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# A remark on Besov spaces interpolation over the 2-adic group

Diego Chamorro

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## Abstract

Motivated by a recent result which identifies in the special setting of the 2-adic group the Besov space  $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$  with  $BV(\mathbb{Z}_2)$ , the space of function of bounded variation, we study in this article some functional relationships between Besov spaces.

**Keywords:** Besov spaces, interpolation,  $p$ -adic groups.

**MSC 2010:** 22E35, 46E35

## 1 Introduction

The starting point of this article is given by the following inequality proved by A. Cohen, W. Dahmen, I. Daubechies & R. De Vore in [4]. For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f \in BV \cap \dot{B}_\infty^{-1,\infty}$  we have

$$\|f\|_{L^2}^2 \leq C \|f\|_{BV} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (1)$$

Here  $BV$  denotes the space of functions of bounded variation and  $\dot{B}_\infty^{-1,\infty}$  stands for an homogeneous Besov space. In the article [3], we proved that in the special setting of the 2-adic group  $\mathbb{Z}_2$ , the space  $BV(\mathbb{Z}_2)$  can be identified to the Besov space  $\dot{B}_1^{1,\infty}(\mathbb{Z}_2)$  and therefore, inequality (1) becomes

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,\infty}} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (2)$$

Note that the previous estimate is *false* in  $\mathbb{Z}_2$ , see [3] for a counterexample. The identification between these two functional spaces and the consequences on the inequality (1) are very surprising in the sense that these estimates *depend* on the underlying group structure: compare the topological properties of  $\mathbb{R}^n$  to the totally discontinuous setting of  $\mathbb{Z}_2$ .

However, one may think that the Besov norm  $\|\cdot\|_{\dot{B}_1^{1,\infty}}$  in the right hand side of (2) is *too* small to achieve the inequality. Thus, it is a natural question to study the validity of (2) if we replace this norm by a *bigger* one (just think on the inclusion of Besov spaces  $\dot{B}_1^{1,q} \subset \dot{B}_1^{1,\infty}$  valid for  $q \geq 1$ ). The answer to this question is given by the next result

**Theorem 1** *If  $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$  is a function such that  $f \in \dot{B}_1^{1,q} \cap \dot{B}_\infty^{-1,\infty}(\mathbb{Z}_2)$  with  $q > 2$ , then the following inequality is false:*

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,q}} \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad (3)$$

This is the main theorem of this article and we will construct a counterexample in the section 4 below, but before, it would be interesting to compare inequality (3) to the general estimates given by the interpolation theory<sup>1</sup>.

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<sup>1</sup>see the book [2] for more details.

Indeed, following this general theory, we can obtain inequalities of the form

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_{p_0}^{s_0, q_0}} \|f\|_{\dot{B}_{p_1}^{s_1, q_1}} \quad (4)$$

for some special values of the real parameters  $s_0, s_1, p_0, p_1, q_0, q_1$ .

Perhaps the most popular case is given by the real method: set  $p_0 = p_1 = p$ , fix  $0 < \theta < 1$  and suppose  $s_0 \neq s_1$  with the relationship  $s = (1 - \theta)s_0 + \theta s_1$ . We obtain the following expression

$$(\dot{B}_p^{s_0, q_0}, \dot{B}_p^{s_1, q_1})_{\theta, q} = \dot{B}_p^{s, q}$$

which gives us the estimate

$$\|f\|_{\dot{B}_p^{s, q}} \leq C \|f\|_{\dot{B}_p^{s_0, q_0}}^{1-\theta} \|f\|_{\dot{B}_p^{s_1, q_1}}^{\theta} \quad (5)$$

It is very important to remark that in this particular case no relationship between  $q_0, q_1$  and  $q$  is asked. Obviously, inequality (3) can not be obtained from (5), since  $p_0 \neq p_1$ .

The case when  $p_0 \neq p_1$  is more restrictive and following the complex method we have for  $1 \leq p_0, q_0 \leq +\infty$  and  $1 \leq p_1, q_1 < +\infty$  the formula

$$[\dot{B}_{p_0}^{s_0, q_0}, \dot{B}_{p_1}^{s_1, q_1}]_{\theta} = \dot{B}_p^{s, q}$$

which gives us an estimate of the type (4) with  $s = (1 - \theta)s_0 + \theta s_1$ ,  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ , and  $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ . Note that we have in this case a relationship between  $q_0, q_1$  and  $q$ . Again, this method can not be applied to inequality (3).

It seems of course that inequality (3) cannot be obtained by an simple interpolation argument -actually this inequality is false in  $\mathbb{R}^n$ -, but what it would make it plausible in the setting of  $\mathbb{Z}_2$  is the special relationship between inequalities (1) and (2) and this is the main reason why theorem 1 is relevant.

The plan of the article is the following. In section 2 we recall some properties of the  $p$ -adic spaces, in section 3 we give the definition of Besov spaces over the 2-adic group  $\mathbb{Z}_2$  and in section 4 we prove theorem 1.

## 2 $p$ -adic groups

Our main reference here are the books [10], [8] and [1] where more details concerning the topological structure of the  $p$ -adic groups can be found.

We write  $a|b$  when  $a$  divide  $b$  or, equivalently, when  $b$  is a multiple of  $a$ . Let  $p$  be any prime number, for  $0 \neq x \in \mathbb{Z}$ , we define the  $p$ -adic valuation of  $x$  by  $\gamma(x) = \max\{r : p^r | x\} \geq 0$  and, for any rational number  $x = \frac{a}{b} \in \mathbb{Q}$ , we write  $\gamma(x) = \gamma(a) - \gamma(b)$ . Furthermore if  $x = 0$ , we agree to write  $\gamma(0) = +\infty$ .

Let  $x \in \mathbb{Q}$  and  $p$  be any prime number, with the  $p$ -adic valuation of  $x$  we can construct a norm by writing

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0 \\ p^{-\infty} = 0 & \text{if } x = 0. \end{cases} \quad (6)$$

This expression satisfy the following properties

- a)  $|x|_p \geq 0$ , and  $|x|_p = 0 \iff x = 0$ ;
- b)  $|xy|_p = |x|_p |y|_p$ ;
- c)  $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ , with equality when  $|x|_p \neq |y|_p$ .

When a norm satisfy *c*) it is called a non-Archimedean norm and an interesting fact is that over  $\mathbb{Q}$  all the possible norms are equivalent to  $|\cdot|_p$  for some  $p$ : this is the so-called Ostrowski theorem, see [1] for a proof.

**Definition 2.1** *Let  $p$  be a any prime number. We define the field of  $p$ -adic numbers  $\mathbb{Q}_p$  as the completion of  $\mathbb{Q}$  when using the norm  $|\cdot|_p$ .*

We present in the following lines the algebraic structure of the set  $\mathbb{Q}_p$ . Every  $p$ -adic number  $x \neq 0$  can be represented in a unique manner by the formula

$$x = p^\gamma(x_0 + x_1p + x_2p^2 + \dots), \quad (7)$$

where  $\gamma = \gamma(x)$  is the  $p$ -adic valuation of  $x$  and  $x_j$  are integers such that  $x_0 > 0$  and  $0 \leq x_j \leq p - 1$  for  $j = 1, 2, \dots$ . Remark that this canonical representation implies the identity  $|x|_p = p^{-\gamma}$ .

Let  $x, y \in \mathbb{Q}_p$ , using the formula (7) we define the sum of  $x$  and  $y$  by  $x + y = p^{\gamma(x+y)}(c_0 + c_1p + c_2p^2 + \dots)$  with  $0 \leq c_j \leq p - 1$  and  $c_0 > 0$ , where  $\gamma(x + y)$  and  $c_j$  are the unique solution of the equation

$$p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots) + p^{\gamma(y)}(y_0 + y_1p + y_2p^2 + \dots) = p^{\gamma(x+y)}(c_0 + c_1p + c_2p^2 + \dots).$$

Furthermore, for  $a, x \in \mathbb{Q}_p$ , the equation  $a + x = 0$  has a unique solution in  $\mathbb{Q}_p$  given by  $x = -a$ . In the same way, the equation  $ax = 1$  has a unique solution in  $\mathbb{Q}_p$ :  $x = 1/a$ .

We take now a closer look at the topological structure of  $\mathbb{Q}_p$ . With the norm  $|\cdot|_p$  we construct a distance over  $\mathbb{Q}_p$  by writing

$$d(x, y) = |x - y|_p \quad (8)$$

and we define the balls  $B_\gamma(x) = \{y \in \mathbb{Q}_p : d(x, y) \leq p^{-\gamma}\}$  with  $\gamma \in \mathbb{Z}$ . Remark that, from the properties of the  $p$ -adic valuation, this distance has the *ultra-metric* property (*i.e.*  $d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq |x|_p + |y|_p$ ).

We gather with the next proposition some important facts concerning the balls in  $\mathbb{Q}_p$ .

**Proposition 2.1** *Let  $\gamma$  be an integer, then we have*

- 1) *the ball  $B_\gamma(x)$  is a open and a closed set for the distance (8).*
- 2) *every point of  $B_\gamma(x)$  is its center.*
- 3)  *$\mathbb{Q}_p$  endowed with this distance is a complete Hausdorff metric space.*
- 4)  *$\mathbb{Q}_p$  is a locally compact set.*
- 5) *the  $p$ -adic group  $\mathbb{Q}_p$  is a totally discontinuous space.*

### 3 Functional spaces

In this article, we will work with the subset  $\mathbb{Z}_2$  of  $\mathbb{Q}_2$  which is defined by  $\mathbb{Z}_2 = \{x \in \mathbb{Q}_2 : |x|_2 \leq 1\}$ , and we will focus on real-valued functions over  $\mathbb{Z}_2$ . Since  $\mathbb{Z}_2$  is a locally compact commutative group, there exists a Haar measure  $dx$  which is translation invariant *i.e.*:  $d(x + a) = dx$ , furthermore we have the identity  $d(ax) = |a|_2 dx$  for  $a \in \mathbb{Z}_2^*$ . We will normalize the measure  $dx$  by setting

$$\int_{\{|x|_2 \leq 1\}} dx = 1.$$

This measure is then unique and we will note  $|E|$  the measure for any subset  $E$  of  $\mathbb{Z}_2$ .

Lebesgue spaces  $L^p(\mathbb{Z}_2)$  are thus defined in a natural way:  $\|f\|_{L^p} = \left( \int_{\mathbb{Z}_2} |f(x)|^p dx \right)^{1/p}$  for  $1 \leq p < +\infty$ , with the usual modifications when  $p = +\infty$ .

Let us now introduce the Littlewood-Paley decomposition in  $\mathbb{Z}_2$ . We note  $\mathcal{F}_j$  the Boole algebra formed by the equivalence classes  $E \subset \mathbb{Z}_2$  modulo the sub-group  $2^j\mathbb{Z}_2$ . Then, for any function  $f \in L^1(\mathbb{Z}_2)$ , we call  $S_j(f)$  the conditionnal expectation of  $f$  with respect to  $\mathcal{F}_j$ :

$$S_j(f)(x) = \frac{1}{|B_j(x)|} \int_{B_j(x)} f(y) dy.$$

The dyadic blocks are thus defined by the formula  $\Delta_j(f) = S_{j+1}(f) - S_j(f)$  and the Littlewood-Paley decomposition of a function  $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$  is given by

$$f = S_0(f) + \sum_{j=0}^{+\infty} \Delta_j(f) \quad \text{where } S_0(f) = \int_{\mathbb{Z}_2} f(x) dx. \quad (9)$$

We will need in the sequel some very special sets noted  $Q_{j,k}$ . Here is the definition and some properties:

**Proposition 3.1** *Let  $j \in \mathbb{N}$  and  $k = \{0, 1, \dots, 2^j - 1\}$ . Define the subset  $Q_{j,k}$  of  $\mathbb{Z}_2$  by*

$$Q_{j,k} = \{k + 2^j\mathbb{Z}_2\}. \quad (10)$$

Then

- 1) We have the identity  $\mathcal{F}_j = \bigcup_{0 \leq k < 2^j} Q_{j,k}$ ,
- 2) For  $k = \{0, 1, \dots, 2^j - 1\}$  the sets  $Q_{j,k}$  are mutually disjoint,
- 3)  $|Q_{j,k}| = 2^{-j}$  for all  $k$ ,
- 4) the 2-adic valuation is constant over  $Q_{j,k}$ .

The verifications are easy and left to the reader.

With the Littlewood-Paley decomposition given in (9), we obtain the following equivalence for the Lebesgue spaces  $L^p(\mathbb{Z}_2)$  with  $1 < p < +\infty$ :

$$\|f\|_{L^p} \simeq \|S_0(f)\|_{L^p} + \left\| \left( \sum_{j \in \mathbb{N}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p}.$$

See the book [9], chapter IV, for a general proof.

For Besov spaces we will define them by the norm

$$\|f\|_{B_p^{s,q}} \simeq \|S_0 f\|_{L^p} + \left( \sum_{j \in \mathbb{N}} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q} \quad (11)$$

where  $s \in \mathbb{R}$ ,  $1 \leq p, q < +\infty$  with the necessary modifications when  $p, q = +\infty$ .

**Remark 1** For homogeneous functional spaces  $\dot{B}_p^{s,q}$ , we drop out the term  $\|S_0 f\|_{L^p}$  in (11).

## 4 Proof of the theorem 1

To begin the construction of the counterexample we consider  $0 < j_0 < j_1$  two integers and we fix  $\alpha, \beta \in \mathbb{R}$  such that

$$2^{2j_0} \leq \frac{\beta}{\alpha}. \quad (12)$$

Take now a decreasing sequence  $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N})$  with  $q > 2$  such that  $\varepsilon_0 = 1$  and  $(\varepsilon_j)_{j \in \mathbb{N}} \notin \ell^2(\mathbb{N})$ .

Define  $N_j$  in the following form

$$N_j = \begin{cases} 2^j & \text{if } 0 < j < j_0, \\ 2^{-j} \frac{\beta}{\alpha} & \text{if } j_0 \leq j \leq j_1. \end{cases} \quad (13)$$

We construct a function  $f : \mathbb{Z}_2 \rightarrow \mathbb{R}$  by considering his values over the dyadic blocs and we will use for this the sets  $Q_{j,k}$  defined in (10):

$$\Delta_j f(x) = \begin{cases} \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,0}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,1}, \\ \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,3}, \\ \vdots & \\ \varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2N_j-2}, \\ -\varepsilon_j \alpha 2^j & \text{over } Q_{j+1,2N_j-1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark that, with this definition of  $\Delta_j f(x)$  we have the identities

- $\|\Delta_j f\|_{L^\infty} = \varepsilon_j \alpha 2^j,$
- $\|\Delta_j f\|_{L^1} = \varepsilon_j \alpha N_j,$
- $\|\Delta_j f\|_{L^2}^2 = \varepsilon_j^2 \alpha^2 2^j N_j.$

From this quantities we construct the following norms

(a) for the Besov space  $\dot{B}_\infty^{-1,\infty}$  we have

$$\|f\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{j \in \mathbb{N}} 2^{-j} \|\Delta_j f\|_{L^\infty} = \alpha, \text{ since the sequence } (\varepsilon_j)_{j \in \mathbb{N}} \text{ is decreasing and } \varepsilon_0 = 1.$$

(b) for the Besov space  $\dot{B}_1^{1,q}$  we write

$$\|f\|_{\dot{B}_1^{1,q}}^q = \sum_{j=0}^{j_1} (2^j \|\Delta_j f\|_{L^1})^q = \sum_{j=0}^{j_1} 2^{jq} \varepsilon_j^q \alpha^q N_j^q = \alpha^q \left( \sum_{j=0}^{j_0} 2^{jq} \varepsilon_j^q N_j^q + \sum_{j>j_0}^{j_1} 2^{jq} \varepsilon_j^q N_j^q \right)$$

We use now the values of  $N_j$  given in (13) and the relationship (12) to obtain

$$= \alpha^q \left( \sum_{j=0}^{j_0} 2^{2jq} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \frac{\beta^q}{\alpha^q} \right) = \beta^q \left( \sum_{j=0}^{j_0} 2^{2jq} \frac{\alpha^q}{\beta^q} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right) \simeq \beta^q \left( \sum_{j=0}^{j_0} 2^{q(2j-2j_0)} \varepsilon_j^q + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right).$$

$$\text{Then we have } \|f\|_{\dot{B}_1^{1,q}} \simeq \beta \left( C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right)^{1/q}.$$

(c) For the Lebesgue space  $L^2$  we use the same arguments above to obtain

$$\|f\|_{L^2}^2 = \sum_{j=0}^{j_1} \varepsilon_j^2 \alpha^2 2^j N_j = \alpha^2 \left( \sum_{j=0}^{j_0} 2^{2j} \varepsilon_j^2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \frac{\beta}{\alpha} \right) \simeq \alpha\beta \left( C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \right).$$

Once these norms are computed, we go back to the inequality

$$\|f\|_{L^2}^2 \leq C \|f\|_{\dot{B}_1^{1,q}} \|f\|_{\dot{B}_\infty^{-1,\infty}}$$

and we have

$$\alpha\beta \left( C_2 + \sum_{j>j_0}^{j_1} \varepsilon_j^2 \right) \leq C \times \alpha \times \beta \left( C_1 + \sum_{j>j_0}^{j_1} \varepsilon_j^q \right)^{1/q}.$$

But, by hypothesis, we have  $(\varepsilon_j)_{j \in \mathbb{N}} \notin \ell^2(\mathbb{N})$  and  $(\varepsilon_j)_{j \in \mathbb{N}} \in \ell^q(\mathbb{N})$ , thus, for  $j_1$  big enough it is impossible to find an universal constant  $C$  such that the above inequality is true. ■

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