ON THE NEWMAN CONJECTURE

Alexander Bulinski

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Alexander BULINSKI[1,2]

Abstract

We consider a random field, defined on an integer-valued d-dimensional lattice $\mathbb{Z}^d$, with covariance function satisfying a condition more general than summability. Such condition appeared in the well-known Newman's conjecture concerning the central limit theorem (CLT) for stationary associated random fields. As was demonstrated by Herrndorf and Shashkin, the conjecture fails already for $d = 1$. In the present paper, we show the validity of modified conjecture leaving intact the mentioned condition on covariance function. Thus we establish, for any integer $d \geq 1$, a criterion of the CLT validity for the wider class of positively associated stationary fields. The uniform integrability for the squares of normalized partial sums, taken over growing parallelepipeds or cubes in $\mathbb{Z}^d$, plays the key role in deriving their asymptotic normality. So our result extends the Lewis theorem proved for sequences of random variables. A representation of variances of partial sums of a field using the slowly varying functions in several arguments is employed in essential way.

Keywords and phrases: stationary random fields, positive association, central limit theorem, uniform integrability, slowly varying functions, the Newman conjecture.

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1 Introduction

The study of asymptotical behavior of the (normalized) sums of random variables is the vast research domain of Probability Theory having various applications. The limit theorems established for independent summands form here the classical core. In this regard one can refer to the monographs [6], [15], [8], [13]; see also references therein.

Stochastic models described by means of families of dependent random variables arose at the beginning of the last century. Thus the Gaussian and Markov processes, martingales, solutions of the stochastic differential equations, mixing processes appeared as well as other important classes (see, e.g., [3], [9]). Moreover, much attention has been paid to studying of random fields.

Since the 1960s due to the problems of mathematical statistics, reliability theory, percolation and statistical physics there arose the stochastic models based on the families of variables possessing various forms of positive or negative dependence (see, e.g., [2]). The key role in these models belongs to the notion of association (in statistical physics the well-known FKG-inequalities imply the association). We will use the following concept extending that introduced in [5].

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[2]Lomonosov Moscow State University and University Paris-6 – Pierre and Marie Curie.

Definition 1 ([12]) *A real-valued random field $X = \{X_t, t \in T\}$ is called positively associated*[3] *(one writes $X \in \mathsf{PA}$) if, for any finite disjoint sets $I = \{s_1, \ldots, s_m\} \subset T$, $J = \{t_1, \ldots, t_n\} \subset T$ and all bounded coordinate-wise nondecreasing Lipschitz functions $f : \mathbb{R}^m \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, one has*

$$\mathsf{cov}(f(X_{s_1}, \ldots, X_{s_m}), g(X_{t_1}, \ldots, X_{t_n})) \geq 0. \tag{1}$$

Recall that a random field X is called associated ([5]), if the definition above is satisfied without the hypothesis $I \cap J = \varnothing$. Obviously association implies positive association. Note that any family of (real-valued) independent random variables is automatically associated. Many other important examples can be found in [2].

For a random field $X = \{X_t, t \in T\}$ and a finite set $U \subset T$ introduce

$$S(U) = \sum_{t \in U} X_t.$$

Further on we will consider random fields defined on a lattice $T = \mathbb{Z}^d$ and a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. In the seminal paper by Newman [12] the central limit theorem (CLT) was established for associated (strictly) stationary random field $X = \{X_t, t \in \mathbb{Z}^d\}$ under *finite susceptibility condition* that is when the covariance function is summable:

$$\sigma^2 := \sum_{j \in \mathbb{Z}^d} \mathsf{cov}(X_0, X_j) < \infty. \tag{2}$$

Namely, these simple assumptions imply for a field X the following relation

$$\frac{S_n - \mathsf{E}S_n}{\sqrt{\langle n \rangle}} \xrightarrow{law} Z \sim \mathcal{N}(0, \sigma^2) \ \text{ as } \ n \to \infty, \ n = (n_1, \ldots, n_d) \in \mathbb{N}^d, \tag{3}$$

here $S_n = S([0, n] \cap \mathbb{Z}^d)$, $[0, n] = [0, n_1] \times \ldots \times [0, n_d]$, $\langle n \rangle = n_1 \ldots n_d$, $\mathcal{N}(0, \sigma^2)$ is a Gaussian law with parameters 0 and σ^2, $\xrightarrow{law}$ stands for weak convergence of distributions.

The goal of this work is to provide the criteria of the CLT validity for positively associated stationary random fields with finite second moment (and in general without condition (2)).

2 Main results

At first it is reasonable to recall several definitions.

Definition 2 *A function $L : \mathbb{R}^d_+ \to \mathbb{R} \setminus \{0\}$ is called slowly varying (at infinity) if, for any vector $a = (a_1, \ldots, a_d)^\top$ with positive coordinates,*

$$\frac{L(a_1 x_1, \ldots, a_d x_d)}{L(x_1, \ldots, x_d)} \to 1 \ \text{ as } \ x = (x_1, \ldots, x_d)^\top \to \infty, \tag{4}$$

i.e. $x_1 \to \infty, \ldots, x_d \to \infty$. For such functions we write $L \in \mathcal{L}(\mathbb{R}^d_+)$.

[3] or weakly associated

We operate with column vectors and use the symbol $\top$ for transposition. A function $L : \mathbb{N}^d \to \mathbb{R} \setminus \{0\}$ is called *slowly varying* (at infinity) if, for any vector $a = (a_1, \ldots, a_d)^\top \in \mathbb{N}^d$, relation (4) holds with additional assumption that $x \in \mathbb{N}^d$. Then we write $L \in \mathcal{L}(\mathbb{N}^d)$.

For example the function $\prod_{k=1}^d \log(x_k \vee 1)$ where $x \in \mathbb{R}_+^d$ belongs to $\mathcal{L}(\mathbb{R}_+^d)$.

Remark 1 *It is well-known that not every function belonging to $\mathcal{L}(\mathbb{N}^d)$ admits extension to a function from the class $\mathcal{L}(\mathbb{R}_+^d)$ even for $d = 1$. However, it is not difficult to verify that if a coordinate-wise nondecreasing function $L \in \mathcal{L}(\mathbb{N}^d)$, then $H(x) := L([\widetilde{x}])$ belongs to $\mathcal{L}(\mathbb{R}_+^d)$. Here $\widetilde{x} = (x_1 \vee 1, \ldots, x_d \vee 1)^\top$ for $x \in \mathbb{R}^d$, and $[x] = ([x_1], \ldots, [x_d])^\top$, i.e. one takes the integer part of each component of x.*

During a long time there was no solution to the Newman conjecture on possible replacement of requirement (2) appearing in CLT by a milder condition. Namely, he considered the partial sums $S(U)$ taken over "integer cubes" U and believed that instead of (2) it suffices to assume that for associated strictly stationary random field $X = \{X_j, j \in \mathbb{Z}^d\}$ with $\mathsf{E}X_0^2 < \infty$ the function

$$\mathsf{K}(r) = \sum_{j \in \mathbb{Z}^d : \|j\| \le r} \mathsf{cov}(X_0, X_j), \quad r \in \mathbb{N}, \tag{5}$$

belongs to $\mathcal{L}(\mathbb{N})$ where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

Unfortunately it turned out that this beautiful hypothesis is not true even for $d = 1$. The first counterexample was constructed by Herrndorf [7], and then Shashkin [14] showed that condition (2) has in a sense the optimal character.

It is worth mentioning also that the Newman CLT was generalized in [4] for partial sums $S(U)$ taken over regularly growing subsets of $\mathbb{Z}^d$. Further extensions are discussed in Chapter 3 of [2].

Definition 3 *A family $X = \{X_j, j \in \mathbb{N}^d\}$ is called uniformly integrable if*

$$\lim_{c \to \infty} \sup_{j \in \mathbb{N}^d} \mathsf{E}|X_j| \mathbb{I}\{|X_j| \ge c\} = 0.$$

For a (wide sense) stationary random field $X = \{X_j, j \in \mathbb{Z}^d\}$ introduce the function

$$\mathsf{K}_X(n) = \sum_{j \in \mathbb{Z}^d : -n \le j \le n} \mathsf{cov}\,(X_0, X_j), \quad n \in \mathbb{N}^d.$$

If $a = (a^{(1)}, \ldots, a^{(d)})^\top$ and $b = (b^{(1)}, \ldots, b^{(d)})^\top$ are vectors in $\mathbb{R}^d$, the notation $a \le b$ means that $a^{(k)} \le b^{(k)}$ for all $k = 1, \ldots, d$. We write $a < b$ whenever $a^{(k)} < b^{(k)}$ for any $k = 1, \ldots, d$.

The following result extends the Lewis theorem proved in [10] for a sequence of random variables.

Theorem 1 *Let a strictly stationary random field $X = \{X_j, j \in \mathbb{Z}^d\} \in \mathsf{PA}$, $0 < \mathsf{E}X_0^2 < \infty$ and $\mathsf{K}_X(\cdot) \in \mathcal{L}(\mathbb{N}^d)$. Then X satisfies CLT, i.e.*

$$\frac{S_n - \mathsf{E}S_n}{\sqrt{\mathsf{var} S_n}} \xrightarrow{law} Z \sim \mathcal{N}(0,1) \ \ as \ \ n \to \infty, \tag{6}$$

if and only if the family $\{(S_n - \mathsf{E}S_n)^2 / (\langle n \rangle \mathsf{K}_X(n)), n \in \mathbb{N}^d\}$ is uniformly integrable.

Consider now a sequence of growing "integer cubes" $C_r = (0, r]^d \cap \mathbb{Z}^d$, $r \in \mathbb{N}$.

Theorem 2 *Let a strictly stationary random field $X = \{X_j, j \in \mathbb{Z}^d\} \in \mathsf{PA}$, $0 < \mathsf{E}X_0^2 < \infty$ and $\mathsf{K}(\cdot) \in \mathcal{L}(\mathbb{N})$. Then*

$$\frac{S(C_r) - \mathsf{E}S(C_r)}{\sqrt{\mathsf{var}S(C_r)}} \stackrel{law}{\longrightarrow} Z \sim \mathcal{N}(0,1) \ \ as \ \ r \to \infty, \tag{7}$$

if and only if the sequence $((S(C_r) - \mathsf{E}S(C_r))^2/(r^d\,\mathsf{K}(r)))_{r\in\mathbb{N}}$ is uniformly integrable.

Theorem 2 shows what one has to assume additionally, for a class of positively associated strictly stationary random fields, besides the condition that the function $\mathsf{K}(\cdot)$ is slowly varying to guarantee that the Newman conjecture holds true for any dimension $d \in \mathbb{N}$. In [11] the author discussed his conjecture and noted without proof that the "mild version" of that hypothesis takes place under the additional condition of uniform integrability of the sequence appearing in Theorem 2 above. Therefore Theorems 1 and 2 show that in fact we do not change the initial problem but clarify its essential feature. We do not deal here with a renorm group approach (do not consider the partition of $\mathbb{R}^d$ by the congruent cubes) but study the partial sums S_n taken over any growing "integer blocks".

3 Proofs of the main results

We start with simple auxiliary statements.

Lemma 1 *Let a function L belonging to $\mathcal{L}(\mathbb{N}^d)$ be coordinate-wise nondecreasing. Then there exist non-random vectors $q_n = (q_n^{(1)}, \dots, q_n^{(d)})^\top \in \mathbb{N}^d$, where $n = (n_1, \dots, n_d)^\top \in \mathbb{N}^d$, such that*

$$q_n^{(k)} \le n_k, \ \frac{q_n^{(k)}}{n_k} \to 0 \ \ for \ \ k = 1, \dots, d, \ \ q_n \to \infty \quad and \quad \frac{L(n)}{L(q_n)} \to 1 \ as \ n \to \infty. \tag{8}$$

Proof. According to Remark 1 we can assume without loss of generality that L is extended to a function belonging to the class $\mathcal{L}(\mathbb{R}_+^d)$. For any $R = (R^{(1)}, \dots, R^{(d)})^\top \in \mathbb{N}^d$ we can choose $N_0(R) \in \mathbb{N}^d$ in such a way that

$$\frac{L(n_1, \dots, n_d)}{L\Big(\frac{n_1}{R^{(1)}}, \dots, \frac{n_d}{R^{(d)}}\Big)} - 1 \le \frac{1}{\langle R \rangle}$$

for all $n \ge N_0(R)$. Now we take a sequence $(R(r))_{r\in\mathbb{N}}$ such that $R(r) \in \mathbb{N}^d$ and $R(r) < R(r+1)$ for each $r \in \mathbb{N}$. Introduce $M_0(1) = N_0(R(1))$ and $M_0(r+1) = (M_0(r) \vee N_0(R(r+1))) + \mathbf{1}$ for $r \in \mathbb{N}$ where, as usual, $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^d$ and

$$(a^{(1)}, \dots, a^{(d)}) \vee (b^{(1)}, \dots, b^{(d)}) = (a^{(1)} \vee b^{(1)}, \dots, a^{(d)} \vee b^{(d)}).$$

Then $M_0(r) < M_0(r+1)$ for $r \in \mathbb{N}$. For arbitrary $r \in \mathbb{N}$ and $n \ge M_0(r)$

$$\frac{L(n_1, \dots, n_d)}{L\Big(\frac{n_1}{R^{(1)}(r)}, \dots, \frac{n_d}{R^{(d)}(r)}\Big)} - 1 \le \frac{1}{\langle R(r) \rangle}.$$

Let us define non-random sequences $(\varepsilon_j^{(k)})_{j\in\mathbb{N}}$ where $k=1,\dots,d$, putting $\varepsilon_j^{(k)}=1/R^{(k)}(r)$ for $M_0^{(k)}(r)\le j<M_0^{(k)}(r+1)$.

For any $\varepsilon>0$ take $r_0\in\mathbb{N}$ in such a way that $1/\langle R(r_0)\rangle<\varepsilon$. Further on, for n such that $M_0(r)\le n<M_0(r+1)$ where $r\ge r_0$, one has

$$1\le\frac{L(n_1,\dots,n_d)}{L(n_1\varepsilon_{n_1}^{(1)},\dots,n_d\varepsilon_{n_d}^{(d)})}=\frac{L(n_1,\dots,n_d)}{L\Big(\frac{n_1}{R^{(1)}(r)},\dots,\frac{n_d}{R^{(d)}(r)}\Big)}$$

$$\le 1+\frac{1}{\langle R(r)\rangle}\le 1+\frac{1}{\langle R(r_0)\rangle}\le 1+\varepsilon.$$

Then we can take $q_n=([n_1\varepsilon_{n_1}^{(1)}],\dots,[n_d\varepsilon_{n_d}^{(d)}])\vee([\log n_1],\dots,[\log n_d])\vee\mathbf{1}$, to ensure the validity of (8). $\square$

Lemma 2 *Let $X=\{X_j,j\in\mathbb{Z}^d\}$ be a wide sense stationary random field with nonnegative covariance function. Assume that $\mathsf{K}_X(\cdot)\in\mathcal{L}(\mathbb{N}^d)$. Then*

$$\mathsf{var}S(U_n)\sim\langle n\rangle\,\mathsf{K}_X(n)\quad as\ \ n\to\infty \tag{9}$$

where $U_n=\{j\in\mathbb{Z}^d:\mathbf{1}\le j\le n\}$, $n\in\mathbb{N}^d$. Conversely, if $\mathsf{var}S(U_n)\sim\langle n\rangle\,L(n)$ as $n\to\infty$, where $L\in\mathcal{L}(\mathbb{N}^d)$, then $L(n)\sim\mathsf{K}_X(n)$ as $n\to\infty$.

Proof. Let $\mathsf{K}_X(\cdot)\in\mathcal{L}(\mathbb{N}^d)$. Due to the (wide-sense) stationarity of X one has $\mathsf{cov}(X_i,X_j)=\mathsf{R}(i-j)$ for $i,j\in\mathbb{Z}^d$. Thus

$$\mathsf{var}S(U_n)=\sum_{i,j\in U_n}\mathsf{cov}(X_i,X_j)=\sum_{i,j\in U_n}\mathsf{R}(i-j)$$

$$=\sum_{m\in\mathbb{Z}^d:-(n-\mathbf{1})\le m\le n-\mathbf{1}}(n_1-|m_1|)\dots(n_d-|m_d|)\mathsf{R}(m)$$

$$\le\langle n\rangle\sum_{m\in\mathbb{Z}^d:-(n-\mathbf{1})\le m\le n-\mathbf{1}}\mathsf{R}(m)\le\langle n\rangle\,\mathsf{K}_X(n), \tag{10}$$

as the function R is nonnegative.

Take any $c\in(0,1)$ and $n\ge\frac{1}{1-c}\mathbf{1}$ (i.e. $cn\le n-\mathbf{1}$, $n\in\mathbb{N}^d$). Using again nonnegativity of R we can write

$$\mathsf{var}S(U_n)=\sum_{m\in\mathbb{Z}^d:-(n-\mathbf{1})\le m\le n-\mathbf{1}}(n_1-|m_1|)\dots(n_d-|m_d|)\mathsf{R}(m)$$

$$\ge(1-c)^d\langle n\rangle\sum_{m\in\mathbb{Z}^d:-cn\le m\le cn}\mathsf{R}(m)=(1-c)^d\langle n\rangle\,\mathsf{K}_X([cn]).$$

In view of Remark 1 we come to the relation

$$(1-c)^d\langle n\rangle\,\mathsf{K}_X([cn])\sim(1-c)^d\langle n\rangle\,\mathsf{K}_X(n),\ \ n\to\infty,\ \ n\in\mathbb{N}^d.$$

Consequently, $\mathsf{var}S(U_n)\sim\langle n\rangle\,\mathsf{K}_X(n)$ as $n\to\infty$, because c can be taken arbitrary close to zero.

Now suppose that $\mathsf{var}S(U_n) \sim \langle n\rangle\, L(n)$ as $n \to \infty$, where $L \in \mathcal{L}(\mathbb{N}^d)$. Then for any $\varepsilon > 0$ and all n sufficiently large (i.e. each component of n is large enough), application of (10) leads to the inequality

$$\mathsf{K}_X(n) \geq \frac{\mathsf{var}S(U_n)}{\langle n\rangle} \geq (1-\varepsilon)L(n). \tag{11}$$

For a fixed $q \in \mathbb{N}$, $q > 1$, and $n_r \in \mathbb{N}$, $m_r \in \mathbb{Z}$ such that $|m_r| \leq n_r$ where $r = 1, \ldots, d$, one has

$$\frac{q}{q-1}\left(1 - \frac{|m_r|}{n_r q}\right) \geq \frac{q}{q-1}\left(1 - \frac{n_r}{n_r q}\right) = 1.$$

Therefore, taking into account condition $\mathsf{R} \geq 0$ we verify that

$$\mathsf{K}_X(n) \leq \left(\frac{q}{q-1}\right)^d \sum_{m\in\mathbb{Z}^d:\, -n\leq m\leq n} \mathsf{R}(m) \prod_{r=1}^{d} \frac{(n_r q - |m_r|)}{n_r q}$$

$$\leq \left(\frac{q}{q-1}\right)^d \left(\prod_{r=1}^{d} n_r q\right)^{-1} \sum_{m\in\mathbb{Z}^d:\, -nq\leq m\leq nq} \mathsf{R}(m) \prod_{r=1}^{d} (n_r q - |m_r|)$$

$$= \left(\frac{q}{q-1}\right)^d \frac{\mathsf{var}S(U_{qn})}{\langle qn\rangle} \sim \left(\frac{q}{q-1}\right)^d L(qn), \quad n \to \infty. \tag{12}$$

As q can be chosen arbitrary large, using (11) and (12) we conclude that the desired statement holds. $\square$

Proof of Theorem 1. Necessity. Suppose that (6) is satisfied. Then

$$\frac{(S_n - \mathsf{E}S_n)^2}{\mathsf{var}S_n} \xrightarrow{law} Z^2 \ \text{ as } \ n \to \infty.$$

Indeed, if the random variables $Y_n \xrightarrow{law} Y$, then for any bounded continuous function $h : \mathbb{R} \to \mathbb{R}$ one has $h(Y_n) \xrightarrow{law} h(Y)$ as $n \to \infty$. Obviously,

$$\frac{(S_n - \mathsf{E}S_n)^2}{\mathsf{var}S_n} \geq 0 \ \text{ and } \ \frac{\mathsf{E}(S_n - \mathsf{E}S_n)^2}{\mathsf{var}S_n} = 1.$$

Thus uniform integrability of the family $\{(S_n - \mathsf{E}S_n)^2/\mathsf{var}S_n, n \in \mathbb{N}^d\}$ follows from the analogue of Theorem 1.5.4 established in [1] for a sequence of random variables indexed by points of $\mathbb{N}$. In view of Lemma 2 we can claim that (9) holds. Consequently, the family $\{(S_n - \mathsf{E}S_n)^2/(\langle n\rangle \mathsf{K}_X(n)), n \in \mathbb{N}^d\}$ is also uniformly integrable.

Sufficiency. If the function K_X is bounded we see that (2) is valid and Theorem 3.1.12 of [2] implies that (6) is satisfied. Thus we will assume further that a function K_X is unbounded. Set $\mathsf{K}_X(t) := \mathsf{K}_X([t] \vee \mathbf{1})$ for $t = (t_1, \ldots, t_d)^\top \in \mathbb{R}^d_+$ where $[t] = ([t_1], \ldots, [t_d])^\top$. This extension of the initial function K_X belongs to $\mathcal{L}(\mathbb{R}^d_+)$ as K_X is coordinate-wise nondecreasing on $\mathbb{N}^d$ (a field $X \in \mathsf{PA}$, therefore its covariance function is nonnegative). Further on we assume that the function K_X is extended on $\mathbb{R}^d_+$ as indicated above.

Let the vectors q_n, $n \in \mathbb{N}^d$, be constructed according to Lemma 1. It is not difficult to find a non-random family of vectors $\{p_n, n \in \mathbb{N}^d\}$, where p_n takes values in $\mathbb{N}^d$, such that

$$q_n^{(k)} \le p_n^{(k)} \le n_k,\ q_n^{(k)}/p_n^{(k)} \to 0 \ \text{ and } \ p_n^{(k)}/n_k \to 0 \ \text{ for } k = 1, \dots, d \ \text{ as } n \to \infty. \tag{13}$$

Now we apply the Bernstein partitioning method. For $n, j \in \mathbb{N}^d$ and introduced p_n and q_n consider the blocks

$$U_n^{(j)} = \{u \in \mathbb{N}^d : (j_k - 1)(p_n^{(k)} + q_n^{(k)}) < u_k \le j_k p_n^{(k)} + (j_k - 1)q_n^{(k)},\ k = 1, \dots, d\},$$

where $u = (u_1, \dots, u_d)$. Let $J_n = \{j \in \mathbb{N}^d : U_n^{(j)} \subset U_n\}$ and

$$W_n = \bigcup_{j \in J_n} U_n^{(j)}, \quad G_n = U_n \setminus W_n, \ \ n \in \mathbb{N}^d.$$

In other words W_n consists of "large blocks" (having the "size" $p_n^{(k)}$ along each of the $k-$th axis for $k = 1, \dots, d$), separated by "corridors" belonging to the set G_n. Put $v_n = \sqrt{\langle n \rangle \mathsf{K}_X(n)}$. Then, for each $t \in \mathbb{R}$ and $n \in \mathbb{N}^d$ we obtain

$$\left| \mathsf{E} \exp\left\{ \frac{it}{v_n} S_n \right\} - e^{-\frac{t^2}{2}} \right| \le \left| \mathsf{E} \exp\left\{ \frac{it}{v_n} S_n \right\} - \mathsf{E} \exp\left\{ \frac{it}{v_n} \sum_{j \in J_n} S(U_n^{(j)}) \right\} \right|$$

$$+ \left| \mathsf{E} \exp\left\{ \frac{it}{v_n} \sum_{j \in J_n} S(U_n^{(j)}) \right\} - \prod_{j \in J_n} \mathsf{E} \exp\left\{ \frac{it}{v_n} S(U_n^{(j)}) \right\} \right|$$

$$+ \left| \prod_{j \in J_n} \mathsf{E} \exp\left\{ \frac{it}{v_n} S(U_n^{(j)} \right\} - e^{-\frac{t^2}{2}} \right| =: \sum_{r=1}^{3} Q_r,$$

here $i^2 = -1$, $Q_r = Q_r(n, t)$ and $S_n = S(U_n)$ as previously. Taking into account that $|e^{ix} - e^{iy}| \le |x - y|$ for all $x, y \in \mathbb{R}$, and using the Lyapunov inequality we get

$$Q_1 \le \frac{|t|}{v_n} \mathsf{E}|S(G_n)| \le \frac{|t|}{v_n} (\mathsf{E} S(G_n)^2)^{1/2}.$$

A random field $X \in \mathsf{PA}$, therefore $\mathsf{cov}\,(X_j, X_u) \ge 0$ for any $j, u \in \mathbb{N}^d$. Thus in view of wide-sense stationarity of X we come to the relations

$$\mathsf{E} S(G_n)^2 \le \sum_{j \in G_n} \sum_{u : -n \le u - j \le n} \mathsf{cov}(X_j, X_u) \le \mathsf{card}\, G_n\, \mathsf{K}_X(n)$$

$$\le \mathsf{K}_X(n) \sum_{k=1}^{d} (m_n^{(k)} q_n^{(k)} + p_n^{(k)} + q_n^{(k)}) \prod_{1 \le l \le d, l \ne k} n_l$$

where $\mathsf{card}\, G$ stands for the cardinality of a set G, $m_n^{(k)} = [n_k/(p_n^{(k)} + q_n^{(k)})]$, $k = 1, \dots, d$. Due to (8) and (13) we get the inequality

$$\frac{\mathsf{E} S(G_n)^2}{\langle n \rangle\, \mathsf{K}_X(n)} \le \sum_{k=1}^{d} \frac{m_n^{(k)} q_n^{(k)} + p_n^{(k)} + q_n^{(k)}}{n_k} \to 0, \quad n \to \infty.$$

Consequently, $Q_1(n,t) \to 0$ for each $t \in \mathbb{R}$ as $n \to \infty$.

For any $n \in \mathbb{N}^d$ the family $\{S(U_n^{(j)}), j \in J_n\} \in \mathsf{PA}$ (see, e.g., Theorem 1.1.8 in [2]). Enumerate elements of this family to obtain the collection of random variables $\{Y_{n,s}, s = 1, \dots, M_n\}$ where $M_n = \mathsf{card}\, J_n$. It is easily seen that

$$\prod_{k=1}^{d} m_n^{(k)} \le M_n \le \prod_{k=1}^{d} (m_n^{(k)} + 1).$$

Recall that for complex-valued random variables Y and V (absolute square integrable) the covariance $\mathsf{cov}(Y,V) := \mathsf{E}(Y - \mathsf{E}Y)\overline{(V - \mathsf{E}V)}$, where the bar denotes the conjugation. Due to Theorem 1.5.3 of [2] one has

$$Q_2 \le \sum_{s=1}^{M_n - 1} \left| \mathsf{cov}\left(\exp\left\{ \frac{it}{v_n} Y_{n,s} \right\}, \exp\left\{ -\frac{it}{v_n} \sum_{l=s+1}^{M_n} Y_{n,l} \right\} \right) \right|$$

$$\le \frac{4t^2}{v_n^2} \sum_{1 \le s,l \le M_n, s \ne l} \mathsf{cov}(Y_{n,s}, Y_{n,l}) \le \frac{4t^2}{\langle n \rangle\, \mathsf{K}_X(n)} \sum_{j \in U_n} \sum_{u \in U_n, |u-j| > q_n} \mathsf{cov}(X_j, X_u)$$

where $|u| = \max_{k=1,\dots,d} |u_k|$. Obviously, for $j \in U_n$

$$\{u \in U_n, |u-j| > q_n\} \subset \{u \in \mathbb{N}^d : j - n \le u \le j + n\} \setminus \{u \in \mathbb{N}^d : |u-j| \le q_n\}.$$

Therefore, the inequality

$$\sum_{j \in U_n} \sum_{u \in U_n, |u-j| > q_n} \mathsf{cov}(X_j, X_u) \le \langle n \rangle (\mathsf{K}_X(n) - \mathsf{K}_X(q_n))$$

and (13) imply that $Q_2(n,t) \to 0$ for each $t \in \mathbb{R}$ as $n \to \infty$.

For any $n \in \mathbb{N}$ introduce a vector $(Z_{n,1}, \dots, Z_{n,M_n})^\top$ having the independent components and such that the law of $Z_{n,k}$ coincides with the law of $Y_{n,k}/v_n$, $k = 1, \dots, d$. Due to Lemma 2 for all $s = 1, \dots, M_n$

$$\mathsf{var} Z_{n,s} = \mathsf{var} Z_{n,1} \sim \langle p_n \rangle\, \mathsf{K}_X(p_n) / \langle n \rangle\, \mathsf{K}_X(n), \quad n \to \infty. \tag{14}$$

Thus

$$\sum_{s=1}^{M_n} \mathsf{var} Z_{n,s} = M_n \mathsf{var} Z_{n,1} \to 1, \quad n \to \infty, \tag{15}$$

since

$$M_n \langle p_n \rangle \sim \prod_{k=1}^{d} [n_k / (p_n^{(k)} + q_n^{(k)})] p_n^{(k)} \sim \langle n \rangle$$

and $\mathsf{K}_X(p_n)/\mathsf{K}_X(n) \to 1$ as $n \to \infty$. For arbitrary $\varepsilon > 0$, taking into account the stationarity of X, we have

$$\sum_{s=1}^{M_n} \mathsf{E} Z_{n,s}^2 \mathbb{I}\{|Z_{n,s}| > \varepsilon\} = \frac{M_n}{\langle n \rangle\, \mathsf{K}_X(n)} \mathsf{E} Y_{n,1}^2 \mathbb{I}\{Y_{n,1}^2 > \varepsilon^2 \langle n \rangle\, \mathsf{K}_X(n)\}$$

$$= \frac{M_n\langle p_n\rangle \mathsf{K}_X(p_n)}{\langle n\rangle \mathsf{K}_X(n)}\mathsf{E}\frac{S(U_n^{(1)})^2}{\langle p_n\rangle \mathsf{K}_X(p_n)}\mathbb{I}\left\{\frac{S(U_n^{(1)})^2}{\langle p_n\rangle \mathsf{K}_X(p_n)} > \varepsilon^2\frac{\langle n\rangle \mathsf{K}_X(n)}{\langle p_n\rangle \mathsf{K}_X(p_n)}\right\} \to 0, \quad n\to\infty,$$

in view of (14), (15) and because

$$\frac{\langle n\rangle \mathsf{K}_X(n)}{\langle p_n\rangle \mathsf{K}_X(p_n)} \to \infty \ \text{ as } \ n\to\infty.$$

We also used uniform integrability of $\{S(U_n^{(1)})^2/(\langle p_n\rangle \mathsf{K}_X(p_n)), n\in\mathbb{N}^d\}$. Indeed, this is a subfamily of the uniformly integrable family $\{S_n^2/(\langle n\rangle \mathsf{K}_X(n)), n\in\mathbb{N}^d\}$. The Lindeberg theorem (see, e.g., [9], p. 69) implies that

$$\sum_{s=1}^{M_n} Z_{n,s} \stackrel{law}{\to} Z\sim\mathcal{N}(0,1), \quad n\to\infty.$$

Therefore,

$$\prod_{s=1}^{M_n}\mathsf{E}\exp\{itZ_{n,s}\} - \exp\left\{-\frac{t^2}{2}\right\}\to 0, \quad n\to\infty.$$

It remains to note that

$$\prod_{j\in J_n}\mathsf{E}\exp\left\{\frac{it}{v_n}S(U_n^{(j)})\right\} = \prod_{s=1}^{M_n}\mathsf{E}\exp\{itZ_{n,s}\}.$$

Thus $Q_3(n,t)\to 0$ for each $t\in\mathbb{R}$ as $n\to\infty$. The proof is complete. □

Proof of Theorem 2. For a wide-sense stationary random field $X=\{X_j, j\in\mathbb{Z}^d\}$ introduce the function

$$\mathsf{R}_X(r) = \sum_{j\in\mathbb{Z}^d:|j|\le r}\mathsf{cov}(X_0,X_j), \quad r\in\mathbb{N}.$$

This function $\mathsf{R}_X(\cdot)$ is close in a sense to $\mathsf{K}(\cdot)$ defined in (5). They coincide if $d=1$. Clearly, for $d\ge 1$

$$\mathsf{K}(r)\le \mathsf{R}_X(r)\le \mathsf{K}(r\sqrt{d}), \quad r\in\mathbb{N}.$$

Consequently, if $\mathsf{K}\in\mathcal{L}(\mathbb{N})$, then $\mathsf{R}_X\in\mathcal{L}(\mathbb{N})$, and vise versa if $\mathsf{R}_X\in\mathcal{L}(\mathbb{N})$, then $\mathsf{K}\in\mathcal{L}(\mathbb{N})$. Now for a sequence $(C_r)_{r\in\mathbb{N}}$ it is not difficult to obtain the desired result following the scheme of the proof of Theorem 1 and using R_X instead of K_X. □

Remark 2 *Lemma* 2 *shows that in Theorems* 1 *and* 2 *instead of normalizations* $\sqrt{\mathsf{var}S_n}$ *and* $\sqrt{\mathsf{var}S(C_r)}$ *for partial sums one can use* $\sqrt{\langle n\rangle K_X(n)}$ *and* $r^{d/2}\sqrt{K(r)}$, *respectively.*

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Alexander BULINSKI,

Faculty of Mathematics and Mechanics, Lomonosov Moscow State University,
Moscow 119991, Russia
and
LPMA, UPMC University Paris-6,
4 Place Jussieu, 75252 Paris CEDEX 05, France

E-mail address: bulinski@mech.math.msu.su