



A remark on Besov spaces interpolation over the 2-adic group

Diego Chamorro

► To cite this version:

| Diego Chamorro. A remark on Besov spaces interpolation over the 2-adic group. 2011. hal-00587059

HAL Id: hal-00587059

<https://hal.science/hal-00587059>

Preprint submitted on 19 Apr 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A remark on Besov spaces interpolation over the 2-adic group

Diego Chamorro

April 19, 2011

Abstract

Motivated by a recent result which identifies in the special setting of the 2-adic group the Besov space $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$ with $BV(\mathbb{Z}_2)$, the space of function of bounded variation, we study in this article some functional relationships between Besov spaces.

Keywords: Besov spaces, interpolation, p -adic groups.

MSC 2010: 22E35, 46E35

1 Introduction

The starting point of this article is given by the following inequality proved by A. Cohen, W. Dahmen, I. Daubechies & R. De Vore in [4]. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f \in BV \cap \dot{B}_\infty^{-1,\infty}$ we have

$$\|f\|_{L^2}^2 \leq C \|f\|_{BV} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (1)$$

Here BV denotes the space of functions of bounded variation and $\dot{B}_\infty^{-1,\infty}$ stands for an homogeneous Besov space. In the article [3], we proved that in the special setting of the 2-adic group \mathbb{Z}_2 , the space $BV(\mathbb{Z}_2)$ can be identified to the Besov space $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$ and therefore, inequality (1) becomes

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,\infty}} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (2)$$

Note that the previous estimate is *false* in \mathbb{Z}_2 , see [3] for a counterexample. The identification between these two functional spaces and the consequences on the inequality (1) are very surprising in the sense that these estimates *depend* on the underlying group structure: compare the topological properties of \mathbb{R}^n to the totally discontinuous setting of \mathbb{Z}_2 .

However, one may think that the Besov norm $\|\cdot\|_{\dot{B}_1^{1,\infty}}$ in the right hand side of (2) is *too* small to achieve the inequality. Thus, it is a natural question to study the validity of (2) if we replace this norm by a *bigger* one (just think on the inclusion of Besov spaces $\dot{B}_1^{1,q} \subset \dot{B}_1^{1,\infty}$ valid for $q \geq 1$). The answer to this question is given by the next result

Theorem 1 *If $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ is a function such that $f \in \dot{B}_1^{1,q} \cap \dot{B}_\infty^{-1,\infty}(\mathbb{Z}_2)$ with $q > 2$, then the following inequality is false:*

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,q}} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (3)$$

This is the main theorem of this article and we will construct a counterexample in the section 4 below, but before, it would be interesting to compare inequality (3) to the general estimates given by the interpolation theory¹.

¹see the book [2] for more details.

Indeed, following this general theory, we can obtain inequalities of the form

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_{p_0}^{s_0, q_0}} \|f\|_{\dot{B}_{p_1}^{s_1, q_1}} \quad (4)$$

for some special values of the real parameters $s_0, s_1, p_0, p_1, q_0, q_1$.

Perhaps the most popular case is given by the real method: set $p_0 = p_1 = p$, fix $0 < \theta < 1$ and suppose $s_0 \neq s_1$ with the relationship $s = (1 - \theta)s_0 + \theta s_1$. We obtain the following expression

$$(\dot{B}_p^{s_0, q_0}, \dot{B}_p^{s_1, q_1})_{\theta, q} = \dot{B}_p^{s, q}$$

which gives us the estimate

$$\|f\|_{\dot{B}_p^{s, q}} \leq C \|f\|_{\dot{B}_p^{s_0, q_0}}^{1-\theta} \|f\|_{\dot{B}_p^{s_1, q_1}}^{\theta} \quad (5)$$

It is very important to remark that in this particular case no relationship between q_0, q_1 and q is asked. Obviously, inequality (3) can not be obtained from (5), since $p_0 \neq p_1$.

The case when $p_0 \neq p_1$ is more restrictive and following the complex method we have for $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p_1, q_1 < +\infty$ the formula

$$[\dot{B}_{p_0}^{s_0, q_0}, \dot{B}_{p_1}^{s_1, q_1}]_{\theta} = \dot{B}_p^{s, q}$$

which gives us an estimate of the type (4) with $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Note that we have in this case a relationship between q_0, q_1 and q . Again, this method can not be applied to inequality (3).

It seems of course that inequality (3) cannot be obtained by an simple interpolation argument -actually this inequality is false in \mathbb{R}^n -, but what it would make it plausible in the setting of \mathbb{Z}_2 is the special relationship between inequalities (1) and (2) and this is the main reason why theorem 1 is relevant.

The plan of the article is the following. In section 2 we recall some properties of the p -adic spaces, in section 3 we give the definition of Besov spaces over the 2-adic group \mathbb{Z}_2 and in section 4 we prove theorem 1.

2 p -adic groups

Our main reference here are the books [10], [8] and [1] where more details concerning the topological structure of the p -adic groups can be found.

We write $a|b$ when a divide b or, equivalently, when b is a multiple of a . Let p be any prime number, for $0 \neq x \in \mathbb{Z}$, we define the p -adic valuation of x by $\gamma(x) = \max\{r : p^r | x\} \geq 0$ and, for any rational number $x = \frac{a}{b} \in \mathbb{Q}$, we write $\gamma(x) = \gamma(a) - \gamma(b)$. Furthermore if $x = 0$, we agree to write $\gamma(0) = +\infty$.

Let $x \in \mathbb{Q}$ and p be any prime number, with the p -adic valuation of x we can construct a norm by writing

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0 \\ p^{-\infty} = 0 & \text{if } x = 0. \end{cases} \quad (6)$$

This expression satisfy the following properties

- a) $|x|_p \geq 0$, and $|x|_p = 0 \iff x = 0$;
- b) $|xy|_p = |x|_p |y|_p$;
- c) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, with equality when $|x|_p \neq |y|_p$.

When a norm satisfy c) it is called a non-Archimedean norm and an interesting fact is that over \mathbb{Q} all the possible norms are equivalent to $|\cdot|_p$ for some p : this is the so-called Ostrowski theorem, see [1] for a proof.

Definition 2.1 *Let p be a any prime number. We define the field of p -adic numbers \mathbb{Q}_p as the completion of \mathbb{Q} when using the norm $|\cdot|_p$.*

We present in the following lines the algebraic structure of the set \mathbb{Q}_p . Every p -adic number $x \neq 0$ can be represented in a unique manner by the formula

$$x = p^\gamma(x_0 + x_1p + x_2p^2 + \dots), \quad (7)$$

where $\gamma = \gamma(x)$ is the p -adic valuation of x and x_j are integers such that $x_0 > 0$ and $0 \leq x_j \leq p-1$ for $j = 1, 2, \dots$. Remark that this canonical representation implies the identity $|x|_p = p^{-\gamma}$.

Let $x, y \in \mathbb{Q}_p$, using the formula (7) we define the sum of x and y by $x + y = p^{\gamma(x+y)}(c_0 + c_1p + c_2p^2 + \dots)$ with $0 \leq c_j \leq p-1$ and $c_0 > 0$, where $\gamma(x+y)$ and c_j are the unique solution of the equation

$$p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots) + p^{\gamma(y)}(y_0 + y_1p + y_2p^2 + \dots) = p^{\gamma(x+y)}(c_0 + c_1p + c_2p^2 + \dots).$$

Furthermore, for $a, x \in \mathbb{Q}_p$, the equation $a + x = 0$ has a unique solution in \mathbb{Q}_p given by $x = -a$. In the same way, the equation $ax = 1$ has a unique solution in \mathbb{Q}_p : $x = 1/a$.

We take now a closer look at the topological structure of \mathbb{Q}_p . With the norm $|\cdot|_p$ we construct a distance over \mathbb{Q}_p by writing

$$d(x, y) = |x - y|_p \quad (8)$$

and we define the balls $B_\gamma(x) = \{y \in \mathbb{Q}_p : d(x, y) \leq p^\gamma\}$ with $\gamma \in \mathbb{Z}$. Remark that, from the properties of the p -adic valuation, this distance has the *ultra-metric* property (i.e. $d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq |x|_p + |y|_p$).

We gather with the next proposition some important facts concerning the balls in \mathbb{Q}_p .

Proposition 2.1 *Let γ be an integer, then we have*

- 1) *the ball $B_\gamma(x)$ is a open and a closed set for the distance (8).*
- 2) *every point of $B_\gamma(x)$ is its center.*
- 3) *\mathbb{Q}_p endowed with this distance is a complete Hausdorff metric space.*
- 4) *\mathbb{Q}_p is a locally compact set.*
- 5) *the p -adic group \mathbb{Q}_p is a totally discontinuous space.*

3 Functional spaces

In this article, we will work with the subset \mathbb{Z}_2 of \mathbb{Q}_2 which is defined by $\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 : |x|_2 \leq 1\}$, and we will focus on real-valued functions over \mathbb{Z}_2 . Since \mathbb{Z}_2 is a locally compact commutative group, there exists a Haar measure dx which is translation invariant i.e.: $d(x+a) = dx$, furthermore we have the identity $d(xa) = |a|_2 dx$ for $a \in \mathbb{Z}_2^*$. We will normalize the measure dx by setting

$$\int_{\{|x|_2 \leq 1\}} dx = 1.$$

This measure is then unique and we will note $|E|$ the measure for any subset E of \mathbb{Z}_2 .

Lebesgue spaces $L^p(\mathbb{Z}_2)$ are thus defined in a natural way: $\|f\|_{L^p} = \left(\int_{\mathbb{Z}_2} |f(x)|^p dx \right)^{1/p}$ for $1 \leq p < +\infty$, with the usual modifications when $p = +\infty$.

Let us now introduce the Littlewood-Paley decomposition in \mathbb{Z}_2 . We note \mathcal{F}_j the Boole algebra formed by the equivalence classes $E \subset \mathbb{Z}_2$ modulo the sub-group $2^j \mathbb{Z}_2$. Then, for any function $f \in L^1(\mathbb{Z}_2)$, we call $S_j(f)$ the conditionnal expectation of f with respect to \mathcal{F}_j :

$$S_j(f)(x) = \frac{1}{|B_j(x)|} \int_{B_j(x)} f(y) dy.$$

The dyadic blocks are thus defined by the formula $\Delta_j(f) = S_{j+1}(f) - S_j(f)$ and the Littlewood-Paley decomposition of a function $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ is given by

$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \quad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x) dx. \quad (9)$$

We will need in the sequel some very special sets noted $Q_{j,k}$. Here is the definition and some properties:

Proposition 3.1 *Let $j \in \mathbb{N}$ and $k = \{0, 1, \dots, 2^j - 1\}$. Define the subset $Q_{j,k}$ of \mathbb{Z}_2 by*

$$Q_{j,k} = \{k + 2^j \mathbb{Z}_2\}. \quad (10)$$

Then

- 1) *We have the identity $\mathcal{F}_j = \bigcup_{0 \leq k < 2^j} Q_{j,k}$,*
- 2) *For $k = \{0, 1, \dots, 2^j - 1\}$ the sets $Q_{j,k}$ are mutually disjoint,*
- 3) *$|Q_{j,k}| = 2^{-j}$ for all k ,*
- 4) *the 2-adic valuation is constant over $Q_{j,k}$.*

The verifications are easy and left to the reader.

With the Littlewood-Paley decomposition given in (9), we obtain the following equivalence for the Lebesgue spaces $L^p(\mathbb{Z}_2)$ with $1 < p < +\infty$:

$$\|f\|_{L^p} \simeq \|S_0(f)\|_{L^p} + \left\| \left(\sum_{j \in \mathbb{N}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p}.$$

See the book [9], chapter IV, for a general proof.

For Besov spaces we will define them by the norm

$$\|f\|_{B_p^{s,q}} \simeq \|S_0 f\|_{L^p} + \left(\sum_{j \in \mathbb{N}} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q} \quad (11)$$

where $s \in \mathbb{R}$, $1 \leq p, q < +\infty$ with the necessary modifications when $p, q = +\infty$.

Remark 1 For homogeneous functional spaces $\dot{B}_p^{s,q}$, we drop out the term $\|S_0 f\|_{L^p}$ in (11).

4 Proof of the theorem 1

To begin the construction of the counterexample we consider $0 < j_0 < j_1$ two integers and we fix $\alpha, \beta \in \mathbb{R}$ such that

$$2^{2j_0} \leq \frac{\beta}{\alpha}. \quad (12)$$

Take now a decreasing sequence $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N})$ with $q > 2$ such that $\varepsilon_0 = 1$ and $(\varepsilon_j)_{j \in \mathbb{N}} \notin \ell^2(\mathbb{N})$.

Define N_j in the following form

$$N_j = \begin{cases} 2^j & \text{if } 0 < j < j_0, \\ 2^{-j} \frac{\beta}{\alpha} & \text{if } j_0 \leq j \leq j_1. \end{cases} \quad (13)$$

We construct a function $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$ by considering his values over the dyadic blocs and we will use for this the sets $Q_{j,k}$ defined in (10):

$$\Delta_j f(x) = \begin{cases} \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,0}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,1}, \\ \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,3}, \\ \vdots & \\ \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2N_j-2}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2N_j-1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark that, with this definition of $\Delta_j f(x)$ we have the identities

- $\|\Delta_j f\|_{L^\infty} = \varepsilon_j \alpha 2^j$,
- $\|\Delta_j f\|_{L^1} = \varepsilon_j \alpha N_j$,
- $\|\Delta_j f\|_{L^2}^2 = \varepsilon_j^2 \alpha^2 2^j N_j$.

From this quantities we construct the following norms

- (a) for the Besov space $\dot{B}_\infty^{-1,\infty}$ we have

$$\|f\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{j \in \mathbb{N}} 2^{-j} \|\Delta_j f\|_{L^\infty} = \alpha, \text{ since the sequence } (\varepsilon_j)_{j \in \mathbb{N}} \text{ is decreasing and } \varepsilon_0 = 1.$$

- (b) for the Besov space $\dot{B}_1^{1,q}$ we write

$$\|f\|_{\dot{B}_1^{1,q}}^q = \sum_{j=0}^{j_1} (2^j \|\Delta_j f\|_{L^1})^q = \sum_{j=0}^{j_1} 2^{jq} \varepsilon_j^q \alpha^q N_j^q = \alpha^q \left(\sum_{j=0}^{j_0} 2^{jq} \varepsilon_j^q N_j^q + \sum_{j>j_0}^{j_1} 2^{jq} \varepsilon_j^q N_j^q \right)$$

We use now the values of N_j given in (13) and the relationship (12) to obtain

$$= \alpha^q \left(\sum_{j=0}^{j_0} 2^{2jq} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \frac{\beta^q}{\alpha^q} \right) = \beta^q \left(\sum_{j=0}^{j_0} 2^{2jq} \frac{\alpha^q}{\beta^q} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right) \simeq \beta^q \left(\sum_{j=0}^{j_0} 2^{q(2j-2j_0)} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right).$$

$$\text{Then we have } \|f\|_{\dot{B}_1^{1,q}} \simeq \beta \left(C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right)^{1/q}.$$

(c) For the Lebesgue space L^2 we use the same arguments above to obtain

$$\|f\|_{L^2}^2 = \sum_{j=0}^{j_1} \varepsilon_j^2 \alpha^2 2^j N_j = \alpha^2 \left(\sum_{j=0}^{j_0} 2^{2j} \varepsilon_j^2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \frac{\beta}{\alpha} \right) \simeq \alpha \beta \left(C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \right).$$

Once these norms are computed, we go back to the inequality

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,q}} \|f\|_{\dot{B}_\infty^{-1,\infty}}$$

and we have

$$\alpha \beta \left(C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \right) \leq C \times \alpha \times \beta \left(C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right)^{1/q}.$$

But, by hypothesis, we have $(\varepsilon_j)_{j \in \mathbb{N}} \notin \ell^2(\mathbb{N})$ and $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N})$, thus, for j_1 big enough it is impossible to find an universal constant C such that the above inequality is true. ■

References

- [1] Yvette AMICE. *Les nombres p-adiques*. PUF, 1975.
- [2] J. BERGH & J. LÖFSTRÖM. *Interpolation Spaces*. Grundlehren der mathematischen Wissenschaften, 223. Springer Verlag (1976).
- [3] D. CHAMORRO. *A counterexample for Improved Sobolev Inequalities over the 2-adic group*, HAL : hal-00531555, (2011).
- [4] A. COHEN, W. DAHMEN, I. DAUBECHIES & R. DE VORE. *Harmonic Analysis of the space BV*. Rev. Mat. Iberoamericana 19, n°1, 235-263 (2003).
- [5] P. GÉRARD, Y. MEYER & F. ORU. *Inégalités de Sobolev Précisées*. Equations aux Dérivées Partielles, Séminaire de l'Ecole Polytechnique, exposé n° IV (1996-1997).
- [6] K. IKEDA, T. KIM, TK. SHIRATANI *On p-adic bounded functions*. Mem. Fac. Sci. Kyushu Univ. Ser. A 46, no 2, 341-349 (1992).
- [7] L.C. JANG, T. KIM, J.-W. SON, S.-H. RIM *On p-adic bounded functions II*. J.Math. Anal. Appl. 264, no 1, 21-31 (2001).
- [8] N. KOBLITZ, *p-adic Numbers, p-adic Analysis and Zeta-functions*. GTM 58. Springer Verlag, 1977.
- [9] E. M. STEIN. *Topics in Harmonic analysis*. Annals of mathematics studies, 63. Princeton University Press (1970).
- [10] V. S. VLADIMIROV, I. V. VOLOVICH and E. I. ZELENOV. *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore (1994).

Diego CHAMORRO

Laboratoire d'Analyse et de Probabilités
Université d'Evry Val d'Essonne & ENSIIE

1 square de la résistance,
91025 Evry Cedex

diego.chamorro@m4x.org