B(0, R))$, we obtain that, for ε sufficiently small, $\tilde{U}_\varepsilon^2 \geq \dfrac{b^2 + 1}{2} - s$ in $B(0, R) \cap \left\{ \dfrac{\lambda\delta^j}{\varepsilon} \cdot [(\mathbb{R}^2 \setminus \omega) - z_\varepsilon^0] \right\}$.

Step 2. Fix $j \in \{1, ..., P\}$ s.t. $\mathcal{M}_j^\varepsilon \neq \emptyset$ and fix $i \in \mathcal{M}_j^\varepsilon$. Note that if we are in the periodic case then $j = 1$ and we fix $y_{k,l} = (\delta k, \delta l) \in \delta\mathbb{Z} \times \delta\mathbb{Z}$ s.t. $y_{k,l} + \delta \cdot Y \subset \Omega$.

We denote $\xi := \dfrac{\delta^j}{\varepsilon}$. For $x \in Y$, consider $W(x) = V_\xi(x) - U_\varepsilon(y_{i,j}^\varepsilon + \delta^j x)$ which satisfies (using (1.5))

$$\begin{cases} -\xi^2 \Delta W(x) = W(x) \left\{ a^\lambda(x)^2 - [V_\xi(x)^2 + U_\varepsilon(y_{i,j}^\varepsilon + \delta^j x)V_\xi(x) + U_\varepsilon(y_{i,j}^\varepsilon + \delta^j x)^2] \right\} & \text{in } Y \\ 0 \leq W \leq Ce^{-\frac{\gamma}{\xi}} & \text{on } \partial Y \end{cases} .$$

Here $\gamma = \alpha \cdot \text{dist}(\partial Y, \omega)$, C and α given by (1.5).

By Step 1, taking $s = b^2$, for sufficiently small ε, we have for $x \in Y \setminus \omega^\lambda$

$$U_\varepsilon^2(y_{i,j}^\varepsilon + \delta^j x), V_\xi^2(x) \geq \max\left(b^2, \frac{1 - b^2}{2}\right) \geq \frac{1}{3}.$$

Thus, using the weak maximum principle, we find that $W \geq 0$ in Y. Consequently, since W is subharmonic, we deduce that $W \leq Ce^{-\frac{\gamma}{\xi}}$. $\qquad\square$

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D.2 Behavior of almost minimizers of $I_{\rho,\varepsilon}$

We recall that for $x_0 \in \mathbb{R}^2$ and $R > r > 0$, we denoted $\mathscr{R}(x_0, R, r) := B(x_0, R) \setminus \overline{B(x_0, r)}$.

D.2.1 Useful results for the periodic situation

We establish three preliminary results for the periodic situation represented Figure 1. Thus in this subsection we assume that U_ε is **the** unique global minimizer of E_ε in H_1^1 with the periodic pinning term a_ε represented Figure 1.

Energetic estimates in rings and global energetic upper bounds

From Lemma 47 (U_ε is close to a periodic function) we obtain

Lemma 48. *For all $1 \geq R > r \geq \varepsilon$, $x, x_0 \in \mathbb{R}^2$ s.t. $B(x_0, R) \subset \Omega_\delta^{\mathrm{incl}}$ and $x - x_0 \in \delta \cdot \mathbb{Z}^2$, we have*

$$\mu_\varepsilon(\mathscr{R}(x, R, r), 1) \geq \mu_\varepsilon(\mathscr{R}(x_0, R, r), 1) - o_\varepsilon(1).$$

Adding the condition that $B(x, R) \subset \Omega_\delta^{\mathrm{incl}}$, we have

$$|\mu_\varepsilon(\mathscr{R}(x, R, r), 1) - \mu_\varepsilon(\mathscr{R}(x_0, R, r), 1)| \leq o_\varepsilon(1).$$

Moreover the $o_\varepsilon(1)$ may be considered independent of x, x_0, R, r.

Lemma 48 implies easily the following estimate.

Proposition 49. *Let $\eta > 0$ and $\eta > \rho \geq \varepsilon$. Then there is $C = C(\Omega, \Omega', g, b, \eta) > 0$ s.t. for $x_0 \in \mathbb{R}^2$ we have*

$$I_{\rho,\varepsilon} \leq d\,\mu_\varepsilon(\mathscr{R}(x_0, \eta, \rho), 1) + C(\eta),$$

where $C(\eta)$ is independent of x_0 and ρ.

From Lemma 47 we get the almost periodicity of $\mu_\varepsilon(\mathscr{R}(\cdot, R, r), 1)$ w.r.t. a $\delta \times \delta$-grid (expressed in Lemma 47). Therefore, the "best points" to minimize $\mu_\varepsilon(\mathscr{R}(\cdot, R, r), 1)$ should be almost periodic.

Another important result is the next proposition. It expresses that the center of an inclusion is not too far to a good point to minimize $\mu_\varepsilon(\mathscr{R}(\cdot, R, r), 1)$. This proposition may be seen as a first step in the proof of the pinning effect of ω_ε.

Proposition 50. *There is C_* which depends only on ω, b and Ω s.t. for sufficiently small ε, for $x \in \Omega$ and $x_{\mathrm{per}} \in B(x, 3\delta\sqrt{2}/2) \cap (\delta\mathbb{Z} \times \delta\mathbb{Z}) \cap \omega_\varepsilon$ we have for $1 > R > r > \varepsilon$*

$$\mu_\varepsilon(\mathscr{R}(x_{\mathrm{per}}, R, r), 1) \leq \mu_\varepsilon(\mathscr{R}(x, R, r), 1) + C_*. \tag{D.4}$$

Proof. If $R \leq 10^2 r$, then the result is obvious with $C_* = 2\pi \ln 10$. Thus we assume that $R > 10^2 r$.

We share the proof in three cases:

Case 1. $r \geq \delta$,

Case 2. $\delta \geq R > r \geq \lambda\delta$,

Case 3. $R \leq \lambda\delta$.

Assume for the moment that

There exists $\tilde{C}_* > 0$ s.t. (D.4) yields in the three previous cases with $C_* = \tilde{C}_*$.   (D.5)

For the general case, we divide $\mathscr{R}(x, R, r)$ into $R_1(x) \cup R_2(x) \cup R_3(x)$ with

$$R_1(x) = \mathscr{R}\left(x, R, \min\{\max(\delta, r), R\}\right),$$

$$R_2(x) = \mathscr{R}\left(x, \min\{\max(\delta, r), R\}, \min\{R, \max(\lambda\delta, r)\}\right),$$

$$R_3(x) = \mathscr{R}\left(x, \min\{R, \max(\lambda\delta, r)\}, r\right).$$

Remark 51.   1. For $k \in \{1, 2, 3\}$, we have $\emptyset \subseteq R_k \subseteq \mathscr{R}(x, R, r)$.

2. It is easy to check that

- $[R_1 = \mathscr{R}(x, R, r) \Leftrightarrow r \geq \delta$ (Case 1.)] and $[R_1 = \emptyset \Leftrightarrow R \leq \delta]$,
- $[R_2 = \mathscr{R}(x, R, r) \Leftrightarrow \lambda\delta \leq r < R \leq \delta$ (Case 2.)]
  and $[R_2 = \emptyset \Leftrightarrow \{\lambda = 1 \text{ or } r \geq \delta \text{ or } R \leq \lambda\delta\}]$,
- $[R_3 = \mathscr{R}(x, R, r) \Leftrightarrow R \leq \lambda\delta$ (Case 3.)] and $[R_3 = \emptyset \Leftrightarrow r \geq \lambda\delta]$.

3. If $\lambda \equiv 1$ then Case 2. never occurs and $R_2 = \emptyset$.

Therefore we have (using Propositions 45, 46.3 and (D.5))

$$
\begin{aligned}
\mu_\varepsilon(\mathscr{R}(x, R, r), 1) &\geq \mu_\varepsilon(R_1(x), 1) + \mu_\varepsilon(R_2(x), 1) + \mu_\varepsilon(R_3(x), 1) \\
\text{(D.5)} \quad &\geq \mu_\varepsilon(R_1(x_{\mathrm{per}}), 1) + \mu_\varepsilon(R_2(x_{\mathrm{per}}), 1) + \mu_\varepsilon(R_3(x_{\mathrm{per}}), 1) - 3\tilde{C}_* \\
\text{(Prop. 45)} \quad &\geq \mu_\varepsilon^{\mathrm{Dir}}(R_1(x_{\mathrm{per}}), 1) + \mu_\varepsilon^{\mathrm{Dir}}(R_2(x_{\mathrm{per}}), 1) + \mu_\varepsilon^{\mathrm{Dir}}(R_3(x_{\mathrm{per}}), 1) - 3(\tilde{C}_* + C_b) \\
&\geq \mu_\varepsilon(\mathscr{R}(x_{\mathrm{per}}, R, r), 1) - 3(\tilde{C}_* + C_b).
\end{aligned}
$$

The last line is obtained by constructing a test function. Therefore, it suffices to take $C_* := 3(\tilde{C}_* + C_b)$.

We now turn to the proof of (D.5) in Case 1, 2 and 3. Recall that we assumed that $R > 10^2 r$.

We treat Case 1. $(R > r \geq \delta)$:

$$
\begin{aligned}
\mu_\varepsilon(\mathscr{R}(x, R, r), 1) \geq \text{(Prop. 45)} &\geq \mu_\varepsilon^{\mathrm{Dir}}(\mathscr{R}(x, R, r), 1) - C_b \\
&\geq \mu_\varepsilon^{\mathrm{Dir}}(\mathscr{R}(x, 10R, 10^{-1}r), 1) - 2\pi\ln 10 - C_b \\
\{\mathscr{R}(x_{\mathrm{per}}, R, 10r) \subset \mathscr{R}(x, 10R, 10^{-1}r)\} &\geq \mu_\varepsilon(\mathscr{R}(x_{\mathrm{per}}, R, 10r), 1) - 2\pi\ln 10 - C_b \\
\geq \text{(Prop. 45)} &\geq \mu_\varepsilon(\mathscr{R}(x_{\mathrm{per}}, R, r), 1) - 3\pi\ln 10 - 2C_b.
\end{aligned}
$$

Thus we may take $\tilde{C}_* = 3\pi\ln 10 + 2C_b$.

We treat Case 2. Note that from Remark 51.3, we may assume that $\lambda \to 0$. On the one hand, it is clear that

$$\mu_\varepsilon(\mathscr{R}(x_{\mathrm{per}}, R, r), 1) \leq \pi\ln\frac{R}{r}.$$

On the other hand, letting

$$
\begin{aligned}
\alpha_\varepsilon: \quad \mathbb{R}^2 &\to \quad\quad\quad\quad \{b^2, 1\} \\
x &\mapsto \begin{cases} b^2 & \text{if } x \in \cup_{M \in \mathbb{Z}^2} \overline{B(\delta M, \lambda\delta)} \\ 1 & \text{otherwise} \end{cases},
\end{aligned}
$$

we have from Proposition 3 that $\alpha_\varepsilon \leq U_\varepsilon^2 + V_\varepsilon$ with $\|V_\varepsilon\|_{L^\infty} = o(\varepsilon^2)$.

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If $\mathscr{R}(x, R, r) \cap \{\alpha_\varepsilon = b^2\} = \emptyset$, then we have $\mu_\varepsilon(\mathscr{R}(x, R, r), 1) \geq \pi \ln \dfrac{R}{r} + o(\varepsilon^2 \ln \lambda)$. And thus the result holds with $\tilde{C}_* = 1$ (for sufficiently small ε).

Otherwise we have $\mathscr{R}(x, R, r) \cap \{\alpha_\varepsilon = b^2\} \neq \emptyset$. In this situation, because $R \leq \delta$, we get that $\mathscr{R}(x, R, r) \cap \{\alpha_\varepsilon = b^2\}$ is a union of at most four connected components. Therefore $S = \{\rho \in (r, R) \,|\, \partial B(x, \rho) \cap \{\alpha_\varepsilon = b^2\}\}$ is a union of at most four segments whose length is lower than $8\lambda\delta$. Consequently, denoting $\overline{S} = \cup_{i=1}^{k}[s_i, t_i]$ (with $s_i < s_{i+1}$), we have for $w_* \in H^1(\mathscr{R}(x, R, r), \mathbb{S}^1)$ which minimizes $\mu_\varepsilon(\mathscr{R}(x, R, r), 1)$

$$
\begin{aligned}
\frac{1}{2} \int_{\mathscr{R}(x,R,r)} U_\varepsilon^2 |\nabla w_*|^2 + o_\varepsilon(1) \;\; &\geq \;\; \frac{1}{2} \int_{\mathscr{R}(x,R,r)} \alpha_\varepsilon |\nabla w_*|^2 \\
(t_0 = r \,\&\, s_{k+1} = R) \;\; &\geq \;\; \sum_{i=0}^{k} \frac{1}{2} \int_{t_i}^{s_{i+1}} \frac{\mathrm{d}\rho}{\rho} \int_{0}^{2\pi} |\partial_\theta w_*|^2 \\
&\geq \;\; \pi \sum_{i=0}^{k} \ln \frac{s_{i+1}}{t_i} = \pi \ln \frac{R}{r} - \pi \sum_{i=1}^{k} \ln \frac{t_i}{s_i}
\end{aligned}
$$

Since $\lambda\delta \leq s_i \leq t_i \leq s_i + 8\lambda\delta$, we have $1 \leq \dfrac{t_i}{s_i} \leq 1 + \dfrac{8\lambda\delta}{s_i} \leq 9$. Therefore we may take $\tilde{C}_* = 4\pi \ln 9 + 1$.

We treat the last case. Since $R \leq \lambda\delta$ and $x_{\mathrm{per}} \in (\delta\mathbb{Z} \times \delta\mathbb{Z}) \cap \omega_\varepsilon$, there is $\tilde{C}_*$ s.t. we have (for sufficiently small ε) $\mu_\varepsilon(\mathscr{R}(x_{\mathrm{per}}, R, r), 1) \leq \pi b^2 \ln \dfrac{R}{r} + \tilde{C}_*$. On the other hand (Proposition 46.2) we have $\mu_\varepsilon(\mathscr{R}(x, R, r), 1) \geq \pi b^2 \ln \dfrac{R}{r}$. Therefore the estimate in the third case is proved. $\qquad\square$

Estimates for almost minimizers

In this subsection we establish a fundamental result: fix an almost minimal configuration $\{\mathbf{x}, \mathbf{1}\}$ for $I_{\rho,\varepsilon}$ (the existence of such configuration is proved Section D.3) and a map which *almost* minimizes $\frac{1}{2} \int_{\Omega' \setminus \overline{B(x_i, \rho)}} U_\varepsilon^2 |\nabla \cdot|^2$. Then the map *almost* minimizes the weighted Dirichlet functional $\frac{1}{2} \int_{\mathscr{R}(x_i, \rho', \rho)} U_\varepsilon^2 |\nabla \cdot|^2$, $10^{-2} \min_{i \neq j} |x_i - x_j| > \rho' > \rho$.

Lemma 52. 1. Let $x \in \mathbb{R}^2$, $0 < r < R$, $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, $C_0 > 0$ and a map $w \in H^1(\mathscr{R}(x, R, r), \mathbb{S}^1)$ s.t. $\deg_{\partial B(x, R)}(w) = 1$ and

$$
\frac{1}{2} \int_{\mathscr{R}(x,R,r)} \alpha |\nabla w|^2 - \mu_\alpha(\mathscr{R}(x, R, r), 1) \leq C_0.
$$

Then for all r', R' s.t. $r < r' < R' < R$ one has

$$
\frac{1}{2} \int_{\mathscr{R}(x,R',r')} \alpha |\nabla w|^2 - \mu_\alpha(\mathscr{R}(x, R', r'), 1) \leq 4C_b + C_0,
$$

where C_b depends only on b and is given by Proposition 45.

2. Let $x_1, ..., x_d \in \Omega$ ($x_i \neq x_j$ for $i \neq j$), $d_i = 1$, $\varepsilon < \rho < 10^{-2}\eta$, $\eta := 10^{-2} \cdot \min\{|x_i - x_j|, \mathrm{dist}(x_i, \partial\Omega)\}$, $C_0 > 0$ and $w \in H^1(\Omega'_\rho, \mathbb{S}^1)$ s.t.

$$
\frac{1}{2} \int_{\Omega'_\rho} U_\varepsilon^2 |\nabla w|^2 \leq I_{\rho,\varepsilon} + C_0.
$$

Then for $\rho \leq r < R < \eta$ one has for all i

$$\frac{1}{2} \int_{\mathscr{R}(x_i, R, r)} U_\varepsilon^2 |\nabla w|^2 - \mu_\varepsilon(\mathscr{R}(x_i, R, r), 1) \leq C_0 + C(\eta);$$

here $C(\eta)$ depends only on b, g, Ω, Ω' and η.

3. *Under the hypotheses of 2., we also have for $\eta > \rho_0 > \rho$*

$$\frac{1}{2} \int_{\Omega'_{\rho_0}} U_\varepsilon^2 |\nabla w|^2 \leq C(\rho_0, C_0);$$

here $C(\rho_0, C_0)$ depends only on $b, g, \Omega, \Omega', C_0, \rho_0$ and η.

Proof. Using the third part of Proposition 46, we have

$$\frac{1}{2} \int_{\mathscr{R}(x, R, r)} \alpha |\nabla w|^2 \leq \mu_\alpha(\mathscr{R}(x, R, R'), 1) + \mu_\alpha(\mathscr{R}(x, R', r'), 1)$$
$$+ \mu_\alpha(\mathscr{R}(x, r', r), 1) + 4C_b + C_0.$$

We easily obtain

$$\frac{1}{2} \int_{\mathscr{R}(x, R, r)} \alpha |\nabla w|^2 \geq \mu_\alpha(\mathscr{R}(x, R, R'), 1) + \frac{1}{2} \int_{\mathscr{R}(x, R', r')} \alpha |\nabla w|^2 + \mu_\alpha(\mathscr{R}(x, r', r), 1)$$

which proves the first assertion.

The second assertion is obtained by using the same argument combined with Proposition 49.

Last assertion is a straightforward consequence of Proposition 49 and both previous assertions. $\square$

D.2.2   Lower bound on circles

In this subsection we prove an estimate for the minimization of weighted 1-dimensional Dirichlet functionals. In the following this estimate will be used to get lower bounds in rings.

Lemma 53. *Let $\theta_0 \in (0, 2\pi)$ and let $\alpha \in L^\infty([0, 2\pi], \{b^2, 1\})$ be s.t. $\mathscr{H}^1(\{\alpha = b^2\}) = \theta_0$. Let $\varphi \in H^1([0, 2\pi], \mathbb{R})$ s.t. $\varphi(2\pi) - \varphi(0) = 2\pi$. The following lower bound holds*

$$\frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi(\theta)|^2 \, \mathrm{d}\theta \geq \frac{2\pi^2}{\int_0^{2\pi} \frac{1}{\alpha}} = \frac{2\pi^2}{2\pi + \theta_0(b^{-2} - 1)}.$$

Here $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure.

Proof. The proof of this lower bound is based on the computation of the minimal energy.

It is easy to check that a minimal function $\varphi_{\min} \in H^1([0, 2\pi], \mathbb{R})$ for $\frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \cdot|^2 \, \mathrm{d}\theta$ under the constraint $\varphi(2\pi) - \varphi(0) = 2\pi$ exists and satisfies $\partial_\theta(\alpha \partial_\theta \varphi_{\min}) = 0$. Thus $\partial_\theta \varphi_{\min} = \frac{\mathrm{Cst}}{\alpha}$ with $\mathrm{Cst} = \frac{2\pi}{\int_0^{2\pi} \alpha^{-1}}$. Therefore

$$\frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi(\theta)|^2 \, \mathrm{d}\theta \geq \frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi_{\min}(\theta)|^2 \, \mathrm{d}\theta = \frac{2\pi^2}{\int_0^{2\pi} \frac{1}{\alpha}} = \frac{2\pi^2}{2\pi + \theta_0(b^{-2} - 1)}.$$

$\square$

D.3 Proof of the first part of Proposition 12

Let $x_1^n, ..., x_N^n \in \Omega$ s.t. $|x_i^n - x_j^n| \geq 8\rho$ and $d_1, ..., d_N > 0$, $\sum d_i = d$ (up to a subsequence the degrees may be considered independent of n).

Assume that

$$\text{there is } i_0 \in \{1, ..., N\} \text{ s.t. } d_{i_0} \neq 1 \text{ or there are } i \neq j \text{ s.t. } |x_i^n - x_j^n| \to 0. \tag{D.6}$$

Up to pass to a subsequence, there are $a_1, ..., a_M \in \overline{\Omega}$ and $\{\Lambda_1, ..., \Lambda_M\}$ a partition of $\{1, ..., N\}$ s.t.

$$i \in \Lambda_l \iff x_i^n \to a_l.$$

For sake of simplicity, we drop the superscript n for the points, i.e., we write x_i instead of x_i^n.

We let $\rho_0 := 10^{-2} \min\{\min_{k \neq l} |a_k - a_l|, \text{dist}(\partial\Omega, \partial\Omega')\}$ with $\min_{k \neq l} |a_k - a_l| = +\infty$ if $M = 1$.

Note that since $d_i > 0$, (D.6) is equivalent to

$$\text{there exists } l_0 \in \{1, ..., M\} \text{ s.t. } \tilde{d}_{l_0} = \sum_{i \in \Lambda_{l_0}} d_i > 1. \tag{D.7}$$

We are going to prove that (D.7) is not possible for almost minimal configurations. In order to do this, for $l \in \{1, ..., M\}$, we obtain a lower bound for the weighted Dirichlet functional defined around a_l. Then using Proposition 49 we will conclude.

For $l \in \{1, ..., M\}$, there are two cases:

1. $\text{Card}(\Lambda_l) > 1$,

2. $\text{Card}(\Lambda_l) = 1$.

In the first case ($\text{Card}(\Lambda_l) > 1$), we apply the separation process (defined Section C.1) in $\Omega_l^n = B(a_l, 2\rho_0) \setminus \cup_{i \in \Lambda_l} \overline{B(x_i, \rho)}$ with $\eta_{\text{stop}} = 10^{-2}\rho_0$.

By construction, the process stops after K steps. For $k \in \{1, ..., K\}$ we denote:

- $\{x_1^k, ..., x_{N_k}^k\}$ the selection of points made in Step k ($x_i^0 = x_i$, $i \in \Lambda_l$),

- η_k' the radius of the intermediate balls in Step k ($\eta_k' = \frac{1}{4} \min_{i \neq j} |x_i^{k-1} - x_j^{k-1}|$),

- η_k the radius of the final balls in Step k ($\eta_k = 2\kappa_k \eta_k'$, $\kappa_k \in \{9^0, ..., 9^d\}$ and $\eta_0 = \rho$).

Since for $i, j \in \Lambda_l$ we have $|x_i - x_j| \to 0$, then, in the end of the process (after K steps), we obtain a unique $x_1^K = y_l \in \{x_i \mid i \in \Lambda_l\}$ in the final selection of points and $\eta_K \to 0$.

From (C.3) and (C.4), the following rings are mutually disjoint (denoting $\eta_0 = \rho$)

$$R_0^l = \mathscr{R}(y_l, \rho_0, \eta_K) \text{ and } R_{j,k} = \mathscr{R}(x_j^k, \eta_{k+1}', \eta_k) \text{ for } k \in \{0, ..., K-1\}, j \in \{1, ..., N_k\}.$$

We let

- for $k \in \{0, ..., K-1\}$ and $j \in \{1, ..., N_k\}$, $\tilde{d}_{j,k} := \sum_{x_i \in B(x_j^k, \eta_{k+1}')} d_i$,

- for $n \geq 1$ we let $x_0 = x_0(n) \in (\delta\mathbb{Z} \times \delta\mathbb{Z}) \cap \omega_{\varepsilon_n}$ be s.t. $B(x_0, 2\rho_0) \subset \Omega$. Thus combining Lemma 48 with Proposition 50, we get that (for sufficiently large n) and for $\rho_0 \geq R > r \geq \rho$

$$\mu_{\varepsilon_n}(\mathscr{R}(x_0, R, r), 1) \leq \inf_{x \in \Omega} \mu_{\varepsilon_n}(\mathscr{R}(x, R, r), 1) + C_* + 1. \tag{D.8}$$

For $w \in H^1_g(\Omega'_\rho, \mathbb{S}^1)$ we have

$$\frac{1}{2} \int_{\Omega^n_l} U^2_{\varepsilon_n} |\nabla w|^2 \geq \frac{1}{2} \int_{R^l_0} U^2_{\varepsilon_n} |\nabla w|^2 + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \frac{1}{2} \int_{R_{j,k}} U^2_{\varepsilon_n} |\nabla w|^2$$

$$\geq \frac{1}{2} \int_{R^l_0} U^2_{\varepsilon_n} |\nabla w|^2 + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \mu_{\varepsilon_n}(\mathscr{R}(x^k_j, \eta'_{k+1}, \eta_k), \tilde{d}_{j,k})$$

$$(\text{D.8}) \geq \frac{1}{2} \int_{\mathscr{R}(x_0, \rho_0, \eta_K)} U^2_{\varepsilon_n} |\nabla w|^2 + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \mu_{\varepsilon_n}(\mathscr{R}(x_0, \eta'_{k+1}, \eta_k), \tilde{d}_{j,k}) - \mathcal{O}(1)$$

$$(\text{C.6}) \geq \tilde{d}^2_l \, \mu_{\varepsilon_n}(\mathscr{R}(x_0, \rho_0, \eta_K), 1) + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \tilde{d}_{j,k} \, \mu_{\varepsilon_n}(\mathscr{R}(x_0, \eta_{k+1}, \eta_k), 1) - \mathcal{O}(1)$$

$$(\text{Prop. } 46.3) \geq \tilde{d}_l \, \mu_{\varepsilon_n}(\mathscr{R}(x_0, \rho_0, \rho), 1) + (\tilde{d}^2_l - \tilde{d}_l)\pi b^2 |\ln \eta_K| - \mathcal{O}(1). \tag{D.9}$$

In the second case $(\text{Card}(\Lambda_l) = 1)$ the computations are direct

$$\frac{1}{2} \int_{\Omega^n_l} U^2_{\varepsilon_n} |\nabla w|^2 \geq \frac{1}{2} \int_{\mathscr{R}(x_i, \rho_0, \rho)} U^2_{\varepsilon_n} |\nabla w|^2$$

$$\geq \tilde{d}_l \, \mu_{\varepsilon_n}(\mathscr{R}(x_0, \rho_0, \rho), 1) + (\tilde{d}^2_l - \tilde{d}_l)\pi b^2 |\ln \rho| - \mathcal{O}(1). \tag{D.10}$$

Summing the lower bounds (D.9) and (D.10) over l and applying Proposition 49, we obtain that if (D.7) occurs, then the configuration $\{\mathbf{x}, \mathbf{d}\}$ cannot be almost minimal because $\eta_K, \rho \to 0$ and $\tilde{d}_{l_0} > 1$. Therefore (D.7) cannot occur for almost minimal configurations.

D.4 Proof of the second part of Proposition 12

We now prove the second part of Proposition 12: we establish the repelling effect of $\partial \Omega$ on the centers x_i's.

Let $x^n_1, ..., x^n_d \in \Omega$ and $\rho = \rho(\varepsilon_n) \downarrow 0$ be s.t. $|x^n_i - x^n_j| \geq 8\rho \ (i \neq j)$ and $\text{dist}(x^n_1, \partial \Omega) \to 0$. From the previous subsection we may assume that there is $\eta_0 > 0$ (independent of n) s.t.

$$\min \left\{ \min_{i \neq j} |x^n_i - x^n_j|, \, \text{dist}(\Omega, \partial \Omega') \right\} \geq 10^2 \eta_0.$$

Up to pass to a subsequence, we may assume that $x^n_i \to a_i \in \overline{\Omega}$ with $a_i \neq a_j$ for $i \neq j$ and that $\eta = \max\{\sqrt{\text{dist}(x^n_1, \partial \Omega)}, \rho\} \to 0$.

For sake simplicity, we assume that for $i = 2, ..., d$ we have $a_i \in \Omega$. If this condition is not satisfied, then a direct adaptation of the following argument may be done. We assume that η_0 is s.t. for $i = 2, ..., d$ we have $\text{dist}(x^n_i, \partial \Omega) \geq 10^2 \eta_0$.

We fix $x_0 = x_0(\varepsilon_n) \in \Omega$ s.t.

$$x_0 - x^n_1 \in \delta\mathbb{Z} \times \delta\mathbb{Z}, \, \text{dist}(x^n_0, \partial \Omega) \geq 10^2 \eta_0 \text{ and } \min_{i=1,...,d} |x_0 - x^n_j| \geq 10^2 \eta_0.$$

We are going to prove that for $w \in H^1_g(\Omega' \setminus \cup_i \overline{B(x^n_i, \rho)}, \mathbb{S}^1)$ we have

$$\frac{1}{2} \int_{\mathscr{R}(x^n_1, \sqrt{\eta}, \eta)} U^2_{\varepsilon_n} |\nabla w|^2 - \mu_{\varepsilon_n}(\mathscr{R}(x_0, \sqrt{\eta}, \eta), 1) \to \infty. \tag{D.11}$$

Remark 54. Estimate (D.11) implies that $\{x_1^n, ..., x_d^n\}$ can not be an almost minimal configuration of points.

Indeed, we may construct a suitable test function $\tilde{w}$ as follows:

Construction 55. The test function $\tilde{w} \in H_g^1(\Omega' \setminus (\overline{B(x_0, \rho)} \cup \cup_{i=2}^d \overline{B(x_i^n, \rho)}), \mathbb{S}^1)$

- For $i = 2, ..., d$, we define $\tilde{w}_{|\mathscr{R}(x_i^n, \eta_0, \rho)}$ by taking a minimal map for $\frac{1}{2} \int_{\mathscr{R}(x_i^n, \eta_0, \rho)} U_{\varepsilon_n}^2 |\nabla \cdot|^2$ in $H^1(\mathscr{R}(x_i^n, \eta_0, \rho), \mathbb{S}^1)$ with the boundary conditions $\tilde{w}(x_i^n + \eta_0 e^{\imath\theta}) = e^{\imath\theta}$ and $\tilde{w}(x_i^n + \rho e^{\imath\theta}) = \mathrm{Cst}_i e^{\imath\theta}$, $\mathrm{Cst}_i \in \mathbb{S}^1$. From Proposition 45 we have

$$\frac{1}{2} \int_{\mathscr{R}(x_i^n, \eta_0, \rho)} U_{\varepsilon_n}^2 |\nabla \tilde{w}|^2 \leq \mu_{\varepsilon_n}(\mathscr{R}(x_i^n, \eta_0, \rho), 1) + C_b.$$

- We divide $\mathscr{R}(x_0, \eta_0, \rho)$ into $\mathscr{R}(x_0, \eta_0, \sqrt{\eta})$, $\mathscr{R}(x_0, \sqrt{\eta}, \eta)$ and $\mathscr{R}(x_0, \eta, \rho)$. In each of these rings we consider the minimal maps for $\frac{1}{2} \int_{\mathrm{ring}} U_{\varepsilon_n}^2 |\nabla \cdot|^2$ with the boundary conditions $\tilde{w}(x_0 + R e^{\imath\theta}) = e^{\imath\theta}$ and $\tilde{w}(x_0 + r e^{\imath\theta}) = \mathrm{Cst}_i e^{\imath\theta}$, $\mathrm{Cst}_i \in \mathbb{S}^1$ where ring $\in \{\mathscr{R}(x_0, \eta_0, \sqrt{\eta}), \mathscr{R}(x_0, \sqrt{\eta}, \eta), \mathscr{R}(x_0, \eta, \rho)\}$, $r < R$ and ring $= \mathscr{R}(x_0, R, r)$.

  Up to consider suitable rotations, we glue these functions to get a map $\tilde{w}_{|\mathscr{R}(x_0, \eta_0, \rho)} \in H^1(\mathscr{R}(x_0, \eta_0, \rho), \mathbb{S}^1)$ which is s.t. $\tilde{w}(x_0 + \eta_0 e^{\imath\theta}) = e^{\imath\theta}$ and (from Proposition 45)

$$\frac{1}{2} \int_{\mathrm{ring}} U_{\varepsilon_n}^2 |\nabla \tilde{w}|^2 \leq \mu_{\varepsilon_n}(\mathrm{ring}, 1) + C_b$$

  with ring $\in \{\mathscr{R}(x_0, \eta_0, \sqrt{\eta}), \mathscr{R}(x_0, \sqrt{\eta}, \eta), \mathscr{R}(x_0, \eta, \rho)\}$.

- We extend $\tilde{w}$ in $\Omega \setminus (\overline{B(x_0, \eta_0)} \cup \cup_{i=2}^d \overline{B(x_i^n, \eta_0)})$ using Lemma 44.1. Then we finally obtain $\tilde{w} \in H_g^1(\Omega' \setminus (\overline{B(x_0, \rho)} \cup \cup_{i=2}^d \overline{B(x_i^n, \rho)}), \mathbb{S}^1)$.

From Lemma 48, (D.11) and by construction of $\tilde{w}$, for $w_n \in H_g^1(\Omega' \setminus \cup_i \overline{B(x_i^n, \rho)}, \mathbb{S}^1)$ we have easily that

$$\int_{\Omega' \setminus \cup_i \overline{B(x_i^n, \rho)}} U_{\varepsilon_n}^2 |\nabla w_n|^2 - \int_{\Omega' \setminus (\overline{B(x_0, \rho)} \cup \cup_{i=2}^d \overline{B(x_i^n, \rho)})} U_{\varepsilon_n}^2 |\nabla \tilde{w}|^2 \to +\infty$$

which implies that $\{x_1^n, ..., x_d^n\}$ can not be an almost minimal configuration of points.

We now turn to the proof of (D.11). We argue by contradiction and we assume that there is $w_* = w_*^{\varepsilon_n} \in H_g^1(\Omega' \setminus \cup_i \overline{B(x_i^n, \rho)}, \mathbb{S}^1)$ s.t.

$$\frac{1}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla w_*|^2 \leq \mu_{\varepsilon_n}(\mathscr{R}(x_0, \sqrt{\eta}, \eta), 1) + \mathcal{O}(1). \tag{D.12}$$

In particular (using Lemma 48) we have

$$\frac{1}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla w_*|^2 = \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), 1) + \mathcal{O}(1).$$

The key ingredient to get a contradiction is the fact that the map w_* is almost constant in the "half" ring $\mathscr{R}(x_1^n, \sqrt{\eta}, \eta) \setminus \Omega$.

By smoothness of Ω, we may assume that the cone $K_{\sqrt{\eta}, \eta} := \{x = x_1^n + \rho e^{\imath\theta} \,|\, \theta \in [0, \pi/2], \eta \leq \rho \leq \sqrt{\eta}\}$ does not intersect Ω: $K_{\sqrt{\eta}, \eta} \cap \Omega = \emptyset$.

We consider the map

$$w_0(x_1^n + r\mathrm{e}^{\imath\theta}) = \begin{cases} \mathrm{e}^{\imath 4\theta} & \text{if } \theta \in [0, \pi/2] \\ 1 & \text{otherwise} \end{cases}, \ r > 0$$

which is s.t. $w_0 \in H^1(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), \mathbb{S}^1)$ and $\deg_{\partial B(x_1^n, \sqrt{\eta})}(w_0) = 1$.

For $\tilde{d} \in \mathbb{N}^*$ (to be fixed later) we define the map $w_{\text{test}} = w_*^{\tilde{d}} w_0 \in H^1(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), \mathbb{S}^1)$ and $\deg_{\partial B(x_1^n, \sqrt{\eta})}(w_{\text{test}}) = \tilde{d} + 1$.

Thus, we have

$$\frac{1}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla w_{\text{test}}|^2 \geq \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), \tilde{d} + 1) = (\tilde{d} + 1)^2 \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), 1).$$

On the other hand, letting $\varphi_*, \varphi_0 : \mathscr{R}(x_1^n, \sqrt{\eta}, \eta) \to \mathbb{R}$ s.t. $w_* = \mathrm{e}^{\imath\varphi_*}$ and $w_0 = \mathrm{e}^{\imath\varphi_0}$, (note that φ_*, φ_0 are locally defined and those gradients are globally defined and lie in $L^2(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), \mathbb{R})$), we have (using (D.12)),

$$\begin{aligned}
\frac{1}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla w_{\text{test}}|^2 &= \frac{1}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\tilde{d}\nabla\varphi_* + \nabla\varphi_0|^2 \\
&= \frac{\tilde{d}^2}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla\varphi_*|^2 + \frac{1}{2} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla\varphi_0|^2 + \\
&\qquad\qquad\qquad\qquad + \tilde{d} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 \nabla\varphi_* \cdot \nabla\varphi_0 \\
&\leq \tilde{d}^2 \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), 1) + 2\pi |\ln\eta| + \\
&\qquad\qquad\qquad\qquad + \tilde{d} \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 \nabla\varphi_* \cdot \nabla\varphi_0 + \mathcal{O}(1).
\end{aligned}$$

Since $w_* = g$ in $\mathscr{R}(x_1^n, \sqrt{\eta}, \eta) \setminus \Omega$ and $\|\nabla\varphi_0\|_{L^2(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta) \cap \Omega)} = 0$, we have (using Cauchy-Schwarz inequality)

$$\int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta)} U_{\varepsilon_n}^2 |\nabla\varphi_*||\nabla\varphi_0| = \int_{\mathscr{R}(x_1^n, \sqrt{\eta}, \eta) \setminus \overline{\Omega}} U_{\varepsilon_n}^2 |\nabla\varphi_*||\nabla\varphi_0| = \mathcal{O}(\sqrt{|\ln\eta|}).$$

Therefore we obtain

$$\tilde{d}^2 \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), 1) + 2\pi |\ln\eta| + \mathcal{O}(\sqrt{|\ln\eta|}) \geq (\tilde{d} + 1)^2 \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), 1)$$

which implies that $2\pi|\ln\eta| + \mathcal{O}(\sqrt{|\ln\eta|}) \geq (2\tilde{d}+1)\mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \sqrt{\eta}, \eta), 1) \geq (2\tilde{d}+1)b^2\pi|\ln\eta|$. Clearly we obtain a contradiction taking $\tilde{d} > (2 - b^2)/(2b^2)$.

Thus, by using Remark 54, the second part of Proposition 12 is proved.

D.5 Proof of the third part of Proposition 12

In this subsection, we prove the third part of Proposition 12: the attractive effect of the inclusions.

Assume that there exist $C_0 > 0$, sequences $\varepsilon_n, \rho \downarrow 0$, $\rho = \rho(\varepsilon_n) \geq \varepsilon_n$ s.t. $\rho/(\lambda\delta) \to 0$ and distinct points $x_1^n, ..., x_d^n$, satisfying

$$\inf_{\substack{w \in H_g^1(\Omega_\rho', \mathbb{S}^1) \\ \deg_{\partial B(x_i, \rho)}(w) = 1}} \frac{1}{2} \int_{\Omega_\rho'} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho, \varepsilon_n} \leq C_0. \tag{D.13}$$

We denote $\mathbf{x}_n = (x_1^n, ..., x_d^n)$. From the first and the second assertion, there exists $\eta_0 > 0$ (independent of n) s.t.

$$\min\left\{\min_{i \neq j} |x_i^n - x_j^n|, \min_i \operatorname{dist}(x_i^n, \partial\Omega)\right\} \geq 10^2 \cdot \eta_0 > 0.$$

We want to prove that there is some $c > 0$ s.t. for $i = 1, ..., d$ we have (for large n) $B(x_i^n, c\lambda\delta) \subset \omega_{\varepsilon_n}$.

To this end, we argue by contradiction and we assume that either $x_1^n \notin \omega_{\varepsilon_n}$ or $x_1^n \in \omega_{\varepsilon_n}$ and $\dfrac{\operatorname{dist}(x_1^n, \partial\omega_{\varepsilon_n})}{\lambda\delta} \to 0$.

We are going to prove that letting $y_n \in \delta \cdot (\mathbb{Z} \times \mathbb{Z})$ s.t. $x_1^n, y_n \in \overline{Y_{k,l}^\delta}$, then,

$$\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1}) - \widehat{\mathcal{I}}_{\rho,\varepsilon_n}((y_n, x_2^n, ..., x_d^n), \mathbf{1}) \to \infty. \tag{D.14}$$

Up to a subsequence, we may assume that $\lim_n \frac{\operatorname{dist}(x_1^n, \omega_{\varepsilon_n})}{\lambda\delta}$ exists. We divide the proof into two steps:

Step 1. if $x_1^n \notin \omega_{\varepsilon_n}$ and $\dfrac{\operatorname{dist}(x_1^n, \omega_{\varepsilon_n})}{\lambda\delta} \to c \in (0, \infty]$, then (D.14) holds;

Step 2. if $\dfrac{\operatorname{dist}(x_1^n, \partial\omega_{\varepsilon_n})}{\lambda\delta} \to 0$, then (D.14) holds.

We now prove Step 1. Assume that $x_1^n \notin \omega_{\varepsilon_n}$, $\dfrac{\operatorname{dist}(x_1^n, \omega_{\varepsilon_n})}{\lambda\delta} \to c \in (0, \infty]$ and

$$\inf_{\substack{w \in H_g^1(\Omega_\rho', \mathbb{S}^1) \\ \deg_{\partial B(x_i, \rho)}(w) = 1}} \frac{1}{2} \int_{\Omega_\rho'} U_{\varepsilon_n}^2 |\nabla w|^2 - I_{\rho,\varepsilon_n} \leq C_0.$$

Denote w_n a minimizer for $\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1})$ (see Proposition 8). Using Lemma 52.2, for $\rho \leq r < R < \eta_0$, one has

$$\frac{1}{2} \int_{B(x_1^n, R) \setminus \overline{B(x_1^n, r)}} U_{\varepsilon_n}^2 |\nabla w_n|^2 - \mu_{\varepsilon_n}(B(x_1^n, R) \setminus \overline{B(x_1^n, r)}, 1) \leq C_0 + C(\eta_0).$$

Let $\kappa \in (0, 10^{-2} \cdot c)$ be s.t. $B(0, 10^2\kappa) \subset \omega \subset Y$ and $\operatorname{dist}(\omega, \partial Y) \geq 10^2\kappa$.

From Lemma 52 (Assertions 2 and 3), we have

$$\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1}) = \sum_{i=1}^d \mu_{\varepsilon_n}(\mathscr{R}(x_i^n, \eta_0, \rho), 1) + \mathcal{O}(1)$$

and

$$\widehat{\mathcal{I}}_{\rho,\varepsilon_n}((y_n, x_2^n, ..., x_d^n), \mathbf{1}) = \mu_{\varepsilon_n}(\mathscr{R}(y_n, \eta_0, \rho), 1) + \sum_{i=2}^d \mu_{\varepsilon_n}(\mathscr{R}(x_i^n, \eta_0, \rho), 1) + \mathcal{O}(1).$$

Recall that $y_n \in \delta \cdot \mathbb{Z}^2$ is s.t. $x_1^n, y_n \in \overline{Y_{k,l}^\delta}$. Since $|x_1^n - y_n| \leq \delta$, using Lemma 44.2 and Propositions 45, 46.3, we have

$$\mu_{\varepsilon_n}(\mathscr{R}(y_n, \eta_0, \rho), 1) = \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \eta_0, \delta), 1) + \mu_{\varepsilon_n}(\mathscr{R}(y_n, \kappa\delta, \rho), 1) + \mathcal{O}(1)$$

Therefore

$$
\begin{aligned}
\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1}) &- \widehat{\mathcal{I}}_{\rho,\varepsilon_n}((y_n, x_2^n, ..., x_d^n), \mathbf{1}) \\
&= \mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa\delta, \rho), 1) - \mu_{\varepsilon_n}(\mathcal{R}(y_n, \kappa\delta, \rho), 1) + \mathcal{O}(1).
\end{aligned} \tag{D.15}
$$

Thus it suffices to estimate the energies in the rings with radii $\kappa\delta$ and ρ. We have (using (1.5))

$$
\mu_{\varepsilon_n}(\mathcal{R}(y_n, \kappa\delta, \rho), 1) = \pi|\ln\lambda| + b^2\pi\ln\frac{\lambda\delta}{\rho} + \mathcal{O}(1). \tag{D.16}
$$

In order to estimate $\mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa\delta, \rho), 1)$, we divide the argument according to the asymptotic of λ. If $\lambda \equiv 1$, then $c \in (0, \infty)$ and thus $\mathrm{dist}(B(x_i, c\delta/3, \omega_{\varepsilon_n}) \geq c\delta/3$. Consequently, from Proposition 3, we have

$$
\mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa\delta, \rho), 1) = \pi\ln\frac{\delta}{\rho} + \mathcal{O}(1).
$$

Therefore (D.14) holds.

If $\lambda \to 0$, we let $\chi = \begin{cases} \dfrac{c}{2}\lambda\delta & \text{if } c < \infty \\ \kappa\lambda\delta & \text{otherwise} \end{cases}$ and $\eta = \mathrm{dist}(x_1^n, \partial\omega_{\varepsilon_n})$. Note that $\dfrac{\eta + 2\lambda\delta}{\eta - \chi} = \mathcal{O}(1)$ and that $U_{\varepsilon_n} = 1 + V_n$ in $\mathcal{R}(x_1^n, \kappa\delta, \eta + 2\lambda\delta) \cup \mathcal{R}(x_1^n, \eta - \chi, \rho)$, $\|V_n\|_{L^\infty} = o(\varepsilon_n^2)$ (from (1.5)).

Thus we obtain

$$
\begin{aligned}
\mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa\delta, \rho), 1) &\geq \pi\ln\frac{\delta}{\eta + 2\lambda\delta} + \pi b^2\ln\frac{\eta + 2\lambda\delta}{\eta - \chi} + \pi\ln\frac{\eta - \chi}{\rho} + \mathcal{O}(1) \\
&= \pi\ln\frac{\delta}{\rho} + \mathcal{O}(1).
\end{aligned} \tag{D.17}
$$

Therefore if $c \in (0, \infty]$, then (D.17) holds. Estimates (D.15),(D.16) and (D.17) contradict (D.13) (because (D.14) holds).

We now turn to Step 2. Arguing as in Step 1., it suffices to prove that

$$
\mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa\delta, \rho), 1) - \mu_{\varepsilon_n}(\mathcal{R}(y_n, \kappa\delta, \rho), 1), 1) \to \infty \text{ for some fixed } \kappa. \tag{D.18}
$$

(And $y_n \in \delta \cdot \mathbb{Z}^2$ s.t. $x_i^n, y_n \in \overline{Y_{k,l}^\delta}$)

We let $\kappa > 0$ (depending only on ω) be s.t.

$$
\kappa < 10^{-2} \cdot \mathrm{dist}(\omega, \partial Y) \text{ and } B(0, 10^2 \cdot \kappa) \subset \omega.
$$

In order to prove (D.18), we divide the annular $\mathcal{R}(x_1^n, \kappa\delta, \rho)$ into three regions :

$$
\mathcal{R}(y_n, \kappa\delta, \rho) = \mathcal{R}(x_1^n, \kappa\delta, \kappa\lambda\delta) \cup \overline{\mathcal{R}(x_1^n, \kappa\lambda\delta, r_n)} \cup \mathcal{R}(x_1^n, r_n, \rho)
$$

with

$$
r_n = \max\left\{\varepsilon_n^{1/4}, \rho, \sqrt{\lambda\delta \cdot \mathrm{dist}(x_n, \partial\omega_\varepsilon)}\right\} + \sqrt{\varepsilon_n}.
$$

We are going to prove that $\mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa\lambda\delta, r_n), 1)$ is too large.

We consider K_n the cone of vertex x_1^n and aperture $\pi/2$ which admits the line $(x_1^n, \Pi_{\partial\omega_{\varepsilon_n}} x_1^n)$ for symmetry axis and s.t. $K_n \cap \omega_{\varepsilon_n} \cap \mathcal{R}(x_1^n, \kappa\lambda\delta, r_n) = \emptyset$. Here $\Pi_{\partial\omega_{\varepsilon_n}}(x_1^n)$ is the orthogonal projection of x_1^n on $\partial\omega_{\varepsilon_n}$.
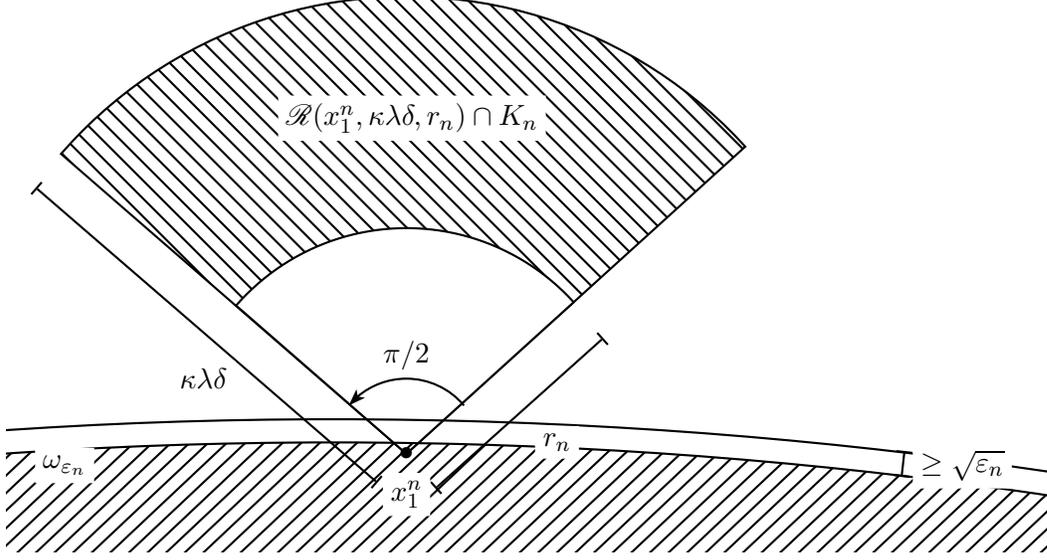
Figure 6: The domain $\mathscr{R}(x_1^n, \kappa\lambda\delta, r_n) \cap K_n$

Note that since $\dfrac{\text{dist}(x_1^n, \omega_{\varepsilon_n})}{\lambda\delta} \to 0$, for large n and small κ (independently of n), by smoothness of ω, K_n is well defined (see Figure 6).

We have $U_{\varepsilon_n} = 1 + V_n$ in $\mathscr{R}(x_1^n, \kappa\lambda\delta, r_n) \cap K_n$ where, $\|V_n\|_{L^\infty} = o(\varepsilon_n^2)$. Thus, if we define $\alpha_n = \begin{cases} 1 & \text{in } K_n \\ b^2 & \text{otherwise} \end{cases}$, then, from Lemma 53 with $\theta_0 = 3\pi/2$, for $w \in H^1(\mathscr{R}(x_1^n, \kappa\lambda\delta, r_n), \mathbb{S}^1)$ s.t. $\deg_{\partial B(x_1^n, r_n)}(w) = 1$, we have

$$\frac{1}{2}\int_{\mathscr{R}(x_1^n, \kappa\lambda\delta, r_n)} \alpha_n |\nabla w|^2 \geq b^2 \frac{4\pi}{b^2+3} \ln \frac{\kappa\lambda\delta}{r_n}.$$

Clearly, from construction, $U_{\varepsilon_n}^2 \geq \alpha_n + o(\varepsilon_n^2)$, thus if w_n is a minimal map for $\widehat{\mathcal{I}}_{\rho,\varepsilon_n}(\mathbf{x}_n, \mathbf{1})$, then we have

$$\frac{1}{2}\int_{\mathscr{R}(x_1^n, \kappa\lambda\delta, r_n)} U_{\varepsilon_n}^2 |\nabla w_n|^2 \geq b^2 \frac{4\pi}{b^2+3} \ln \frac{\kappa\lambda\delta}{r_n} + o_n(1).$$

Now the computations are direct

$$\begin{aligned}
\mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \kappa\delta, \rho), 1) &= \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \kappa\delta, 3\lambda\delta), 1) + \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, \kappa\lambda\delta, r_n), 1) \\
&\qquad + \mu_{\varepsilon_n}(\mathscr{R}(x_1^n, r_n, \rho), 1) + \mathcal{O}(1) \\
&\geq \pi|\ln\lambda| + b^2 \frac{4\pi}{b^2+3} \ln \frac{\lambda\delta}{r_n} + b^2\pi \ln \frac{r_n}{\rho} + \mathcal{O}(1). \qquad \text{(D.19)}
\end{aligned}$$

Therefore, (D.18) is a direct consequence of (D.16) and (D.19) since $\dfrac{\lambda\delta}{r_n} \to +\infty$.

E   Proof of Propositions 16 and 17

We now prove the results specific to th pinning terms with dilution. We begin these proofs by their key ingredient.

E.1 An important effect of the dilution of inclusions

We first state a result which establishes that a "sufficiently large" circle has a small intersection with ω_ε if $\lambda \to 0$.

Lemma 56. *We denote C_ρ a circle with radius ρ.*

1. *Assume that the pinning term is periodic. Once $\lambda \leq 1/8\pi$, for $\rho \geq \delta/3$ we have $\mathscr{H}^1(C_\rho \cap \omega_\varepsilon) \leq 16\pi^2\,\lambda\rho$.*

2. *Assume that the pinning term is not periodic, and recall that the inclusions with size $\lambda\delta^j$ is $\omega_\varepsilon^j = \cup_{i \in \mathcal{M}_j^\varepsilon}\{y_{i,j}^\varepsilon + \lambda\delta^j \cdot \omega\}$, $j \in \{1, ..., P\}$.*
   Once $\lambda \leq 1/8\pi$, for $\rho \geq \delta^j/3$ we have $\mathscr{H}^1(C_\rho \cap \omega_\varepsilon^j) \leq 16\pi^2\,\lambda\rho$.
   Here $\mathscr{H}^1(C_\rho \cap \omega_\varepsilon)$ is the 1-dimensional Hausdorff measure of $C_\rho \cap \omega_\varepsilon$.

Proof. In order to unify the notations, we fix $j = 1$ if we are in the periodic case (and $j \in \{1, ..., P \mid M_j^\varepsilon \in \mathbb{N}^*\}$ if we are in the non-periodic case).
Assume that $C_\rho \cap \omega_\varepsilon \neq \emptyset$ and let

$$
S_j := \begin{cases} \left\{ \tilde{Y}_\varepsilon = (\delta k, \delta l) + \delta \cdot Y \;\middle|\; \begin{array}{c} k, l \in \mathbb{Z}^2,\ \tilde{Y}_\varepsilon \subset \Omega \\ \text{and } \tilde{Y}_\varepsilon \cap C_\rho \neq \emptyset \end{array} \right\} & \text{in the periodic case} \\[2em] \left\{ \tilde{Y}_\varepsilon = B(y_{i,j}^\varepsilon, \delta^j) \;\middle|\; y_{i,j}^\varepsilon \in \mathcal{M}_j^\varepsilon \text{ and } \tilde{Y}_\varepsilon \cap C_\rho \neq \emptyset \right\} & \text{in the non-periodic case} \end{cases}.
$$

For $\tilde{Y}_\varepsilon \in S_j$, we denote

- $\tilde{\omega}_\varepsilon$ **the** connected component of ω_ε which is included in $\tilde{Y}_\varepsilon$

- z_j the center of $\tilde{Y}_\varepsilon = \begin{cases} z_1 + \delta \cdot Y,\ z_1 \in \delta \cdot \mathbb{Z}^2 & \text{in the periodic case} \\ B(z_j, \delta^j),\ z_j \in \mathcal{M}_j^\varepsilon & \text{in the non-periodic case} \end{cases}$.

We first treat the case where $C_\rho \subset \tilde{Y}_\varepsilon \in S_j$: since $\rho \geq \delta^j/3$ and $\tilde{\omega}_\varepsilon \subset B(z_j, \lambda\delta^j)$ (because $\omega \subset B(0,1)$), we have

$$
\mathscr{H}^1(C_\rho \cap \tilde{\omega}_\varepsilon) = \mathscr{H}^1(C_\rho \cap \omega_\varepsilon^j) \leq \mathscr{H}^1(\partial B(z_j, \lambda\delta^j)) = 2\pi\lambda\delta^j \leq 6\pi\lambda\rho.
$$

Otherwise, for $\tilde{Y}_\varepsilon \in S_j$, $C_\rho \nsubseteq \tilde{Y}_\varepsilon$ and thus

$$
\mathscr{H}^1(C_\rho \cap \tilde{\omega}_\varepsilon) \leq \mathscr{H}^1(C_\rho \cap \overline{B(y_{i,j}^\varepsilon, \lambda\delta^j)}) \leq 2\pi\lambda\delta^j \text{ (because } \omega \subset B(0,1))
$$

and

$$
\mathscr{H}^1(C_\rho \cap \tilde{Y}_\varepsilon \setminus \tilde{\omega}_\varepsilon) \geq \delta^j \cdot \left(\frac{1}{2} - 2\pi\lambda \right).
$$

Last estimate comes from the fact that $C_\rho \nsubseteq \tilde{Y}_\varepsilon$. Thus $\mathscr{H}^1(C_\rho \cap \tilde{Y}_\varepsilon \setminus \tilde{\omega}_\varepsilon)$ is at least a radius of $\tilde{Y}_\varepsilon$ (if we are in the non periodic or half of side length of $\tilde{Y}_\varepsilon$ otherwise) minus the previous upper bound. Thus we obtain (for $\lambda \leq 1/8\pi$)

$$
\mathscr{H}^1(C_\rho \cap \tilde{\omega}_\varepsilon) \leq 2\pi\lambda \frac{\mathscr{H}^1(C_\rho \cap \tilde{Y}_\varepsilon \setminus \tilde{\omega}_\varepsilon)}{\dfrac{1}{2} - 2\pi\lambda} \leq 8\pi\lambda\mathscr{H}^1(C_\rho \cap \tilde{Y}_\varepsilon \setminus \tilde{\omega}_\varepsilon).
$$

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Consequently,

$$
\begin{aligned}
\mathscr{H}^1(C_\rho \cap \omega_\varepsilon) &= \sum_{\tilde{Y}_\varepsilon \in S_j} \mathscr{H}^1(C_\rho \cap \tilde{\omega}_\varepsilon) \\
&\leq 8\pi\lambda \sum_{\tilde{Y}_\varepsilon \in S_j} \mathscr{H}^1(C_\rho \cap \tilde{Y}_\varepsilon \setminus \tilde{\omega}_\varepsilon) \leq 8\pi\lambda \mathscr{H}^1(C_\rho) = 16\pi^2\lambda\rho.
\end{aligned}
$$

$\square$

E.2   Proof of Proposition 16

We are now in position to prove Proposition 16. The proof is done in 3 steps.

Let $\varepsilon_n \downarrow 0$, $\rho = \rho(\varepsilon_n) \downarrow 0$, $\rho \geq \varepsilon_n$ and let $\mathbf{x}_n$ be a quasi-minimizer for J_{ρ,ε_n} (defined Notation 15).

From Corollaries 13 & 14, up to pass to a subsequence, there are $\eta_0 > 0$ and $\mathbf{a} = (a_1, ..., a_d) \in \Omega^d$ s.t. $x_i^n \to a_i$, $|a_i - a_j|, \mathrm{dist}(a_i, \partial\Omega) > 10^2\eta_0$.

We prove that $W_g(a_1, ..., a_d) = \min_{b_1, ..., b_n \in \Omega} W_g(b_1, ..., b_n)$. We argue by contradiction and we assume that, up to consider a smaller value for η_0 if necessary, we have the existence of $\mathbf{b} = (b_1, ..., b_d) \in \Omega^d$ s.t. $|b_i - b_j| \geq 10^2\eta_0$, $\mathrm{dist}(b_i, \partial\Omega) > 10^2\eta_0$ and

$$
W_g(\mathbf{b}) < W_g(\mathbf{a}) - 10^2\eta_0.
$$

Step 1. We estimate the energies in perforated domains with a fixed perforation size

The goal of this step is to prove the existence of small ρ_0 (independent of n) s.t. we have for $\mathbf{c} \in \{\mathbf{a}, \mathbf{b}\}$ and $\mathbf{x} \in \Omega^d$ satisfying $\max_i |x_i - c_i| \leq \rho_0$

$$
\hat{\mathcal{J}}_{\rho_0, \mathbb{1}}(\mathbf{x}) - \hat{\mathcal{J}}_{\rho_0, \varepsilon_n}(\mathbf{x}) \leq 2\eta_0. \tag{E.1}
$$

From [10] ((15) and Lemma 2), we may fix $\eta_0 > \rho_0 > 0$ independent of n s.t. for $\mathbf{c} \in \{\mathbf{a}, \mathbf{b}\}$, we have

$$
\hat{\mathcal{J}}_{\rho_0, \mathbb{1}}(\mathbf{x}) - \hat{\mathcal{I}}_{\rho_0, \mathbb{1}}(\mathbf{x}) \leq \eta_0 \text{ for all } \mathbf{x} \in \Omega^d \text{ s.t. } \max_i |x_i - c_i| \leq \rho_0,
$$

$$
\left| \hat{\mathcal{I}}_{\rho_0, \mathbb{1}}(\mathbf{x}) - \pi d |\ln \rho_0| - W_g(\mathbf{x}) \right| \leq \eta_0 \text{ for all } \mathbf{x} \in \Omega^d \text{ s.t. } \max_i |x_i - c_i| \leq \rho_0
$$

and

$$
|W_g(\mathbf{c}) - W_g(\mathbf{x})| \leq \eta_0 \text{ for all } \mathbf{x} \in \Omega^d \text{ s.t. } \max_i |x_i - c_i| \leq \rho_0.
$$

For $\mathbf{c} \in \{\mathbf{a}, \mathbf{b}\}$ and $\mathbf{x} \in \Omega^d$ s.t. $\max_i |x_i - c_i| \leq \rho_0$:

- We let $\theta_\mathbf{x} = \sum_{i=1}^d \theta_{x_i}$ where $\theta_{x_i} \in (-\pi, \pi]$, $\dfrac{x - x_i}{|x - x_i|} = \mathrm{e}^{\imath\theta_{x_i}}$ $(x \neq x_i)$ is main determination of the argument of $x - x_i$.

- We fix $\phi_0^\mathbf{x} \in C^\infty(\partial\Omega, \mathbb{R})$ s.t. $\mathrm{e}^{\imath\phi_0^\mathbf{x}} = g\mathrm{e}^{-\imath\theta_\mathbf{x}}$. Clearly, since $\deg_{\partial\Omega}(g\mathrm{e}^{-\imath\theta_\mathbf{x}}) = 0$, and since $g\mathrm{e}^{-\imath\theta_\mathbf{x}} \in C^\infty(\partial\Omega, \mathbb{S}^1)$, $\phi_0^\mathbf{x} \in C^\infty(\partial\Omega, \mathbb{R})$ is well defined [3].

- We let $\phi_* = \phi_*^\mathbf{x}, \phi = \phi^\mathbf{x} \in H^1$ be **the** solutions of

$$
\begin{cases}
-\Delta\phi_* = 0 & \text{in } \Omega \setminus \cup \overline{B(x_i, \rho_0)} \\
\phi_* = \phi_0 & \text{on } \partial\Omega \\
\partial_\nu\phi_* = -\sum_{j\neq i} \partial_\nu\theta_{x_j} & \text{on } \partial B(x_i, \rho_0), \ i = 1, ..., d
\end{cases}
$$

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and

$$\begin{cases} -\mathrm{div}(U_\varepsilon^2 \nabla\phi) = \mathrm{div}(U_\varepsilon^2 \nabla\theta_\mathbf{x}) & \text{in } \Omega \setminus \cup\overline{B(x_i,\rho_0)} \\ \phi = \phi_0 & \text{on } \partial\Omega \\ \partial_\nu\phi = -\sum_{j\neq i}\partial_\nu\theta_{x_j} & \text{on } \partial B(x_i,\rho_0),\ i=1,...,d \end{cases}.$$

- We let $\psi = \phi - \phi_*$ be **the** solution of

$$\begin{cases} -\mathrm{div}(U_\varepsilon^2 \nabla\psi) = \mathrm{div}[(U_\varepsilon^2 - 1)(\nabla\theta_\mathbf{x} - \nabla\phi_*)] & \text{in } \Omega \setminus \cup\overline{B(x_i,\rho_0)} \\ \psi = 0 & \text{on } \partial\Omega \\ \partial_\nu\psi = 0 & \text{on } \partial B(x_i,\rho_0),\ i=1,...,d \end{cases}.$$

Remark 57. 1. From Proposition 8, the functions ϕ_*, ϕ are s.t. $w_* = e^{\imath(\theta_\mathbf{x}+\phi_*)}, w = e^{\imath(\theta_\mathbf{x}+\phi_*)} \in \mathcal{I}_{\rho_0}(\mathbf{x})$ satisfy

$$\hat{\mathcal{I}}_{\rho_0,\mathbb{1}}(\mathbf{x}) = \frac{1}{2}\int_{\Omega\setminus\overline{\cup B(x_i,\rho_0)}} |\nabla w_*|^2 = \frac{1}{2}\int_{\Omega\setminus\overline{\cup B(x_i,\rho_0)}} |\nabla(\theta_\mathbf{x} + \phi_*)|^2$$

and

$$\hat{\mathcal{I}}_{\rho_0,\varepsilon}(\mathbf{x}) = \frac{1}{2}\int_{\Omega\setminus\overline{\cup B(x_i,\rho_0)}} U_\varepsilon^2|\nabla w|^2 = \frac{1}{2}\int_{\Omega\setminus\overline{\cup B(x_i,\rho_0)}} U_\varepsilon^2|\nabla(\theta_\mathbf{x} + \phi)|^2.$$

2. $\nabla\phi$ and $\nabla\phi_*$ are bounded independently of $\mathbf{x}$ and ε_n in $L^2(\Omega \setminus \cup\overline{B(x_i,\rho_0)})$.

3. From a Poincaré inequality we have the existence of C_0 independent of $\mathbf{x}$ s.t.

$$\|\psi\|_{L^2(\Omega\setminus\cup\overline{B(x_i,\rho_0)})} \leq C_0\|\nabla\psi\|_{L^2(\Omega\setminus\cup\overline{B(x_i,\rho_0)})}.$$

Therefore, using a trace inequality in $\mathscr{R}(x_i, 2\rho_0, \rho_0)$ we obtain $\|\psi\|_{L^2(\partial B(x_i,\rho_0))} \leq C_0'$, C_0' is independent of $\mathbf{x}, n$.

4. We have $|\nabla\phi_*|$ which is bounded in $L^\infty(\Omega \setminus \cup\overline{B(x_i,\rho_0)})$:

$$|\nabla\phi_*| \leq C_0 \text{ with } C_0 \text{ independent of } \mathbf{x}.$$

Indeed, with standard result of elliptic interior regularity, we have

$$\|\phi_*\|_{C^2(\partial B(x_i,8\rho_0))}, \|\phi_*\|_{C^2(\partial B(c_i,4\rho_0))} \leq C_0'.$$

Thus, from global regularity for the Laplacian, we have

$$\|\nabla\phi_*\|_{L^\infty(\Omega\setminus\cup\overline{B(c_i,4\rho_0)})}, \|\nabla\phi_*\|_{L^\infty(\mathscr{R}(x_i,8\rho_0,\rho_0))} \leq C_0''.$$

We let $\Omega_{\rho_0} = \Omega_{\rho_0}(\mathbf{x}) := \Omega\setminus\cup\overline{B(x_i,\rho_0)}$. We are now in position to prove that $\int_{\Omega_{\rho_0}} |\nabla\psi|^2 \to 0$ when $n \to \infty$ uniformly on $\mathbf{x}$. This estimate will easily imply (E.1). Indeed

$$\hat{\mathcal{I}}_{\rho_0,\mathbb{1}}(\mathbf{x}) - \hat{\mathcal{I}}_{\rho_0,\varepsilon_n}(\mathbf{x}) = \frac{1}{2}\int_{\Omega_{\rho_0}} U_{\varepsilon_n}^2 \left[|\nabla(\theta_\mathbf{x} + \phi_*)|^2 - |\nabla(\theta_\mathbf{x} + \phi)|^2\right]$$

$$+ \frac{1}{2}\int_{\Omega_{\rho_0}} (1 - U_{\varepsilon_n}^2)|\nabla(\theta_\mathbf{x} + \phi_*)|^2$$

$$\left(\begin{array}{c}\text{Cauchy-Schwarz}\\ \text{inequality}\end{array}\right) \leq \tilde{C}_0\left(\|\nabla\psi\|_{L^2(\Omega_{\rho_0})} + \|1 - U_{\varepsilon_n}^2\|_{L^2(\Omega_{\rho_0})}\right) \to 0.$$

Consequently we obtain

$$\hat{\mathcal{J}}_{\rho_0,\mathbb{1}}(\mathbf{x}) - \hat{\mathcal{J}}_{\rho_0,\varepsilon_n}(\mathbf{x}) \leq \hat{\mathcal{I}}_{\rho_0,\mathbb{1}}(\mathbf{x}) - \hat{\mathcal{I}}_{\rho_0,\varepsilon_n}(\mathbf{x}) + \eta_0 \leq \eta_0 + o_n(1) \leq 2\eta_0$$

which is exactly (E.1).

Thus it remains to establish that $\displaystyle\int_{\Omega_{\rho_0}} |\nabla\psi|^2 \to 0$ when $n \to \infty$ uniformly on $\mathbf{x}$:

$$
\begin{aligned}
\int_{\Omega_{\rho_0}} U_{\varepsilon_n}^2 |\nabla\psi|^2 &= \int_{\Omega_{\rho_0}} \operatorname{div}[(U_{\varepsilon_n}^2 - 1)(\nabla\theta_{\mathbf{x}} - \nabla\phi_*)]\psi \\
&= \int_{\Omega_{\rho_0}} (1 - U_{\varepsilon_n}^2)(\nabla\theta_{\mathbf{x}} - \nabla\phi_*) \cdot \nabla\psi + \int_{\partial\Omega_{\rho_0}} (U_{\varepsilon_n}^2 - 1)\partial_\nu(\theta_{\mathbf{x}} - \phi_*)\psi.
\end{aligned}
$$

From the L^2 bound on $\nabla\psi$ and the L^∞ bounds on $\nabla\phi_*, \nabla\theta_{\mathbf{x}}$ we have (with C_0 independent of $\mathbf{x}$)

$$
\begin{aligned}
\int_{\Omega_{\rho_0}} U_{\varepsilon_n}^2 |\nabla\psi|^2 &\leq \left(\int_{\Omega_{\rho_0}} |1 - U_{\varepsilon_n}^2|^2 |\nabla\theta_{\mathbf{x}} - \nabla\phi_*|^2\right)^{1/2} \left(\int_{\Omega_{\rho_0}} |\nabla\psi|^2\right)^{1/2} + \\
&\quad + \left(\int_{\partial\Omega_{\rho_0}} (U_{\varepsilon_n}^2 - 1)^2 |\partial_\nu(\theta_{\mathbf{x}} - \phi_*)|^2\right)^{1/2} \left(\int_{\partial\Omega_{\rho_0}} |\psi|^2\right)^{1/2} \\
&\leq C_0 \left(\|1 - U_{\varepsilon_n}^2\|_{L^2(\Omega_{\rho_0})} + \|1 - U_{\varepsilon_n}^2\|_{L^2(\partial\Omega_{\rho_0})}\right).
\end{aligned}
$$

From Proposition 3 and Lemma 56 we have $\|1 - U_{\varepsilon_n}^2\|_{L^2(\partial\Omega_{\rho_0})} = \mathcal{O}(\lambda)$ uniformly in $\mathbf{x}$.

Therefore $\displaystyle\int_{\Omega\setminus\overline{\cup B(x_i,\rho_0)}} |\nabla\psi|^2 \to 0$ when $n \to \infty$ uniformly on $\mathbf{x}$ and (E.1) holds.

Step 2. We study the energies in $\mathscr{R}(x_i, \rho_0, \max(\delta, \lambda^2))$

Let

- $\kappa = \max(\lambda, \sqrt{\delta})$

- $\mathbf{x}_n$ be a quasi minimizer for $J_{\rho,\varepsilon}$

- $w_n = e^{\iota\varphi_n}$ be a minimizer of $\hat{\mathcal{J}}_{\rho,\varepsilon_n}(\mathbf{x}_n)$ (φ_n is locally defined and its gradient is globally defined in $\Omega \setminus \overline{\cup B(x_i, \rho)}$).

We prove that there is $r \in (\kappa^2, \kappa)$ s.t.

$$\frac{1}{2}\int_0^{2\pi} |\partial_\theta\varphi_n(x_i^n + re^{\iota\theta})|^2 d\theta \leq \pi + \frac{1}{\sqrt{|\ln\kappa|}} \text{ for } i = 1, ..., d. \tag{E.2}$$

This estimate is obtained via a mean value argument. We first prove that

$$\mu_{\varepsilon_n}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) = \mu_{\mathbb{1}}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) + o_{\varepsilon_n}(1).$$

Indeed we let ω' be a smooth open set s.t. $\overline{\omega} \subset \omega'$ and $\overline{\omega'} \subset B(0,1)$. We define $\alpha'_\varepsilon = \begin{cases} b^2 & \text{in } \delta\mathbb{Z} \times \delta\mathbb{Z} + \lambda\delta \cdot \omega' \\ 1 & \text{otherwise} \end{cases}$. From Proposition 3, we have $\alpha'_\varepsilon \leq U_\varepsilon^2 + V_\varepsilon$ with $\|V_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^2)$.

For $\rho \geq \delta$ and $x \in \mathbb{R}^2$, from Lemma 56, we have $\mathscr{H}^1[\{\alpha'_\varepsilon = b^2\} \cap \partial B(x,\rho)] \leq 16\pi^2 \lambda \rho$. Therefore, using Lemma 53 we obtain

$$
\begin{aligned}
\mu_{\mathbb{1}}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) + \mathcal{O}(\lambda|\ln\kappa|) \quad &\leq (\text{Lemma 53}) \leq \quad \mu_{\alpha'_{\varepsilon_n}}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) \\
&\leq (\alpha'_\varepsilon \leq U_\varepsilon^2 + V_\varepsilon) \leq \quad \mu_{\varepsilon_n}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) + o_{\varepsilon_n}(1) \\
&\leq (U_\varepsilon^2 \leq 1) \leq \quad \mu_{\mathbb{1}}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) + o_{\varepsilon_n}(1).
\end{aligned}
$$

Since for $s \in (\kappa^2, \kappa)$ we have $s \geq \delta$, we obtain (because $\kappa \geq \lambda$)

$$
\mu_{\varepsilon_n}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) = \mu_{\mathbb{1}}(\mathscr{R}(x_i^n, \kappa, \kappa^2), 1) + \mathcal{O}(\lambda|\ln\kappa|) = \pi|\ln\kappa| + o_{\varepsilon_n}(1).
$$

Therefore from Corollary 14 and Lemma 52.2, $\dfrac{1}{2}\displaystyle\int_{\mathscr{R}(x_i^n,\kappa,\kappa^2)} U_{\varepsilon_n}^2 |\nabla w_n|^2 = \pi|\ln\kappa| + \mathcal{O}(1)$. On the other hand, from a standard estimate, we have

$$
\frac{1}{2}\int_0^{2\pi} |\partial_\theta \varphi_n(x_i^n + se^{i\theta})|^2 \mathrm{d}\theta \geq \pi, \ \forall\, s \in (\kappa^2, \kappa).
$$

We deduce that

$$
\pi d|\ln\kappa| + \mathcal{O}(1) \geq \frac{1}{2}\int_{\cup\mathscr{R}(x_i^n,\kappa,\kappa^2)} |\nabla w_n|^2 \geq \frac{1}{2}\int_{\kappa^2}^{\kappa} \frac{\mathrm{d}s}{s} \sum_i \int_0^{2\pi} |\partial_\theta \varphi_n(x_i^n + se^{i\theta})|^2 \mathrm{d}\theta.
$$

Assume that $r \in (\kappa^2, \kappa)$ s.t. (E.2) holds does not exist. Then we obtain that for $s \in (\kappa^2, \kappa)$

$$
\sum_i \frac{1}{2}\int_0^{2\pi} |\partial_\theta \varphi_n(x_i^n + se^{i\theta})|^2 \mathrm{d}\theta > \pi d + \frac{1}{\sqrt{|\ln\kappa|}}
$$

and consequently

$$
\frac{1}{2}\int_{\cup\mathscr{R}(x_i^n,\kappa,\kappa^2)} |\nabla w_n|^2 \geq |\ln\kappa|\left(\pi d + \frac{1}{\sqrt{|\ln\kappa|}}\right) = \pi d|\ln\kappa| + \sqrt{|\ln\kappa|}.
$$

Clearly this lower bound contradicts $\dfrac{1}{2}\displaystyle\int_{\mathscr{R}(x_i^n,\kappa,\kappa^2)} U_{\varepsilon_n}^2 |\nabla w_n|^2 = \pi|\ln\kappa| + \mathcal{O}(1)$. We are

now in position to estimate the energy in $\mathscr{R}(x_i^n, \rho_0, r)$. Let $h_i^n : \mathbb{S}^1 \to \mathbb{S}^1$, $h_i^n(e^{i\theta}) = w_n(x_i^n + re^{i\theta})$. We have $h_i^n \times \partial_\tau [h_i^n(e^{i\theta})] = \partial_\tau [\varphi_n(x_i^n + re^{i\theta})]$.

Thus from (E.2): $\|h_i^n \times \partial_\tau h_i^n\|_{L^2(\mathbb{S}^1)}^2 \leq 2\pi + 2/\sqrt{|\ln\kappa|}$. Consequently

$$
\int_{\mathbb{S}^1} |h_i^n \times \partial_\tau h_i^n - 1|^2 = \int_{\mathbb{S}^1} \left\{|h_i^n \times \partial_\tau h_i^n|^2 + 1 - 2h_i^n \times \partial_\tau h_i^n\right\} \leq 2/\sqrt{|\ln\kappa|} \to 0.
$$

Therefore $h_i^n \times \partial_\tau h_i^n \to 1$ in $L^2(\mathbb{S}^1)$. Consequently, up to pass to a subsequence, we have the existence of $\alpha_i \in \mathbb{S}^1$ s.t. $\alpha_i^{-1} h_i^n e^{-i\theta} \to 1$ in $H^1(\mathbb{S}^1)$.

From Propositions 12 and 13 in [13] we have

$$
\begin{aligned}
\inf_{\substack{w \in H^1(\mathscr{R}(x_i^n,\rho_0,r),\mathbb{S}^1) \\ w(x_i^n + \rho_0 e^{i\theta}) = \alpha_i e^{i\theta} \\ w(x_i^n + re^{i\theta}) = h_i^n(e^{i\theta})}} \frac{1}{2}\int_{\mathscr{R}(x_i^n,\rho_0,r)} |\nabla w|^2 \ &= \ \inf_{\substack{w \in H^1(\mathscr{R}(x_i^n,\rho_0,r),\mathbb{S}^1) \\ w(x_i^n + \rho_0 e^{i\theta}) = \alpha_i e^{i\theta} \\ w(x_i^n + re^{i\theta}) = \alpha_i e^{i\theta}}} \frac{1}{2}\int_{\mathscr{R}(x_i^n,\rho_0,r)} |\nabla w|^2 + o_n(1) \\
&= \ \pi \ln\frac{\rho_0}{r} + o_n(1).
\end{aligned}
$$

Step 3. We conclude

We are going to construct a map $\tilde{w}_n \in \mathcal{J}_\rho(\mathbf{y}_n)$, $\max |y_i - b_i| \leq \delta$ and s.t.

$$\int_{\Omega \setminus \overline{B(y_i,\rho)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 + \eta_0 \leq \hat{\mathcal{J}}_{\rho,\varepsilon_n}(\mathbf{x}_n). \tag{E.3}$$

Clearly (E.3) is in contradiction with the assumption: $J_{\rho,\varepsilon_n} - \hat{\mathcal{J}}_{\rho,\varepsilon_n}(\mathbf{x}_n) \to 0$. Then this contradiction will imply that $\mathbf{a} = \lim \mathbf{x_n}$ minimizes W_g.

We let $\mathbf{y}_n$ be s.t. $\max |y_i^n - b_i| \leq \delta$ and $x_i^n - y_i^n \in \delta\mathbb{Z} \times \delta\mathbb{Z}$ and we define

$$\tilde{w}_n(x) = \begin{cases} w_{\rho_0}^{\mathbf{y}_n}(x) & \text{if } x \in \Omega \setminus \cup \overline{B(y_i^n,\rho_0)} \\ \mathrm{Cst}_{i,n} w^i(x - y_i^n + x_i^n) & \text{if } x \in \mathscr{R}(y_i^n,\rho_0,r) \\ \mathrm{Cst}_{i,n} w_n[x - y_i^n + x_i^n] & \text{if } x \in \mathscr{R}(y_i^n,r,\rho) \end{cases}$$

Here:

- $w_{\rho_0}^{\mathbf{y}_n}$ is a minimizer of $\hat{\mathcal{J}}_{\rho_0,\mathbb{1}}(\mathbf{y}_n)$,

- w^i is a minimizer of $\displaystyle \inf_{\substack{w \in H^1(\mathscr{R}(x_i^n,\rho_0,r),\mathbb{S}^1) \\ w(x_i^n + \rho_0 e^{i\theta}) = \alpha_i e^{i\theta} \\ w(x_i^n + r e^{i\theta}) = h_i^n(e^{i\theta})}} \frac{1}{2} \int_{\mathscr{R}(x_i^n,\rho_0,r)} |\nabla w|^2$

- $\mathrm{Cst}_{i,n} \in \mathbb{S}^1$ is a constant s.t. $\tilde{w}_n \in H^1(\Omega \setminus \cup \overline{B(y_i^n,\rho)}, \mathbb{S}^1)$

- w_n is the minimizer of $\hat{\mathcal{J}}_{\rho,\varepsilon_n}(\mathbf{x}_n)$ used in Step 2..

We now compare the energies of $\tilde{w}_n$ and w_n.

$$\int_{\Omega \setminus \overline{\cup B(y_i^n,\rho)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 = \int_{\Omega \setminus \overline{\cup B(y_i^n,\rho_0)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 + \int_{\cup_i \mathscr{R}(y_i^n,\rho_0,r)} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 +$$

$$+ \int_{\cup_i \mathscr{R}(y_i^n,r,\rho)} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2.$$

From Step 1. (the definition of ρ_0 and Estimate (E.1)), we have

$$\frac{1}{2} \int_{\Omega \setminus \overline{\cup B(y_i^n,\rho_0)}} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 \leq \pi d |\ln \rho_0| + W_g(\mathbf{y}_n) + \eta_0 + o_n(1)$$

$$\leq \pi d |\ln \rho_0| + W_g(\mathbf{x}_n) - 10\eta_0$$

$$\leq \frac{1}{2} \int_{\Omega \setminus \overline{\cup B(x_i^n,\rho_0)}} U_{\varepsilon_n}^2 |\nabla w_n|^2 - 2\eta_0.$$

From Step 2., letting $\alpha_\varepsilon' = \begin{cases} b^2 & \text{in } \delta\mathbb{Z} \times \delta\mathbb{Z} + \lambda\delta \cdot \omega' \\ 1 & \text{otherwise} \end{cases}$, we have

$$\frac{1}{2} \int_{\cup_i \mathscr{R}(y_i^n,\rho_0,r)} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 \quad = (\text{Step 2.}) = \quad \pi d \ln \frac{\rho_0}{r} + o_n(1)$$

$$\leq (\text{Lem. 53 & 56}) \leq \quad \frac{1}{2} \int_{\cup_i \mathscr{R}(x_i^n,\rho_0,r)} \alpha' |\nabla w_n|^2 + o_n(1)$$

$$\leq (\alpha_\varepsilon' \leq U_\varepsilon^2 + V_\varepsilon) \leq \quad \frac{1}{2} \int_{\cup_i \mathscr{R}(x_i^n,\rho_0,r)} U_{\varepsilon_n}^2 |\nabla w_n|^2 + o_n(1).$$

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From Lemma 47

$$\int_{\cup_i \mathscr{R}(y_i^n, r, \rho)} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2 = \int_{\cup_i \mathscr{R}(x_i^n, r, \rho)} U_{\varepsilon_n}^2 |\nabla w_n|^2 + o_n(1).$$

Therefore we obtain (E.3) and consequently Proposition 16 holds.

E.3   Proof of Proposition 17

The strategy to prove Proposition 17 is the following:

Step 1. We let $\kappa = \max(\lambda, \delta)$. We first characterize almost minimal configurations for $I_{\kappa, \varepsilon}$ (i.e the domain Ω is perforated by discs with radius κ).

Step 2. We make the description of almost minimal points $(x_\varepsilon)_\varepsilon$ for $\mu_\varepsilon(\mathscr{R}(\cdot, \kappa, \lambda \delta^{3/2}), 1)$.

Step 3. We estimate $\inf_{x_0 \in \mathbb{R}^2} \mu_\varepsilon(\mathscr{R}(x_0, \lambda \delta^{3/2}, \rho), 1)$ and we conclude.

Step 1. We study almost minimal configurations for $I_{\kappa, \varepsilon}$, $\kappa = \max(\lambda, \delta)$

We prove that $\{\mathbf{x}, \mathbf{d}\} = \{(x_1^\varepsilon, d_1), ..., (x_N^\varepsilon, d_N)\}$ is an almost minimal configuration for $I_{\kappa, \varepsilon}$ **if and only if** $N = d$, $d_i = 1$ and there is $\eta_0 > 0$ s.t. $\mathrm{dist}(x_i^\varepsilon, \partial\Omega), |x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0$.

First note that for $\eta_0 > 0$ and $x_1^\varepsilon, ..., x_d^\varepsilon \in \Omega$ s.t. $\mathrm{dist}(x_i, \partial\Omega), |x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0$ we have easily

$$I_{\kappa, \varepsilon} \leq \hat{\mathcal{I}}_{\kappa, \varepsilon}(\mathbf{x}, \mathbf{d}) \leq \pi d |\ln \kappa| + C(\eta_0) \tag{E.4}$$

with $C(\eta_0)$ which is independent of ε.

We consider $\{\mathbf{x}, \mathbf{d}\}$ which is almost minimal for $I_{\kappa, \varepsilon}$. We argue as in the proof of Proposition 12 (Assertions 1 and 2, see Subsections D.3 & D.4). We use the separation process defined Subsection C.1 and the associated natural partition of $\Omega_\kappa := \Omega \setminus \cup \overline{B(x_i^\varepsilon, \kappa)}$ (see Subsection C.2).

Here the key ingredients are Lemmas 53 & 56 (which replace the periodic structure of the pinning term). Combining both lemmas we get that if $R > r \geq \kappa$, then

$$\mu_\varepsilon(\mathscr{R}(x_0, R, r), 1) = \pi \ln \frac{R}{r} + \mathcal{O}(\lambda \ln \frac{R}{r}).$$

The rings $\mathscr{R}(x_0, R, r)$ which occur in the partition of Ω_κ are all s.t. $C(\Omega) \geq R > r \geq \kappa$ and thus $\dfrac{R}{r} = \mathcal{O}(\kappa^{-1})$. Which infer that $\mathcal{O}(\lambda \ln \frac{R}{r}) = o_\varepsilon(1)$ and consequently $\mu_\varepsilon(\mathscr{R}(x_0, R, r), 1) = \pi \ln \frac{R}{r} + o_\varepsilon(1)$

Therefore we get: If $\{\mathbf{x}, \mathbf{d}\}$ is an almost minimal configuration for $I_{\kappa, \varepsilon}$ then $N = d$, $d_i = 1$ and there is $\eta_0 > 0$ s.t. $\mathrm{dist}(x_i^\varepsilon, \partial\Omega), |x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0$. This is proved by contradiction exactly as in Subsections D.3 & D.4 and using (E.4).

Moreover, if $\{\mathbf{x}, \mathbf{d}\}$ is an almost minimal configuration for $I_{\kappa, \varepsilon}$, then the arguments of Subsections D.3 & D.4 in conjunction with (E.4), yield $|\hat{\mathcal{I}}_{\kappa, \varepsilon}(\mathbf{x}, \mathbf{d}) - \pi d |\ln \kappa|| \leq C(\eta_0)$. Here η_0 is obtained in the previous paragraph. Therefore we get $I_{\kappa, \varepsilon} = \pi d |\ln \kappa| + \mathcal{O}(1)$.

Conversely, from (E.4), for $\eta_0 > 0$ and $x_1^\varepsilon, ..., x_d^\varepsilon \in \Omega$ s.t. $\mathrm{dist}(x_i^\varepsilon, \partial\Omega), |x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0$, we have $\{x_1^\varepsilon, ..., x_d^\varepsilon\}$ which is almost minimal for $I_{\kappa, \varepsilon}$.

Step 2. We study almost minimal configurations for $\mu_\varepsilon(\mathscr{R}(\cdot, \kappa, \lambda \delta^{3/2}), 1)$

For $j \in \{1, ..., P\}$, we denote: $\omega_\varepsilon^j := \cup_{i \in \mathcal{M}_j^\varepsilon} \{y_{i,j}^\varepsilon + \lambda \delta^j \omega\}$. And recall that the set of centers of connected components of ω_ε^j is $\widehat{\mathcal{M}}_j^\varepsilon := \{y_{i,j}^\varepsilon \mid i \in \mathcal{M}_j^\varepsilon\}$.

Letting $x_\varepsilon^0 \in \omega_\varepsilon$ et $c > 0$ (independent of ε) s.t. $B(x_\varepsilon^0, c\lambda\delta) \subset \omega_\varepsilon^1$, on the one hand we may easily prove that

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon^0, \delta, \lambda\delta^{3/2}), 1) = \pi b^2 |\ln \delta^{1/2}| + \pi |\ln \lambda| + \mathcal{O}(1), \tag{E.5}$$

and on the other hand, applying Lemmas 53 & 56, we have

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon^0, \kappa, \delta), 1) = \pi[1 + \mathcal{O}(\lambda)] \ln \frac{\kappa}{\delta}. \tag{E.6}$$

Therefore, from (E.5) and (E.6), we get

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon^0, \kappa, \lambda\delta^{3/2}), 1) = \pi \left[\frac{b^2}{2} + 1 + \mathcal{O}(\lambda) \right] |\ln \delta| + \pi \ln \frac{\kappa}{\lambda} + \mathcal{O}(1). \tag{E.7}$$

We are going to prove that this situation $(B(x_\varepsilon^0, c\lambda\delta) \subset \omega_\varepsilon^1)$ is the only way to get the minimal energy. More precisely we prove that for a fixed constant $C_0 > 0$, if we have $(x_\varepsilon)_\varepsilon \subset \Omega$ which is s.t.

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) \le \inf_{x_0 \in \Omega} \mu_\varepsilon(\mathscr{R}(x_0, \kappa, \lambda\delta^{3/2}), 1) + C_0, \tag{E.8}$$

then there is $c > 0$ independent of ε s.t. for sufficiently small ε we have $B(x_\varepsilon, c\lambda\delta) \subset \omega_\varepsilon$, i.e. $B(x_\varepsilon, c\lambda\delta) \subset y_{i_\varepsilon,1}^\varepsilon + \lambda\delta\omega$ with $y_{i_\varepsilon,1}^\varepsilon \in \widehat{\mathcal{M}}_1^\varepsilon$.

We let $C_0 > 0$ and $(x_\varepsilon)_\varepsilon \subset \Omega$ s.t. (E.8) holds.

Up to pass to a sequence $\varepsilon_n \downarrow 0$, dropping the subscript n (we write ε instead of ε_n), we may assume that one of these cases occurs

Case 0. $\exists c > 0$ s.t. $B(x_\varepsilon, c\lambda\delta) \subset \omega_\varepsilon$,

Case 1. $x_\varepsilon \notin \cup_{j=1}^P \cup_{i \in \mathcal{M}_j^\varepsilon} B(y_{i,j}^\varepsilon, \delta^j)$,

Case 2. $x_\varepsilon \in \cup_{j=2}^P \cup_{i \in \mathcal{M}_j^\varepsilon} B(y_{i,j}^\varepsilon, \delta^j)$,

Case 3. $\{x_\varepsilon \in \cup_{i \in \mathcal{M}_1^\varepsilon} B(y_{i,1}^\varepsilon, \delta) \setminus \overline{\omega_\varepsilon^1}\}$ or $\{x_\varepsilon \in \omega_\varepsilon^1 \ \& \ \text{dist}(x_\varepsilon, \partial\omega_\varepsilon^1)/\lambda\delta \to 0\}$.

We want to prove that only Case 0. occurs if (E.8) holds.
Case 1. From Lemmas 53 & 56, it is direct to prove that

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) \ge \pi \left[1 + \mathcal{O}(\lambda) \right] \ln \frac{\kappa}{\lambda\delta^{3/2}} = \pi \left[\frac{3}{2} + \mathcal{O}(\lambda) \right] |\ln \delta| + \pi \ln \frac{\kappa}{\lambda}.$$

Using (E.7) we get

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\varepsilon(\mathscr{R}(x_0, \kappa, \lambda\delta^{3/2}), 1) \to +\infty.$$

Therefore, if $(x_\varepsilon)_\varepsilon$ satisfies (E.8), then Case 1. does not occur.
Case 2. We let $j_0 \in \{2, ..., P\}$ be s.t. $x_\varepsilon \in \cup_{i \in \mathcal{M}_{j_0}^\varepsilon} B(y_{i,j_0}, \delta^{j_0})$. We define $\kappa' := \max\{\delta^{j_0}, \lambda\delta^{3/2}\}$ and we denote $y_0 = y_{i,j_0}^\varepsilon \in \widehat{\mathcal{M}}_{j_0}$ be s.t. $x_\varepsilon \in B(y_0, \delta^{j_0})$.

We first assume that $x_\varepsilon \notin \overline{\omega_\varepsilon}$ and we let $\text{æ} = \max\{\lambda\delta^{3/2}, \text{dist}(x_\varepsilon, \partial\omega_\varepsilon^{j_0}) - \lambda\delta^{j_0}\}$. In order to estimate $\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1)$, we divide $\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2})$ into

$$\mathscr{R}(x_\varepsilon, \kappa, \kappa' + 2\lambda\delta^{j_0}) \cup \mathscr{R}(x_\varepsilon, \kappa' + 2\lambda\delta^{j_0}, \text{æ} + 2\lambda\delta^{j_0}) \cup \mathscr{R}(x_\varepsilon, \text{æ} + 2\lambda\delta^{j_0}, \text{æ}) \cup \mathscr{R}(x_\varepsilon, \text{æ}, \lambda\delta^{3/2}).$$

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From Lemmas 53 & 56 we have

$$\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \kappa' + 2\lambda\delta^{j_0}), 1) \geq \pi[1 + \mathcal{O}(\lambda)] \ln \frac{\kappa}{\kappa' + 2\lambda\delta^{j_0}}.$$

Note that $\mathrm{dist}(\mathcal{R}(x_\varepsilon, \kappa' + 2\lambda\delta^{j_0}, \ae + 2\lambda\delta^{j_0}), \omega_\varepsilon^{j_0}) \geq \lambda\delta^{j_0}$ and if for some j we have $\mathcal{R}(x_\varepsilon, \kappa' + 2\lambda\delta^{j_0}, \ae + 2\lambda\delta^{j_0}) \cap \omega_\varepsilon^j \neq \emptyset$, then $\mathrm{dist}(x_\varepsilon, \omega_\varepsilon^j) \geq \delta^j$ (because $x_\varepsilon \in B(y_0, \delta^{j_0})$). Therefore, using Proposition 3 and Lemmas 53 & 56 we get

$$\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa' + 2\lambda\delta^{j_0}, \ae + 2\lambda\delta^{j_0}), 1) \geq \pi[1 + \mathcal{O}(\lambda)] \ln \frac{\kappa' + 2\lambda\delta^{j_0}}{\ae + 2\lambda\delta^{j_0}}.$$

It is obvious that

$$\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \ae + 2\lambda\delta^{j_0}, \ae), 1) \geq b^2\pi \ln \frac{\ae + 2\lambda\delta^{j_0}}{\ae} \geq b^2\pi \ln\left(1 + 2\delta^{j_0 - 3/2}\right) = o_\varepsilon(1).$$

By definition of $\ae$, from Proposition 3, we have

$$\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \ae, \lambda\delta^{3/2}), 1) \geq \pi \ln \frac{\ae}{\lambda\delta^{3/2}} - o_\varepsilon(1).$$

Summing these lower bounds we have

$$
\begin{aligned}
\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) &\geq \pi[1 + \mathcal{O}(\lambda)]\left[\ln \frac{\kappa}{\kappa' + 2\lambda\delta^{j_0}} + \ln \frac{\kappa' + 2\lambda\delta^{j_0}}{\ae + 2\lambda\delta^{j_0}} + \ln \frac{\ae}{\lambda\delta^{3/2}}\right] + o_\varepsilon(1) \\
&\geq \pi[1 + \mathcal{O}(\lambda)] \ln \frac{\kappa}{\lambda\delta^{3/2}} + o_\varepsilon(1) \\
&= \pi[1 + \mathcal{O}(\lambda)]\left(\frac{3}{2}|\ln\delta| + \ln\frac{\kappa}{\lambda}\right) + o_\varepsilon(1)
\end{aligned}
$$

and therefore $\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\varepsilon(\mathcal{R}(x_0, \kappa, \lambda\delta^{3/2}), 1) \to +\infty$ (because $0 \leq \ln(\kappa/\lambda) \leq |\ln\delta|$ and from (E.7)).

We now assume that $x_\varepsilon \in \omega_\varepsilon$. Because $j_0 \geq 2$ and $x_\varepsilon \in B(y_0, \lambda\delta^{j_0})$, we have $B(y_0, 2\lambda\delta^{j_0}) \cap \mathcal{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}) = \emptyset$. Therefore, from the dilution of the inclusion, if there is $\tilde{\omega}_\varepsilon$, a connected component of ω_ε^j s.t. $\mathcal{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}) \cap \tilde{\omega}_\varepsilon$, then $\mathrm{dist}(x_\varepsilon, \tilde{\omega}_\varepsilon) \geq \delta^j/3$. Consequently, from Lemmas 53 & 56, we have

$$\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) \geq \pi[1 + \mathcal{O}(\lambda)] \ln \frac{\kappa}{\lambda\delta^{3/2}} = \pi[1 + \mathcal{O}(\lambda)]\left(\frac{3}{2}|\ln\delta| + \ln\frac{\kappa}{\lambda}\right).$$

From (E.7), we obtain that $\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\varepsilon(\mathcal{R}(x_0, \kappa, \lambda\delta^{3/2}), 1) \to +\infty$.

We deduce that if $(x_\varepsilon)_\varepsilon$ satisfies (E.8), then Case 2. does not occur.

Case 3. We denote $y_0 := y_{i,1}^\varepsilon \in \widehat{\mathcal{M}}_1^\varepsilon$ be s.t. $x_\varepsilon \in B(y_0, \delta)$.

On the one hand, if $\kappa \leq 10^{-2}\delta$, then we have $\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \delta), 1), \mu_\varepsilon(\mathcal{R}(y_0, \kappa, \delta), 1) \leq 2\pi \ln 10$.

On the other hand, if $\kappa > 10^{-2}\delta$, then we have $\mathcal{R}(y_0, \kappa, 10\delta) \subset \mathcal{R}(x_\varepsilon, 10\kappa, 10^{-1}\delta)$ and thus (using Proposition 45) we get

$$
\begin{aligned}
\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \delta), 1) &\geq \mu_\varepsilon(\mathcal{R}(x_\varepsilon, 10\kappa, 10^{-1}\delta), 1) - (2\pi \ln 10 + C_b) \\
&\geq \mu_\varepsilon(\mathcal{R}(y_0, \kappa, 10\delta), 1) - (2\pi \ln 10 + C_b) \\
&\geq \mu_\varepsilon(\mathcal{R}(y_0, \kappa, 10\delta), 1) - (3\pi \ln 10 + 2C_b).
\end{aligned}
$$

Moreover, following the argument of Subsection D.5, we have (because $\dfrac{\lambda\delta^{3/2}}{\lambda\delta} \to 0$)

$$\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \delta, \lambda\delta^{3/2}), 1) - \mu_\varepsilon(\mathscr{R}(y_0, \delta, \lambda\delta^{3/2}), 1) \to +\infty.$$

Therefore we have the existence of $H_\varepsilon \to +\infty$ as $\varepsilon \to 0$ s.t.

$$
\begin{aligned}
\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) &\geq \mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \delta), 1) + \mu_\varepsilon(\mathscr{R}(x_\varepsilon, \delta, \lambda\delta^{3/2}), 1) \\
&\geq \mu_\varepsilon(\mathscr{R}(y_0, \kappa, \delta), 1) + \mu_\varepsilon(\mathscr{R}(y_0, \delta, \lambda\delta^{3/2}), 1) + H_\varepsilon \\
\text{(Prop. 46.3)} \quad &\geq \mu_\varepsilon(\mathscr{R}(y_0, \kappa, \lambda\delta^{3/2}), 1) + H_\varepsilon - 2C_b.
\end{aligned}
$$

Consequently $\mu_\varepsilon(\mathscr{R}(x_\varepsilon, \kappa, \lambda\delta^{3/2}), 1) - \inf_{x_0\in\Omega} \mu_\varepsilon(\mathscr{R}(x_0, \kappa, \lambda\delta^{3/2}), 1) \to +\infty$ and since $(x_\varepsilon)_\varepsilon$ satisfies (E.8), Case 3 does not occur.

Step 3. We study $\inf_{x_0\in\mathbb{R}^2} \mu_\varepsilon(\mathscr{R}(x_0, \lambda\delta^{3/2}, \rho), 1)$ and we conclude

It is obvious that $\inf_{x_0\in\mathbb{R}^2} \mu_\varepsilon(\mathscr{R}(x_0, \lambda\delta^{3/2}, \rho), 1) = \pi b^2 \ln \dfrac{\lambda\delta^{3/2}}{\rho} + o_\varepsilon(1)$. Now we are in position to conclude. On the one hand, from the previous steps, for $\eta_0, c > 0$ and a configuration of points/degrees $\{\mathbf{x}_\varepsilon, \mathbf{1}\} = \{(x_1^\varepsilon, 1), ..., (x_d^\varepsilon, 1)\}$ s.t. $|x_i^\varepsilon - x_j^\varepsilon|, \mathrm{dist}(x_i^\varepsilon, \partial\Omega) \geq \eta_0$ and $B(x_i^{\rho,\varepsilon}, c\lambda\delta) \subset \omega_\varepsilon^1$ for all $i \neq j$, $i, j \in \{1, ..., N\}$, we have $\hat{\mathcal{I}}_{\rho,\varepsilon}(\mathbf{x}_\varepsilon) = I_{\rho,\varepsilon} + \mathcal{O}(1)$.

On the other hand, for $\varepsilon_n \downarrow 0$, if either there is $i \in \{1, ..., N\}$ s.t. $d_i > 1$ or $\mathrm{dist}(x_i^{\rho,\varepsilon}, \partial\Omega) \to 0$ or there are $i \neq j$ s.t. $|x_i - x_j| \to 0$, then the configuration of points/degrees cannot be almost minimal for I_{δ,ε_n} and thus it cannot be almost minimal for I_{ρ,ε_n}.

Moreover, if there is i s.t. $x_i^{\varepsilon_n} \notin \omega_{\varepsilon_n}^1$ or $\mathrm{dist}(x_i^{\varepsilon_n}, \partial\omega_{\varepsilon_n}^1)/(\lambda\delta) \to 0$, then $(x_i^{\varepsilon_n})_n$ cannot be an almost minimal configuration for $\mu_\varepsilon(\mathscr{R}(\cdot, \kappa, \lambda\delta^{3/2}), 1)$. And thus $\{\mathbf{x}, \mathbf{d}\}$ cannot be an almost minimal configuration for I_{ρ,ε_n}.

Therefore Assertions 1. and 2. of Proposition 17 holds.

The rest of the proposition is obtained exactly as Corollary 14.

F Proof of Proposition 32

We use the *unfolding operator* (see [8], definition 2.1). We define, for $\Omega_0 \subset \mathbb{R}^2$ an open set, $p \in (1, \infty)$ and $\delta > 0$:

$$
\begin{aligned}
\mathcal{T}_\delta : \quad L^p(\Omega_0) \quad &\to \quad L^p(\Omega_0 \times \tilde{Y}) \\
\phi \quad &\mapsto \quad \mathcal{T}_\delta(\phi)(x,y) = \begin{cases} \phi\left(\delta\left[\dfrac{x}{\delta}\right] + \delta y\right) & \text{for } (x,y) \in \tilde{\Omega}_\delta^{\mathrm{incl}} \times \tilde{Y} \\ 0 & \text{for } (x,y) \in \Lambda_\delta \times \tilde{Y} \end{cases}
\end{aligned}
$$

and

$$\tilde{Y} = (0,1) \times (0,1), \qquad \tilde{\Omega}_\delta^{\mathrm{incl}} := \bigcup_{\substack{\tilde{Y}_\delta^K \subset \Omega_0 \\ \tilde{Y}_\delta^K = \delta(K+\tilde{Y}),\, K\in\mathbb{Z}^2}} \overline{\tilde{Y}_\delta^K},$$

$$\Lambda_\delta := \Omega_0 \setminus \tilde{\Omega}_\delta^{\mathrm{incl}} \quad \text{and} \quad \left[\dfrac{x}{\delta}\right] := \left(\left[\dfrac{x_1}{\delta}\right], \left[\dfrac{x_2}{\delta}\right]\right) \in \mathbb{Z}^2.$$

Here, for $s \in \mathbb{R}$, $[s]$ is the integer part of s.

We will use the following results:

$$\mathcal{T}_\delta \text{ is linear and continuous, of norm at most 1 ([8], Proposition 2.5),} \qquad \text{(F.1)}$$

$$\mathcal{T}_\delta(\phi\psi) = \mathcal{T}_\delta(\phi)\mathcal{T}_\delta(\psi) \text{ ([8], equation (2.2))}, \tag{F.2}$$

$$\delta\mathcal{T}_\delta(\nabla\phi)(x,y) = \nabla_y\mathcal{T}_\delta(\phi)(x,y) \text{ for } \phi \in W^{1,p}(\Omega_0) \text{ ([8], equation (3.1))}, \tag{F.3}$$

$$\text{for } \phi \in L^1(\Omega_0), \text{ we have } \int_{\tilde{\Omega}_\delta^{\text{incl}}} \phi = \int_{\Omega_0 \times \tilde{Y}} \mathcal{T}_\delta(\phi_\delta) \text{ ([8], Proposition. 2.5 (i))}. \tag{F.4}$$

If $\phi_\delta \in H^1(\Omega_0)$ is such that $\phi_\delta \rightharpoonup \phi_0$ in H^1, then, up to subsequence, there exists $\hat{\phi} \in L^2(\Omega_0, H^1_{\text{per}}(\tilde{Y}))$ s.t.:

$$\mathcal{T}_\delta(\phi_\delta) \to \phi_0 \text{ and } \mathcal{T}_\delta(\nabla\phi_\delta) \rightharpoonup \nabla\phi_0 + \nabla_y\hat{\phi} \text{ in } L^2(\Omega_0 \times \tilde{Y}) \text{ ([8], Theorem 3.5)}. \tag{F.5}$$

Here $H^1_{\text{per}}(\tilde{Y})$ stands for the set of functions $\phi \in H^1(\tilde{Y})$ s.t. the extending of ϕ by $\tilde{Y}$-periodicity is in $H^1_{\text{loc}}(\mathbb{R}^2)$ (see [9], section 3.4).

In order to define properly the homogenized matrix $\mathcal{A}$ we recall a classical result (see Theorem 4.27 in [9]).

Proposition 58. *Let $H_0 \in L^\infty(\tilde{Y}, [b^2, 1])$. For all $f \in (H^1_{\text{per}}(\tilde{Y}))'$ s.t. f annihilates the constants there exists a unique solution $h \in H^1_{\text{per}}(\tilde{Y})$ of*

$$\text{div}(H_0\nabla_y h) = f \text{ and } \mathcal{M}_{\tilde{Y}}(h) = \int_{\tilde{Y}} h = 0.$$

Using the previous theorem we denote $\chi_j \in H^1_{\text{per}}(\tilde{Y})$ the unique solution of

$$\text{div}(H_0\nabla_y\chi_j) = \partial_{y_j}(H_0) \text{ and } \mathcal{M}_{\tilde{Y}}(\chi_j) = 0. \tag{F.6}$$

With these auxiliary functions, we can give an explicit expression of $\mathcal{A}$ the homogenized matrix of $H_0(\frac{\cdot}{\delta})\text{Id}_{\mathbb{R}^2}$ (see Theorem 6.1 in [9]):

$$\mathcal{A} = \int_{\tilde{Y}} H_0 \begin{pmatrix} 1 - \partial_{y_1}\chi_1 & -\partial_{y_1}\chi_2 \\ -\partial_{y_2}\chi_1 & 1 - \partial_{y_2}\chi_2 \end{pmatrix} = \int_{\tilde{Y}} H_0(\text{Id}_{\mathbb{R}^2} - \nabla_y\chi), \ \chi = (\chi_1, \chi_2).$$

For the convenience of the reader we restate, in larger detail, Proposition 32.

Proposition. *Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth bounded open set and let $v_n \in H^2(\Omega_0, \mathbb{C})$ be s.t.*

1. *$|v_n| \leq 1$ and $\int_{\Omega_0} (1 - |v_n|^2)^2 \to 0$,*

2. *$v_n \rightharpoonup v_*$ in $H^1(\Omega_0)$ and $v_* \in H^1(\Omega_0, \mathbb{S}^1)$,*

3. *there is $H_n \in W^{1,\infty}(\Omega_0, [b^2, 1])$ and $\delta_n \downarrow 0$ s.t. $\mathcal{T}_{\delta_n}(H_n) \to H_0$ in $L^2(\Omega_0 \times \tilde{Y})$ with H_0 independent of $x \in \Omega_0$,*

4. *$-\text{div}(H_n\nabla v_n) = v_n f_n(x), f_n \in L^\infty(\Omega_0, \mathbb{R})$.*

Then v_* is a solution of

$$-\text{div}(\mathcal{A}\nabla v_*) = (\mathcal{A}\nabla v_* \cdot \nabla v_*)v_*.$$

Here $\mathcal{A}$ is the homogenized matrix of $H_0(\frac{\cdot}{\delta})\text{Id}_{\mathbb{R}^2}$ given by

$$\mathcal{A} = \int_{\tilde{Y}} H_0 \begin{pmatrix} 1 - \partial_{y_1}\chi_1 & -\partial_{y_1}\chi_2 \\ -\partial_{y_2}\chi_1 & 1 - \partial_{y_2}\chi_2 \end{pmatrix}.$$

Proof. In order to keep notations simple, we write, in what follows, δ rather than δ_n.

Since f_n is real valued, we have that $\operatorname{div}(H_n \nabla v_n) \times v_n = 0$. From (F.1) and (F.2), we obtain

$$\operatorname{div}_y \left[\mathcal{T}_\delta(H_n)(x,y) \mathcal{T}_\delta(\nabla v_n)(x,y) \right] \times \mathcal{T}_\delta(v_n)(x,y) = 0 \text{ in } \Omega_0 \times \tilde{Y}. \tag{F.7}$$

Note that from the assumptions and (F.1),(F.5), passing to a subsequence, there is $\hat{w} \in L^2(\Omega_0, H^1_{\mathrm{per}}(\tilde{Y}))$ s.t.

$$\mathcal{T}_\delta(v_n)(x,y) \to v_*(x), \ \mathcal{T}_\delta(\nabla v_n)(x,y) \rightharpoonup \nabla v_*(x) + \nabla_y \hat{v}(x,y) \text{ in } L^2(\Omega_0 \times \tilde{Y})$$

and

$$\mathcal{T}_\delta(H_n)(x,y) \to H_0(y) \text{ in } L^2(\Omega_0 \times \tilde{Y}).$$

Thus we obtain the convergence:

$$\operatorname{div}_y \left[\mathcal{T}_\delta(H_n)(x,y) \mathcal{T}_\delta(\nabla v_n)(x,y) \right] \times \mathcal{T}_\delta(v_n)(x,y) \rightharpoonup \operatorname{div}_y \left[H_0(\nabla v_* + \nabla_y \hat{v}) \right] \times v_* \text{ in } L^2(\Omega_0 \times H^{-1}(\tilde{Y})).$$

Consequently,

$$\operatorname{div}_y \left[H_0(\nabla v_* + \nabla_y \hat{v}) \right] \times v_* = 0.$$

Since v_* is independent of $y \in \tilde{Y}$, the previous assertion is equivalent to

$$-\operatorname{div}_y \left[H_0 \nabla_y (\hat{v} \times v_*) \right] = (\nabla_y H_0 \cdot \nabla v_*) \times v_*,$$

which in turn is equivalent to

$$-\operatorname{div}_y \left[H_0 \nabla_y (\hat{v} \times v_*) \right] = \sum_i \partial_{y_i} H_0 (\partial_i v_* \times v_*).$$

Hence, from Proposition 58 and (F.6), we obtain

$$\hat{v} \times v_* = -\sum_i \chi_i (\partial_i v_* \times v_*) = -\chi \cdot (\nabla v_* \times v_*), \ \chi = (\chi_1, \chi_2). \tag{F.8}$$

Let $\psi \in \mathcal{D}(\Omega_0)$ and n sufficiently large s.t. $\operatorname{Supp}(\psi) \subset \tilde{\Omega}_\delta^{\mathrm{incl}}$. Since $-\operatorname{div}\left[H_n \nabla v_n \times v_n\right] = 0$, we have

$$\int_{\tilde{\Omega}_\delta^{\mathrm{incl}}} H_n \nabla v_n \times v_n \cdot \nabla \psi = 0.$$

This identity combined with (F.4) implies that

$$\int_{\Omega_0 \times \tilde{Y}} \mathcal{T}_\delta [H_n (\nabla v_n \times v_n) \cdot \nabla \psi] = 0.$$

Therefore, using (F.3) and (F.5), we obtain:

$$0 = \int_{\Omega_0 \times \tilde{Y}} \mathcal{T}_\delta \left[H_n (\nabla v_n \times v_n) \cdot \nabla \psi \right] \quad = \quad \int_{\Omega_0 \times \tilde{Y}} \mathcal{T}_\delta(H_n) \mathcal{T}_\delta(\nabla v_n) \times \mathcal{T}_\delta(v_n) \cdot \mathcal{T}_\delta(\nabla \psi)$$

$$\xrightarrow[n \to \infty]{} \int_{\Omega_0 \times \tilde{Y}} H_0 \left[\nabla v_* \times v_* + \nabla_y (\hat{v} \times v_*) \right] \cdot \nabla \psi.$$

Finally, for all $\psi \in \mathcal{D}(\Omega_0)$, using (F.8), we have

$$0 = \int_{\Omega_0 \times \tilde{Y}} H_0 \nabla v_* \times v_* \left[\mathrm{Id}_{\mathbb{R}^2} - \nabla_y \chi \right] \cdot \nabla \psi \quad = \quad \int_{\Omega_0} \left(\left\{ \int_{\tilde{Y}} H_0 \left[\mathrm{Id}_{\mathbb{R}^2} - \nabla_y \chi \right] \right\} \nabla v_* \times v_* \right) \nabla \psi$$

$$= \quad -\int_{\Omega_0} -\operatorname{div}\left(\mathcal{A} \nabla v_* \times v_* \right) \psi.$$

Here $\mathcal{A} = \int_{\tilde{Y}} H_0 \left(\mathrm{Id}_{\mathbb{R}^2} - \nabla_y \chi \right)$.

Thus $-\mathrm{div}\left(\mathcal{A} \nabla v_* \times v_* \right) = 0$. Note that, since H_0 and χ are independent of x, $\mathcal{A}$ is a constant matrix. This fact combined with the equation $-\mathrm{div}\left(\mathcal{A} \nabla v_* \times v_* \right) = 0$ implies that v_* satisfies

$$- \mathrm{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_*. \tag{F.9}$$

Indeed, we can always consider φ_* which is locally defined in Ω_0 and whose gradient is globally defined and in $L^2(\Omega_0, \mathbb{R}^2)$ s.t. $v_* = \mathrm{e}^{\iota \varphi_*}$.

Since $v_* \times \nabla v_* = \nabla \varphi_*$ we obtain that $\mathrm{div}(\mathcal{A} \nabla \varphi_*) = 0$. Identity (F.9) follows from the equation of φ_* and the fact that $|\nabla \varphi_*|^2 = |\nabla v_*|^2$.

$\square$

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