Gaussian stationary processes over graphs, general frame and maximum likelihood identification

Thibault Espinasse, Fabrice Gamboa, Jean-Michel Loubes

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T. Espinasse, F. Gamboa and J-M. Loubes

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Abstract

In this paper, we give a construction of some stationary Gaussian fields indexed by graphs as image of adequate invariant functions. This construction also relies on spectral theory of Hilbertian operators defined on a graph. We then extend Whittle maximum likelihood estimators of the parameters of the corresponding *spectral density* and provide their asymptotic behavior.

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Introduction

In the past few years, much interest has been paid to the study of random fields over graphs, driven by the growing needs for both theoretical and practical results for data indexed by graphs. On one side, the definition of graphical models by J.N. Darroch, S.L. Lauritzen and T.P. Speed in 1980 [9] fostered new interest in Markov fields, and many tools have been developed in this direction (see, for instance [26] and [25]). On another side, the industrial demand linked to graphical problems has risen with the apparition of new technologies. In very particular, the Internet and social networks provide a huge field of applications, but biology, economy, geography or image analysis also benefit from models taking into account a graph structure.

The analysis of road traffic is at the root of this work. Actually, prediction of road traffic deals with the forecast of speed of vehicles which may be seen as a spatial random field over the traffic network. Some work has been done without taking into account the particular graph structure of the speed process (see for example [11] and [19] for related statistical issues). In this paper, we build a model for Gaussian random fields over graphs and study statistical properties of such stochastic processes.

A random field over graph is a spatial process indexed by the vertices of a graph, namely $(X_i)_{i\in G}$, where G is a given graph. Many models already exist in the probabilistic literature, ranging from Markov fields to autoregressive processes, which are based on two general kinds of construction. On the one hand, graphical models are Markov fields (see for instance [15]). They are built by specifying a dependency structure for X_i and X_j, conditionally to the other variables, as soon as the locations $i \in G$ and $j \in G$ are connected. This particular dependency structure must be chosen by the statistician. For graphical models, we refer for instance to [9] and references therein. On the other hand, the graph itself, through the adjacency operator, can provide the dependency. This is the case, for example, of autoregressive models (see [16]). Here, the local form of the graph is strongly used for statistical inference.

In this paper, we will provide a new framework that takes into account advantages of both points of views. Hence, we extend some classical results from time series to spatial fields over general graph. In particular, we study regular ARMA processes on graphs. For this, we will make use of spectral analysis of time series and extend to our framework some classical results. In particular, the notion of spectral density may be extended to graphs. This will enable us to construct a maximum likelihood estimate for the parameters of the spectral density. This also leads to an extension of the Whittle approximation (see [13], [2]). Actually, many extensions of this approximation have been performed, even in non-stationary cases (see [8], [22], [12]). The extension proposed in this paper concerns spatial autoregressive processes. We point out that we will compare throughout all the paper our new framework with the $G = \mathbb{Z}^d, d \geq 1$ case.

The paper falls into the following parts. Section 1 is devoted to recall some definitions of graphs and spectral theory for time series. Then we state the definition of general spatial $ARMA$ processes over a graph in Section 2, and one of our main theorems is given in Section

3. Section 4 provides some way to modify the graph itself, in order to consider a larger class of processes, which leads to the construction of a large class of Markov fields. Section 5 gives the definition of the spectral measure for the graph, and a sufficient condition for its existence. Some simulations are provided in Section 6. The last section provides all necessary tools to prove the main theorem, in particular Szegö Lemmas are given in Section 7.1, while the proofs of the technical Lemmas are postponed in Section 7.3.

1 Definitions and useful properties for spectral analysis and Toeplitz operators

In the whole paper, we will consider a Gaussian spatial process $(X_i)_{i\in G}$ indexed by the vertices of an infinite undirected weighted graph.

We will call $\mathbf{G} = (G, W)$ this graph, where

- G is the set of vertices. $\mathbf{G}$ is said to be infinite as soon as G is infinite (but countable).
- $W \in [-1,1]^{G\times G}$ is the symmetric weighted adjacency operator. That is, $|W_{ij}| \neq 0$ when $i \in G$ and $j \in G$ are connected.

We assume that W is symmetric ($W_{ij} = W_{ji},\ i, j \in G$) since we deal only with undirected graphs.

For any vertex $i \in G$, a vertex $j \in G$ is said to be a neighbor of i if, and only if, $W_{ij} \neq 0$. The degree $\deg(i)$ of i is the number of neighbors of the vertex i, and the degree of the graph $\mathbf{G}$ is defined as the maximum degree of the vertices of the graph $\mathbf{G}$:

$$\deg(\mathbf{G}) := \max_{i\in G} \deg(i).$$

From now, we assume that the degree of the graph $\mathbf{G}$ is bounded :

$$\deg(\mathbf{G}) < +\infty.$$

Assume also that the entries of W belong to $[-1, 1]$. This is not restrictive since renormalizing the adjacency operator does not change the objects introduced later. In particular, the spectral representation of Hilbertian operator is not sensitive to a renormalization.

When $G = \mathbb{Z}$, we will use the renormalized adjacency operator

$$W_{ij}^{(\mathbb{Z})} = \frac{1}{2}\mathbb{1}_{\{|i-j|=1\}}, (i, j \in \mathbb{Z}).$$

Here, $\deg(\mathbb{Z}) = 2$. This case will be used in all the paper as an illustration example.

To introduce the spectral decomposition, consider the action of the adjacency operator on $l^2(G)$ as

$$\forall u \in l^2(G), (Wu)_i := \sum_{j\in G} W_{ij}u_j, (i \in G).$$

We denote by B_G the set of all bounded Hilbertian operators on $l^2(G)$. The operator space B_G will be endowed with the classical operator norm

$$\forall A \in B_G, \|A\|_{2,op} := \sup_{u\in l^2(G), \|u\|_2\leq 1} \|Au\|_2 ,$$

where $\|.\|_2$ stands for the usual norm on $l^2(G)$.

Notice that, as the degree of $\mathbf{G}$ and the entries of W are both bounded, W lies in B_G.

Recall that for any bounded Hilbertian operator $A \in B_G$, the spectrum $\mathrm{Sp}(A)$ is defined as the set of complex numbers λ such that $\lambda \mathrm{Id}_G - A$ is not invertible (here Id_G stands for the identity on $l^2(G)$). Since W is bounded and self-adjoint, $\mathrm{Sp}(W)$ is a compact non-empty subset of $\mathbb{R}$ [23].

We aim now at providing a spectral representation of any bounded normal Hilbertian operator. For this, recall first the definition of a resolution of identity (see [23]):

Definition 1.1. *Let $\mathcal{M}$ be a σ-algebra over a set Ω. We call identity resolution (on $\mathcal{M}$) an application*

$$E : \mathcal{M} \to B_G$$

such that,

1. *$E(\emptyset) = 0, E(\Omega) = I$.*
2. *For any $\omega \in \mathcal{M}$, the operator $E(\omega)$ is a self-adjoint projection operator.*
3. *For any $\omega, \omega' \in \mathcal{M}$, we have*

$$E(\omega \cap \omega') = E(\omega)E(\omega') = E(\omega')E(\omega).$$

4. *For any $\omega, \omega' \in \mathcal{M}$ such that $\omega \cap \omega' = \emptyset$, we have*

$$E(\omega \cup \omega') = E(\omega) + E(\omega').$$

Therefore, for all $x \in l^2(G)$ and $y \in l^2(G)$, the functional $E_{x,y}$ defined by

$$E_{xy}(\omega) = \langle E(\omega)x, y\rangle_{l^2(G)}$$

is a complex measure on $\mathcal{M}$.

We can now recall the fundamental decomposition theorem (see [23])

Theorem 1.1 (Spectral decomposition)**.** *If $A \in B_G$ is normal (A commutes with its adjoint), then there exists a unique identity resolution E over all Borelian subsets of* $\mathrm{Sp}(A)$*, such that*

$$A = \int_{\mathrm{Sp}(A)} \lambda \mathrm{d}E(\lambda).$$

Moreover, for any $U \in B_S$ such that $UA = AU$, every projector $E(\omega), \omega \subset \mathrm{Sp}(A)$ commutes with U

Since W is self-adjoint in our case, it is a normal operator, so Theorem 1.1 may be applied.

We obtain the spectral representation of the adjacency operator W thanks to an identity resolution E over the Borelians of $\mathrm{Sp}(W)$

$$W = \int_{\mathrm{Sp}(W)} \lambda \mathrm{d}E(\lambda).$$

Moreover, with this decomposition, we can give a spectral representation for the powers $W^k, k \in \mathbb{Z}$ of W. Define first, for any $i \in G$, the sequences δ_i in $l^2(G)$ by

$$\delta_i := (\mathbb{1}_{k=i})_{k \in G}.$$

For any $i, j \in G$, the sequences δ_i and δ_j define a real measure μ_{ij} by

$$\forall \omega \subset \mathrm{Sp}(W), \mu_{ij}(\omega) := E_{\delta_i \delta_j}(\omega) = \langle E(\omega)\delta_i, \delta_j \rangle_{l^2(G)}.$$

Hence, we can write :

$$\forall k \in \mathbb{N}, \forall i, j \in G, \left(W^k\right)_{ij} = \int_{\mathrm{Sp}(W)} \lambda^k \mathrm{d}\mu_{ij}.$$

In the usual case of $\mathbb{Z}$, an explicit expression for μ_{ij} can be given. Denote $T_k(X)$ the k^{th}-Chebychev polynomial ($k \in \mathbb{N}$). We can provide the spectral decomposition of $W^{(\mathbb{Z})}$

$$\forall i, j \in \mathbb{Z}, \left(\left(W^{(\mathbb{Z})}\right)^k\right)_{ij} = \int_{[-1,1]} \lambda^k \frac{T_{|j-i|}(\lambda)}{\sqrt{1-\lambda^2}} \mathrm{d}\lambda.$$

This shows that, in this case, and for any $i, j \in G$, the measure $\mathrm{d}\mu_{ij}$ is absolutely continuous with respect to the Lebesgue measure, and its density is given by

$$\frac{\mathrm{d}\mu_{ij}}{\mathrm{d}\lambda} = \frac{T_{|j-i|}(\lambda)}{\sqrt{1-\lambda^2}}.$$

Note that we recover the usual spectral decomposition by setting :

$$\forall i, j \in G, \mathrm{d}\hat{\mu}_{ij}(t) := \frac{1}{2\pi} \cos\left((j-i)t\right) \mathrm{d}t.$$

We get

$$\forall i, j \in \mathbb{Z}, \left(\left(W^{(\mathbb{Z})}\right)^k\right)_{ij} = \int_{[0,2\pi]} \cos(t)^k \mathrm{d}\hat{\mu}_{ij}(t).$$

This corresponds to another choice for the identity resolution. Further on, this enables to handle the usual case of processes indexed by $\mathbb{Z}$.

Since our aim is to study some kind of stationary processes indexed by the vertices G of the graph $\mathbf{G}$, recall once again what happens for the usual case of $\mathbb{Z}$. In particular, let us introduce Toeplitz operators associated to stationary processes.

Let $\mathbf{X} = (X_i)_{i \in \mathbb{Z}}$ be a stationary Gaussian process indexed by $\mathbb{Z}$. Since $\mathbf{X}$ is Gaussian, stationarity is equivalent to second order stationarity, that is, $\forall i, k \in \mathbb{Z}, \mathrm{Cov}(X_i, X_{i+k})$ does not depend on i. Thus, we can define

$$r_k := \mathrm{Cov}(X_i, X_{i+k}).$$

In this example, we assume that $(r_k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$. This leads to a particular form of the covariance operator Γ defined on $l^2(\mathbb{Z})$ by

$$\forall i, j \in \mathbb{Z}, \Gamma_{ij} := r_{i-j}.$$

Recall that $B_{\mathbb{Z}}$ denotes here the set of bounded Hilbertian operators on $l^2(\mathbb{Z})$. Notice that, since $(r_k)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$, we have $\Gamma \in B_{\mathbb{Z}}$ (see for instance [6] for more details). This bounded operator is constant over each diagonals, and so is called a Toeplitz operator (see also [5] for a general introduction to Toeplitz operators).

Toeplitz operators enjoy the following representation,

$$\forall i,j \in \mathbb{Z}, \mathcal{T}(g)_{ij} := \Gamma_{ij} = \frac{1}{2\pi}\int_{[0,2\pi]} g(t)\cos\left((i-j)t\right)\mathrm{d}t,$$

where g is the spectral density of the process $\mathbf{X}$, defined by

$$g(t) := 2\sum_{k\in\mathbb{N}^*} r_k \cos(kt) + r_0.$$

This expression can be written, using the Chebychev polynomials $(T_k)_{k\in\mathbb{N}}$,

$$g(t) := 2\sum_{k\in\mathbb{N}^*} r_k T_k\left(\cos(t)\right) + r_0 T_0\left(\cos(t)\right).$$

This falls in the framework of Theorem 3.1, by setting

$$\forall \lambda \in [-1,1], f(\lambda) := 2\sum_{k\in\mathbb{N}^*} r_k T_k(\lambda) + r_0 T_0(\lambda).$$

We get, using the family $(\hat{\mu}_{ij})_{i,j\in\mathbb{Z}}$ defined above,

$$\forall i,j \in \mathbb{Z}, \Gamma_{ij} = \int_{[0,2\pi]} f\left(\cos(t)\right)\mathrm{d}\hat{\mu}_{ij}(t).$$

Notice that the last expression may also be written as $\Gamma = f(W^{(\mathbb{Z})})$, and the convergence is ensured by the boundedness of $W^{(\mathbb{Z})}$ together with the boundedness of the Chebychev polynomials, $T_k([-1,1]) \subset [-1,1], \forall k \in \mathbb{Z}$, and the summability of the sequence $(r_k)_{k\in\mathbb{Z}}$.

Finally, we will extend usual MA processes to any graph, using this theory. This will be the purpose of Section 2.

We first recall some properties about the moving average representation MA_∞ of a process on $\mathbb{Z}$. This representation exists as soon as the log of the spectral density is integrable (see for instance [6]). In this case, there exists a sequence $(a_k)_{k\in\mathbb{N}}$, with $a_0 = 1$, and a Gaussian white noise $\epsilon = (\epsilon_k)_{k\in\mathbb{Z}}.$, such that the process $\mathbf{X}$ may be written as

$$\forall i \in \mathbb{Z}, X_i = \sum_{k\in\mathbb{N}} a_k \epsilon_{i-k}.$$

Defining the function h over the unit circle $\mathcal{C}$ by

$$\forall x \in \mathcal{C}, h(x) = \sum_{k\in\mathbb{N}} a_k x^k,$$

we recover, with a few computations, the spectral decomposition of the covariance operator Γ of $\mathbf{X}$:

$$\forall i,j \in \mathbb{Z}, \Gamma_{ij} = \int_{[0,2\pi]} \left|h(e^{it})\right|^2 \mathrm{d}\hat{\mu}_{ij}(t).$$

This implies the equality

$$f\left(\cos(t)\right) = \left|h(e^{it})\right|^2.$$

Recall that when h is a polynomial of degree p (with non null first coefficient), the process is said to be MA_p. In this case, f is also a polynomial of degree p. Reciprocally, if f is a real polynomial of degree p, and as soon as $f\left(\cos(t)\right)$ is even, and non-negative for any $t \in [0, 2\pi]$, the Fejér-Riesz theorem provides a factorization of $f\left(\cos(t)\right)$ such that $f\left(\cos(t)\right) = |h(e^{it})|^2$ (see for instance [18]). This proves that $\mathbf{X}$ is MA_p if, and only if, its covariance operator may be written $f(W^{(\mathbb{Z})})$, where f is a polynomial of degree p.

This remark is fundamental for the construction we provide in the following section (see Definition 2.1).

2 Analytic construction

In this section, we will define moving average and autoregressive processes over the graph $\mathbf{G}$. In the whole section, we deal with the adjacency weighted operator W of the graph $\mathbf{G}$.

As explained in the last section, since W is bounded and self-adjoint, $\mathrm{Sp}(W)$ is a non-empty compact subset of $\mathbb{R}$, and W admits a spectral decomposition thanks to an identity resolution E, given by

$$W = \int_{\mathrm{Sp}(W)} \lambda \mathrm{d}E(\lambda).$$

We define here MA and AR Gaussian processes, with respect to the operator W, by defining the corresponding classes of covariance operators, since the covariance operator fully characterizes any Gaussian process.

Definition 2.1. *Let $(X_i)_{i\in G}$ be a Gaussian process, indexed by the vertices G of the graph $\mathbf{G}$, and Γ its covariance operator.*

If there exists an analytic function f defined on the convex hull of $\mathrm{Sp}(W)$, such that

$$\Gamma = \int_{\mathrm{Sp}(W)} f(\lambda) \mathrm{d}E(\lambda),$$

we will say that X is

- *$MA_q^{(W)}$ if f is a polynomial of degree q.*
- *$AR_p^{(W)}$ if $\frac{1}{f}$ is a polynomial of degree p which has no root in the convex hull of $\mathrm{Sp}(W)$.*
- *$ARMA_{p,q}^{(W)}$ if $f = \frac{P}{Q}$ with P a polynomial of degree p and Q a polynomial of degree q with no roots in the convex hull of $\mathrm{Sp}(W)$.*

Otherwise, we will talk about the $MA_\infty^{(W)}$ representation of the process $\mathbf{X}$*. We call* f *the* ***spectral density*** *of the process* $\mathbf{X}$*, and denote its attached covariance operator by*

$$\Gamma = \mathcal{K}(f).$$

Remark Actually, this last construction may also be understood as

$$\Gamma = \mathcal{K}(f) = f(W),$$

in the sense of normal convergence of the associated power serie. However, the spectral representation will be useful in the following (see Section 5) and allows weaker regularity.

This kind of modeling is interesting when the interactions are propagated time by time and locally (that may be for instance a good modeling for traffic problems).

The notation $\mathcal{K}(.)$ has to be understood by analogy with the notation $\mathcal{T}(.)$ used for Toeplitz operators.

Notice that, in the usual case of $\mathbb{Z}$, and for finite order $ARMA$, we recover the usual definition as shown in the end of the previous section. So, the last definition may be seen as an extension of isotropic $ARMA$ for any graph $\mathbf{G}$. Besides, note that this extension is given by the equivalence, for any $g \in \mathbb{L}^2([0, 2\pi])$, such that $\int_{[0,2\pi]} \log(g) < +\infty$,

$$\forall f \in \mathbb{L}^2([-1, 1]), (g = f(\cos(t)) \Leftrightarrow \mathcal{T}(g) = \mathcal{K}(f)).$$

This means that, in the usual case $\mathbf{G} = \mathbb{Z}$, the definition of spectral density in our framework is the usual one, up to an change of variable $\lambda = \cos(t)$ (see Section 1).

Now, we get a representation of moving average processes over any graph $\mathbf{G}$. The following section gives the main result of this paper. It deals with the maximum likelihood identification, and Section 4 proposes a general definition of stationary processes indexed by the vertices of a graph, and shows the stationarity of MA representation thanks to this definition. We underline here that the definition given in Section 4 allows constructions of MA processes built with some "isotropic modifications" of W.

3 Convergence of maximum approximated likelihood estimators

In this section as before, $\mathbf{G} = (G, W)$ is a graph with bounded degree. Let also $(X_i)_{i \in G}$ be a Gaussian spatial process indexed by the vertices of $\mathbf{G}$ with spectral density f_{θ_0} (defined in Section 2) depending on an unknown parameter $\theta_0 \in \Theta$. We aim at estimating θ_0. For this, we will generalize classical maximum likelihood estimation of time series.

We will also develop a Whittle approximation for $ARMA$ processes indexed by the vertices of a graph. That is an approximation of the likelihood that provides convergence of likelihood estimate. We follow here the guidelines of the proof given in [2] for the usual case of time series.

3.1 Framework and Assumptions

Let us now specify the framework of our study. Let $(\mathbf{G}_n)_{n\in\mathbb{N}}$ be a growing sequence of finite nested subgraphs. This means that if $\mathbf{G}_n = (G_n, W_n)$, we have $G_n \subset G_{n+1} \subset G$ and that for any $i, j \in G_n$, it holds that $W_n(i,j) = W(i,j)$.

Let $m_n = \mathrm{Card}(G_n)$. We set also

$$\delta_n = \mathrm{Card}\left\{i \in G_n, \exists j \in G\backslash G_n, W_{ij} \neq 0\right\}.$$

The sequence $(m_n)_{n\in\mathbb{Z}}$ may actually be seen as the "volume" of the graph $\mathbf{G}_n$, and δ_n as the size of the boundary of G_n. For the special case $G = \mathbb{Z}^d$ and $G_n = [-n, n]^d$, we get $m_n = (2n+1)^d$ and $\delta_n = 2d(2n+1)^{d-1}$.

The ratio $\frac{\delta_n}{m_n}$ is a natural quantity associated to the expansion of the graph that also appears in isoperimetrical [21] and graph expander issues. We will assume here that this ratio goes to 0 when the size of the graph goes to infinity. We also assume that W is normalized. In short, we set

Assumption 3.1.

- $\sup_{i,j\in G} W_{ij} \leq \frac{1}{D}$
- $\delta_n = o(m_n)$

Notice here that the first assumption implies that $\mathrm{Sp}(W) \subset [-1, 1]$. The second assumption is a non-expansion criterion which is satisfied for the last examples $G = \mathbb{Z}^d$ and $G_n = [-n, n]^d$, but not for a homogeneous tree, whatever the choice of the sequence of subgraphs $(\mathbf{G_n})_{n\in\mathbb{N}}$ is.

We will now choose a parametric family of covariance operators of $MA^{(W)}$ processes as defined in the last section. First, let Θ be a compact subspace of $\mathbb{R}$.

We point out that for sake of simplicity, we choose a one-dimensional parameter space Θ. Nevertheless, all the results could be easily extended to the case $\Theta \subset \mathbb{R}^k, k \geq 1$.

Define $\mathcal{F}$ as the set of positive analytic functions over the convex hull of $\mathrm{Sp}(W)$.

Let also $(f_\theta)_{\theta\in\Theta}$ be a parametric family of functions of $\mathcal{F}$. They define a parametric set of covariances on G (see Section 2) by

$$\mathcal{K}(f_\theta) = f_\theta(W).$$

As in [2], we will need a strong regularity for this family of spectral densities.

Let us introduce a regularity factor for any analytic function $f \in \mathcal{F}$, $f(x) = \sum_k f_k x^k$ $(x \in \mathrm{Sp}(W))$, by setting

$$\alpha(f) := \sum_{k\in\mathbb{N}} |f_k|\,(k+1).$$

Now, let $\rho > 0$ and define,

$$\mathcal{F}_\rho := \{f \in \mathcal{F}, \alpha(\log(f)) \leq \rho\}.$$

Notice that for any $f \in \mathcal{F}_\rho$, we have $\alpha(f) \leq e^\rho, \alpha(\frac{1}{f}) \leq e^\rho$.

We need the following assumption

Assumption 3.2.

- *The application $\theta \to f_\theta$ is injective.*
- *For any $\lambda \in Sp(W)$, the application $\theta \to f_\theta(\lambda)$ is continuous.*
- $\forall \theta \in \Theta, f_\theta \in \mathcal{F}_\rho$.

From now, consider $\theta_0 \in \mathring{\Theta}$. Let $\mathbf{X}$ be a centered Gaussian MA_∞ process over $\mathbf{G}$ with covariance operator $\mathcal{K}(f_{\theta_0})$ (see section 2).

We observe the restriction of this process on the subgraph $\mathbf{G}_n$ defined before. Our aim is to compute the maximum likelihood estimator of θ_0. Let $X_n = (\mathbf{X}_i)_{i \in G_n}$ be the observed process and $\mathcal{K}_n(f_\theta)$ be its covariance :

$$X_n \sim \mathcal{N}\left(0, \mathcal{K}_n(f_{\theta_0})\right).$$

The corresponding log-likelihood at θ is

$$L_n(\theta) := -\frac{1}{2}\left(m_n \log(2\pi) + \log\det\left(\mathcal{K}_n(f_\theta)\right) + X_n^T\left(\mathcal{K}_n(f_\theta)\right)^{-1} X_n\right).$$

Recall what happens in the usual case of time series ($G = \mathbb{Z}$, $G_n = [-n, n]$). Using the notations of Section 1, we assume that $\mathbf{X}$ is a stationary process indexed by $\mathbb{Z}$ with covariance operator Γ. Recall that, as explained in Section 2, if the usual spectral density $g \in \mathbb{L}^2([0, 2\pi])$ is such that $\log(g)$ is integrable, then the function $f \in \mathbb{L}^2([-1, 1])$ defined by $g(t) = f(\cos(t))$, is the spectral density of our framework. That means that

$$\Gamma = \mathcal{K}(f) = f(W^{(\mathbb{Z})}) = \mathcal{T}(g).$$

It is usual to maximize an approximation of the likelihood. The classical approximation is the Whittle's one ([13]), where

$$\frac{1}{n}\log\det\left(\mathcal{T}_n(g)\right)$$

is replaced by

$$\frac{1}{2\pi}\int_{[0,2\pi]} \log\left(g\left(t\right)\right) \mathrm{d}t.$$

Back to the general case, we aim at performing the same kind of approximation. For this, we will need the following assumption to ensure the convergence of $\log\det\left(\mathcal{K}_n(f_\theta)\right)$ (see Section 2 for the definition of μ_{xx}) :

Assumption 3.3. *There exists a positive measure μ, such that*

$$\frac{1}{m_n}\sum_{x \in G_n} \mu_{xx} \underset{n\to\infty}{\overset{\mathcal{D}}{\to}} \mu.$$

Here, $\mathcal{D}$ stands for the convergence in distribution

The limit measure μ is classically called the spectral measure of $\mathbf{G}$ with respect to the sequence of subgraphs $(\mathbf{G}_n)_{n\in\mathbb{Z}}$ (see [20] for example). In Section 5, we give some sufficient conditions to ensure the existence of the spectral measure.

As in the case of time series (for $G = \mathbb{Z}$), we can approximate the log-likelihood. It avoids an inversion of matrix and a computation of a determinant. Indeed, we will consider the two following approximations.

$$\bar{L}_n(\theta) := -\frac{1}{2}\left(m_n \log(2\pi) + m_n \int \log(f_\theta(x))\mathrm{d}\mu(x) + X_n^T \left(\mathcal{K}_n(f_\theta)\right)^{-1} X_n\right).$$

$$\tilde{L}_n(\theta) := -\frac{1}{2}\left(m_n \log(2\pi) + m_n \int \log(f_\theta(x))\mathrm{d}\mu(x) + X_n^T \left(\mathcal{K}_n\left(\frac{1}{f_\theta}\right)\right) X_n\right).$$

Generally, approximated maximum likelihood estimators are not asymptotically normal. Indeed, the derivate of the approximated log-likelihood has to be asymptotically unbiased [2].

To overcome this problem in $\mathbb{Z}^d$, the tapered periodogram can be used (see [15], [14], [7]). The following corresponds to graph extensions of standard time series models.

This is tractable only in two cases :

- The MA_P case : There exists $P > 0$ such that the true spectral density f_{θ_0} is a polynomial of degree bounded by P.
- The AR_P case : There exists $P > 0$ such that *all* the spectral densities (for any $\theta \in \Theta$) of the parametric set are such that $\frac{1}{f_\theta}$ is a polynomial of degree bounded by P.

So, to define the unbiased approximated log-likelihood, we first introduce the unbiased periodogram in one of this cases. Now, let $P > 0$.

Define a subset V_P of signed measures on $\mathbb{R}$ as

$$V_P := \{\mu_{ij}, i, j \in G, d_{\mathbf{G}}(i, j) \leq P\},$$

where $d_{\mathbf{G}}(i, j), i, j \in G$ stands for the usual distance on the graph $\mathbf{G}$, i.e. the length of the shortest path going from i to j.

This set gives all the possible local measures $\mu_{ij}, i, j \in G$ (see Section 1) over the graph $\mathbf{G}$.

To get the asymptotic normality, we need the following assumption

Assumption 3.4. *The set V_P of possible local measures over G is finite, and n is large enough to ensure that*

$$\forall v \in V_P, \exists (i, j) \in G_n^2, \mu_{ij} = v.$$

Remark This assumption is quite strong, and holds for instance for graphs built by reproducing a finite graph (the pattern) at each vertex of a distance transitive graph. In particular, this holds for any Cayley graph or distance transitive graph. This assumption may be relaxed, but it is a hard and technical work that will be the issue of a forthcoming paper.

Define now the matrix $B^{(n)}$ (the dependency on P is omitted, for clarity) by

$$\begin{aligned} B_{ij}^{(n)} &:= \frac{\text{Card}\left\{(k,l) \in G_n \times G, \mu_{kl} = \mu_{ij}\right\}}{\text{Card}\left\{(k,l) \in G_n \times G_n, \mu_{kl} = \mu_{ij}\right\}}, \text{ if }, d_{\mathbf{G}}(k,l) \leq P \\ &:= 1 \text{ if } d_{\mathbf{G}}(k,l) > P. \end{aligned}$$

The matrix $B^{(n)}$ gives a boundary correction, comparing, for any $v \in V_P$ the frequency of the interior couples of vertices with local measure v with the boundary couples of vertices with local measure v. Actually, this way to deal with the edge effect is very similar to the one used for $\mathbf{G} = \mathbb{Z}^d$ (see [7], [14]).

As example, let us now describe the case $G = \mathbb{Z}^2$, for $P = 2$. In this case $W^{(\mathbb{Z}^2)}$ is

$$\forall i,j,k,l \in \mathbb{Z}, W^{(\mathbb{Z}^2)}\left((i,j),(k,l)\right) := \frac{1}{4}\mathbb{1}_{|i-j|+|k-l|=1}.$$

In this example, we set $G_n = [1,n]^2$, and we can compute the matrix $B^{(n)}$. Indeed, it only is needed to notice that

$$\mu_{(i_1,j_1),(i_1+k,j_1+l)} = \mu_{(i_2,j_2),(i_2+\epsilon_1 k, j_2+\epsilon_2 l)}, i_1, i_2, j_1, j_2, k, l \in \mathbb{Z}, \epsilon_1, \epsilon_2 \in \{-1,1\}.$$

This means that the local measure of a couple of vertices depends only of their relative positions (stationarity and isotropy of this set of measure). So, we need to count the configurations given by Figure 1 since we consider only couples of vertices $u, v \in \mathbb{Z}^2$ such that $d_{\mathbb{Z}^2}(u,v) \leq 2$.

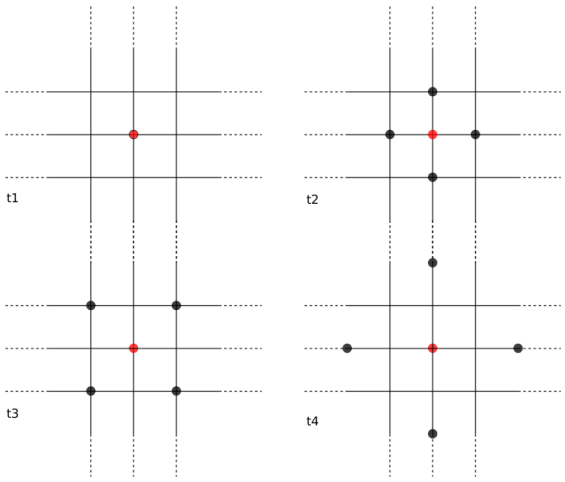


Figure 1: Possible configurations for couple of vertices

We get, for any $i, j \in \mathbb{Z}$,

- $B_{(i,j),(i,j)}^{(n)} = \frac{n^2}{n^2} = 1.$
- $B_{(i,j),(i,j\pm1)}^{(n)} = B_{(i,j),(i\pm1,j)}^{(n)} = \frac{4n(n-1)}{4n^2}.$
- $B_{(i,j),(i\pm1,j\pm1)}^{(n)} = \frac{4(n-1)^2}{n^2}.$

- $B^{(n)}_{(i,j),(i,j\pm 2)} = B^{(n)}_{(i,j),(i\pm 2,j)} = \frac{4n(n-2)}{4n^2}$

One can notice that

$$\sup_{ij} \left| B^{(n)}_{ij} - 1 \right| \underset{n\to\infty}{\to} 0.$$

The assumption 3.5 ensure that this property holds for the graph we consider.

Back to the general case, let $f \in \mathcal{F}_\rho$. We define the unbiased periodogram as

$$X_n^T \mathcal{Q}_n(\frac{1}{f}) X_n.$$

where

$$\mathcal{Q}_n(f) := B^{(n)} \odot \mathcal{K}_n(f).$$

Here, the operation $\odot$ denotes the Hadamard product for the matrix i.e.

$$\forall i,j \in G_n, \left(B^{(n)} \odot \mathcal{K}_n(f)\right)_{ij} = \left(B^{(n)}\right)_{ij} \mathcal{K}_n(f)_{ij}.$$

Notice that this is actually a way to extend the so called tapered periodogram (see for instance [14]).

We now define the unbiased empirical log-likelihood, for any $\theta \in \Theta$

$$L_n^{(u)}(\theta) := -\frac{1}{2}\left(m_n \log(2\pi) + m_n \int \log(f_\theta(x)) \mathrm{d}\mu(x) + X_n^T \left(\mathcal{Q}_n(\frac{1}{f_\theta}) \right) X_n \right)$$

We denote respectively $\hat{\theta}_n, \tilde{\theta}_n, \bar{\theta}_n, \theta^{(u)}$ the maximum likelihood estimators associated respectively to $L_n, \tilde{L}_n, \bar{L}_n, L_n^{(u)}$.

We will need the following assumption,

Assumption 3.5. *There exists a positive sequence $(u_n)_{n\in\mathbb{N}}$ such that,*

$$u_n \underset{n\to\infty}{\to} 0,$$

and

$$\sup_{ij} \left| B^{(n)}_{ij} - 1 \right| \leq u_n.$$

Notice that the last assumption holds for example in the case $\mathbf{G} = \mathbb{Z}^d, d > 1$.

To prove asymptotic normality and efficiency of the last estimator $\theta_n^{(u)}$, we will need also the following assumption.

Assumption 3.6. *Assume that*

- *There exists a positive sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n = o(\sqrt{m_n})$ and*

$$\forall f \in \mathcal{F}_\rho, \left| \frac{1}{m_n} \operatorname{Tr}(\mathcal{K}_{G_n}(f)) - \int f \mathrm{d}\mu \right| \leq \alpha(f) v_n.$$

- *For any $\theta \in \Theta$, f_θ is twice differentiable on Θ and*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(f_\theta) \in \mathcal{F}_\rho, \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}(f_\theta) \in \mathcal{F}_\rho.$$

The first assumption means that the convergence of the empirical measure of eigenvalues of $\mathcal{K}(f)$ to the spectral measure μ is faster than $\sqrt{m_n}$. The second assumption is more classical. For example it is required in the case $\mathbf{G} = \mathbb{Z}$ (see [2]).

3.2 Convergence and asymptotic efficiency

Let $\rho > 0$. We can now state the fundamental theorem :

Theorem 3.1. *Under Assumptions 3.1, 3.2 and 3.3, the sequences $(\hat{\theta}_n)_{n\in\mathbb{N}}$, $(\bar{\theta}_n)_{n\in\mathbb{N}}$, $(\tilde{\theta}_n)_{n\in\mathbb{N}}$ converge, as n goes to infinity, $P_{f_{\theta_0}}$-a.s. to the true value θ_0. If moreover Assumption 3.5 holds, this is also true for $(\theta_n^{(u)})_{n\in\mathbb{N}}$.*

Proof. The proof follows the guidelines of [2]. We highlight the main changes performed here. First, we define the Kullback information on G_n of f_{θ_0} with respect to $f \in \mathcal{F}_\rho$, by

$$\mathbb{IK}_n(f_{\theta_0}, f) := \mathbb{E}_{P_{f_{\theta_0}}}\left[-\log(\frac{\mathrm{d}P_f}{\mathrm{d}P_{f_{\theta_0}}})\right].$$

and the asymptotic information (on $\mathbf{G}$) by

$$\mathbb{IK}(f_{\theta_0}, f) = \lim_n \frac{1}{m_n}\mathbb{IK}_n(f_{\theta_0}, f)$$

whenever it is finite.

The convergence of the estimators of the maximum approximated likelihood is a direct consequence of the following lemmas :

Lemma 3.1. *For any $f \in \mathcal{F}_\rho$, and under Assumptions 3.1, 3.2 and 3.3, the asymptotic Kullback information exists and may be written as*

$$\mathbb{IK}(f_{\theta_0}, f) = \frac{1}{2}\int\left(-\log(\frac{f_{\theta_0}}{f}) - 1 + \frac{f_{\theta_0}}{f}\right)\mathrm{d}\mu$$

Furthermore, if we set $l_n(\theta, X_n) = \frac{1}{m_n}L_n(\theta, X_n)$, we have that $P_{f_{\theta_0}}$-a.s.,

$$l_n(\theta_0, X_n) - l_n(\theta, X_n) \underset{n\to\infty}{\to} \mathbb{IK}(f_{\theta_0}, f_\theta)$$

uniformly in $\theta \in \Theta$.

This property also holds for $\bar{l}_n := \frac{1}{m_n}\bar{L}_n$ and $\tilde{l}_n := \frac{1}{m_n}\tilde{L}_n$

Furthermore, for $P > 0$, and for both the AR_P or the MA_P case (see above), this also holds for $l_n^{(u)} := \frac{1}{m_n}L_n^{(u)}$.

Lemma 3.2. *Let f_{θ_0} be the true spectral density, and $(\ell_n)_{n\in\mathbb{N}}$ be a deterministic sequence of continuous functions such that*

$$\forall\theta \in \Theta, \ell_n(\theta_0) - \ell_n(\theta) \underset{n\to\infty}{\to} \mathbb{IK}(f_{\theta_0}, f_\theta)$$

uniformly as n tends to infinity. Then, if $\theta_n = \arg\max_\theta \ell_n(\theta)$, we have

$$\theta_n \underset{n\to\infty}{\to} \theta_0.$$

The proof of these lemmas are given in Appendix (Subsection 7.2). □

Theorem 3.2. *In both the AR_P or MA_P cases, and and under all previous assumptions 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, the estimator $\theta_n^{(u)}$ of θ_0 is asymptotically normal:*

$$\sqrt{m_n}(\theta_n^{(u)} - \theta_0) \underset{n\to\infty}{\overset{\mathcal{D}}{\to}} \mathcal{N}\left(0, \left(\frac{1}{2}\int \left(\frac{f'_{\theta_0}}{f_{\theta_0}}\right)^2 \mathrm{d}\mu\right)^{-1}\right).$$

Furthermore, the Fisher information of the model is

$$J(\theta_0) := \frac{1}{2}\int \left(\frac{f'_{\theta_0}}{f_{\theta_0}}\right)^2 \mathrm{d}\mu.$$

Hence, previous estimator is asymptoticly efficient.

Proof. Here again, we mimic the usual proof by extending the result of [2] to the graph case.

Using a Taylor expansion, we get

$$(l_n^{(u)})'(\theta_0) = (l_n^{(u)})'(\theta_n^{(u)}) + (\theta_0 - \theta_n^{(u)})(l_n^{(u)})''(\breve{\theta}_n),$$

where $\breve{\theta}_n \in \left]\theta_n^{(u)}, \theta_0\right[$. As $\theta_n^{(u)} = \arg\max l_n^{(u)}$, we have

$$(l_n^{(u)})'(\theta_n^{(u)}) = 0.$$

So that,

$$\sqrt{m_n}(\theta_0 - \theta_n^{(u)}) = \left((l_n^{(u)})''(\breve{\theta}_n)\right)^{-1} \sqrt{m_n}(l_n^{(u)})'(\theta_0).$$

The end of the proof relies on three lemmas :

Lemma 3.3 provides the asymptotic normality for $\sqrt{m_n}(l_n^{(u)})'(\theta_0)$. Combined with Lemma 3.4, we get the asymptotic normality for $\sqrt{m_n}(\theta_0 - \theta_n^{(u)})$. Finally, Lemma 3.5 gives the Fisher information.

Lemma 3.3.

$$\sqrt{m_n}(l_n^{(u)})'(\theta_0) \underset{n\to\infty}{\overset{\mathcal{D}}{\to}} \mathcal{N}\left(0, \frac{1}{2}\int \left(\frac{f'_{\theta_0}}{f_{\theta_0}}\right)^2 \mathrm{d}\mu\right).$$

Lemma 3.4.

$$\left((l_n^{(u)})''(\breve{\theta}_n)\right)^{-1} \underset{n\to\infty}{\to} 2\left(\int \left(\frac{f'_{\theta_0}}{f_{\theta_0}}\right)^2 \mathrm{d}\mu\right)^{-1}, P_{f_{\theta_0}} - \ a.s.$$

Lemma 3.5. *The asymptotic Fisher information is :*

$$J(\theta_0) = \frac{1}{2}\int \left(\frac{f'_{\theta_0}}{f_{\theta_0}}\right)^2 \mathrm{d}\mu.$$

The proof of these lemmas are given in Appendix (Subsection 7.3)

□

4 Algebraical construction of covariance operators on a graph

In Section 2, we dealt with the case of $ARMA^{(W)}$ processes indexed by the vertices G of a graph $\mathbf{G}$. We built all processes using the weighted adjacency operator without specifying a general notion of stationarity. In this section, we give a definition of stationarity for Gaussian processes on a graph, compatible with the construction of $ARMA^{(W)}$ processes given in Section 2.

This leads to a large class of covariance operators. In this class, we lay a special interest in general $ARMA$, built with operators different from W. Actually, that may be seen as a modification of the graph on which we build $ARMA$ processes. So this section is devoted to modify the weights in an "isotropic" way, and locally, for finite order operators (see Definition 4.2). Now, let us give a sense to this "isotropic" modification.

For clarity, each time we will define a quantity of interest we will illustrate it for the case $\mathbf{G} = \mathbb{Z}$.

To define a notion of stationarity (with respect to W) for Gaussian processes indexed by G, a first idea is to use the set of all automorphisms of $\mathbf{G}$. Recall that, a permutation σ on G is an automorphism if it leaves W invariant :

$$\forall i, j \in G,\ W_{\sigma(i)\sigma(j)} = W_{ij}.$$

In the case of $\mathbb{Z}$, the automorphisms are the symmetry and the translation operators. Stationarity is defined through invariance by these transformations of the covariance function. More generally, the definition of stationarity on $\mathbb{Z}^d$, homogeneous trees or distance transitive graph may be set in the same way [14], [1], [17]. Unfortunately, as the generic situation for a graph is to a have a trivial set of automorphisms (reduced to the identity operator), this way to define stationarity is a dead-end. Indeed, in this approach any covariance operator would be stationary. To get away from this dead-end we choose to take another path **defining stationary covariance operators as images of W by adequate invariant functions**. We recall here that $W \in B_G$ where B_G is the set of all bounded Hilbertian operators on $l^2(G)$. We denote also by Σ_G the set of permutation of G. Let us define the operator M_σ attached to the permutation $\sigma \in \Sigma_G$ by

$$\forall i, j \in G, (M_\sigma)_{ij} = \mathbb{1}_{i=\sigma(j)}.$$

Let F_G be the set of all continuous linear operators from $l^1(G)$ to $l^\infty(G)$. We first define the class of invariant functions.

Definition 4.1. *We call invariant a function*

$$\Phi : \mathrm{Dom}(\Phi) \subset B_G \mapsto F_G,$$

that satisfies the following assumptions:

- $\mathrm{Dom}(\Phi)$ *is stable by any permutations and by transposition*

$$\forall A \in \mathrm{Dom}(\Phi), \forall \sigma \in \Sigma_G, M_\sigma^{-1} A M_\sigma \in \mathrm{Dom}(\Phi), A^T \in \mathrm{Dom}(\Phi).$$

- *For any permutation $\sigma \in \Sigma_G$, Φ commutes with the conjugation by M_σ*

$$\forall \sigma \in \Sigma_G, \forall A \in \mathrm{Dom}(\Phi), \Phi(M_\sigma^{-1} A M_\sigma) = M_\sigma^{-1} \Phi(A) M_\sigma.$$

- *Φ commutes with the transposition*

$$\forall A \in\in \mathrm{Dom}(\Phi), \Phi(A^T) = \Phi(A)^T.$$

We will denote by $\mathcal{I}_G$ the set of invariant functions from a subset of B_G to F_G.

Remark Notice that any invariant function $\Phi \in \mathcal{I}_G$ is given by a family of functions Φ_{ij} all defined on a subset $\mathrm{Dom}(\Phi)$ of B_G. Let $1_G, 2_G \in G$ be two vertices of $\mathbf{G}$. Actually, thanks to the invariance, the functions $(\Phi_{ij})_{i,j \in G}$ are completely determined by their domain $\mathrm{Dom}(\Phi)$, and two real-valued functions $\phi := \Phi_{1_G 1_G}$ and $\psi := \Phi_{1_G 2_G}$ from B_G to $\mathbb{R}$ such that

- For any $k \in G$, for all $A \in \mathrm{Dom}(\Phi)$, and for any permutation $\sigma \in \Sigma_G$ such that $\sigma(1_G) = k$,
$$\Phi_{kk}(A) = \phi(M_\sigma^{-1} A M_\sigma).$$
- For any $k, l \in G, k \neq l$, for all $A \in \mathrm{Dom}(\Phi)$, and for any permutation $\sigma \in \Sigma_G$ such that $\sigma(1_G) = k, \sigma(2_G) = l$,
$$\Phi_{kl}(A) = \psi(M_\sigma^{-1} A M_\sigma).$$
- For all $A \in \mathrm{Dom}(\Phi)$, and for any permutation $\sigma \in \Sigma_G$ permuting 1_G and 2_G
$$\psi(A) = \psi(M_\sigma^{-1} A^T M_\sigma).$$

Note that we also get

- For any $k \in G$, for all $A \in \mathrm{Dom}(\Phi)$, and for any permutation $\sigma \in \Sigma_G$ letting 1_G invariant
$$\phi(A) = \phi(M_\sigma^{-1} A M_\sigma).$$
- For any $k \in G$, for all $A \in \mathrm{Dom}(\Phi)$, and for any permutation $\sigma \in \Sigma_G$ letting 1_G and 2_G invariant,
$$\psi(A) = \psi(M_\sigma^{-1} A M_\sigma).$$

An example is given by the discrete Laplacian. Recall that the discrete Laplacian $L^{(W^{(G)})}$ of $W^{(G)}$ on the graph $\mathbf{G}$ is defined by :

$$\forall i, j \in G, L_{ij}^{(W^{(G)})} := \mathbb{1}_{i=j} \sum_{k \in G} W_{ik}^{(G)} - W_{ij}^{(G)}.$$

Set, for $A \in B_G$,

$$\phi_L(A) := \frac{1}{2} \left(\sum_{k \in G} A_{k 1_G} + \sum_{k \in G} A_{1_G k} \right),$$

and

$$\psi_L(A) := -A_{1_G 2_G},$$

where ϕ and ψ are defined in the last remark. We get

$$\Phi_L(A)_{ij} = \mathbb{1}_{(i=j)} \frac{1}{2} \left(\sum_{k \in G} A_{k1_G} + \sum_{k \in G} A_{1_G k} \right) - A_{ij} = L_{ij}^{(A)}.$$

The domain of Φ_L is the set of the operators $A \in B_G$ such that, for any $i \in G$, the sequences $(A_{ik})_{k \in G}$ and $(A_{ki})_{k \in G}$ are summable. This domain is stable by permutations and transposition. Moreover, we can verify the invariance property, writing, for any permutation $\sigma \in \Sigma_G$,

$$\begin{aligned}
\Phi_L(A)_{\sigma(i)\sigma(j)} &= \mathbb{1}_{(\sigma(i)=\sigma(j))} \frac{1}{2} \left(\sum_{k \in G} A_{k\sigma(i)} + \sum_{k \in G} A_{\sigma(i)k} \right) - A_{\sigma(i)\sigma(j)} \\
&= \mathbb{1}_{i=j} \frac{1}{2} \left(\sum_{\sigma(k) \in G} A_{\sigma(k)\sigma(i)} + \sum_{\sigma(k) \in G} A_{\sigma(i)\sigma(k)} \right) - A_{\sigma(i)\sigma(j)} \\
&= \Phi_L(M_\sigma^{-1} A M_\sigma)_{ij}.
\end{aligned}$$

and

$$\begin{aligned}
\forall \sigma \in \Sigma_G, \Phi_L(A^T)_{ij} &= \mathbb{1}_{i=j} \frac{1}{2} \left(\sum_{k \in G} A_{ik} + \sum_{k \in G} A_{ki} \right) - A_{ji} \\
&= \mathbb{1}_{j=i} \sum_{k \in G} \frac{1}{2} \left(\sum_{k \in G} A_{jk} + \sum_{k \in G} A_{kj} \right) - A_{ji} \\
&= \Phi_L(M_\sigma^{-1} A M_\sigma)_{ij}.
\end{aligned}$$

Then, this shows that

$$\Phi_L \in \mathcal{I}_{\mathcal{G}}.$$

We go back to the general construction. One can have the intuition that the functions $\Phi_{ii}, i \in G$ associated to an invariant function Φ contains information on the exploration of the graph from the vertex $i \in G$. This leads to the definition of the order of an invariant function. This order may be either finite or infinite and roughly speaking depends on the exploration size. Recall that the natural distance $d_{\mathbf{G}}$ on $\mathbf{G}$ measure the length of the shortest path going a vertex to another. This distance depends only on the edges, and not on the weights.

For any $r > 0$, let us define $B^{(W)}(i, r)$ as the ball of radius r (for the natural distance $d_{\mathbf{G}}$) centered on a vertex $i \in G$:

$$B^{(W)}(i, r) := \{ j \in G, d_{\mathbf{G}}(i, j) \leq r \}.$$

We now precisely define the order of an invariant function.

Definition 4.2. *Let Φ be an invariant function in $\mathcal{I}_G$, we will say that Φ is r-local for some $r \geq 0$ if for any $W \in \mathrm{Dom}(\Phi)$, $\phi(W)$ depends only on $\left(W_{jk}, j, k \in B^{(W)}(1_G, r)\right)$, and $\psi(W)$ depends only on*

$$\left(W_{jk}, j, k \in B^{(W)}(1_G, r) \cup B^{(W)}(2_G, r)\right),$$

where the functions ϕ and ψ have been defined in the previous Remark. The order of Φ is define as the smallest $r \geq 0$ such that Φ is r-local.

The order of an invariant function is an important notion. Indeed, Definition 4.1 builds a large class of invariant functions. This class of function will lead to a class of covariance operators of stationary processes (see Definition 4.3). Therefore, this last class of stationary covariance operators will be very large too. Defining the order of an invariant function gives a way to classify stationary processes.

Back to the general construction, we recall that F_G is the set of linear continuous operators from $l^1(G)$ into $l^\infty(G)$. Any covariance operator lies in this set, so we can define stationarity over the graph $\mathbf{G}$ with some subclasses of F_G using the invariant functions $\mathcal{I}_G$.

We are now able to state an extension of isotropic stationary Gaussian processes to any graph.

Definition 4.3. *We say that a Gaussian process $(X_i)_{i \in G}$ is stationary of order r, if its covariance operator Γ verifies*

$$\Gamma \in \mathcal{S}_+^r(W) := \begin{Bmatrix} \Gamma = \Phi(W), \Phi \in \mathcal{I}_G, \Phi \text{ has order } r \\ \Gamma \text{ is positive definite} \end{Bmatrix}$$

We say that an operator W' is an admissible modification of the graph $\mathbf{G}$ of order r if we have

$$W' \in \mathcal{S}_1^r(W) := \begin{Bmatrix} W' = \Phi(W), \Phi \in \mathcal{I}_G, \Phi \text{ has order } r \\ \left(W_{ij} = 0 \Rightarrow W'_{ij} = 0\right) \end{Bmatrix}$$

Remark A stationary process of finite order $r > 0$ has the following property.

Let $(i_1, j_1), (i_2, j_2) \in G^2$ be two couples of vertices such that $B^{(W^{(G)})}(i_1, r)$ is isomorph to $B^{(W^{(G)})}(i_2, r)$ and $B^{(W^{(G)})}(j_1, r)$ is isomorph to $B^{(W^{(G)})}(j_2, r)$. Then for any stationary covariance operator Γ of order less or equal to r,

$$\Gamma_{i_1 j_1} = \Gamma_{i_2 j_2}.$$

That means that, if the graph is locally the isomorphic in two different regions, then the correlations will be also identical in this areas.

To close this section, let us notice that if the entries of W takes their values in a finite set, then the entries of any admissible modification of finite order $r > 0$ take also their values in a finite set, since there is a finite number of possible subgraphs of size bounded by $\deg(\mathbf{G})^k$.

Remark All the previous work (Sections 2 3) may be applied to a admissible modification W' of the graph instead of W.

In very particular, any AR_1 model built with a isotropic modification of $\mathbf{G}$ provides a graphical model [9]. That was one of the main motivation for this section. Indeed the definition of stationarity we propose here leads, on one hand, to usual stationary processes when it is well defined ($\mathbb{Z}^d$, the homogeneous tree...).

On the other hand, for a graph with a trivial set of automorphism, we recover many classical kind of modeling as $ARMA$ processes and Markov fields. This falls into a very classical point of view on Gaussian fields, and allow us to used all general tools developed in this framework for our case.

To end this section, let us deal with an example with the usual case of $G = \mathbb{Z}$. In this case, the last construction recover any covariance operator Γ, even non regular. Indeed, set

$$\forall i, j \in \mathbb{Z}, \Gamma_{ij} = r_{|i-j|},$$

and define, for any $A \in B_{\mathbb{Z}}$ such that the following is well defined, and for any $p \in \mathbb{N}, i, j \in \mathbb{Z}$,

$$\Phi^{(p)}(A)_{ij} = \sum_{k_1 \neq i,j \in \mathbb{Z}} \sum_{k_2 \neq i,j,k_1 \in \mathbb{Z}} \cdots \sum_{k_{p-1} \neq i,j,k_1,\cdots,k_{p-2} \in \mathbb{Z}} A_{ik_1} A_{k_1 k_2} \cdots A_{k_{p-1} j} 2^p.$$

We can define $\Phi \in \mathcal{I}_G$, for any A such it is well defined, as

$$\Phi(A) = \sum_{p \geq 0} r_p \Phi^{(p)}(A).$$

Notice that $W^{(\mathbb{Z})} \in \mathrm{Dom}(\Phi)$. Thus we get that

$$\Gamma = \Phi(W^{(\mathbb{Z})}),$$

which proves the statement.

Remark Back to the general case, if Γ is the covariance of an ARMA process over the graoh $\mathbf{G}$, build with any admissible modification W' of the graph, then $\Gamma \in \mathcal{S}_+^\infty$, so ARMA processes are always stationary processes with respect to Definition 4.3.

In the framework of raod traffic modelling, as in other modelling issues, W' may be seen as a structural generating operator. That means that the physical evolution of the underlying temporal process (diffusion...) is given by this operator. The parameters of the ARMA process gives the particular state of the field at a given time.

One example of this construction will be given in an applied forthcoming paper.

5 Homogeneity and spectral measure

This section is entirely devoted to the study of the convergence of the spectral measure (see 3.3). As before, we will work with the adjacency operator W and a sequence of finite nested subgraphs $(G_n)_{n \in \mathbb{N}}$, but this work can be extended to any admissible modification $W' \in S_1^\infty$ of the graph $\mathbf{G}$.

Recall that m_n denotes the cardinal of the subgraph G_n. Let W_{G_n} denote the restriction of W over the subgraph G_n (see Section 3), and $\lambda_1^{(n)}, \cdots, \lambda_{m_n}^{(n)}$ its eigenvalues (written with their multiplicity orders). Define

$$\mu_n^{[1]} := \frac{1}{m_n} \sum_{i=1}^{m_n} \delta_{\lambda_i^{(n)}},$$

and

$$\mu_n^{[2]} = \frac{1}{m_n} \sum_{x \in G_n} \mu_{xx},$$

where the measure μ_{xx} is defined in Section 1. Define also

Recall that Assumption 3.3 deals with the convergence of $\mu_n^{[2]}$ to the spectral measure μ. Actually, under Assumption 3.1, this last assumption is equivalent to the following.

Assumption 5.1. *Weak convergence of the spectral measure*

$$\mu_n^{[1]} \underset{n \to \infty}{\to} \mu.$$

in the sense of the weak convergence.

To prove the equivalence between Assumption 3.3 and Assumption 5.1, we just have to notice that :

$$\begin{aligned} \int_{\mathrm{Sp}(W)} \lambda^k \mathrm{d}\mu_n^{(1)}(\lambda) &- \int_{\mathrm{Sp}(W)} \lambda^k \mathrm{d}\mu_n^{(2)}(\lambda) \\ &= \frac{1}{m_n} \sum_{i=1}^{m_n} \left(\lambda^{(n)}\right)_i^k - \frac{1}{m_n} \sum_{x \in G_n} (W^k)_{xx} \\ &= \frac{1}{m_n} \mathrm{Tr}\left((W_{G_n})^k\right) - \frac{1}{m_n} \mathrm{Tr}\left((W^k)_{G_n}\right). \end{aligned}$$

So that, we get the result by lemma 7.1.

This is a classical graph theory assumption, usually verified in the frame of random graph (in which $(G_n)_{n \in \mathbb{N}}$ is not a growing sequence of nested subgraphs, but a sequence of graphs of size n whose spectral measure converges weakly to a particular law.). For instance, the spectral measure of some random graphs with given degree distribution converges to the spectral measure of regular tree (see [3] [4], [20]).

In the frame of road traffic, we provide also another assumption about the sequence of graphs, that ensures the convergence of the spectral measure.

We first define, for any $l \geq 0$, the l-type of a vertex $k \in G$ by

$$t_l(k) = W_{kk}^l.$$

The l-type of a vertex is the number of loops around k, counted with their weight. Define also a subset U_l of $\mathbb{R}$ as

$$U_l := \{t_l(k), k \in G\}.$$

This set gives all the possible values among the vertices $k \in G$ of the l-type of k. Notice first that when the entries of W takes their values in a finite set, U_l is finite for any $l \geq 0$. In particular it is also true for any admissible modification of W of finite order (see Section 4).

Assumption 5.2 (Homogeneity assumption)**.** *The sequence* $(G_n)_{n\in\mathbb{N}}$ *and the operator* W *satisfy :*

$$\forall l \geq 0, \forall n \geq 1, \forall v \in U_l, \frac{\sharp\{j \in G_n, t_l(j) = v\}}{m_n} \underset{n\to\infty}{\to} p_v^{(l)}.$$

Remark : This last assumption may be understood as an homogeneity hypothesis on the frequency of the type for the vertices of G_n. Figure 2 shows an example of admissible sequences of subgraphs.

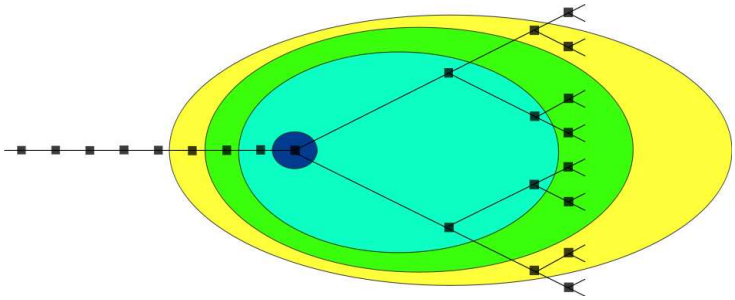


Figure 2: Example of admissible sequences of subgraphs.

Proposition 5.1. *Under Assumption 5.2, the spectral measure is convergent. That is, there exists* μ *with*

$$\mu_n^{[2]} \underset{n\to\infty}{\overset{\mathcal{D}}{\to}} \mu, \text{ in distribution.}$$

Proof. Under homogeneity assumption, we have

$$\frac{1}{m_n}\mathrm{Tr}\left((W^l)_{G_n}\right) \underset{n\to\infty}{\to} \sum_{v\in U_l} p_v^{(l)} v.$$

Define

$$\mu^{(l)} := \sum_{v\in J_l} p_v^{(l)} v.$$

Since $\sup_{i,j\in S}|W_{ij}| \leq \frac{1}{d}$, we also have

$$\|W\|_{2,in} \leq 1.$$

So that

$$\forall l \geq 0, U_l \subset \mathrm{Sp}(W) \subset [-1, 1].$$

And

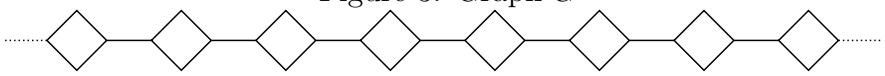
$$\mu^{(l)} \leq 1.$$

Since the series $t \mapsto \sum \mu^{(l)} l \frac{t^l}{l!}$ has a positive convergence radius, there exists a probability measure μ with moments $(\mu^{(l)})_{l\geq 0}$. The weak convergence of the measure is derived from the convergence of the moments, and the tightness of μ. This finishes the proof of the last proposition. □

6 Simulations

In this section, we give some simulations over a very simple case, where the graph G is built taking some rhombus connected by a simple edge both on the left and right (see Figure 3).

Figure 3: Graph G

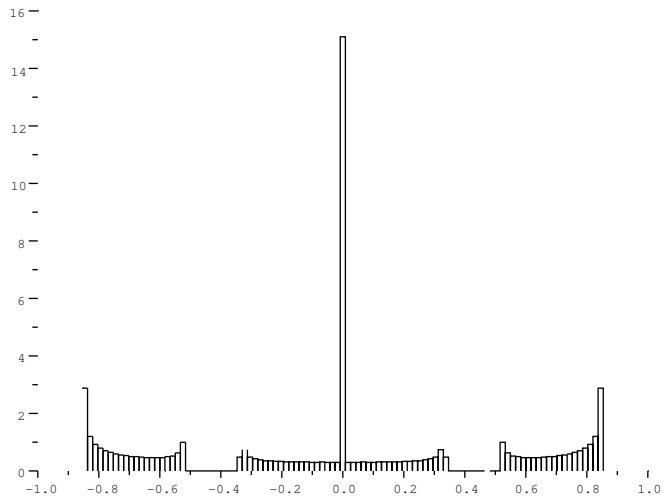


The sequence of nested subgraphs chosen here is the growing neighborhood sequence (we chose a point x and we take $G_n = \{y \in G, d_{\mathbf{G}}(x,y) \leq n\}$). We study an AR_2 model, where,

$$\Theta =]-1,1[\,,$$
$$f_\theta(x) = \left(\frac{1}{1-\theta x}\right)^2 (\theta \in \Theta).$$

Here, we take for W the adjacency operator of G normalized in order to get $\sup_{i,j\in G} W_{ij} \leq \frac{1}{\deg(G)}$. We chose $\theta_0 = \frac{1}{2}$, $m_n = 724$. We approximate the spectral measure of G by the spectral measure of a very large graph (around 10000 vertices) built in the same way. Figure 4 shows the empirical spectrum of the graph G with respect to the sequence of subgraphs $(G_n)_{n\in\mathbb{N}}$.

Figure 4: Empirical spectrum



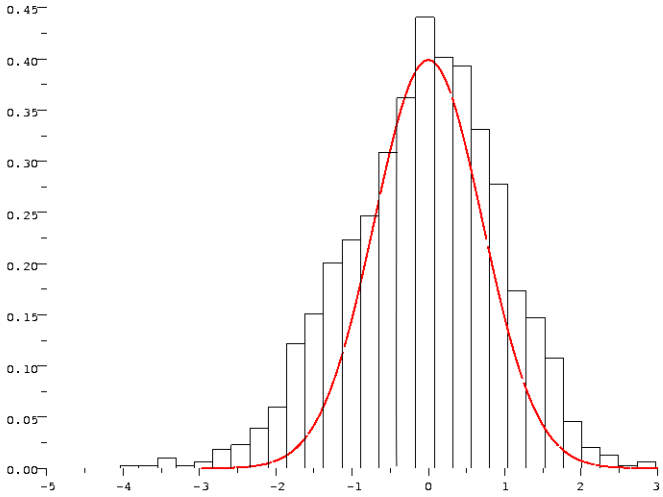
To compute $(\mathcal{K}_n(f_\theta))^{-1}$, we use the power series representation of f_θ, and truncate this expression after the 15 first coefficient. This choice ensures that the simulation errors are neglectible with respect to the theoretical ones.

Figure 5 gives the empirical distribution of

$$\sqrt{m_n}\sqrt{\int_{\mathrm{Sp}(A)} \left(\frac{f'_\theta}{f_\theta}\right)^2}\left(\tilde{\theta}_n - \theta_0\right).$$

We observe that given the size of the subgraphs chosen, the error is a little less concentrated than the asymptotic error (in red) which is a $\mathcal{N}(0, 1)$.

Figure 5: Empirical distribution



7 Appendix

7.1 Szegö Lemmas

Szegö Lemmas [13] are useful in time series analysis. Indeed, they provide good approximations for the likelihood. As explained in Section 3, these approximations of the likelihood are easier to compute.

In this section, we generalize a weak version of the Szegö Lemmas, for a general graph, under Assumption 3.1 (non expansion criterion for G_n), and Assumption 3.3 (existence of the spectral measure μ).

For any matrix $(B_{ij})_{i,j\in G_n}$, we define the block norm

$$b_N(B) = \frac{1}{\delta_N} \sum_{i,j\in G_N} |B_{ij}| .$$

We can state the equivalent version of the first Szegö lemma for time-series

Lemma 7.1. *Asymptotic homomorphism*

Let k, n be positive integers, and let $g_1, \cdots, g_k$ be analytic functions over $[-1, 1]$ having finite regularity factors (i.e. $\alpha(g_i) < +\infty, i = 1, \cdots, k$). Then,

$$b_n \left(\mathcal{K}_n(g_1) \cdots \mathcal{K}_n(g_k) - \mathcal{K}_n(g_1 \cdots g_k)\right) \leq \frac{k-1}{2} \alpha(g_1) \cdots \alpha(g_k).$$

Corollary 7.1. *For any $g \in \mathcal{F}_\rho$ (see Section 3), and under Assumptions 3.1 and 3.3,*

$$\frac{1}{m_n} \log \det(\mathcal{K}_n(g)) \underset{n\to\infty}{\to} \int \log(g) \mathrm{d}\mu.$$

Proof. of Lemma 7.1 This proof follows again the one of [2]. We will prove the result by induction on k.

First we deal with the case $k = 2$. Let f and g analytic functions over $[-1, 1]$ such that $\alpha(f) < +\infty$ and $\alpha(g) < +\infty$. We write

$$\begin{aligned} & b_n(\mathcal{K}_n(f)\mathcal{K}_n(g) - \mathcal{K}_n(fg)) \\ = & \frac{1}{\delta_n} \sum_{i,j\in G_n} \left| \sum_{k\in G_n} \left(\mathcal{K}_n(f)\right)_{ik} \left(\mathcal{K}_n(g)\right)_{kj} - \sum_{k\in G} \left(\mathcal{K}_n(f)\right)_{ik} \left(\mathcal{K}_n(g)\right)_{kj} \right| \\ = & \frac{1}{\delta_n} \sum_{i,j\in G_n} \sum_{k\in G\setminus G_n} |\mathcal{K}(f)_{ik}| \, |\mathcal{K}(g)_{kj}| . \end{aligned}$$

Using $\mathcal{K}(g) = \sum_{h=0}^{\infty} g_h W^h$, Fubini's theorem gives, since all the previous sequences are

in $l^1(G)$,

$$\begin{aligned} & b_n(\mathcal{K}_n(f)\mathcal{K}_n(g) - \mathcal{K}_n(fg)) \\ \leq & \frac{1}{\delta_n} \sum_{i,j \in G_n} \sum_{k \in G \setminus G_n} \left| (\mathcal{K}_n(f))_{ik} (\mathcal{K}_n(g))_{kj} \right| \\ \leq & \left(\sup_{k \in G \setminus G_n} \sum_{i \in G_n} |\mathcal{K}(f)_{ik}| \right) \times \frac{1}{\delta_n} \sum_{k \in G \setminus G_n} \sum_{j \in G_n} \sum_{h=0}^{\infty} |g_h| \left| (W^h)_{kj} \right| \\ \leq & \left(\sup_{k \in G} \sum_{i \in G} |\mathcal{K}(f)_{ik}| \right) \times \sum_{h=0}^{\infty} |g_h| \frac{1}{\delta_n} \sum_{k \in G \setminus G_n} \sum_{j \in G_n} \left| (W^h)_{kj} \right|. \end{aligned}$$

Introducing

$$\Delta_h = \sup_{N \in \mathbb{N}} \frac{1}{\delta_N} \sum_{k \in G \setminus G_N} \sum_{j \in G_N} \left| \left(W^h\right)_{kj} \right|,$$

we get

$$b_n(\mathcal{K}_n(f)\mathcal{K}_n(g) - \mathcal{K}_n(fg)) \leq \sup_{k \in G} \sum_{i \in G} |\mathcal{K}(f)_{ik}| \sum_{h=0}^{\infty} |g_h| \Delta_h.$$

The coefficient Δ_h is a porosity factor. It measures the weight of the paths of length h going from the interior of G_n to outside.

Note that $\Delta_h \leq h+1$, so we get

$$\sum_{h=0}^{\infty} |g_h| \Delta_h \leq \alpha(g).$$

Now, we define another norm on B_G :

$$\|B\|_{\infty,in} := \sup_{k \in G} \sum_{i \in G} |B_{ik}|, (B \in B_G).$$

We thus obtain

$$\begin{aligned} \|\mathcal{K}(f)\|_{\infty,in} &= \sup_{k \in G} \sum_{i \in G} |\mathcal{K}(f)_{ik}| \\ &\leq \sum_{h=0}^{\infty} |f_h| \left\|W^h\right\|_{\infty,in} \\ &\leq \sum_{h=0}^{\infty} |f_h| \|W\|_{\infty,in}^h \\ &\leq \sum_{h=0}^{\infty} |f_h| := \|f\|_{1,pol}. \end{aligned}$$

Finally, we get

$$b_n(\mathcal{K}_{G_n}(f)\mathcal{K}_{G_n}(g) - \mathcal{K}_{G_n}(fg)) \leq \|f\|_{1,pol} \alpha(g)$$

To conclude the proof of the lemma, by symmetrization of the last inequality, and since $1 \leq (h+1)$, we have,

$$b_n\left(\mathcal{K}_n(f)\mathcal{K}_n(g) - \mathcal{K}_n(fg)\right) \leq \frac{1}{2}\alpha(f)\alpha(g).$$

To perform the inductive step, we need the following inequalities [24]:

$$\begin{aligned}
\alpha(fg) &\leq \alpha(f)\alpha(g), \\
b_n(BC) &\leq \|B\|_{\infty,in}\, b_n(C), \\
b_n(B+C) &\leq b_n(B) + b_n(C), \\
\|\mathcal{K}_n(f)\|_{\infty,in} &= \|f\|_{1,pol} \leq \alpha(f).
\end{aligned}$$

Let $k > 1$, and assume that for all $j \leq k-1$, Lemma 7.1 holds. Under the previous assumptions, and the inductive hypothesis for $k-1$ we get,

$$\begin{aligned}
b_n\left(\mathcal{K}_n(g_1)\times \quad \cdots \quad \times\mathcal{K}_n(g_k) - \mathcal{K}_n(g_1\cdots g_k)\right) & \\
\leq \|\mathcal{K}_n(g_1)\|_{\infty,in}\, b_n\left(\mathcal{K}_n(g_2)\cdots\mathcal{K}_n(g_k) - \mathcal{K}_n(g_2\cdots g_k)\right) & \\
+ b_n\left(\mathcal{K}_n(g_1)\mathcal{K}_n(g_2\cdots g_k) - \mathcal{K}_n(g_1\cdots g_k)\right) & \\
\leq \alpha(g_1)\frac{k-2}{2}\alpha(g_2)\cdots\alpha(g_k) + \frac{1}{2}\alpha(g_1)\alpha(g_2\cdots g_k) & \\
\leq \frac{k-1}{2}\alpha(g_1)\cdots\alpha(g_k), &
\end{aligned}$$

which achieves the induction step and proves the result. □

Proof. of Corollary 7.1

Let $g \in \mathcal{F}_\rho$, and k be a positive integer. Using Lemma 7.1, we have

$$\operatorname{Tr}\left(\mathcal{K}_n(g)^k - \mathcal{K}_n(g^k)\right) \leq \frac{\delta_n}{m_n} b_n\left(\mathcal{K}_n(g)^k - \mathcal{K}_n(g^k)\right). \tag{1}$$

Thus, we have, thanks to Assumption 3.1

$$\frac{1}{m_n}\operatorname{Tr}\left(\mathcal{K}_n(g)^k - \mathcal{K}_n(g^k)\right) \underset{n\to+\infty}{\to} 0.$$

Denote $\mu_g^{[1]}$ the real measure whose k^{th}-moment is given by

$$\int x^k \mathrm{d}\mu_g^{[1]} = \lim_n \frac{1}{m_n}\operatorname{Tr}\left(\mathcal{K}_n(g)^k\right),$$

and $\mu_g^{[2]}$ the real measure whose k^{th}-moment is given by

$$\int x^k \mathrm{d}\mu_g^{[2]} = \lim_n \frac{1}{m_n}\operatorname{Tr}\left(\mathcal{K}_n(g^k)\right).$$

Notice that both of this measures have support between $\inf g \geq e^{-\rho} > 0$ and $\sup g \leq e^{\rho} < +\infty$, since $\alpha(\log(g)) < \rho$ (see Section 3). Therefore, the equality of the moments given by Equation 1 gives the equality of the measures $\mu_g^{[1]}$ and $\mu_g^{[2]}$.

So that, we get

$$\frac{1}{m_n}\log\left(\det\left(\mathcal{K}_n(g)\right)\right)-\frac{1}{m_n}\mathrm{Tr}\left(\mathcal{K}_n\left(\log(g)\right)\right)\underset{n\to+\infty}{\to}0.$$

Assumption 3.3 achieves the proof of the Corollary since it implies that

$$\frac{1}{m_n}\mathrm{Tr}\left(\mathcal{K}_n\left(\log(g)\right)\right)\underset{n\to+\infty}{\to}\int\log(g)\mathrm{d}\mu.$$

□

The following lemma enables to replace $\mathcal{K}_n(g)$ by the unbiased version $\mathcal{Q}_n(g)$ (see 3).

Lemma 7.2. *Under assumptions 3.1,3.3, 3.4 and 3.5, and if f or g is a polynomial having degree less than or equal to P, we have*

$$\left|\frac{1}{m_n}\mathrm{Tr}\left((\mathcal{K}_n(f)\mathcal{K}_n(g))^p-(\mathcal{K}_n(f)\mathcal{Q}_n(g))^p\right)\right|\leq 2^p u_n\alpha(f)^p\alpha(g)^p$$

Proof. We define, for any f,

$$f_{abs}(x)=\sum_k|f_k|\,x^k$$

Actually, the proof is based of the following idea: as soon as f or g is a polynomial having degree less than or equal to P, we have to control only the number of path of length less than or equal to P (counted with their weights).

Let p be a positive number. Recall that $\mathcal{Q}_n(\frac{1}{g})=B^{(n)}\odot\mathcal{K}_n(\frac{1}{g})$ (see Section 3), we have,

$$
\begin{aligned}
\tfrac{1}{m_n} & \left|\mathrm{Tr}\left(\left(\mathcal{K}_n(f)\mathcal{K}_n(\frac{1}{g})\right)^p - \left(\mathcal{K}_n(f)\mathcal{Q}_n(\frac{1}{g})\right)^p\right)\right| \\
\leq & \frac{1}{m_n}\left|\sum_{i\in G_n}\sum_{i_0=i,i_1,\cdots,i_{2p}=i}\prod_{l=0\cdots p} B^{(n)}_{i_{2l}i_{2l+1}}\mathcal{K}_n(\frac{1}{g})_{i_{2l}i_{2l+1}}\mathcal{K}_n(f)_{i_{2l+1}i_{2l+2}}\right. \\
& \left. - \frac{1}{m_n}\sum_{i\in G_n}\sum_{i_0=i,i_1,\cdots,i_{2p}=i}\prod_{l=0\cdots p}\mathcal{K}_n(\frac{1}{g})_{i_{2l}i_{2l+1}}\mathcal{K}_n(f)_{i_{2l+1}i_{2l+2}}\right| \\
\leq & \frac{1}{m_n}\sup_{i_1,i_2,\cdots,i_{2p+1}}\left|\prod_{l=0\cdots p-1} B^{(n)}_{i_{2l+1}i_{2l+2}} - 1\right| \\
& \times\sum_{i\in G_n}\sum_{i_0=i,i_1,\cdots,i_{2p}=i}\prod_{l=0\cdots p}\left|\mathcal{K}_n(\frac{1}{g})_{i_{2l}i_{2l+1}}\mathcal{K}_n(f)_{i_{2l+1}i_{2l+2}}\right| \\
\leq & \frac{1}{m_n}\sup_{i_1,i_2,\cdots,i_{2p+1}}\left|\prod_{l=0\cdots p-1} B^{(n)}_{i_{2l+1}i_{2l+2}} - 1\right| \\
& \times\sum_{i\in G_n}\sum_{i_0=i,i_1,\cdots,i_{2p}=i}\prod_{l=0\cdots p}\mathcal{K}_n((\frac{1}{g})_{abs})_{i_{2l}i_{2l+1}}\mathcal{K}_n(f_{abs})_{i_{2l+1}i_{2l+2}} \\
\leq & \sup_{i_1,i_2,\cdots,i_{2p+1}}\left|\prod_{l=0\cdots p-1} B^{(n)}_{i_{2l+1}i_{2l+2}} - 1\right|\left\|\left(K_{G_n}(f_{abs})K_{G_n}((\frac{1}{g})_{abs})\right)^p\right\|_{2,in} \\
\leq & \sup_{i_1,i_2,\cdots,i_{2p+1}}\left|\prod_{l=0\cdots p-1} B^{(n)}_{i_{2l+1}i_{2l+2}} - 1\right|\alpha(f)^p\alpha(\frac{1}{g})^p.
\end{aligned}
$$

Using Assumption 3.5, we get,

$$
\begin{aligned}
\tfrac{1}{m_n} & \left|\mathrm{Tr}\left(\left(\mathcal{K}_n(f)\mathcal{K}_n(\frac{1}{g})\right)^p - \left(\mathcal{K}_n(f)\mathcal{Q}_n(\frac{1}{g})\right)^p\right)\right| \\
\leq & \left|(1+u_n)^p - 1\right|\alpha(f)^p\alpha(\frac{1}{g})^p \\
\leq & \left|(1+u_n-1)\left((1+u_n)^{p-1} + (1+u_n)^{p-2} + \cdots + 1\right)\right|\alpha(f)^p\alpha(\frac{1}{g})^p \\
\leq & \left|u_n\left(2^p-1\right)\right|\alpha(f)^p\alpha(\frac{1}{g})^p \\
\leq & u_n 2^p\alpha(f)^p\alpha(\frac{1}{g})^p
\end{aligned}
$$

This ends the proof of the Lemma. □

Finally, the following lemma explains the choice of $B^{(n)}$. The unbiased quadratic form $\mathcal{Q}_n$ is no more than a correction of the error between $\mathcal{K}_n(f)\mathcal{K}_n(g)$ and $\mathcal{K}_n(fg)$.

Lemma 7.3 (Exact correction)**.** *Let $f, g \in \mathcal{F}_\rho$, and assume that either f or g is a polynomial of degree less than or equal to P (see section 3). Then, the unbiased quadratic form $\mathcal{Q}_n(f_\theta)$ verify*

$$\mathrm{Tr}\left(\mathcal{K}_n(f)\mathcal{Q}_n(g)\right) = \mathrm{Tr}\left(\mathcal{K}_n(fg)\right).$$

Proof. of Lemma 7.3

First, notice that

$$\mathrm{Tr}\left(\mathcal{K}_n(f)\mathcal{Q}_n(g)\right) = \sum_{i,j\in G_n} \mathcal{K}_n(f)_{ij}\mathcal{K}_n(g)_{ij}B_{ij}^{(n)}.$$

Since this expression is symmetric on f, g, we can now consider the case where f is a polynomial of degree less than or equal to P.

Actually, since f is a polynomial, $\mathcal{K}_n(f)_{ij} = 0$ as soon as $d(i,j) > P$ $(i,j \in G)$. Then, if $i, j, k, l \in G$ are such that $\mu_{ij} = \mu_{kl}$, we have

$$\mathcal{K}_n(f)_{ij}\mathcal{K}_n(g)_{ij} = \mathcal{K}_n(f)_{kl}\mathcal{K}_n(g)_{kl}.$$

So that, we may here denote, for convenience, $K(f)_{\mu_{ij}}$.

Using Assumption 3.4, this leads to

$$\begin{aligned}
\mathrm{Tr}\left(\mathcal{K}_n(f)\mathcal{Q}_n(g)\right) &= \sum_{i,j\in G_n} \mathcal{K}_n(f)_{ij}\mathcal{K}_n(g)_{ij}B_{ij}^{(n)} \\
&= \sum_{v\in V_P} \sum_{\substack{i,j\in G_n \\ \mu_{ij}=v, d_{\mathbf{G}}(i,j)\le P}} \mathcal{K}_n(f)_v\mathcal{K}_n(g)_v B_v^{(n)} \\
&= \sum_{v\in V_P} \mathcal{K}_n(f)_v\mathcal{K}_n(g)_v \mathrm{Card}\left\{(i,j)\in G_n\times G_n, \mu_{ij}=v\right\} \\
&\quad \times \frac{\mathrm{Card}\left\{(i,j)\in G_n\times G, \mu_{ij}=v\right\}}{\mathrm{Card}\left\{(i,j)\in G_n\times G_n, \mu_{ij}=v\right\}}, \\
&= \sum_{v\in V_P} \sum_{\substack{(i,j)\in G_n\times G, \\ \mu_{ij}=v, d_{\mathbf{G}}(i,j)\le P}} \mathcal{K}_n(f)_v\mathcal{K}_n(g)_v B_v^{(n)} \\
&= \sum_{(i,j)\in G_n\times G} \mathcal{K}_n(f)_{ij}\mathcal{K}_n(g)_{ij}B_{ij}^{(n)} \\
&= \mathrm{Tr}\left(\mathcal{K}_n(fg)\right).
\end{aligned}$$

That finishes the proof of Lemma 7.3, and ends this section. □

7.2 Proofs of the Lemmas of theorem 3.1

Recall that the theorem relies on two lemmas. Lemma 3.2 states a condition on deterministic sequences to provide the convergence of the maximizer of these sequences.

Proof. of Lemma 3.2 Recall that f_{θ_0} denotes the true spectral density. Let $(\ell_n)_{n\in\mathbb{N}}$ be a deterministic sequence of continuous functions such that

$$\forall\theta\in\Theta, \ell_n(\theta_0) - \ell_n(\theta) \underset{n\to\infty}{\to} \frac{1}{2}\int\left(-\log(\frac{f_{\theta_0}}{f_\theta}) - 1 + \frac{f_{\theta_0}}{f_\theta}\right)\mathrm{d}\mu.$$

uniformly as n tends to infinity. Denotes moreover $\theta_n = \arg\max_\theta \ell_n(\theta)$. We aim at proving that

$$\theta_n \underset{n\to\infty}{\to} \theta_0.$$

Using the compactness of Θ, let θ_∞ be an accumulation point of the sequence $(\theta_n)_{n\in\mathbb{N}}$, and $(\theta_{n_k})_{k\in\mathbb{N}}$ be a subsequence converging to θ_∞. As the function $\theta \mapsto \frac{1}{2}\int \left(-\log(\frac{f_{\theta_0}}{f_\theta}) - 1 + \frac{f_{\theta_0}}{f_\theta}\right) \mathrm{d}\mu$ is continuous on Θ, and the convergence of $(\ell_n(\theta_0) - \ell_n(\theta))_{n\in\mathbb{N}}$ is uniform in θ, we have

$$\ell_{n_k}(\theta_0) - \ell_{n_k}(\theta_{n_k}) \xrightarrow{k\to\infty} \frac{1}{2}\int -\log(\frac{f_{\theta_0}}{f_{\theta_\infty}}) - 1 + \frac{f_{\theta_0}}{f_{\theta_\infty}}\mathrm{d}\mu.$$

But we can notice that, thanks to the definition of θ_n, $\ell_{n_k}(\theta_0) - \ell_{n_k}(\theta_{n_k}) \leq 0$ So, since the function $x \mapsto -\log(x) + x - 1$ is non negative and vanishes if, and only if, $x = 1$, we get that $f_{\theta_0} = f_{\theta_\infty}$. By injectivity of the function $\theta \to f_\theta$, we get $\theta_\infty = \theta_0$, for any accumulation point θ_∞ of the sequence $(\theta_n)_{n\in\mathbb{N}}$, which ends the proof of this first lemma. □

Lemma 3.1 provides the uniform convergence of the contrasts of maximum likelihood and approximated maximum likelihood to the Kullback information. The proof may be cut into several lemmas.

Proof. of Lemma 3.1

First, notice that by construction, we have, for any $\theta \in \Theta$,

$$\mathbb{IK}(f_{\theta_0}, f_\theta) = \lim_n \mathbb{E}\left[\frac{1}{m_n}\left(L_n(f_{\theta_0}, X_n) - L_n(f_\theta, X_n)\right)\right]$$

when it exists. Then, we can compute

$$\begin{aligned} l_n(f_{\theta_0}, X_n) - l_n(f_\theta, X_n) &= -\frac{1}{2m_n}\left(\log\det(\mathcal{K}_n(f_{\theta_0})) - \log\det(\mathcal{K}_n(f_\theta))\right) \\ &\quad -\frac{1}{2m_n}\left(X_n^T\mathcal{K}_n(f_{\theta_0})^{-1}X_n - X_n^T\mathcal{K}_n(f_\theta)^{-1}X_n\right) \end{aligned}$$

Corollary 7.1 of Lemma 7.1 provides the following convergence

$$\frac{1}{m_n}\left(\log\det(\mathcal{K}_n(f_{\theta_0})) - \log\det(\mathcal{K}_n(f_\theta))\right) \underset{n\to\infty}{\to} \int \log\left(\frac{f_{\theta_0}}{f_\theta}\right)\mathrm{d}\mu$$

To prove the existence of $\mathbb{IK}(f_{\theta_0}, f_\theta)$, it only remains to prove the $\mathbb{P}_{f_{\theta_0}}$-a.s. convergence of $\frac{1}{m_n}X_n^T\mathcal{K}_n(f_\theta)^{-1}X_n$ to $\int \frac{f_{\theta_0}}{f_\theta}\mathrm{d}\mu$ as n goes to infinity.

This is ensured by the following Lemma.

Lemma 7.4 (Concentration lemma)**.** *For respectively* $\Lambda = \mathcal{K}_n(\frac{1}{f_\theta})$, $\Lambda = (\mathcal{K}_n(f_\theta))^{-1}$ *or* $\Lambda = \mathcal{Q}_n(\frac{1}{f_\theta})$*, we have,*

$$\frac{1}{m_n}X_n^T\Lambda X_n \underset{n\to\infty}{\to} \int \frac{f_{\theta_0}}{f_\theta}\mathrm{d}\mu, \mathbb{P}_{f_{\theta_0}} - a.s..$$

Lemma 7.4 combined with Corollary 7.1 ensures the $\mathbb{P}_{f_{\theta_0}}$ − a.s. convergence of $\tilde{l}_n(f_{\theta_0}) - \tilde{l}_n(f_\theta)$, $\bar{l}_n(f_{\theta_0}) - \bar{l}_n(f_\theta)$ to $\mathbb{IK}(f_{\theta_0}, f_\theta)$. It provides also the $\mathbb{P}_{f_{\theta_0}}$ − a.s. convergence of $l_n^{(u)}(f_{\theta_0}) - l_n^{(u)}(f_\theta)$ to $\mathbb{IK}(f_{\theta_0}, f_\theta)$ in the AR_P or MA_P cases (see Section 3). To achieve the assertion of 3.1, it only remains to show the uniform convergences on Θ of the last quantities. This will be done using an equicontinuity argument given by the following Lemma.

Lemma 7.5 (Equicontinuity lemma)**.** *For all $n \geq 0$, the sequences of functions $(l_n(f_{\theta_0}, X_n) - l_n(f_\theta, X_n))_{n\in\mathbb{N}}$ is an $\mathbb{P}_{f_{\theta_0}}$-a.s. equicontinuous sequence on $(\{f_\theta, \theta \in \Theta\}, \|.\|_\infty)$. This property also holds for $\bar{l_n}$,$\tilde{l}_n$. Furthermore, the sequence $\left(l_n^{(u)}(f_{\theta_0}, X_n - l_n^{(u)}(f_\theta, X_n)\right)_{n\in\mathbb{N}}$ is also $\mathbb{P}_{f_{\theta_0}}$-a.s. equicontinuous, on $\left(\{f_\theta, \theta \in \Theta\}, \|.\|_{1,pol}\right)$.*

We can now achieve the proof of Lemma 3.1 :

First, notice that the space $\{f_\theta, \theta \in \Theta\}$ is compact for the topology of the uniform convergence. This also holds for $\left(\{f_\theta, \theta \in \Theta\}, \|.\|_{1,pol}\right)$. So, there exists a dense sequence $(f_{\theta_p})_{p\in\mathbb{N}}$. Then, using Lemma 7.1 and Corollary 7.1, the sequence $\left(l_n(f_{\theta_0}, X_n) - l_n(f_{\theta_p}, X_n)\right)_{n\in\mathbb{N}}$ converges $\mathbb{P}_{f_{\theta_0}}$-a.s. to $\mathbb{IK}(f_{\theta_0}, f_{\theta_p})$.

If a sequence of functions is equicontinuous and converges pointwise on a dense subset of its domain, and if its co-domain is a complete space, then the sequence converges pointwise on all the domain [23].

Using this well known property, we obtain, $\mathbb{P}_{f_{\theta_0}}$-a.s., the pointwise convergence of $(l_n(f_{\theta_0}, X_n) - l_n(f_\theta, X_n$ to $\mathbb{IK}(f_{\theta_0}, f_\theta)$, for any $\theta \in \Theta$.

Furthermore, if a sequence of functions is equicontinuous and converges pointwise on its domain, then this convergence is uniform on any compact subspace of the domain [23].

Thus, we get, $\mathbb{P}_{f_{\theta_0}}$-a.s., the uniform convergence on Θ of the sequence $(l_n(f_{\theta_0}, X_n) - l_n(f_\theta, X_n))_{n\in\mathbb{N}}$ to $\mathbb{IK}(f_{\theta_0}, f_\theta)$.

Using the same kind of arguments, this uniform convergence also holds for $\bar{l_n}$,$\tilde{l}_n$ and $l_n^{(u)}$. This finishes the proof of Lemma 3.1.

□

7.3 Proof of the technical lemmas

Proof. of Lemma 7.4

Let $\theta \in \Theta$. First, consider the case $\Lambda_n = \mathcal{K}_n\left(\frac{1}{f_\theta}\right)$. We aim at proving that

$$\frac{1}{m_n} X_n^T \Lambda_n X_n \underset{n\to\infty}{\to} \int \frac{f_{\theta_0}}{f_\theta} \mathrm{d}\mu, \mathbb{P}_{f_{\theta_0}} - \text{a.s.}.$$

To do that, we make use of classical tools of large deviation (see [10]). We compute the

Laplace transform of $X_n^T \Lambda_n X_n$:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{f_{\theta_0}}} & \left[e^{\lambda X_n^T \mathcal{K}_n(\frac{1}{f_\theta}) X_n} \right] \\
= & \frac{1}{(\sqrt{2\pi})^{m_n} \sqrt{\det(\mathcal{K}_n(f_{\theta_0}))}} \int e^{\frac{1}{2} X_n^T \left(\left(\mathcal{K}_n(f_{\theta_0}) \right)^{-1} - 2\lambda \mathcal{K}_n(\frac{1}{f_\theta}) \right) X_n} \\
= & \frac{1}{\sqrt{\det(\mathcal{K}_n(f_{\theta_0}))}} \sqrt{\det \left(\left[(\mathcal{K}_n(f_{\theta_0}))^{-1} - 2\lambda \mathcal{K}_n(\frac{1}{f_\theta}) \right]^{-1} \right)} \\
= & \frac{1}{\sqrt{\det \left(I_{G_n} - 2\lambda \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{K}_n(\frac{1}{f_\theta}) \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \right)}}.
\end{aligned}$$

These last equalities hold as soon as $I_{G_n} - 2\lambda \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{K}_n(\frac{1}{f_\theta}) \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}$ is positive. This is true whenever $\lambda \leq 0$ or small enough.

Now, for $\lambda \leq 0$, define

$$\phi_n(\lambda) := \frac{1}{m_n} \log \left(\mathbb{E}_{\mathbb{P}_{f_{\theta_0}}} \left[e^{\lambda X_n^T \mathcal{K}_n(\frac{1}{f_\theta}) X_n} \right] \right),$$

This function verifies

$$\phi_n(\lambda) = -\frac{1}{2m_n} \log \det \left(I_{G_n} - 2\lambda \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{K}_n(\frac{1}{f_\theta}) \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \right).$$

Define also

$$\phi(\lambda) = \lim_n \phi_n(\lambda),$$

We get, using Corollary 7.1,

$$\phi(\lambda) = -\frac{1}{2} \int \log \left(1 - 2\lambda \frac{f_{\theta_0}}{f_\theta} \right).$$

We can also compute

$$\phi''(\lambda) = \int \frac{2(\frac{f_{\theta_0}}{f_\theta})^2}{(1 - 2\lambda \frac{f_{\theta_0}}{f_\theta})^2} \mathrm{d}\mu > 0.$$

As very usual, we define the convex conjugate of ϕ by

$$\phi^*(t) := \sup_{\lambda \in \mathbb{R}^-} \left[\lambda t - \phi(\lambda) \right], t \in \mathbb{R}.$$

As soon as ϕ is strictly convex, $\phi^*(t) > \phi(0) = 0$, for any $t \neq \phi'(0) = \int \frac{f}{g} \mathrm{d}\mu$.
We can now write, for $\lambda \leq 0$,

$$\begin{aligned}
\frac{1}{m_n} \log(\mathbb{P}(\frac{1}{m_n} X_n^T \Lambda_n X_n \geq t)) &= \frac{1}{m_n} \log(\mathbb{P}(e^{\lambda X_n^T \Lambda_n X_n} \geq e^{m_n \lambda t})) \\
&\leq \frac{1}{m_n} \log \left(e^{-m_n \lambda t} \right) + \frac{1}{m_n} \log \left(\mathbb{E}[e^{\lambda X_n^T \Lambda_n X_n}] \right) \\
&\leq -\lambda t + \phi_n(\lambda).
\end{aligned}$$

Then we get, $\forall t > \int \frac{f}{g} \mathrm{d}\mu$,

$$\limsup_n \left(\frac{1}{m_n} \log(\mathbb{P}(\frac{1}{m_n} X_n^T \Lambda_n X_n \geq t)) \right) \leq -\lambda t + \phi(\lambda)$$

So that, taking the infimum on λ, we get

$$\limsup_n \left(\frac{1}{m_n} \log(\mathbb{P}(\frac{1}{m_n} X_n^T \Lambda_n X_n \geq t)) \right) \leq -\phi^*(t) < 0$$

We can obtain the same bound for $t < \int \frac{f}{g} \mathrm{d}\mu$. By Borel-Cantelli theorem, we get the $\mathbb{P}_{f_{\theta_0}}$-almost sure convergence of $\frac{1}{m_n} X_n^T \Lambda_n X_n$ to $\int \frac{f}{g} \mathrm{d}\mu$. To prove the same convergence with $\Lambda_n = (\mathcal{K}_n(f_\theta))^{-1}$, we have to show that the difference between the spectral empirical measure of $\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{K}_n(\frac{1}{f_\theta}) \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}$ and $\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{K}_n(f_\theta)^{-1} \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}$ converges weakly to zero. It is sufficient to control the convergence of every moment, because this two last measures both have compact support.

For this, we make use of the Schatten norms. For any A, B matrices of $M_{m_n}(\mathbb{R})$, we define

$$\|A\|_{Sch,p} = \left(\sum s_k(A)^p \right)^{\frac{1}{p}},$$

where $s_k(A)$ are the singular values of A.

Note that

$$|Tr(AB)| \leq \|AB\|_{Sch,1} \leq \|A\|_{Sch,1} \|B\|_{Sch,\infty}.$$

Recall that since $f_\theta \in \mathcal{F}_\rho$, we have $e^{-\rho} \leq f_\theta \leq e^\rho$. Hence, for any $p \geq 1$,

$$\begin{aligned}
&\frac{1}{m_n} \left| \mathrm{Tr}\Big(\mathcal{K}_n^p(\frac{1}{f_\theta}) \mathcal{K}_n^p(f_{\theta_0}) - \mathcal{K}_n^{-p}(f_\theta) \mathcal{K}_n^p(f_{\theta_0}) \Big) \right| \\
&\qquad \leq \frac{1}{m_n} \left\| \mathcal{K}_n(f_\theta)^{-p} \mathcal{K}_n^p(f_{\theta_0}) \right\|_{Sch,\infty} \left\| \left(\mathcal{K}_n^p(\frac{1}{\theta}) \mathcal{K}_n^p(f_\theta) - I_{G_n} \right) \right\|_{Sch,1} \\
&\qquad \leq \frac{\delta_n}{m_n} \frac{e^{2\rho p}}{e^{-2\rho p}} \alpha(f_\theta)^{2p} \alpha(\frac{1}{f_\theta})^{2p} \underset{n \to \infty}{\to} 0.
\end{aligned}$$

To obtain the same bound with $\Lambda_n = \mathcal{Q}_n(\frac{1}{f_\theta})$, we have to prove that the difference between the spectral empirical measures of $\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{K}_n(\frac{1}{f_\theta}) \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}$ and $\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}} \mathcal{Q}_n(\frac{1}{f_\theta}) \mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}$ converge weakly to zero. This last assertion is a direct consequence of Lemma 7.2. So, we get

$$\frac{1}{m_n} X_n^T \Lambda_n X_n \to \int \frac{f_{\theta_0}}{f_\theta}, \mathbb{P}_{f_{\theta_0}} - \text{ a.s.}$$

□

Proof. of Lemma 7.5

Recall that we aim at proving that, $\mathbb{P}_{f_{\theta_0}}$-a.s., the sequence of functions $(l_n(f_{\theta_0}, X_n) - l_n(f_\theta, X_n))_{n \in \mathbb{N}}$ is equicontinuous on $\{f_\theta, \theta \in \Theta\}$, and that this property also holds for $\bar{l_n}$,$\tilde{l_n}$ and $l_n^{(u)}$.

First, we will prove the equicontinuity of the sequence

$$\left(\frac{1}{m_n}\log\det(\mathcal{K}_n(f_\theta))\right)_{n\in\mathbb{N}}.$$

Let $\theta,\theta'\in\Theta$.

Denote λ_i the eigenvalues of $\mathcal{K}_n(f_{\theta'})^{-1}\left(\mathcal{K}_n(f_{\theta'})-\mathcal{K}_n(f_\theta)\right)$. Since $f_\theta\in\mathcal{F}_\rho$, we have $e^{-\rho}\leq f_\theta\leq e^\rho$.

Notice that we have

$$\begin{aligned}\sup_{i=1,\cdots,n}|\lambda_i| &= \left\|\mathcal{K}_n(f_{\theta'})^{-1}\left(\mathcal{K}_n(f_{\theta'})-\mathcal{K}_n(f_\theta)\right)\right\|_{2,op}\\ &\leq e^\rho\left\|f_{\theta'}-f_\theta\right\|_\infty.\end{aligned}$$

So that, to prove the equicontinuity, we may assume that θ is close enough to θ' to ensure that $\sup_{i=1,\cdots,n}|\lambda_i|\leq\frac{1}{2}$.

We have

$$\begin{aligned}&\frac{1}{m_n}\left|\log\det(\mathcal{K}_n(f_{\theta'}))-\log\det(\mathcal{K}_n(f_\theta))\right|\\ &\qquad=\frac{1}{m_n}\left|\log\det\left(I_{G_n}-\mathcal{K}_n(f_{\theta_0})^{-1}\left(\mathcal{K}_n(f_{\theta'})-\mathcal{K}_n(f_\theta)\right)\right)\right|\\ &\qquad\leq\frac{1}{m_n}\sum_{i\in G_n}|\log(1+\lambda_i)|\\ &\qquad\leq\frac{1}{m_n}\sup_{i\in G_n}|\log(1+\lambda_i)|\\ &\qquad\leq 2\log(2)\sup_{i\in G_n}|\lambda_i|\\ &\qquad\leq 2\log(2)e^\rho\left\|f_{\theta'}-f_\theta\right\|_\infty.\end{aligned}$$

Furthermore, the sequence $(\int\log(f_\theta)\mathrm{d}\mu)_{n\in\mathbb{N}}$ is also equicontinuous since, using a Taylor formula,

$$\int\left|\log(f_{\theta'})\mathrm{d}\mu-\int\log(f_\theta)\mathrm{d}\mu\right|\leq e^\rho\left\|f_{\theta'}-f_\theta\right\|_\infty.$$

Now we tackle the equicontinuity of the sequences $\left(X_n^T\mathcal{K}_n(f_\theta)^{-1}X_n\right)_{n\in\mathbb{N}}$, $\left(X_n^T\mathcal{K}_n(\frac{1}{f_\theta})X_n\right)_{n\in\mathbb{N}}$ and $\left(X_n^T\mathcal{Q}_n(\frac{1}{f_\theta})X_n\right)_{n\in\mathbb{N}}$.

Notice first that, for any matrix $B\in M_n(\mathbb{R})$,

$$\frac{1}{m_n}\left|X_n^TBX_n\right|\leq\frac{1}{m_n}\left\|B\right\|_{2,op}\left|X_n^TX_n\right|.$$

It is thus sufficient to prove the equicontinuity of the sequences $(\mathcal{K}_n(f_\theta)^{-1})_{n\in\mathbb{N}}$, $(\mathcal{K}_n(\frac{1}{f_\theta}))_{n\in\mathbb{N}}$ and $(\mathcal{Q}_n(f_\theta)^{-1})_{n\in\mathbb{N}}$, for the norm $\|.\|_{2,op}$

Note that

$$\left\|\mathcal{K}_n(\frac{1}{f_{\theta'}}) - \mathcal{K}_n(\frac{1}{f_\theta})\right\|_{2,op} \leq \left|\frac{1}{f_{\theta'}} - \frac{1}{f_\theta}\right|_\infty \leq e^{2\rho} \left\|f_{\theta'} - f_\theta\right\|_\infty .$$

Then,

$$\left\|(\mathcal{K}_n(f_{\theta'}))^{-1} - (\mathcal{K}_n(f_\theta))^{-1}\right\|_{2,op} \leq \left\|(\mathcal{K}_n(f_{\theta'}))^{-1}(\mathcal{K}_n(f_\theta))^{-1}\right\|_{2,op} \left\|(\mathcal{K}_n(f_{\theta'})) - (\mathcal{K}_n(f_\theta))\right\|_{2,op} \leq e^{2\rho} \left\|f_{\theta'} - f_\theta\right\|_\infty .$$

Then, recall that, for any symmetric matrix $B \in M_n(\mathbb{R})$, we have

$$\|B\|_{2,op} \leq \|B\|_{\infty,op} .$$

Recall also that $\mathcal{Q}_n(f_\theta) = B^{(n)} \odot \mathcal{K}_n(f_\theta)$. Denote

$$\begin{aligned}
\left\|\mathcal{Q}_n(\frac{1}{f_{\theta'}}) - \mathcal{Q}_n(\frac{1}{f_\theta})\right\|_{2,op} &\leq \left\|\mathcal{Q}_n(\frac{1}{f_{\theta'}}) - \mathcal{Q}_n(\frac{1}{f_\theta})\right\|_{\infty,op} \\
&\leq \sup_{i,j=1,\cdots n} \left|B_{ij}^{(n)}\right| \left\|\mathcal{K}_n(\frac{1}{f_{\theta'}}) - \mathcal{K}_n(\frac{1}{f_\theta})\right\|_{\infty,op} \\
&\leq (1+u_n) \left\|\frac{1}{f_{\theta'}} - \frac{1}{f_\theta}\right\|_{1,pol} \text{(see Assumption 3.5).}
\end{aligned}$$

Since the map $f_\theta \mapsto \frac{1}{f_\theta}$ is continuous over $\mathcal{F}_\rho$, which is compact, we get the uniform equicontinuity of the map $f_\theta \mapsto X_n^T \mathcal{Q}_n(\frac{1}{f_\theta}) X_n$ (for the norm $\|.\|_{1,pol}$).

This concludes the proof of Lemma 7.5 □

Proof. of Lemma 3.3

We aim at proving the asymptotic normality of $\sqrt{m_n}(l_n^{(u)})'(\theta_0)$.

Using the Fourier transform, it is sufficient to prove that

$$\lim_n \mathbb{E}\left[\exp\left(i\sqrt{m_n}t\left((l_n^{(u)})'(\theta_0)\right)\right)\right] = \exp\left(-\int \frac{1}{4}t^2 \frac{(f'_{\theta_0})^2}{f_{\theta_0}^2}(t)\mathrm{d}\mu(t)\right)$$

Recall that we have

$$(l_n^{(u)})'(\theta) = -\frac{1}{2}\int \frac{f'_\theta}{f_\theta}\mathrm{d}\mu + \frac{1}{2m_n} X_n^T \mathcal{Q}_n(\frac{f'_\theta}{f_\theta^2}) X_n.$$

We can compute

$$\begin{aligned}
\sqrt{m_n}\mathbb{E}\left[(l_n^{(u)})'(\theta_0)\right] &= \sqrt{m_n}\left(-\frac{1}{2}\int \frac{f'_{\theta_0}}{f_{\theta_0}}\mathrm{d}\mu + \frac{1}{2m_n}\operatorname{Tr}\left(\mathcal{K}_n(f_{\theta_0})\mathcal{Q}_n(\frac{f'_\theta}{f_\theta^2})\right)\right) \\
&= \sqrt{m_n}\left(-\frac{1}{2}\int \frac{f'_{\theta_0}}{f_{\theta_0}}\mathrm{d}\mu + \frac{1}{2m_n}\operatorname{Tr}\left(\mathcal{K}_n\left(f_{\theta_0}\frac{f'_{\theta_0}}{f_{\theta_0}^2}\right)\right)\right) \text{(see Lemma 7.3)} \\
&\leq C v_n \sqrt{m_n} \underset{n\to\infty}{\to} 0 \text{ (see Assumption 3.6).}
\end{aligned}$$

If we define

$$Z_n = t\frac{1}{2m_n}X^T\mathcal{Q}_n(\frac{f'_\theta}{f^2_\theta})X,$$

and

$$Z = t\frac{1}{2}\int \frac{f'_\theta}{f_\theta}\mathrm{d}\mu,$$

the last equality means that

$$\sqrt{m_n}\left(\mathbb{E}\left[Z_n\right] - Z\right) \to 0.$$

It holds here only if f_{θ_0} is a polynomial, or if all the $f_\theta, \theta \in \Theta$ are polynomial. This brings out that the second theorem holds for the AR_P or MA_P case. It also explains the denomination of unbiased estimator of the approximated likelihood for $\theta^{(u)}$.

Then, it is sufficient to show

$$\lim_n \mathbb{E}\left[\exp\left(i\sqrt{m_n}\left(Z_n - \mathbb{E}\left[Z_n\right]\right)\right)\right] = \exp\left(-\int \frac{1}{4}t^2\frac{(f'_{\theta_0})^2(t)}{f^2_{\theta_0}(t)}\mathrm{d}\mu(t)\right).$$

If τ_k denotes the eigenvalues of the symmetric matrix

$$M_n := \frac{t}{2}\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\mathcal{Q}_n(\frac{f'_{\theta_0}}{f^2_{\theta_0}})\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}},$$

then we can write

$$Z_n = \frac{1}{m_n}\sum_{k=1}^{m_n}\tau_k Y_k^2.$$

where $(Y_k)_{k\in G_n}$ has the standard Gaussian distribution on $\mathbb{R}^{m_n}$.

The independence of Y_k leads to

$$\log\left(\mathbb{E}\left[\exp\left(i\sqrt{m_n}\left(Z_n - \mathbb{E}\left[Z_n\right]\right)\right)\right]\right) = -\sum_{k=1}^{m_n}\left(i\frac{\tau_k}{\sqrt{m_n}} + \frac{1}{2}\log(1 - 2i\frac{\tau_k}{\sqrt{m_n}})\right).$$

The τ_k are bounded, thanks to the following inequality:

$$\begin{aligned}
\left\|M_n\right\|_{2,op} &= \left\|\frac{t}{2}\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\mathcal{Q}_n(\frac{f'_{\theta_0}}{f^2_{\theta_0}})\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\right\|_{2,op} \\
&\leq \left\|\frac{t}{2}\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\right\|_{2,op}\left\|\mathcal{Q}_n(\frac{f'_{\theta_0}}{f^2_{\theta_0}})\right\|_{2,op}\left\|\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\right\|_{2,op} \\
&\leq \left\|\frac{t}{2}\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\right\|_{2,op}\left\|\mathcal{Q}_n(\frac{f'_{\theta_0}}{f^2_{\theta_0}})\right\|_{1,op}\left\|\mathcal{K}_n(f_{\theta_0})^{\frac{1}{2}}\right\|_{2,op} \\
&\leq e^\rho\alpha(f'_{\theta_0})\alpha(f_{\theta_0})^2(1+u_n).
\end{aligned}$$

The Taylor expansion of $\log(1 - 2\frac{\tau_k}{\sqrt{m_n}})$ gives

$$\log\left(\mathbb{E}\left[\exp\left(i\sqrt{m_n}\left(Z_n - \mathbb{E}\left[Z_n\right]\right)\right)\right]\right) = -\frac{1}{m_n}\sum_{k=1}^{m_n}\tau_k^2 + R_n.$$

With $|R_n| \leq C\frac{1}{m_n\sqrt{m_n}}\sum_{k=1}^{m_n}|\tau_k|^3$

Since the τ_k are bounded the assertion will be proved if we show that

$$\frac{1}{m_n}\operatorname{Tr}(M_n^2) = \frac{1}{m_n}\sum_{k=1}^{m_n}\tau_k^2 \xrightarrow{n\to\infty} \int \frac{1}{4}t^2\frac{(f'_{\theta_0})^2(t)}{f^2_{\theta_O}(t)}\mathrm{d}\mu(t).$$

This last convergence is a consequence of Lemmas 7.1 and 7.2.

This provides the asymptotic normality of $\sqrt{m_n}(l_n^{(u)})'(\theta_0)$ and achieves the proof of Lemma 3.3:

$$\sqrt{m_n}(l_n^{(u)})'(\theta_0) \underset{n\to\infty}{\to} \mathcal{N}(0, \frac{1}{2}\int\left(\frac{f'_{\theta_0}}{f_{\theta_0}}\right)^2\mathrm{d}\mu).$$

□

Proof. of Lemma 3.4

We aim now at proving the $P_{f_{\theta_0}}$-a.s. following convergence:

$$\left((l_n^{(u)})''(\breve{\theta}_n)\right)^{-1} \underset{n\to\infty}{\to} \frac{1}{2}\left(\int\frac{(f'_{\theta_0})^2}{f^2_{\theta_0}}\mathrm{d}\mu\right)^{-1}$$

We have

$$(l_n^{(u)})''(\theta) = -\frac{1}{2m_n}\left(\int\frac{f''_\theta f_\theta - (f'_\theta)^2}{f^2_\theta}\mathrm{d}\mu + X_n^T\mathcal{Q}_n\left(\frac{2(f'_\theta)^2 - f''_\theta f_\theta}{f^3_\theta}\right)X_n\right),$$

which leads to

$$(l_n^{(u)})''(\theta) \underset{n\to\infty}{\to} \frac{1}{2}\int\left(\frac{f''_\theta f_\theta - (f'_\theta)^2}{f^2_\theta} + \frac{f_{\theta_0}\big(2(f'_\theta)^2 - f''_\theta f_\theta\big)}{f^3_\theta}\right)\mathrm{d}\mu, P_{f_{\theta_0}}\text{-a.s.}$$

Since the sequence $l_n^{(u)}$ is equicontinuous and $\breve{\theta}_n \underset{n\to\infty}{\to} \theta_0$, we obtain the desired convergence :

$$(l_n^{(u)})''(\breve{\theta}_n) \underset{n\to\infty}{\to} \frac{1}{2}\int\left(\frac{(f'_{\theta_0})^2}{f^2_{\theta_0}}\right)\mathrm{d}\mu, P_{f_{\theta_0}}\text{-a.s.}$$

□

Proof. of Lemma 3.5

We want to compute the asymptotic Fisher information. As usual, it is sufficient to compute

$$\frac{1}{m_n}\operatorname{Var}\left(L'_n(\theta_0)\right) = \lim_n \frac{1}{2m_n}\operatorname{Tr}(M_n(\theta_0)^2),$$

where $M_n(\theta) = \mathcal{K}_n(f_\theta)^{-1}\mathcal{K}_n(f'_\theta)\mathcal{K}_n(f_\theta)^{-1}\mathcal{K}_n(f_{\theta_0})$.

This leads, together with Lemma 7.1, and Assumption 3.3 to

$$\frac{1}{m_n}\operatorname{Var}\left(L'_n(\theta_0)\right) \to \frac{1}{2}\int\frac{(f'_{\theta_0})^2}{f^2_{\theta_0}}\mathrm{d}\mu.$$

This ends the proof of the last lemma. □

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