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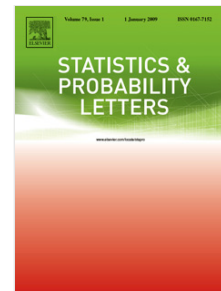
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A version of the Elfving problem with random starting time

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Abstract

An optimal stopping time problem with a Poisson process, discount function and random starting time is considered. Generalizations to a problem with random horizon and to a multi-person stopping game with priorities are presented.

Key words: stopping game, Poisson process, random starting time, random horizon

1 Introduction

An agent, who is waiting for some permit to run the business, is interested in some projects. He will obtain the permit at some random time M . Until M , he observes the market and learns which project should be accepted. The projects appear according to a Poisson process and have assigned values Y_1, Y_2, \dots , which are i.i.d. random variables. At each time only one project is presented. The agent is allowed to accept only one project at the time of its appearance and he is interested in accepting the most profitable one. The profit from the accepted project with value Y_n at time τ_n is $g(Y_n r(\tau_n))$ if $\tau_n \geq M$, and 0 otherwise, where $r(\cdot)$ and $g(\cdot)$ express the change of the value of the projects in time and the agent's subjective value of projects, respectively.

The results of the paper extend those obtained in the paper by Elfving (1967) and Ferenstein and Krasnosielska (2009a, 2009b). Elfving (1967) considered

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the problem of selling a commodity where offers Y_1, Y_2, \dots appear according to a Poisson process. The offers Y_i are i.i.d. random variables independent of the jump times τ_1, τ_2, \dots of the Poisson process. The owner obtains a reward $Y_n r(\tau_n)$ if he accepts the offer Y_n at time τ_n , where $r(\cdot)$ is a discount function. Siegmund (1967) removed several additional assumptions from Elfving solution. Ferenstein and Krasnosielska (2009a) generalized the Elfving problem to the case of a multi-person stopping game with priorities. A model with random horizon based on the Elfving problem was considered in Ferenstein and Krasnosielska (2009b). Stadje (1987) generalized the Elfving problem to a multi stopping time problem. Kühne and Rüschemdorff (2000) obtained the approximations for the stopping of i.i.d. sequences in the domain of max-stable laws. Cowan and Zabczyk (1978) considered a continuous-time version of the secretary problem in which the offers appear at jump times of the Poisson process up to fixed time T . Bruss (1987) extended the model of Cowan and Zabczyk (1978) to the case of a compound Poisson process. Szajowski (2007) generalized the problem considered by Bruss (1987) to the case of a two-person game with random priority. A full-information best choice problem with random starting point was considered by Porosiński and Szajowski (2000). The starting point (an observation which cannot be chosen) is a positive random variable independent of observations which are i.i.d. random variables. The observations appear according to a renewal process. A decision maker obtains some imperfect information about the random starting point. The aim of the decision maker is to choose the largest observation. Sakaguchi (1986) analyzed the full information secretary problem with Poisson stream of offers arriving in a time interval of random length. Samuel-Cahn (1996) considered an optimal stopping time problem with independent random horizon M and a known number of offers. The problem has been translated to a problem with discounting. Multi-person games with offers having uniform distribution and appearing according to a Poisson process were considered in Dixon (1993). A two-person game with random priority and two Poisson streams of offers was considered in Sakaguchi (1991). In this game player i has higher priority at the times when the observations are from stream i . General models of multi-person discrete-time games with priorities were analyzed in Enns and Ferenstein (1987). A multi-person stopping game with a known and finite number of observations was considered in Ramsey and Szajowski (2000). A game with a large number of offers and players was analyzed in Ramsey (2008).

In the paper the notations and the solution of the Elfving problem follow those of Chow et al. (1971, pp. 113-118).

2 Optimal stopping time problems with random starting time

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space on which all random variables are considered. Let Y_1, Y_2, \dots be independent, non-negative random variables with continuous distribution function F , $0 < \tau_1 < \tau_2 < \dots$ be the jump times of a homogeneous Poisson process with intensity 1, $Y_0 = 0, \tau_0 = 0$. Let M be a non-negative random variable with known distribution function F_M . We assume that the sequences $\{Y_n\}$ and $\{\tau_n\}$ are independent and they are independent of M . Let us introduce various σ -fields of events dependent on available observations: $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n, \tau_1, \dots, \tau_n)$, $\mathcal{F}_n^* = \sigma(\mathcal{F}_n, \mathbb{I}(M \leq \tau_0), \dots, \mathbb{I}(M \leq \tau_n))$, $n = 1, 2, \dots$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_0^* = \sigma(\mathcal{F}_0, \mathbb{I}(M \leq \tau_0))$, where $\mathbb{I}(A)$ is the indicator function of the event A . Let \mathcal{T} and \mathcal{T}^* be the sets of Markov times adapted to $\{\mathcal{F}_n\}_{n=0}^\infty$ and $\{\mathcal{F}_n^*\}_{n=0}^\infty$, respectively. Write $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. Our aim is to find $\sup_{t^* \in \mathcal{T}^*} EX_{t^*}^*$, where

$$X_n^* = g(Y_n r(\tau_n)) \mathbb{I}(M \leq \tau_n), \quad (1)$$

$X_\infty^* = \limsup_{n \rightarrow \infty} X_n^*$ and $r : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ and $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ are Borel functions. We assume that r is bounded, continuous from the right, g is increasing, continuous, $g(0) = 0$, and there exist functions g_1 and g_2 such that $|g(yr(x_1)) - g(yr(x_2))| \leq |g_1(y)g_2(x_1, x_2)|$ for $x_1, x_2 \geq 0$ and $y \in \text{supp } Y_1$, $E|g_1(Y_1)| < \infty$, $g_1(0) = 0$, $\int_0^\infty |g_2(x_1 + t, x_2 + t)| dt < c|x_1 - x_2|$ for some constant $c \geq 0$. Moreover, we assume that there exist constants m and U such that $U > m \geq 0$, $F_M(U) > 0$, $r(s) = 0$ for $s \in [0, m) \cup [U, \infty)$ and $r(s) > 0$ for $s \in [m, U)$. If $m > 0$, then $U < \infty$, and if $m = 0$, then $U \leq \infty$. Let g^{-1} denote the inverse function for g . Let S be the set of all points of discontinuity of the functions $r(\cdot)$ and $F_M(\cdot)$ and all points of non-differentiability of the function $F_M(\cdot)$. Assume that $S \cap (m, U) = \{s_1, \dots, s_k\}$, $k < \infty$, $s_1 < \dots < s_k$, $s_0 = m$, $s_{k+1} = U$ and $\int_0^\infty h(x) dx < \infty$, where $h(x) = E(g(Y_1 r(x)))$. Denote $\bar{F}(y) = 1 - F(y)$.

Note that the conditions concerning the functions r and g are satisfied if r is increasing at most on a finite number of intervals and either $g(x) = ax^p$, $p > 0$, $a > 0$, $E(Y_1^p) < \infty$, $\int_0^\infty (r(x))^p dx < \infty$ or $E(Y_1) < \infty$ and $g(x)$ is a Lipschitz function. For example $g(x) = \ln(x+1)$, $x \geq 0$, $r(x) = x$ for $x \in [0, 8)$, $r(x) = 1$ for $x \in [8, U)$, and 0 elsewhere.

Remark 2.1 From Equation (1), the fact that $g(\cdot)$ is nonnegative and the definition of $h(\cdot)$ we get $E(\sum_{n=1}^\infty X_n^*) \leq E(\sum_{n=1}^\infty g(Y_n r(\tau_n))) = \sum_{n=1}^\infty E(h(\tau_n)) < \infty$ and consequently $X_\infty^* = \lim_{n \rightarrow \infty} X_n^* = 0$ and $E(\sup(X_n^*)) < \infty$.

Note that m may be interpreted as the time up to which we plan to observe the market and learn, $\max\{M, m\}$ as the time up to which we will actually learn, r as the discount function if it is non-increasing and $r(0) = 1$, g as the utility function.

To solve the above problem we need to find a solution of a generalization of the Elfving problem without random starting time.

2.1 The generalization of the Elfving problem

Let the above assumptions be satisfied and let our goal be to find $\sup_{t \in \mathcal{T}} EX_t(u)$, where $X_n(u) = g(Y_n r(u + \tau_n))$, $u \geq 0$. We have $E(\sup_n X_n(u)) < \infty$ and $X_\infty(u) = 0$. Note that $\{Z_{n,u}\}_{n=0}^\infty$, $Z_{n,u} = (Y_n, u + \tau_n)$, is a homogeneous Markov chain with respect to $\{\mathcal{F}_n\}_{n=0}^\infty$, and $X_n(u)$ is the function of $Z_{n,u}$ only. Moreover, since Y_i are i.i.d. random variables, $E(\text{ess sup}_{t \in \mathcal{T}_{n+1}} E(X_t(u) | \mathcal{F}_{n+1}) | \mathcal{F}_n)$ is a function of $u + \tau_n$ only, say $V(u + \tau_n)$. Moreover, the Markov time $\sigma(u) = \inf\{n \geq 0 : X_n(u) \geq V(u + \tau_n)\}$ is optimal in \mathcal{T} , where $\inf \emptyset = +\infty$. Since $X_0(u) = 0$, it follows that $\sigma(u) = \inf\{n \geq 1 : X_n(u) \geq V(u + \tau_n)\}$ and $\sup_{t \in \mathcal{T}} E(X_t(u)) = E \max\{X_0(u), V(u + \tau_0)\} = V(u)$. From the inequality $\sup_{t \in \mathcal{T}} |E(X_t(u_1)) - E(X_t(u_2))| \geq |\sup_{t \in \mathcal{T}} E(X_t(u_1)) - \sup_{t \in \mathcal{T}} E(X_t(u_2))|$ and the properties of the functions g and r we deduce that $V(\cdot)$ is continuous (for more details see Appendix A). Set

$$y(s) = \frac{g^{-1}(V(s))}{r(s)} \quad \text{for } s \in [m, U] \quad (2)$$

and $y(s) = 0$ elsewhere. Define $f_u(v) = P(\tau_{\sigma(u)} > v)$. Note that $f_u(\cdot)$ is continuous. Moreover, the assumptions on $r(\cdot)$, $g(\cdot)$ and the form of $\sigma(u)$ ensure that $f_u(m) = 1$. Hence, $f_u(v) = \exp(-\int_{u+m}^{u+v} \bar{F}(y(v')) dv')$ for $m \leq v < U - u$ and $0 \leq u \leq U - m$ (see Chow et al. (1971, pp. 114-115)). Note that $f_u(v)$ is differentiable with respect to v in each of the intervals $(s_i - u, s_{i+1} - u)$, $i = j + 1, \dots, k$, where $u \in (s_j - m, s_{j+1} - m)$, $j = 0, \dots, k$.

The optimal mean reward $V(u)$ may be expressed as follows:

$$V(u) = E(X_{\sigma(u)}(u)) = E(E(g(Y_{\sigma(u)} r(u + \tau_{\sigma(u)})) | \tau_{\sigma(u)}, \sigma(u))).$$

Note that for $\tau_{\sigma(u)} \in [m, U - u)$ and $u \in [0, U - m)$ the conditional distribution of $g(Y_{\sigma(u)} r(u + \tau_{\sigma(u)}))$ given $\tau_{\sigma(u)}, \sigma(u)$ is the same as the conditional distribution of $g(Y_{\sigma(u)} r(u + \tau_{\sigma(u)}))$ given $\{Y \geq y(u + \tau_{\sigma(u)})\}$, where $Y, \tau_{\sigma(u)}$ are independent and Y has the distribution function F . Hence, using the formula for $f_u(v)$ we obtain for $u \in (s_j - m, s_{j+1} - m)$, $j = 0, \dots, k$,

$$V(u) = \int_{u+m}^U \int_{y(v)}^\infty g(xr(v)) dF(x) \exp\left(-\int_{u+m}^v \bar{F}(y(v')) dv'\right) dv. \quad (3)$$

Equation (3) is differentiable with respect to u in each of the intervals $(s_j - m, s_{j+1} - m)$, $j = 0, \dots, k$, hence for $u \in (s_j - m, s_{j+1} - m)$ we have

$$\frac{d}{du}V(u) = \bar{F}(y(m+u))V(u) - \int_{y(m+u)}^{\infty} g(xr(m+u))dF(x). \quad (4)$$

Remark 2.2 Equation (4) can be solved by recursion. Note that $V(u) = 0$ for $u \in [U - m, \infty)$. Hence, using the boundary condition $V(U - m) = 0$ and the fact that $V(u)$ is continuous, we obtain $V(u)$ for $u \in (U - 2m, U - m)$. In the same way, we can compute $V(u)$ for $u \in (\max\{U - nm, 0\}, U - nm + m)$, $n = 3, 4, \dots$. Hence, from continuity of $V(u)$, we obtain $V(u)$ for $u \in [0, \infty)$.

The proof of the theorem below is based on Chow et al. (1971, pp. 115-118).

Theorem 2.1 (i) A continuous function $\tilde{V}(\cdot)$ satisfies (3) if and only if $\tilde{V}(\cdot)$ satisfies (4) and $\tilde{V}(u) \rightarrow 0$ as $u \rightarrow U - m$.
(ii) If a continuous function $\tilde{V}(\cdot)$ satisfies (3) on $[0, U - m)$ and $\tilde{V}(\cdot) = 0$ elsewhere, then $\tilde{V}(\cdot) = V(\cdot)$ and $\sigma(u) = \inf\{n \geq 1 : X_n(u) \geq \tilde{V}(u + \tau_n)\}$.

Proof. (i) The fact that $\tilde{V}(\cdot)$ satisfying (3) has to satisfy (4) has been shown above. Moreover, note that the right-hand side of (3) is non-negative, $\exp(-\int_{u+m}^v \bar{F}(\tilde{y}(v'))dv') \leq 1$ and $0 \leq \int_{\tilde{y}(v)}^{\infty} g(xr(v))dF(x) \leq E(g(Y_1 r(v))) = h(v)$, where $\tilde{y}(s) = g^{-1}(\tilde{V}(s))/r(s)$ for $s \in [m, U)$ and $\tilde{y}(s) = 0$ elsewhere. Hence $\tilde{V}(u) \rightarrow 0$ as $u \rightarrow U - m$. Now, suppose that a continuous function $\tilde{V}(\cdot)$ satisfies (4) and $\tilde{V}(u) \rightarrow 0$ as $u \rightarrow U - m$. For $u \in [0, U - m)$, define $V_1(\cdot)$ by

$$V_1(u) = \int_{u+m}^U \int_{\tilde{y}(v)}^{\infty} g(xr(v))dF(x) \exp\left(-\int_{u+m}^v \bar{F}(\tilde{y}(v'))dv'\right)dv \quad (5)$$

and $V_1(u) = 0$ for $u \geq U - m$. Note that $V_1(u)$ is continuous for $u \geq 0$ and $V_1(u) = \tilde{V}(u) = 0$ for $u \geq U - m$. We wish to show that $V_1(u) = \tilde{V}(u)$ for $u \in [0, U - m]$. Note that $V_1(u) \rightarrow 0$ as $u \rightarrow U - m$. Moreover, differentiating (5) in each of the intervals $(s_j - m, s_{j+1} - m)$, $j = 0, \dots, k$, we obtain

$$\frac{d}{du}V_1(u) = \bar{F}(\tilde{y}(m+u))V_1(u) - \int_{\tilde{y}(m+u)}^{\infty} g(xr(m+u))dF(x). \quad (6)$$

Assume that $V_1(u_0) \neq \tilde{V}(u_0)$ at some point $u_0 \in (s_k - m, U - m)$. Let u_1 be the first point after u_0 such that $u_1 \in (s_k - m, U - m)$ and $V_1(u_1) = \tilde{V}(u_1)$. If such a u_1 does not exist we set $u_1 = U - m$. From (6) and the assumption that $\tilde{V}(\cdot)$ satisfies (4) we obtain $d(V_1(u) - \tilde{V}(u))/du = \bar{F}(\tilde{y}(m+u))(V_1(u) - \tilde{V}(u))$. Hence,

for $u \in (u_0, u_1)$ we have $\ln |V_1(u) - \tilde{V}(u)| = \ln |V_1(u_0) - \tilde{V}(u_0)| + \int_{u_0}^u \bar{F}(\tilde{y}(m + u')) du'$. For $u \rightarrow u_1$ the left-hand side of the above equation goes to $-\infty$ while the right-hand side is finite. This contradiction shows that $V_1(u) = \tilde{V}(u)$ for $u \in (s_k - m, U - m)$. From the continuity of the functions $V_1(u)$ and $\tilde{V}(u)$ we obtain $V_1(u) = \tilde{V}(u)$ for $u \in [s_k - m, U - m]$. Now by recursion we conclude that $V_1(u) = \tilde{V}(u)$ for $u \geq 0$.

(ii) Let $u \geq 0$ be fixed. Suppose that there exists a continuous function $\tilde{V}(u+z)$ such that for $u+z \in (s_j - m, s_{j+1} - m)$, $z \geq 0$, $j = 0, \dots, k$,

$$\tilde{V}(u+z) = \int_{u+z+m}^U \int_{\tilde{y}(v)}^{\infty} g(xr(v)) dF(x) \exp\left(-\int_{u+z+m}^v \bar{F}(\tilde{y}(v')) dv'\right) dv, \quad (7)$$

where $\tilde{y}(s) = g^{-1}(\tilde{V}(s))/r(s)$ for $s \in [m, U)$ and $\tilde{y}(s) = 0$ elsewhere, and $\tilde{V}(u+z) = 0$ for $u+z \geq U$. Define $\tilde{\sigma}_1(u+z) = \inf\{n \geq 1 : Y_n \geq \tilde{y}(u+z+\tau_n)\}$. Considerations similar to those above show that $E(X_{\tilde{\sigma}_1(u+z)}(u+z))$ is equal to the right-hand side of (7). Let $\mathcal{H}_v = \sigma(N(v'), v' \leq v, Y_1, \dots, Y_{N(v)})$, where $N(v)$, $v \geq 0$, is a homogeneous Poisson process with intensity 1. Then $\mathcal{H}_{\tau_n} = \mathcal{F}_n$ for $n \geq 1$, where $\mathcal{H}_{\tau_n} = \{A \in \mathcal{F} : A \cap \{\tau_n \leq v\} \in \mathcal{H}_v \text{ for all } v \geq 0\}$. From the properties of the exponential distribution we see that the conditional joint distribution of $(Y_{N(v)+1}, \tau_{N(v)+1}), (Y_{N(v)+2}, \tau_{N(v)+2}), \dots$ given \mathcal{H}_v is the same as the unconditional joint distribution of $(Y_1, v + \tau_1), (Y_2, v + \tau_2), \dots$. Hence,

$$\begin{aligned} P(X_{\tilde{\sigma}(u+z)}(u+z) \leq t, \tilde{\sigma}(u+z) > N(v) \mid \mathcal{H}_v) \\ = \mathbb{I}(\tilde{\sigma}(u+z) > N(v)) P(X_{\tilde{\sigma}(u+z+v)}(u+z+v) \leq t). \end{aligned}$$

Therefore, for all $u+z \in [0, U-m)$, $v \in [0, U-u-z)$, $z \geq 0$, we have $\mathbb{I}(\tilde{\sigma}(u+z) > N(v)) \tilde{V}(u+z+v) = \mathbb{I}(\tilde{\sigma}(u+z) > N(v)) E(X_{\tilde{\sigma}(u+z)}(u+z) \mid \mathcal{H}_v)$. Hence,

$$\begin{aligned} \tilde{V}(u+z) &= E(X_{\tilde{\sigma}(u+z)}(u+z)) \\ &= E(E(X_1(u+z) \mathbb{I}(X_1(u+z) \geq \tilde{V}(u+z+\tau_1)) \\ &\quad + X_{\tilde{\sigma}(u+z)}(u+z) \mathbb{I}(X_1(u+z) < \tilde{V}(u+z+\tau_1)) \mid \mathcal{F}_1)) \\ &= E(\max\{X_1(u+z), \tilde{V}(u+z+\tau_1)\}). \end{aligned}$$

Define $\tilde{\Gamma}(y, u+z) = \max\{g(yr(u+z)), \tilde{V}(u+z)\}$ for $y \in [0, \infty)$, $u+z \in [0, U-m)$, $z \geq 0$. Then $E(\tilde{\Gamma}(Y_1, u+z+\tau_1)) = \tilde{V}(u+z)$. Hence, using the facts that the sequences $\{Y_n\}$ and $\{\tau_n\}$ are independent and Y_i are i.i.d. random variables, we see that $E(\tilde{\Gamma}(Y_{n+1}, u+\tau_{n+1}) \mid \mathcal{F}_n) = \tilde{V}(u+\tau_n)$. Defining

$\tilde{\gamma}_n(u) = \tilde{\Gamma}(Y_n, u + \tau_n)$ for $n \geq 1$, we get $E(\tilde{\gamma}_{n+1}(u) | \mathcal{F}_n) = \tilde{V}(u + \tau_n)$. Hence,

$$\tilde{\gamma}_n(u) = \max\{X_n(u), E(\tilde{\gamma}_{n+1}(u) | \mathcal{F}_n)\}. \quad (8)$$

Let $\mathcal{T}_n = \{t \in \mathcal{T} : t \geq n\}$, $\gamma_n(u) = \text{ess sup}_{t \in \mathcal{T}_n} E(X_t(u) | \mathcal{F}_n)$ for $n \geq 1$, $\mathcal{T}_n^N = \{t \in \mathcal{T}_n : t \leq N\}$, $\gamma_n^N(u) = \text{ess sup}_{t \in \mathcal{T}_n^N} E(X_t(u) | \mathcal{F}_n)$ for $n = 1, \dots, N$, $\gamma'_n(u) = \lim_{N \rightarrow \infty} \gamma_n^N(u)$ for $n \geq 1$. From (8), we have $\tilde{\gamma}_n(u) \geq X_n(u)$ and $\tilde{\gamma}_\infty(u) = 0$, because $\lim_{n \rightarrow \infty} \tilde{V}(u + \tau_n) = 0$ and $X_\infty(u) = 0$. Define $W(u) = \sum_{n=1}^{\infty} X_n(u)$. Then from Remark 2.1 we have $E(W(u)) < \infty$. Since $\tilde{\gamma}_n(u) \leq X_n(u) + E(\sum_{k=n+1}^{\infty} X_k(u) | \mathcal{F}_n)$ for $n \geq 1$, we have $\tilde{\gamma}_n(u) \leq E(W(u) | \mathcal{F}_n)$. Therefore, the assumptions of Lemma 4.9 from Chow et al. (1971, p. 79)) are satisfied and $\tilde{\gamma}_n(u) \leq \gamma_n(u)$. Moreover, from Theorem 4.6 of Chow et al. (1971, p. 76), we conclude that $\{\gamma'_n(u), \mathcal{F}_n\}_{n=1}^{\infty}$ is the minimal supermartingale dominating $\{X_n(u), \mathcal{F}_n\}_{n=1}^{\infty}$ and from Theorem 4.4 of Chow et al. (1971, p. 69)), we have $\gamma'_n(u) = \gamma_n(u)$ for $n \geq 1$. Hence, $\tilde{\gamma}_n(u) = \gamma_n(u)$ and $\tilde{V}(u + \tau_n) = V(u + \tau_n)$. Moreover, $V(u) = \max\{X_0(u), E(\gamma_1(u) | \mathcal{F}_0)\} = E(\gamma_1(u)) = E(\tilde{\gamma}_1(u)) = \tilde{V}(u)$ for $u \geq 0$. ■

2.2 Solution of the optimal stopping time problem with random starting time

Note that if $\mathbb{I}(M \leq \tau_n) = 1$, then $\mathbb{I}(M \leq \tau_{n+1}) = 1$. Moreover, if $\mathbb{I}(M \leq \tau_n) = 0$, then $\mathbb{I}(M \leq \tau_k) = 0$ for $k = n - 1, \dots, 0$. Hence, $\{Z_n^*, \mathcal{F}_n^*\}_{n=0}^{\infty}$, where $Z_n^* = (Y_n, \tau_n, \mathbb{I}(M \leq \tau_n))$, is a homogeneous Markov chain and X_n^* is a function of Z_n^* only. Therefore, using the fact that the Y_i are i.i.d. random variables we deduce that $E(\text{ess sup}_{t^* \in \mathcal{T}_{n+1}^*} E(X_{t^*}^* | \mathcal{F}_{n+1}^*) | \mathcal{F}_n^*)$ is a function of τ_n and $\mathbb{I}(M \leq \tau_n)$ only, say $V^*(\tau_n, \mathbb{I}(M \leq \tau_n))$. Moreover, since $X_0^* = 0$, the Markov time $\sigma^* = \inf\{n \geq 1 : X_n^* \geq V^*(\tau_n, \mathbb{I}(M \leq \tau_n))\}$ is optimal in \mathcal{T}^* , where $\inf \emptyset = +\infty$. Note that

$$V^*(\tau_i, \mathbb{I}(M \leq \tau_i)) > 0 \quad \text{for } \tau_i < U. \quad (9)$$

Hence, $\{\mathbb{I}(M > v) = 1\} \cap \{X_i \geq V^*(\tau_i, \mathbb{I}(M \leq \tau_i))\} \cap \{\tau_i \leq v\} = \emptyset$ for $v \in [0, U)$. Therefore, $P(M > v, \tau_{\sigma^*} \leq v) = 0$ for $v \in [0, U)$. Hence, $P(M > v) = P(M > v, \tau_{\sigma^*} > v)$ for $v \in [0, U)$ and as a consequence

$$P(\tau_{\sigma^*} > v) = P(M > v) + P(\tau_{\sigma^*} > v, M \leq v).$$

Moreover, note that $P(\tau_{\sigma^*} > v)$ is continuous for $v \geq 0$. Define

$$f^*(v) = P(\tau_{\sigma^*} > v, M \leq v) \quad \text{for } 0 \leq v < U.$$

Note that $f^*(v)$ is continuous in each of the intervals (s_i, s_{i+1}) , $i = 0, \dots, k$. Let $v \in (s_i, s_{i+1})$ and $h > 0$ be such that $(v - h, v + h) \subseteq (s_i, s_{i+1})$. Note that

$$f^*(v + h) = P(\tau_{\sigma^*} > v + h, M \leq v) + P(\tau_{\sigma^*} > v + h, M \in (v, v + h]).$$

It is easy to see that

$$\begin{aligned} \exp(-h)(F_M(v + h) - F_M(v)) &\leq P(\tau_{\sigma^*} > v + h, M \in (v, v + h]) \\ &\leq F_M(v + h) - F_M(v). \end{aligned}$$

To compute $P(\tau_{\sigma^*} > v + h, M \leq v)$, note that

$$\mathbb{I}(M \leq \tau_k) V^*(\tau_k, \mathbb{I}(M \leq \tau_k)) = \mathbb{I}(M \leq \tau_k) V(\tau_k), \quad k = 1, 2, \dots \quad (10)$$

To prove Equation (10), we first use the approximation by finite horizon problem (finite number of events in stream (τ_n)) and Theorem 3.2 of Chow et al. (1971, pp. 50 and 68) and the properties of $\mathbb{I}(M \leq \tau_k)$. Next, we use Theorem 4.3 of Chow et al. (1971, pp. 50 and 68) (for more details see Appendix B). Now, note that from (9) for $\tau_n < U$ we have $X_n^* \geq V^*(\tau_n, \mathbb{I}(M \leq \tau_n))$ iff $\mathbb{I}(M \leq \tau_n) = 1$ and $g(Y_n r(\tau_n)) \geq V^*(\tau_n, \mathbb{I}(M \leq \tau_n))$. Moreover, $X_n^* < V^*(\tau_n, \mathbb{I}(M \leq \tau_n))$ iff $\mathbb{I}(M \leq \tau_n) = 0$ or $\mathbb{I}(M \leq \tau_n) = 1$ and $g(Y_n r(\tau_n)) < V^*(\tau_n, \mathbb{I}(M \leq \tau_n))$. Note that if $N(v + h) - N(v) = 1$, then $\tau_{N(v)+1} \in (v, v + h]$. Hence, using (10) and (2) we get

$$\begin{aligned} \exp(-h)f^*(v) + h \exp(-h)F\left(\inf_{x \in (v, v+h]} y(x)\right)f^*(v) + o(h) \\ \leq P(\tau_{\sigma^*} > v + h, M \leq v) \\ \leq \exp(-h)f^*(v) + h \exp(-h)F\left(\sup_{x \in (v, v+h]} y(x)\right)f^*(v) + o(h). \end{aligned}$$

Using the fact that $\exp(-h) = 1 - h + o(h)$ and $f^*(v) \in [0, 1]$ we obtain

$$\lim_{h \rightarrow 0^+} \frac{f^*(v + h) - f^*(v)}{h} = \lim_{h \rightarrow 0^+} \frac{F_M(v + h) - F_M(v)}{h} - f^*(v)\bar{F}(y(v)).$$

Moreover, taking $f^*(v) = f^*(v - h + h)$ we obtain $\lim_{h \rightarrow 0^+} \frac{f^*(v-h) - f^*(v)}{-h} = \lim_{h \rightarrow 0^+} \frac{F_M(v-h) - F_M(v)}{-h} - f^*(v)\bar{F}(y(v))$. Since $F_M(v)$ is differentiable in each of the intervals (s_i, s_{i+1}) , $i = 0, \dots, k$, we have $\lim_{h \rightarrow 0} \frac{F_M(v+h) - F_M(v)}{h} = F'_M(v)$. Therefore,

$$\frac{df^*(v)}{dv} = F'_M(v) - f^*(v)\bar{F}(y(v)). \quad (11)$$

Moreover, for each interval (s_i, s_{i+1}) , $i = 0, \dots, k$, there exists exactly one solution of (11) with boundary condition $f^*(s_i) = b_i$, $i = 0, \dots, k$, respectively. Hence, writing $E_t^v = \exp(-\int_t^v \bar{F}(y(x))dx)$ we have $f^*(v) = \int_{s_i}^v F'_M(t)E_t^v dt + b_i E_{s_i}^v$. Note that $b_0 = f^*(m) = P(M \leq m)$. The remaining conditions $f^*(s_i) = b_i$, $i = 1, \dots, k$, are obtained recursively from the continuity of the function $P(\tau_{\sigma^*} > v)$. Let $\bar{F}_M(t) = 1 - F_M(t)$. Define $w = \max\{n \geq 0 : s_n \leq v\}$. Hence, for $v \in (m, U)$

$$P(\tau_{\sigma^*} > v) = \bar{F}_M(v) + \int_m^v F'_M(t)E_t^v dt + F_M(m)E_m^v + \sum_{j=1}^w P(M = s_j)E_{s_j}^v.$$

Note that the above function is differentiable in each of the intervals (s_i, s_{i+1}) , $i = 0, \dots, k$. Since $P(M \leq \tau_{\sigma^*}) = 1$, we have

$$\sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*) = E(X_{\sigma^*}^*) = E(E(g(Y_{\sigma^*} r(\tau_{\sigma^*})) \mid \tau_{\sigma^*}, \sigma^*)).$$

Moreover, from (9) and (10), we find that for $\tau_{\sigma^*} \in [m, U)$ the conditional distribution of $g(Y_{\sigma^*} r(\tau_{\sigma^*}))$ given $\tau_{\sigma^*}, \sigma^*$ is the same as the conditional distribution of $g(Y r(\tau_{\sigma^*}))$ given $\{Y \geq y(\tau_{\sigma^*})\}$, where Y, τ_{σ^*} are independent, Y has the distribution function F and $y(u)$ is defined by (2). Hence

$$\begin{aligned} \sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*) &= \int_m^U \int_{y(v)}^{\infty} g(xr(v))dF(x) \\ &\cdot \left[\int_m^v F'_M(t)E_t^v dt + F_M(m)E_m^v + \sum_{j=1}^w P(M = s_j)E_{s_j}^v \right] dv. \end{aligned} \quad (12)$$

Remark 2.3 *In case $r(m) = 0$, the function $y(s)$ is defined by (2) on (m, U) instead of $[m, U)$, but this does not affect the solutions of the problems under consideration. Moreover, we can consider the case $m > 0$ and U infinite or the problems in which the set $S \cap (m, U)$ is countable and does not have accumulation points in (m, U) but in this case an effective solution also does not exist.*

Proposition 2.1 *Suppose that the assumptions of Section 2 are satisfied. Then $\sup_{t \in \mathcal{T}} E(X_t(0)F_M(\tau_t)) \leq \sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*) \leq \sup_{t \in \mathcal{T}} E(X_t(0))$.*

Proof. Since $\mathcal{T} \subset \mathcal{T}^*$ and $X_n^* \leq X_n(0)$, it follows that $\sup_{t \in \mathcal{T}} E(X_t^*) \leq \sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*) \leq \sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*(0))$. Moreover, X_n^* is not measurable with respect to \mathcal{F}_n , hence for $t \in \mathcal{T}$ we have $E(X_t^*) = E(X_t(0)F_M(\tau_t))$. Since $X_n(0)$ is a function of $Z_{n,0}$ only, $\sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*(0)) = \sup_{t \in \mathcal{T}} E(X_t(0))$. ■

Let us note that if $P(M \leq m) = 1$, then in Proposition 2.1 we have equalities instead of inequalities. In Example 1, we will show that in general the inequalities may be strict.

2.3 Stopping time problem with random starting time and random horizon

Suppose that a random variable T , which can be interpreted as a random horizon, has known distribution function F_T and is independent of M , Y 's and τ 's. Denote $\bar{F}_T(s) = 1 - F_T(s)$. Let $\hat{\mathcal{T}}$ be the set of Markov times adapted to $\{\hat{\mathcal{F}}_n\}_{n=0}^\infty$, where $\hat{\mathcal{F}}_n = \sigma(\mathcal{F}_n^*, \mathbb{I}(T > \tau_0), \dots, \mathbb{I}(T > \tau_n))$, $\hat{\mathcal{T}}_n = \{\hat{t} \in \hat{\mathcal{T}} : \hat{t} \geq n\}$, $n = 0, 1, \dots$. Note that $E(\sup X_n^* \mathbb{I}(T > \tau_n)) < \infty$. The theorem below is a generalization of the theorems presented by Samuel-Cahn (1996) and Ferencstein and Krasnosielska (2009b). The idea of the transformation of an optimal stopping time problem with random horizon to an optimal stopping time problem with a modified structure of rewards is based on the paper of Samuel-Cahn (1996).

Theorem 2.2 *Let Y_1, Y_2, \dots be i.i.d. nonnegative random variables independent of the jump times τ_1, τ_2, \dots of a Poisson process, $Y_0 = 0$, $\tau_0 = 0$. Let M and T be independent positive random variables independent of the sequences of Y 's and τ 's, with distribution functions F_M and F_T , respectively. Let $g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ and $r : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be Borel functions, $X_n^* = g(Y_n r(\tau_n)) \mathbb{I}(M \leq \tau_n)$, $E(\sup_n X_n^*) < \infty$ and $X_\infty^* = \limsup_{n \rightarrow \infty} X_n^*$. Then, for any $k \in \mathbb{N} \cup \{0\}$, $\sup_{\hat{t} \in \hat{\mathcal{T}}_k} E(X_{\hat{t}}^* \mathbb{I}(T > \tau_{\hat{t}})) = \sup_{t^* \in \mathcal{T}_k^*} E(X_{t^*}^* \bar{F}_T(\tau_{t^*}))$.*

Proof. Set $\xi_n = \mathbb{I}(M \leq \tau_n)$, $\zeta_n = \mathbb{I}(T > \tau_n)$, $\hat{Z}_n = (Y_n, \tau_n, \xi_n, \zeta_n)$, $n = 0, 1, \dots$. Note that if $\zeta_n = 1$, then $\zeta_k = 1$ for $k = n - 1, \dots, 0$. Moreover, if $\zeta_n = 0$, then $\zeta_{n+1} = 0$. Hence, we see that $\{\hat{Z}_n, \hat{\mathcal{F}}_n\}_{n=0}^\infty$ is a homogeneous Markov chain with the state space $E = \text{supp } Y_1 \times \mathbb{R}_0^+ \times \{0, 1\} \times \{0, 1\}$. Therefore, $E(\text{ess sup}_{\hat{t} \in \hat{\mathcal{T}}_{n+1}} E(X_{\hat{t}}^* \zeta_{\hat{t}} \mid \hat{\mathcal{F}}_{n+1}) \mid \hat{\mathcal{F}}_n)$ is a function of τ_n , ξ_n and ζ_n , say $\hat{V}^*(\tau_n, \xi_n, \zeta_n)$. Hence, the Markov time $\hat{\sigma}_k^* = \inf\{n \geq k : X_n^* \zeta_n \geq \hat{V}^*(\tau_n, \xi_n, \zeta_n)\}$ is optimal in $\hat{\mathcal{T}}_k$. Define $\varphi(y, s, w, v) = g(yr(s))wv$ and $B = \{(y, s, w, v) \in E : \varphi(y, s, w, v) \geq \hat{V}^*(s, w, v)\}$. Hence $\hat{\sigma}_k^* = \inf\{n \geq k : (Y_n, \tau_n, \xi_n, \zeta_n) \in B\}$. Define: $\bar{B} = E \setminus B$, $C = \{(y, s, w) : (y, s, w, 1) \in B\}$, $\bar{C} = \{(y, s, w) : (y, s, w, 1) \in \bar{B}\}$ and $R = \{(y, s, w, 1) \in E\}$. Note that $B \cap R = C \times \{1\}$ and $\bar{B} \cap R = \bar{C} \times \{1\}$. Using the fact that $\mathbb{I}(\zeta_n = 1) = \mathbb{I}(\zeta_n = 1) \cdot \dots \cdot \mathbb{I}(\zeta_0 = 1)$, we obtain $X_{\hat{\sigma}_k^*}^* = X_{\tilde{\sigma}_k^*}^*$, where $\tilde{\sigma}_k^* = \inf\{n \geq k : (Y_n, \tau_n, \xi_n) \in C\}$ and $\tilde{\sigma}_k^* \in \mathcal{T}_k^*$. Since $\mathbb{I}(T > \tau_n)$ is not measurable with respect to \mathcal{F}_n^* , we have $E(X_{t^*}^* \mathbb{I}(T > \tau_{t^*})) = E(X_{t^*}^* \bar{F}_T(\tau_{t^*}))$ for $t^* \in \mathcal{T}^*$. ■

From the above theorem, we see that to find $\sup_{\hat{t} \in \hat{\mathcal{T}}} E(X_{\hat{t}}^* \mathbb{I}(T > \tau_{\hat{t}}))$ it is enough to find $\sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^* \bar{F}_T(\tau_{t^*}))$. Without loss of generality we can assume that $\inf\{s : F_T(s) = 1\} \geq U$ and all points of discontinuity of $F_T(\cdot)$

belong to S . Moreover, we assume that there exist functions \hat{g}_1 and \hat{g}_2 such that $|g(yr(x_1))\bar{F}_T(x_1) - g(yr(x_2))\bar{F}_T(x_2)| \leq |\hat{g}_1(y)\hat{g}_2(x_1, x_2)|$ for $x_1, x_2 \geq 0$ and $y \in \text{supp } Y_1$, $E|\hat{g}_1(Y_1)| < \infty$, $\hat{g}_1(0) = 0$ and $\int_0^\infty |\hat{g}_2(x_1 + t, x_2 + t)| dt < \hat{c}|x_1 - x_2|$ for some constant $\hat{c} \geq 0$. Note that the above conditions are satisfied if $E(Y_1) < \infty$ and $g(x) = x$ and $r(x)$ is non-increasing and continuous for $x \geq 0$. Denote $\hat{E}_t^v = \exp(-\int_t^v \bar{F}(\hat{y}(x))dx)$, where $\hat{y}(s) = g^{-1}(\hat{V}(s)/\bar{F}_T(s))/r(s)$ for $s \in (m, U)$ and $\hat{y}(s) = 0$ elsewhere, and $\hat{V}(u)$ is continuous on $[0, \infty)$ and for $u \in (s_j - m, s_{j+1} - m)$, $j = 0, \dots, k$, satisfies

$$\frac{d}{du}\hat{V}(u) = \bar{F}(\hat{y}(u))\hat{V}(u) - \bar{F}_T(m+u) \int_{\hat{y}(u+m)}^\infty g(xr(m+u))dF(x)$$

with $\lim_{u \rightarrow U} \hat{V}(u) = 0$. Note that $\hat{V}(u) = \sup_{t \in \mathcal{T}} E(X_t(u)\bar{F}_T(u))$. Moreover,

$$\begin{aligned} \sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^* \bar{F}_T(\tau_{t^*})) &= \int_m^U \bar{F}_T(v) \int_{\hat{y}(v)}^\infty g(xr(v))dF(x) \\ &\cdot \left[\int_m^v F'_M(t) \hat{E}_t^v dt + F_M(m) \hat{E}_m^v + \sum_{j=1}^w P(M = s_j) \hat{E}_{s_j}^v \right] dv. \end{aligned}$$

3 Game with known starting time and known horizon

In this section we generalize the problem considered in Subsection 2.1 to a multi-person game considered in Ferenstein and Krasnosielska (2009a). In this case m can be interpreted as the time up to which the players have to wait for a permit to run their businesses. In this section we need additionally the following assumptions $U < \infty$, $r(x+y) \leq r(x)r(y)$ for $x, y \geq m$, $g(xy) \leq g(x)g(y)$. These assumptions are needed to show that relevant stopping times are finite (see Stadje (1987)). Note that these assumptions are satisfied if $g(s) = s^p$, $p > 0$, and $r(s) = \exp(-s)$. Moreover, assume that there are $j > 1$ ordered players who sequentially observe offers Y_n at times τ_n , $n = 1, 2, \dots$. Players' indices correspond to their ordering, called priorities, so that 1 refers to the player with the highest priority and j to the player with the lowest one. Each player who has just decided to make a selection at τ_n gets the reward $X_n(0)$ if and only if he has not obtained any reward before and there is no player with higher priority who has also decided to take the current reward. As soon as the player gets the reward, he quits the game. The remaining players select rewards in the same manner, their priorities remain as previously.

Let $V_{j,l}(u)$ be the optimal mean reward for the game with reward sequence $\{g(Y_n r(u + \tau_n))\}$, $u \geq 0$, for player l in the j -person game, where l is the

player's priority, $l = 1, \dots, j$, $j = 1, 2, \dots$. Note that $V_{1,1}(u) = V(u)$ for $u \geq 0$. As in Ferencstein and Krasnosielska (2009a) we find $V_{j,l}(u)$. Denote $y_l(s) = g^{-1}(V_{l,l}(s))/r(s)$ for $s \in [m, U)$ and 0 elsewhere, $l = 1, \dots, j$.

Theorem 3.1 (i) $V_{j,l}(\cdot)$ is continuous on $[0, \infty)$ for $l = 1, \dots, j$.
(ii) $V_{j,l}(u) = V_{l,l}(u)$, for $l = 1, \dots, j-1$, $u \geq 0$.
(iii) For $u \in (s_i - m, s_{i+1} - m)$, $i = 0, \dots, k$, $j > 1$, we have

$$\begin{aligned} \frac{d}{du} V_{j,j}(u) &= \bar{F}(y_j(u+m))V_{j,j}(u) - V_{j-1,j-1}(u+m)\bar{F}(y_{j-1}(u+m)) \\ &\quad - \int_{y_j(u+m)}^{y_{j-1}(u+m)} g(xr(u+m))dF(x). \end{aligned}$$

The above differential equation can be solved numerically with the condition $\hat{V}_{j,j}(u) \rightarrow 0$ as $u \rightarrow U-m$ in accordance with the rule presented in Remark 2.2. Taking $u = 0$ we obtain the optimal mean rewards for all players in the game with rewards of the form $X_n(0)$.

4 Examples

Let $g(s) = s$ for $s \geq 0$ and $\{Y_i\}$ be a sequence of i.i.d. random variables with exponential distribution with mean 1. Then $\bar{F}(y) = \exp(-y)$.

Example 4.1 Let $r(u) = 1$ for $u \in [0, 1/2)$, $r(u) = 2/3$ for $u \in [1/2, 2)$ and $r(u) = 0$ elsewhere, $P(M = 0) = 1/2$, $P(M = 1) = 1/2$. Then $m = 0$ and $U = 2$ and $\lim_{u \rightarrow 2} y(u) = \lim_{u \rightarrow 2} (V(u)/r(u)) = 0$ and Equation (12) is equivalent to $dy(u)/du = -\exp(-y(u))$ in each of the intervals $(0, 1/2)$, $(1/2, 2)$. Hence, for $u \in (1/2, 2)$ and $u \in (0, 1/2)$ we obtain $y(u) = \ln(1+c_i-u)$, $c_i \in \mathbb{R}$, $i = 1, 2$, respectively. Using condition $y(2) = 0$ we get $c_1 = 2$ and $y(u) = \ln(3-u)$ for $u \in (1/2, 2)$. Since $V(u)$ is continuous, $V(1/2) = \ln((5/2)^{2/3})$. Therefore, $c_2 = (5/2)^{2/3} - 1/2$ and $y(u) = \ln((5/2)^{2/3} + 1/2 - u)$ for $u \in (0, 1/2)$. Hence, $V(0) = \ln((5/2)^{2/3} + 1/2)$. From (12) we get $\sup_{t^* \in \mathcal{T}^*} E(X_{t^*}^*) = \int_0^2 = \int_0^{(1/2)^-} + \int_{(1/2)^+}^1 + \int_1^2 \approx 0.1853 + 0.0948 + 0.3764 = 0.6565$. Now, taking $\tilde{r}(u) = r(u)F_M(u)$ and using (4), we find that $\sup_{t \in \mathcal{T}} E(X_t(0)F_M(\tau_t)) \approx 0.5856$. Hence, in Proposition 2.1 the inequalities cannot be replaced by equalities.

Example 4.2 Let $P(M = 0) = 1$, $P(T = 3) = 1$ and $r(s) = 1$ for $s \in [1, 3)$ and $r(s) = 0$ elsewhere. Hence, $m = 1$, $U = 3$ and (4) becomes

$$\frac{dV(u)}{du} = \exp(-r(u+1)V(u+1))V(u)$$

$$-r^2(u+1)V(u+1)\exp(-r(u+1)V(u+1)) - r(u+1)\exp(-V(u+1)).$$

Since $V(\cdot)$ is continuous and $V(u) = 0$ for $u \in [3, \infty)$ and $r(u+1) = 0$ for $u \in (2, \infty)$, we have $V(u) = 0$ for $u \in [2, \infty)$. Hence $dV(u)/du = V(u) - 1$ for $u \in (1, 2)$. Using the condition $V(2) = 0$, we obtain $|V(u) - 1| = \exp(u - 2)$. Since $V(u)$ is continuous, we have $V(u) = 1 - \exp(u - 2)$ for $u \in [1, 2)$. Therefore, $dV(u)/du = \exp(-(1 - \exp(u - 1)))(V(u) + (\exp(u - 1) - 2))$ for $u \in (0, 1)$. Solving this equation numerically with the boundary condition $V(1) = 1 - \exp(-1)$ we obtain $V(0) \approx 1,0057$.

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Appendix A. Continuity of function $V(\cdot)$

Note that $|V(u)| < \infty$ for all $u \geq 0$. Now, we will show that for all $u_1 \geq 0$ and $u_2 \geq 0$ we have:

$$\sup_{t \in \mathcal{T}} |E(X_t(u_1)) - E(X_t(u_2))| \geq \left| \sup_{t \in \mathcal{T}} E(X_t(u_1)) - \sup_{t \in \mathcal{T}} E(X_t(u_2)) \right|. \quad (\text{A1})$$

Note that $\sigma(u_1)$ and $\sigma(u_2)$ are the optimal stopping times for sequences $X_n(u_1)$ and $X_n(u_2)$, respectively, that is $\sup_{t \in \mathcal{T}} E(X_t(u_i)) = E(X_{\sigma(u_i)}(u_i))$ for $i \in \{1, 2\}$. Moreover, without loss of generality, we can assume that

$$\sup_{t \in \mathcal{T}} E(X_t(u_1)) \geq \sup_{t \in \mathcal{T}} E(X_t(u_2)). \quad (\text{A2})$$

Therefore,

$$\begin{aligned} \sup_{t \in \mathcal{T}} |E(X_t(u_1)) - E(X_t(u_2))| &\geq |E(X_{\sigma(u_1)}(u_1)) - E(X_{\sigma(u_1)}(u_2))| \\ &= |E(X_{\sigma(u_1)}(u_1)) - \sup_{t \in \mathcal{T}} E(X_t(u_2)) + \sup_{t \in \mathcal{T}} E(X_t(u_2)) - E(X_{\sigma(u_1)}(u_2))| \\ &\geq |E(X_{\sigma(u_1)}(u_1)) - \sup_{t \in \mathcal{T}} E(X_t(u_2))| = \left| \sup_{t \in \mathcal{T}} E(X_t(u_1)) - \sup_{t \in \mathcal{T}} E(X_t(u_2)) \right|, \end{aligned}$$

where the last inequality we obtain from Equation (A2) and definition of supremum. Hence, we conclude that Equation (A1) is satisfied for all nonnegative u_1 and u_2 . Now, we will prove that $V(u)$ is continuous for $u \geq 0$. Let $u_1 \geq 0$ and $u_2 \geq 0$, then

$$\begin{aligned}
|V(u_1) - V(u_2)| &= \left| \sup_{t \in \mathcal{T}} E(X_t(u_1)) - \sup_{t \in \mathcal{T}} E(X_t(u_2)) \right| \\
&\leq \sup_{t \in \mathcal{T}} |E(X_t(u_1)) - E(X_t(u_2))| \leq \sup_{t \in \mathcal{T}} E|g(Y_{tr}(u_1 + \tau_t)) - g(Y_{tr}(u_2 + \tau_t))| \\
&\leq \sup_{t \in \mathcal{T}} E|g_1(Y_t)g_2(u_1 + \tau_t, u_2 + \tau_t)| \leq \sum_{n=0}^{\infty} E|g_1(Y_n)g_2(u_1 + \tau_n, u_2 + \tau_n)| \\
&= E(g_1(Y_1)) \sum_{n=1}^{\infty} E|g_2(u_1 + \tau_n, u_2 + \tau_n)| \\
&= E(g_1(Y_1)) \int_0^{\infty} |g_2(u_1 + x, u_2 + x)| dx \leq E(g_1(Y_1))c|u_1 - u_2|.
\end{aligned}$$

Hence, $V(\cdot)$ satisfies the Lipschitz condition, and consequently it is continuous. ■

Appendix B. Derivation of Equation (10)

Let $X_n = X_n(0)$, $\gamma_n = \gamma_n(0) = \text{ess sup}_{t \in \mathcal{T}_n} E(X_t | \mathcal{F}_n)$, $\gamma_n^* = \text{ess sup}_{t^* \in \mathcal{T}_n^*} E(X_{t^*}^* | \mathcal{F}_n^*)$ for $n = 1, 2, \dots$, $\mathcal{T}_n^N = \{\min\{t, N\} : t \in \mathcal{T}_n\}$, $\mathcal{T}_n^{*,N} = \{\min\{t^*, N\} : t^* \in \mathcal{T}_n^*\}$, $\gamma_n^N = \gamma_n^N(0) = \text{ess sup}_{t \in \mathcal{T}_n^N} E(X_t | \mathcal{F}_n)$, $\gamma_n^{*,N} = \text{ess sup}_{t^* \in \mathcal{T}_n^{*,N}} E(X_{t^*}^* | \mathcal{F}_n^*)$ for $n = 1, 2, \dots, N$, $N \in \mathbb{N}$. Let $k < N$. Using the property of $\mathbb{I}(M \leq \tau_k)$ and Theorem 3.2 from Chow, Robbins and Siegmund, 1971, p. 50, we have

$$\gamma_N^N \mathbb{I}(M \leq \tau_k) = X_N \mathbb{I}(M \leq \tau_k) = X_N^* \mathbb{I}(M \leq \tau_k) = \gamma_N^{*,N} \mathbb{I}(M \leq \tau_k).$$

Hence, if $\mathbb{I}(M \leq \tau_k) = 1$, then $\gamma_N^N = \gamma_N^{*,N}$. Moreover, if $\mathbb{I}(M \leq \tau_k) = 0$, then $\mathbb{I}(M \leq \tau_k)E(\gamma_N^N | \mathcal{F}_{N-1}^*) = \mathbb{I}(M \leq \tau_k)E(\gamma_N^{*,N} | \mathcal{F}_{N-1}^*)$. Therefore, for $n = N - 1$

$$\begin{aligned}
\gamma_{N-1}^N \mathbb{I}(M \leq \tau_k) &= \max\{X_{N-1}, E(\gamma_N^N | \mathcal{F}_{N-1})\} \mathbb{I}(M \leq \tau_k) \\
&= \max\{X_{N-1} \mathbb{I}(M \leq \tau_k), \mathbb{I}(M \leq \tau_k) E(\gamma_N^N | \mathcal{F}_{N-1}^*)\} \\
&= \max\{X_{N-1}^* \mathbb{I}(M \leq \tau_k), \mathbb{I}(M \leq \tau_k) E(\gamma_N^{*,N} | \mathcal{F}_{N-1}^*)\} \\
&= \mathbb{I}(M \leq \tau_k) \max\{X_{N-1}^*, E(\gamma_N^{*,N} | \mathcal{F}_{N-1}^*)\} = \mathbb{I}(M \leq \tau_k) \gamma_{N-1}^{*,N}.
\end{aligned}$$

Hence, by recursion we obtain

$$\gamma_n^N \mathbb{I}(M \leq \tau_k) = \mathbb{I}(M \leq \tau_k) \gamma_n^{*,N} \quad \text{for } n = N, \dots, k.$$

Note that $X_n \geq 0$ and $X_n^* \geq 0$ for $n = 1, 2, \dots$, hence $\gamma_k^N \geq 0$ and $\gamma_k^{*,N} \geq 0$ for $k = 1, 2, \dots, N$ and $N \in \mathbb{N}$. Consequently, $\lim_{N \rightarrow \infty} \gamma_k^N \geq 0$ and $\lim_{N \rightarrow \infty} \gamma_k^{*,N} \geq 0$. Hence, $(\lim_{N \rightarrow \infty} \gamma_k^N)^- = 0$ and $(\lim_{N \rightarrow \infty} \gamma_k^{*,N})^- = 0$, where by definition $x^- = -\min\{x, 0\}$. Therefore, $\liminf_{k \rightarrow \infty} \int_{t > k} (\lim_{N \rightarrow \infty} \gamma_k^N)^- d\mathcal{P} = 0$ and $\liminf_{k \rightarrow \infty} \int_{t^* > k} (\lim_{N \rightarrow \infty} \gamma_k^{*,N})^- d\mathcal{P} = 0$ for each $t \in \mathcal{T}_n$ and $t^* \in \mathcal{T}_n^*$, $n = 1, 2, \dots$. Therefore, the assumption of Theorem 4.3 from Chow, Robbins and Siegmund, 1971, p. 68 is satisfied and consequently $\gamma_n = \lim_{N \rightarrow \infty} \gamma_n^N$ and $\gamma_n^* = \lim_{N \rightarrow \infty} \gamma_n^{*,N}$ for $n = 1, 2, \dots$. Hence, for $n \geq k$ we have

$$\mathbb{I}(M \leq \tau_k) \gamma_n = \mathbb{I}(M \leq \tau_k) \lim_{N \rightarrow \infty} \gamma_n^N = \mathbb{I}(M \leq \tau_k) \lim_{N \rightarrow \infty} \gamma_n^{*,N} = \mathbb{I}(M \leq \tau_k) \gamma_n^*.$$

And hence, using the fact that γ_n is \mathcal{F}_n -measurable and independent of M , we have for $n \geq k$

$$\begin{aligned} \mathbb{I}(M \leq \tau_k) V(\tau_n) &= \mathbb{I}(M \leq \tau_k) E(\gamma_{n+1} | \mathcal{F}_n) = \mathbb{I}(M \leq \tau_k) E(\gamma_{n+1} | \mathcal{F}_n^*) \\ &= E(\gamma_{n+1} \mathbb{I}(M \leq \tau_k) | \mathcal{F}_n^*) = E(\gamma_{n+1}^* \mathbb{I}(M \leq \tau_k) | \mathcal{F}_n^*) = \mathbb{I}(M \leq \tau_k) E(\gamma_{n+1}^* | \mathcal{F}_n^*) \\ &= \mathbb{I}(M \leq \tau_k) V^*(\tau_n, \mathbb{I}(M \leq \tau_n)). \end{aligned}$$

To complete the proof it is enough to take $n = k$. ■

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