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DARMON’S POINTS AND QUATERNIONIC SHIMURA VARIETIES

JÉRÔME GÄRTNER

Abstract. In this paper, we generalize a conjecture due to Darmon and Logan (see [DL03] and [Dar04], chapter 8) in an adelic setting. We study the relation between our construction and Kudla’s works on cycles on orthogonal Shimura varieties. This relation allows us to conjecture a Gross-Kohnen-Zagier theorem for Darmon’s points.

1. Introduction

The theory of complex multiplication gives a collection of Heegner points on elliptic curves over \( \mathbb{Q} \), which are defined over class fields of imaginary quadratic fields. These points allow us to prove Birch and Swinnerton-Dyer’s conjecture over \( \mathbb{Q} \) for analytic rank 1 curves, thanks to the work of Gross-Zagier and Kolyvagin.

Let us briefly recall the construction of Heegner points. If \( E \) is an elliptic curve over \( \mathbb{Q} \) then we know that \( E \) is modular. Let \( N \) be the conductor of \( E \). There exists a modular form \( f \in S_2(N) \) such that \( L(E, s) = L(f, s) \). Denote by \( \Phi_N : \Gamma_0(N) \backslash \mathcal{H} \to E(\mathbb{C}) \) the modular uniformization which is obtained by taking the composition of the map \( z_0 \in \mathcal{H} \mapsto c \int_{z_0}^{\infty} 2\pi i f(z) dz \) (here \( c \) denotes the Manin constant) with the Weierstrass uniformization. Let \( z_0 \in \mathcal{H} \cap K \), where \( K/\mathbb{Q} \) is an imaginary quadratic field. A Heegner point is given essentially by \( z_0 \) in \( \mathbb{C}/A_E \cong E(\mathbb{C}) \). The theory of complex multiplication shows that these points are defined over class fields of \( K \).

In [Dar04], Darmon gives a conjectural construction of Stark-Heegner points, which is a generalization of classical Heegner points. These points should help us to understand, on one hand the Birch and Swinnerton-Dyer conjecture, on the other hand Hilbert’s twelfth problem.

In more concrete terms, assume that \( F \) is a totally real number field of narrow class number 1. Let \( \tau_j \) be its archimedean places, and \( K/F \) some quadratic “ATR” extension (i.e. \( K \) has exactly one complex place). Darmon defines a collection of points on elliptic curves \( E/F \) which are expected to be defined over class fields of \( K \). In this case, the (conjectural, but partially proved by Skinner-Wiles) modularity of \( E \) gives the existence of a Hilbert modular form \( f \) on \( \mathcal{H} \) whose periods appear as a tensor product of periods of \( E_{\tau_j} = E \otimes_{F, \tau_j} \mathbb{C} \). The construction explained in [DL03] can be seen as an exotic Abel-Jacobi map.

In this paper, we generalize Darmon’s construction by removing the hypothesis “ATR” on \( K \) (but we assume that \( K \) is not CM) and the technical hypothesis that \( F \) has narrow class number 1. We replace the Hilbert modular variety used in the “ATR” case by a general quaternionic Shimura variety and define a suitable Abel-Jacobi map. We are able to specify the invariants of the quaternion algebra using local epsilon factors and to give a conjectural Gross-Zagier formula for these points. We conclude the paper by establishing a relation to Kudla’s study of cycles on orthogonal Shimura varieties, in order to give a Gross-Kohnen-Zagier type conjecture.

Let us summarize the main construction of this paper. Let \( F \) be a totally real field of degree \( d \) and let \( \tau_1, \ldots, \tau_d \) be its archimedean places. Fix \( r \in \{2, \ldots, d\} \), and a quadratic extension \( K/F \) such that the set of archimedean places of \( F \) that split completely in \( K \) is \( \{\tau_2, \ldots, \tau_r\} \). Let \( B/F \) be a quaternion algebra which splits at \( \tau_1, \ldots, \tau_r \) and ramifies at \( \tau_{r+1}, \ldots, \tau_d \). Let \( G = \text{Res}_{F/Q} B^\times \). We will denote by \( \text{Sh}_H(G, X) \) the quaternionic Shimura variety of level \( H \) (a compact open subgroup of \( G(\mathbb{A}_f) \)) whose complex points are given by

\[
\text{Sh}_H(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathbb{C} \times \mathbb{R}^r) \times G(\mathbb{A}_f)/H.
\]

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Fix an embedding \( q : K \hookrightarrow B \). There is an action of \((K \otimes \mathbf{R})^+/(F \otimes \mathbf{R})^+\) on \((C \otimes \mathbf{R})^+\). By considering a suitable orbit of this action, we obtain a real cycle \( T_b \) of dimension \( r - 1 \) on \( \text{Sh}_H(G, X)(C) \). Using the theorem of Matsushima and Shimura, we deduce that there exists an \( r \)-cycle \( \Delta_b \) on \( \text{Sh}_H(G, X)(C) \) such that \( \partial \Delta_b \) is an integral multiple of \( \beta_0 \).

Let \( E/F \) be an elliptic curve, assumed modular, i.e., there exists a Hilbert modular eigenform \( \tilde{\varphi} \) satisfying \( L(E, s) = L(\tilde{\varphi}, s) \). We will assume that this form corresponds to an automorphic form \( \varphi \) on \( B \) by the Jacquet-Langlands correspondence. There exists a holomorphic differential form \( \omega_\varphi \) of degree \( r \) on \( \text{Sh}_H(G, X)(C) \) naturally attached to \( \varphi \). In general, the set of periods of \( \omega_\varphi \) is a dense subset of \( C \). Fix some character \( \beta \) of the set of connected components of \((K \otimes \mathbf{R})^+/(F \otimes \mathbf{R})^+\).

Following Darmon we define a modified differential form \( \omega_\varphi^\beta \) whose periods are, assuming Yoshida’s period conjecture, a lattice, homothetic to some sublattice of the Neron lattice of \( E \).

The image of (a suitable multiple of) the complex number \( \int_{\Delta_b} \omega_\varphi^\beta \) is independent of the choice of \( \Delta_b \). Hence it defines by Weierstrass uniformization a point \( P_b^\beta \) in \( E(C) \). We conjecture

**Conjecture (5.1.1).** \( P_b^\beta = \Phi \left( \int_{\Delta_b} \omega_\varphi^\beta \right) \in E(C) \) lies in \( E(K^{ab}) \) and

\[
\forall a \in A_K^\times \quad \text{rec}_K(\alpha) P_b^\beta = \beta(\alpha_\infty) P_{\alpha(\alpha)b}^\beta.
\]

Let us assume this conjecture is true and denote by \( K_b^+ \) the field of definition of \( P_b^\beta \). Let \( \pi = \pi(\varphi) \) be the automorphic representation generated by \( \varphi \); fix a character \( \chi : \text{Gal}(K_b^+)/K \rightarrow \mathbf{C}^\times \). Denote by \( \epsilon(\pi \times \chi, \frac{1}{2}) \) the sign in the functional equation of the Rankin-Selberg \( L \)-function \( L(\pi \times \chi, s) \) and by \( \eta_K : F_b^+/F_b^\times N_{F/K}(K_b^+) \rightarrow \{\pm 1\} \) the quadratic character of \( K/F \). The following proposition proves that \( b \) is uniquely determined by \( K \) and the isogeny class of \( E/F \).

**Proposition (5.3.1).** Let \( b \in \hat{B}^\times \) and assume conjecture 5.1.1. If

\[
\epsilon(\pi^\beta) = \sum_{\sigma \in \text{Gal}(K_b^+)/K} \chi(\sigma) \otimes P_b^\beta \in E(K_b^+)^+ \otimes \mathbf{Z}[\chi]
\]

is not torsion, then :

\[
\forall v \mid \infty \quad \eta_{K,v}(-1)\epsilon(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v) \quad \text{and} \quad \epsilon(\pi \times \chi, \frac{1}{2}) = -1.
\]

The last part of this paper is focused on a conjecture in the spirit of the Gross-Kohnen-Zagier theorem. Assume that \( E(F) \) has rank 1. Denote by \( P_b \) some generator modulo torsion. For each totally positive \( t \in O_F \) such that \( (t) \) is square free and prime to \( d_{K/F} \), denote by \( K[t] \) the quadratic extension \( K[t] = F(\sqrt{\pm D_t}) \), where \( D_t \in F \) satisfies \( \tau_j(D_t) > 0 \) if and only if \( j \in \{1, r + 1, \ldots, d\} \).

Let \( P_{t,1} \) be Darmon’s point obtained for \( K[t] \) and \( b = 1 \), and set

\[
P_t = \text{Tr}_{K[t]/F} P_{t,1}.
\]

The point \( P_t \) is in \( E(F) \) and there exists some integer \( [P_t] \in \mathbf{Z} \) such that \( P_t = [P_t] P_0 \). In the spirit of conjecture 5.3 of [DT08] we conjecture that :

**Conjecture (6.3.5).** There exists a Hilbert modular form \( g \) of level \( 3/2 \) such that the \( [P_t] \) are proportional to some Fourier coefficients of \( g \).

In our attempt to adapt Yuan, Zhang and Zhang’s proof in the CM case [YZZ09] to prove this conjecture, we obtained a relation between Darmon’s points and Kudla’s program, see Proposition 5.5.3.2.

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2. Quaternionic Shimura varieties

In this section we recall some properties of Shimura varieties associated to quaternion algebras. The standard references are Reimann’s book [Rei97] and [Mil05]. The content of this section is more or less the transcription to Shimura varieties of what is done for curves in [CV07] and [Nek07].
Let $F$ be a totally real field of degree $d = [F : \mathbb{Q}]$ and $\tau_1, \ldots, \tau_d$ its archimedean places. Denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ so $\tau_j : F \hookrightarrow \overline{\mathbb{Q}}$. Fix $r \in \{2, \ldots, d\}$ and a finite set $S_B$ of non-archimedean primes satisfying

$$|S_B| \equiv d - r \mod 2.$$  

Let $B$ be the unique quaternion algebra over $F$ ramified at the set $\text{Ram}(B) = \{\tau_{r+1}, \ldots, \tau_d\} \cup S_B$.

For each $j \in \{1, \ldots, d\}$ we put $B_{\tau_j} = B \otimes_{F, \tau_j} \mathbb{R}$. It is not necessary but more convenient to fix for each $j \in \{\tau_1, \ldots, \tau_r\}$ an $\mathbb{R}$-algebra isomorphism

$$B_{\tau_j} \sim M_2(\mathbb{R}).$$

The constructions given in this paper are independent on the choice of these isomorphisms, as in the author’s PhD thesis [Gär11].

Let $G$ be the algebraic group over $\mathbb{Q}$ satisfying $G(\mathbb{A}) = (B \otimes_{\mathbb{Q}} A)^{\times}$ for every commutative $\mathbb{Q}$-algebra $A$. We will denote by $\text{nr} : G(\mathbb{A}) \rightarrow (F \otimes_{\mathbb{Q}} A)^{\times}$ the reduced norm and by $Z$ the center of $G$. For $j \in \{1, \ldots, d\}$ let $G_j$ be the algebraic group over $\mathbb{R}$ given by $G_j = G \otimes_{F, \tau_j} \mathbb{R}$; thus $G_\mathbb{R}$ decomposes as $G_1 \times \cdots \times G_d$. For any abelian group $A$, denote by $\hat{A}$ the group $A \otimes \hat{\mathbb{Z}}$.

Let $X$ be the $G(\mathbb{R})$-conjugacy class of the morphism $h : S = \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m, \mathbb{C}}) \rightarrow G(\mathbb{R}) = G_1(\mathbb{R}) \times \cdots \times G_d(\mathbb{R})$ defined by

$$x + iy \mapsto \left( \begin{array}{ccc} x & y & \vdots & \vdots \\ -y & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & y \\ \vdots & \ddots & \ddots & -y \\ 1 & \cdots & 1 & 1 \end{array} \right)_r \times \cdots \times \left( \begin{array}{ccc} 1 & \cdots & 1 \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots & x \\ 1 & \cdots & 1 & 1 \end{array} \right)_r^{-d}. \quad \text{times}$$

The set $X$ has a natural complex structure [Mil90] and the following map is a holomorphic isomorphism between $X$ and $(\mathbb{C} \setminus \mathbb{R})^r$:

$$ghg^{-1} \mapsto g \cdot (i, \ldots, i) = \left( \begin{array}{c} a_1i + b_1 \\ c_1i + d_1 \\ \vdots \\ a_ri + b_r \\ c_ri + d_r \end{array} \right),$$

where $g = (g_1, \ldots, g_d) \in G(\mathbb{R})$ and for $j \in \{1, \ldots, r\}$ $g_j$ is identified with $\left( \begin{array}{c} a_j \\ c_j \\ b_j \\ d_j \end{array} \right)$.

Quaternionic Shimura varieties. Let $H$ be an open-compact subgroup of $\tilde{B}^{\times}$. The quaternionic Shimura varieties considered in this paper are algebraic varieties $\text{Sh}_H(G, X)$ whose complex points are given by

$$\text{Sh}_H(G, X)(\mathbb{C}) = B^{\times} \backslash (X \times \tilde{B}^{\times} / H),$$

where the left-action of $B^{\times}$ and the right-action of $H$ are given by

$$\forall k \in B^{\times} \quad \forall h \in H \quad \forall (x, b) \in X \times \tilde{B}^{\times} \quad k \cdot (x, b) \cdot h = (kx, bkh).$$

Such Shimura varieties are defined over some number field called the reflex field. In our case this number field is

$$F' = \mathbb{Q} \left( \sum_{j=1}^{r} \tau_j(\alpha), \quad \alpha \in F \right) \subset \overline{\mathbb{Q}} \subset \mathbb{C}.$$  

We will denote by $[x, b]_H$ the element of $\text{Sh}_H(G, X)(\mathbb{C})$ represented by $(x, b)$ and by $[x, b]_{H \tilde{F}^{\times}}$ the corresponding element of the modified variety $\text{Sh}_H(G/Z, X)(\mathbb{C}) = B^{\times} \backslash (X \times \tilde{B}^{\times} / HZ)$.

**Remark 2.1.1.** The complex Shimura varieties are compact whenever $B \neq M_2(F)$. The Hilbert modular varieties used by Darmon in [Dar04] chapter 7 and 8 are obtained when $B = M_2(F)$ and $r = d$.

The Shimura varieties form a projective system $\{\text{Sh}_H(G, X)\}_H$ indexed by open compact subgroups in $\tilde{B}^{\times}$. The transition maps $\text{pr} : \text{Sh}_H(G, X) \rightarrow \text{Sh}_H'(G, X)$ are defined on complex points by

$$[x, b]_H \mapsto [x, b]_{H'}.$$
There is an action of $\hat{B}^\times$ on the projective system $\{\text{Sh}_H(G, X)\}_H$. The right multiplication by $g \in B^\times$ induces an isomorphism $[g] : \text{Sh}_H(G, X) \xrightarrow{\sim} \text{Sh}_H(G, X)_g^{-1}H_g$, defined on complex points by

$$[g][x, b]_H = [x, bg]_{g^{-1}H_g}.$$ 

Complex conjugation. Fix $j \in \{1, \ldots, r\}$. Let $h_j : S \to G_{j, R}$ be the morphism obtained by composing $h$ with the $j$-th projection $G_{R} \to G_{j, R}$ and $X_j$ the $G_{j, R}$-conjugacy class of $h_j$. For $x_j = g_j h_j g_j^{-1} \in X_j$, the set $\text{Im}(g_jh_jg_j^{-1})$ is a maximal anisotropic $R$-torus in $G_{j, R}$. The map $t_j : x_j \mapsto \text{Im}(x_j)$ satisfies $|t_j^{-1}(x_j(x_j))| = 2$, thus there exists a unique antiholomorphic and $G_{j, R}$-equivariant involution

$$t_j : X_j \longrightarrow X_j$$

such that

$$\forall x_j \in X_j \quad t_j^{-1}(x_j) = \{x_j, t_j(x_j)\}.$$ 

More precisely, under the identification $X_j \xrightarrow{\sim} C \times R$, the map $t_j$ satisfies $t_j(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$

and $t_j^{-1}(x+iy) = \{x+iy, x-iy\}$. Note that the map $t_j$ can be extended to complex points of the Shimura varieties by $t_j([x, b]_H) = [t_j(x), b]_H$; $t_j$ acts trivially on $X_k$ for $k \neq j$.

Differential forms. In this section we recall some facts concerning differential forms on Shimura varieties. We will denote by $\Omega_H = \Omega_{H^F}$ the sheaf of differentials of degree $r$ on $\text{Sh}_H(G, X)$ and by $\Omega_H^n$ the sheaf of holomorphic $r$-differentials on $\text{Sh}_H(G, X)(C)$, provided that $\text{Sh}_H(G, X)$ is smooth. Recall that the GAGA principle gives us the following isomorphism between global sections

$$\Gamma(\text{Sh}_H(G, X), \Omega_H) \otimes_{F} C \xrightarrow{\sim} \Gamma(\text{Sh}_H(G, X)(C), \Omega^n_H).$$

Notice that in general, $\text{Sh}_H(G, X)$ is not smooth. In this last case we will fix some integer $n \geq 3$ such that for each $p$ in $\text{Ram}(B)$ we have $p \nmid nO_F$ and for each $v \nmid nO_F$ isomorphisms $\iota_v : B_v \xrightarrow{\sim} M_2(F_v)$. The group

$$H' = \left\{ (h_v) \in H, \text{ s.t. } \forall v \mid nO_F \quad h_v \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod } nO_{F_v} \right\}$$

is of finite index in $H$ and $\text{Sh}_{H'}(G, X)$ is smooth. The map $\text{Sh}_{H'}(G, X) \to \text{Sh}_H(G, X)$ is a finite covering. We define $\Omega_{H'} = \prod_{H' \to H} \Omega_{H'} = (\Omega_H)^H$. By abuse of language, we shall call an element of $\Gamma(\Omega_{H'}) = \Gamma(\text{Sh}_{H'}(G, X), \Omega_{H'}) = (\prod_{H' \to H} \Gamma(\text{Sh}_H(G, X), \Omega_{H'}))$ a global $r$-form on $\text{Sh}_H(G, X)$. Remark that the space of global holomorphic $r$-forms $\lim_{\longrightarrow H} \Gamma(\Omega^n_{H'})$ is equipped with a canonical action of $\hat{B}^\times$ given by pull-backs $[g]^*$.

Let $\varepsilon \in \{\pm 1\}^r$ and denote by $\Gamma((\Omega^n_{H'})^\varepsilon)$ the space of $r$-forms on $\text{Sh}_{H'}(G, X)(C)$ which are holomorphic (resp. anti-holomorphic) in $z_j$ if $\varepsilon_j = +1$ (resp. if $\varepsilon_j = -1$). The maps $t_j$ pulled-back on $\Gamma((\Omega^n_{H'})^\varepsilon)$ satisfy

$$t_j^* : \Gamma((\Omega^n_{H'})^\varepsilon) \longrightarrow \Gamma((\Omega^n_{H'})^{\varepsilon^j})$$

where $\varepsilon'_k = \varepsilon_k$ for $k \neq j$ and $\varepsilon'_j = -\varepsilon_j$.

When $\sigma \in \prod_{j=2}^{r} \{\pm 1\}$ we will define $e_j \in \{0, 1\}$ by $\sigma_j = (-1)^{e_j}$ and $t_j^* \prod_{j=2}^{r} (t_j^*)^{e_j}$. Let $\beta : \prod_{j=2}^{r} \{\pm 1\} \to \{\pm 1\}$ be a character and $\omega \in \Gamma((\Omega^n_{H'})^\beta)$. We shall denote by $\omega^\beta$ the element $\omega^\beta = \sum_{\sigma \in \prod_{j=2}^{r} \{\pm 1\}} \beta(\sigma) t_j^{e_j}(\omega)$ of $\bigoplus_{H' \to H} \Gamma((\Omega^n_{H'})^\varepsilon)$.

Automorphic forms. Let $S^H_{2, \ldots, 2, 0, \ldots, 0}(B_{\Lambda})$ be the space $S^H_{2, \ldots, 2, 0, \ldots, 0}(B_{\Lambda})$ of functions $\varphi : B_{\Lambda}^\times \simeq G(R) \times \hat{B}^\times \longrightarrow C$ satisfying the following properties:

1. $\forall g \in B_{\Lambda}^\times \forall b \in B_{\Lambda}^\times \varphi(gb) = \varphi(b)$,
2. $\forall g \in (R^\times)^r \times G_{r+1}(R) \times \cdots \times G_2(R) \subset G(R) \forall b \in B_{\Lambda}^\times \varphi(bg) = \varphi(b),$
3. $\forall h \in H \forall b \in B_{\Lambda}^\times \varphi(hb) = \varphi(b),$
4. $\forall g \in B_{\Lambda}^\times \forall (\theta_1, \ldots, \theta_r) \in R^r$

$$\varphi \left( g \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{pmatrix}, 1, \ldots, 1 \right) = e^{-2i\theta_1} \times \cdots \times e^{-2i\theta_r} \varphi(g),$$
Let \( B_{\mathbb{A}} \) be the ring of adeles over \( \mathbb{Q} \).

There is an action of \( M \) on \( B_{\mathbb{A}} \) defined by:

\[
\forall g \in B_{\mathbb{A}}, \forall \varphi \in S_2, \forall x \in B_{\mathbb{A}}^x, \quad g \cdot \varphi(x) = \varphi(xg);
\]

thus \( S_2^H \) is the space of \( H \)-invariant functions in \( S_2 \).

By modifying the properties 4 and 5 above we obtain the following new definition:

**Definition 2.1.2.** Let \( \varepsilon : \{\tau_1, \ldots, \tau_r\} \rightarrow \{\pm 1\} \) and \( \varepsilon_i = \varepsilon(\tau_i) \). The space \( (S_2^H)^{\varepsilon} \) is the space of maps \( \varphi : B_{\mathbb{A}}^x \approx G(\mathbb{R}) \times B_{\mathbb{A}}^x \rightarrow \mathbb{C} \) satisfying 1-3 above and

4'. for all \( g \in B_{\mathbb{A}}^x \) and \( (\theta_1 \ldots \theta_r) \in \mathbb{R}^r \)

\[
\varphi\left( g \left( \begin{array}{cccc}
\cos \theta_1 & -\sin \theta_1 & \ldots & -\sin \theta_r \\
\sin \theta_1 & \cos \theta_1 & \ldots & \cos \theta_r \\
0 & 0 & \ldots & 1 \\
1 & 1 & \ldots & 1
\end{array} \right) \right) = e^{-2i\varepsilon_1 \theta_1} \ldots \ldots e^{-2i\varepsilon_r \theta_r} \varphi(g)
\]

5'. for all \( g \in B_{\mathbb{A}}^x \) the map

\[
(x_1 + iy_1, \ldots, x_r + iy_r) \mapsto \frac{1}{y_1 \ldots y_r} \varphi\left( g \left( \begin{array}{cccc}
y_1 & x_1 & \ldots & x_r \\
0 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1
\end{array} \right) \right)
\]

is holomorphic (resp. anti-holomorphic) in \( z_j = x_j + iy_j \in \mathcal{H} \) if \( \varepsilon_j = 1 \) (resp. \( \varepsilon_j = -1 \)).

We will denote by \( S_2^{\varepsilon} \) (resp. \( (S_2^H)^{\varepsilon} \)) the space of elements in \( S_2 \) (resp. \( S_2^H \)) which are \( \hat{F}^{\varepsilon} \)-invariant.

We are now able to affirm the existence of relations between \( r \)-forms on \( Sh_{H}(G, X)(\mathbb{C}) \) and automorphic forms:

**Proposition 2.1.3.** There exist bijections compatible with the \( \hat{B}^x \)-action between the following spaces:

\[
\begin{align*}
\Gamma(\Omega_0^{2H}) & \quad and \quad S_2^{\varepsilon} \\
\Gamma((\Omega_0^{2H})^{\varepsilon}) & \quad and \quad (S_2^{H})^{\varepsilon} \\
\Gamma(Sh_{H}(G/Z, X)(\mathbb{C})), (\Omega_0^{2H})^{\varepsilon} & \quad and \quad (S_2^{H})^{\varepsilon} \hat{F}^{\varepsilon}
\end{align*}
\]

This statement is completely analogous to section 3.6 of [CV07], see [Gär11], Propositions 1.2.2.4 and 1.2.2.5 for more details.

Matsushima-Shimura theorem. The decomposition of the cohomology of quaternionic Shimura varieties given by Matsushima-Shimura theorem will be useful in the following sections. Let us recall this result when \( B \neq M_2(F) \) [MS63] and [Pre90]. Denote by \( h_F^+ \) the narrow class number of \( F \).

**Theorem 2.1.4.** Let \( m \in \{0, \ldots, 2r\} \). We have the following decomposition:

\[
H^m(Sh_{H}(G, X)(\mathbb{C})), \mathbb{C}) \simeq \left\{ \begin{array}{ll}
\text{Vect } \bigwedge_{i \in \{1, \ldots, r-1\}} \frac{dz_i \wedge d\bar{z}_i}{y^2_i} \bigg| \frac{a}{m/2} \bigg) & \quad \text{if } m \neq r \\
\text{Vect } \bigwedge_{i \in \{1, \ldots, r-1\}} \frac{dz_i \wedge d\bar{z}_i}{y^2_i} \bigg| \frac{a}{m/2} \bigg) \oplus \bigoplus_{\varepsilon \in \{\pm 1\}^r} (S_2^H)^{\varepsilon} & \quad \text{if } m = r
\end{array} \right.
\]

and

\[
H^m(Sh_{H}(G/Z, X)(\mathbb{C})), \mathbb{C}) \simeq \left\{ \begin{array}{ll}
\text{Vect } \bigwedge_{i \in \{1, \ldots, r-1\}} \frac{dz_i \wedge d\bar{z}_i}{y^2_i} \bigg| \frac{a}{m/2} \bigg) & \quad \text{if } m \neq r \\
\text{Vect } \bigwedge_{i \in \{1, \ldots, r-1\}} \frac{dz_i \wedge d\bar{z}_i}{y^2_i} \bigg| \frac{a}{m/2} \bigg) \oplus \bigoplus_{\varepsilon \in \{\pm 1\}^r} (S_2^H)^{\varepsilon} \hat{F}^{\varepsilon} & \quad \text{if } m = r
\end{array} \right.
\]

where \( s \) (resp. \( s' \)) is the number of connected components of \( Sh_{H}(G, X)(\mathbb{C}) \) (resp. of \( Sh_{H}(G/Z, X)(\mathbb{C}) \)).
3. Periods

3.1. Yoshida’s conjecture. Let $E/F$ be an elliptic curve, assumed modular in the sense that there exists a cuspidal, parallel weight two Hilbert modular form $\tilde{\varphi} \in S_2(\text{GL}_2(F_\lambda))$ satisfying $L(E, s) = L(\tilde{\varphi}, s)$. We shall assume that the automorphic representation generated by $\tilde{\varphi}$ is obtained by the Jacquet-Langlands correspondence from $\varphi \in S_2^J(F_\lambda^\vee)(B_\lambda^\vee)$.

Denote by $\pi = \pi_\infty \otimes \pi_f$ the automorphic representation of $B_\lambda^\vee/F_\lambda^\vee$ generated by $\varphi$. We shall assume until section 3.3, only for simplicity, that $\dim \pi_f^j = 1$.

Let $M = h^1(E)$ be the motive over $F$ with coefficients in $\mathbb{Q}$ associated to $E$. Yoshida [Yos94] conjectures the existence of a rank 2 motive $M'$ over the reflex field $F'$, with coefficients in $\mathbb{Q}$, satisfying $M' = \bigotimes_{\{\tau_1, \ldots, \tau_r\}} \text{Res}_{F/F'} M$. This motivic conjecture is the following:

**Conjecture 3.1.1** (Yoshida, [Yos94]). The motive $M'$ over $F'$ is isomorphic to the motive associated to the part $H^*(\text{Sh}_{HFA}(G, X))^{(E)}$ of the cohomology for which Hecke eigenvalues are the same as $E$.

While looking at the $\ell$-adic realization, this conjecture is in fact the Langlands cohomological conjecture. This case is known, up to semi-simplification, thanks to Brylinski and Labesse in the case $B = M_2(F)$ [BL84], Langlands in the case $B \neq M_2(F)$ for primes of good reduction, [Lan79] and Reimann (+ Zink) [Rei97, RZ91] for a more general cases.

Recall the following decompositions given by Yoshida in [Yos94] section 5.1, when we focus on $\tau' : F' \rightarrow \mathbb{C}$ induced by $\tilde{\tau}' : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$.

Betti cohomology. There exists an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathcal{J} : M_B' \sim \bigotimes_{j=1}^r M_{B, \tau_j}$$

de Rham cohomology. The map

$$\mathcal{J} : M'_{\text{dR}} \sim \left( \bigotimes_{j=1}^r (M_{\text{dR}} \otimes_{F, \tau_j} \overline{\mathbb{Q}}) \right)^{\text{Gal}(\overline{\mathbb{Q}}/F')},$$

is an isomorphism of $F'$-vector-spaces. The right hand side is a tensor product of $\overline{\mathbb{Q}}$-vector spaces and the action of $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$ is given by $\bigotimes_{x \in \{\tau_1, \ldots, \tau_r\}} (x_\sigma \otimes_{F, \tau_j} a_\sigma) \mapsto \bigotimes_{x \in \{\tau_1, \ldots, \tau_r\}} (x_\sigma \otimes_{F, \tau_j} a_\sigma)$.

Comparison isomorphisms. Let $I = \bigotimes_{j=1}^r I_{\tau_j}$, where

$$I_{\tau_j} : M_{B, \tau_j} \otimes_{\mathbb{Q}} \mathbb{C} \sim M_{\text{dR}} \otimes_{F, \tau_j} \mathbb{C}$$

are isomorphisms of $\mathbb{C}$-vector spaces, and $I'$ be the following isomorphism over $\mathbb{C}$:

$$I' : M'_B \otimes_{\mathbb{Q}} \mathbb{C} \sim M'_{\text{dR}} \otimes_{F'} \mathbb{C}.$$

The maps $I \circ (\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_C)$ and $(\mathcal{J} \otimes_{F'} \text{id}_C) \circ I'$ satisfy:

$$(*) \quad I \circ (\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_C) = (\mathcal{J} \otimes_{F'} \text{id}_C) \circ I' : M'_B \otimes_{\mathbb{Q}} \mathbb{C} \sim \bigotimes_{j=1}^r (M_{\text{dR}} \otimes_{F, \tau_j} \mathbb{C}).$$

Yoshida’s period conjecture consists of the isomorphisms $\mathcal{J}$, $\mathcal{J}$, $I$ and $I'$ satisfying $(*)$. It is the Hodge-de Rham realization of the motivic conjecture above.

Complex conjugation : Let $c_{\tau_j}$ be the complex conjugation on $M_{B, \tau_j}$. We will need the following hypothesis, which allows us to compare $c_{\tau_j}$ with $t_{\tau_j}^*$ on $M'_{B, \tau_j} \otimes_{F'} \mathbb{C}$.

**Hypothesis 3.1.2.** The action of $t_{\tau_j}^*$ on $M'_{\text{dR}} \otimes_{F'} \mathbb{C}$ corresponds via the isomorphism

$$(\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_C) \circ (I')^{-1} : M'_{\text{dR}} \otimes_{F'} \mathbb{C} \sim M'_B \otimes_{\mathbb{Q}} \mathbb{C} \sim \left( \bigotimes_{k=1}^r (M_{B, \tau_k} \otimes_{\mathbb{Q}} \mathbb{C}) \right) \otimes_{\mathbb{Q}} \mathbb{C},$$

to the action of $c_{\tau_j}$ on $M_{B, \tau_j}$. 
3.2. Lattices and periods. Fix some $\omega_\varphi \neq 0$ in $F^r M_{dR}'$. By definition of $M'$, there exists a finite set of places $S$ of $F$ such that for $v \notin S$, $T_v \omega_\varphi = a_v(E) \omega_\varphi$.

Let $\Omega_{E/F}$ be the sheaf of differentials on $E/F$. Fix $\eta \neq 0 \in H^0(E, \Omega_{E/F}) = F^1 M_{dR}$. For $j \in \{1, \ldots, n\}$, let

$$\eta_j = \eta \otimes_{F, r_j} 1 \in H^0 \left( E \otimes_{F, r_j} \mathbb{Q} \Omega_{(E \otimes_{F, r_j} \mathbb{Q})/\mathbb{Q}} \right) \cong (F^1 M_{dR}) \otimes_{F, r_j} \mathbb{Q}.$$ 

Then

$$\bigotimes_{j=1}^r \eta_j \in \bigotimes_{j=1}^r (F^1 M_{dR} \otimes_{F, r_j} \mathbb{Q}) = \mathcal{J} (F^r M_{dR}'),$$

and there exists $\alpha \in F^r \times$ such that

$$\mathcal{J} (\alpha \omega_\varphi) = \eta_1 \otimes \cdots \otimes \eta_r.$$

Let $j \in \{1, \ldots, r\}$ and $E_j = E \otimes_{F, r_j} \mathbb{C}$. We shall denote by $H_1(E_j, \mathbb{Z})^\pm$ the eigenspaces of the complex conjugation action on $H_1(E_j, \mathbb{Z})$. Then

$$\left\{ \int_{\mathcal{T}} \eta_j, \: \mathcal{T} \in H_1(E_j, \mathbb{Z})^\pm \right\} = \mathbb{Z} \Omega_j^\pm,$$

where $\Omega_j^+ \in \mathbb{R} \setminus \{0\}$ and $\Omega_j^- \in i \mathbb{R} \setminus \{0\}$ are determined up to a sign. We fix the signs by imposing, e.g., $\text{Re} (\Omega_j^+) > 0$ and $\text{Im} (\Omega_j^+) > 0$.

Fix a character $\beta : \{1\} \times \prod_{j=2}^r \{\pm 1\} \to \{\pm 1\}$, and write $\beta = \prod_{j=2}^r \beta_j$. We set

$$\omega_\varphi^\beta = \left( \sum_{\sigma \in \{1\}} \prod_{j=2}^r \beta_j(\sigma) t_j^* \right) \omega_\varphi = \prod_{j=2}^r (1 + \beta_j(-1) t_j^*) \omega_\varphi$$

and

$$\Omega^\beta = \prod_{j=2}^r \Omega_j^\beta(-1).$$

The following identities

$$\left( \bigotimes_{j=1}^r M_{B, r_j} \right) \otimes_{\mathbb{Q}} \mathbb{C} = \bigotimes_{j=1}^r \text{Hom}_{\mathbb{Z}}(H_1(E_j, \mathbb{Z}), \mathbb{C}) = \text{Hom}_{\mathbb{Z}} \left( \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z}), \mathbb{C} \right)$$

and Yoshida’s conjecture show that the image of $\alpha \omega_\varphi^\beta$ under the map

$$(\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_{\mathbb{C}}) \circ I^{-1} = I^{-1} \circ (\mathcal{J} \otimes_{F^r} \text{id}_{\mathbb{C}}) : M_{dR}' \otimes_{F^r} \mathbb{C} \to \left( \bigotimes_{j=1}^r M_{B, r_j} \right) \otimes_{\mathbb{Q}} \mathbb{C}$$

is identified with the linear form

$$(1) \quad \left\{ \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z}), \mathbb{C} \right\} \to \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z}), \mathbb{C} \quad \text{by} \quad \int_{\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_r} \bigotimes_{j=1}^r (1 + \beta_j(-1) t_j^*) \eta_j$$

Hypothesis 3.1.2 allows us to be more explicit. Let $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_r \in \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z})$, then

$$\int_{\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_r} \bigotimes_{j=1}^r (1 + \beta_j(-1) t_j^*) \eta_j = \left( \int_{\mathcal{T}_1} \eta_1 \right) \prod_{j=2}^r \left( \int_{\mathcal{T}_j} (1 + \beta_j(-1) t_j^*) \eta_j \right)$$

and the linear form (1) takes values in $A_1 \Omega^\beta = (\mathbb{Z} \Omega_1^+ + \mathbb{Z} \Omega_1^-) \Omega^\beta$.

Under the dual isomorphism $\mathcal{J}^*$ of $\mathcal{J}$, the lattices

$$\bigotimes_{j=1}^r \mathbb{Q} H_1(E_j, \mathbb{Z}) \subset \bigotimes_{j=1}^r \mathbb{Q} M_{B, r_j}^*$$

and $\text{Im} (H_r(\text{Sh}_{H}(G/Z, X)(\mathbb{C}), \mathbb{Z}) \to (M_{B})^*)$
are commensurable. Thus there exists $\xi \in \mathbb{Z} \smallsetminus \{0\}$ such that
\[
\xi \text{ Im } (H_r(\text{Sh}_H(G/Z,X)(C),Z) \longrightarrow (M'_B)^* \subset \mathcal{G}_r^*(\bigotimes_{j=1}^r \Omega^j(E_j,Z)).
\]

This proves the following proposition:

**Proposition 3.2.1.** Under the hypothesis made in this section (E is modular, the multiplicity one in Yoshida’s motivic conjecture and 3.1.2), there exist $\alpha \in F'^\infty$ and $\xi \in \mathbb{Z} \smallsetminus \{0\}$ such that
\[
\forall \gamma \in H_r(\text{Sh}_H(G,X)(C),Z), \forall \beta : \prod_{j=2}^r \{\pm 1\} \to \{\pm 1\}, \quad \xi \int_\gamma \alpha \omega_\beta^\beta \in \Lambda_1 \Omega^\beta.
\]

3.3. General case. When $m_H(\pi) = \dim \pi^H_1(\phi) > 1$ Yoshida’s conjecture is the following

**Conjecture 3.3.1.** The motive $H'(\text{Sh}_H(G,X))^{(E)}$ is isomorphic to
\[
\left( \bigotimes_{\{1,\ldots, \tau_r\}} \text{Res}_{\overline{F}/F} M \right)^{m_H(\pi)}.
\]

In general the motive $H'(\text{Sh}_H(G,X))^{(E)}$ has rank $\neq 2^r$. We shall provide Betti and de Rham realizations of a submotive $M' \subset H'(\text{Sh}_H(G,X))^{(E)}$ of rank $2^r$ and an isomorphism $M' \sim \bigotimes_{\{1,\ldots, \tau_r\}} \text{Res}_{\overline{F}/F} M$.

We need $0 \neq \omega_\phi \in F'^rH'_r(\text{Sh}_H(G/Z,X)/F't)^{(E)}$ satisfying de Rham cohomology. The $F'$-vector space
\[
M'_B := \left( \bigoplus_{\omega_\phi \in \{\pm 1\}^r} \text{C}^* \phi (\omega_\phi \otimes 1) \right) \cap H'_r(\text{Sh}_H(G/Z,X)/F'^t)^{(E)}
\]
has dimension $2^r$.

Thus
\[
F'^rM'_B := M'_B \cap F'^rH'_r(\text{Sh}_H(G/Z,X)/F'^t)^{(E)} = F'^r \omega_\phi.
\]

Betti cohomology. Let
\[
I' : H'_{B}(\text{Sh}_H(G/Z,X)(C),Q)(E) \otimes_{\mathbb{Q}} \mathbb{C} \sim \bigotimes_{\{1,\ldots, \tau_r\}} H'_r(\text{Sh}_H(G/Z,X)/F'^t)^{(E)} \otimes_{\mathbb{F}'} \mathbb{C}.
\]

The $\mathbb{Q}$-vector space
\[
M'_B := I'^{-1}(M'_B \otimes_{\mathbb{F}'} \mathbb{C}) \cap H'_r(\text{Sh}_H(G/Z,X)(C),Q)^{(E)}
\]
has dimension $2^r$.

**Definition 3.3.2.** An element $\omega_\phi \in F'^rH'_r(\text{Sh}_H(G/Z,X)/F'^t)^{(E)}$ is said rational if it satisfies the equations above.

Comparison isomorphisms. There exist isomorphisms
\[
\mathcal{G} : M'_B \sim \bigotimes_{j=1}^r \text{M}_{B,\tau_j},
\]
\[
\mathcal{G} : M'_B \sim \left( \bigotimes_{j=1}^r (\text{M}_{B,\tau_j} \otimes_{F',\tau_j} \overline{\mathbb{Q}}) \right)^{\text{Gal}(\overline{\mathbb{Q}}/F')},
\]

and
\[
I_\tau_j : \text{M}_{B,\tau_j} \otimes_{\mathbb{Q}} \mathbb{C} \sim \text{M}_{B,\tau_j} \otimes_{F,\tau_j} \mathbb{C}.
\]

Set $I = \bigotimes_{j=1}^r I_\tau_j$. We have
\[
(*) \quad I \circ (\mathcal{G} \otimes_{\mathbb{Q}} \text{id}_\mathbb{C}) = (\mathcal{G} \otimes_{\mathbb{F}'} \text{id}_{\mathbb{C}}) \circ I' : M'_B \otimes_{\mathbb{Q}} \mathbb{C} \sim \bigotimes_{j=1}^r \text{M}_{B,\tau_j} \otimes_{F,\tau_j} \mathbb{C}.
\]

As in Proposition 3.2.1 we have
Proposition 3.3.3. Let $\omega_\varphi \in F'^{H'_{\text{dR}}}(\text{Sh}_H(G/Z, X)/F')^{(E)}$ be rational. If $E$ is modular and if Yoshida's conjecture is true, then there exist $\alpha \in F'^{X}$ and $\xi \in \mathbb{Z} \smallsetminus \{0\}$ such that

$$\forall \gamma \in H_r(\text{Sh}_H(G, X)(C), Z), \forall \beta : \prod_{j=2}^r \{\pm 1\} \to \{\pm 1\}, \quad \xi \int_\gamma \alpha \omega_\varphi^\alpha \in \Lambda_1 \Omega^\beta.$$

Example. Let $H_1, H_2 \subset \hat{B}^\times$ be compact open subgroups such that there exists $g \in \hat{B}^\times$ satisfying $g^{-1}H_1g \subset H_2$. Let $\omega_\varphi \in F'^{H'_{\text{dR}}}(\text{Sh}_H(G/Z, X)/F')^{(E)}$ be rational. Let us explain a way to obtain $\omega_\varphi \in F'^{H'_{\text{dR}}}(\text{Sh}_H(G/Z, X)/F')^{(E)}$ rational.

Let

$$\text{pr} : \text{Sh}_{g^{-1}H_1g}(G/Z, X) \to \text{Sh}_{H_2}(G/Z, X)$$

be the map given by

$$[x, b]_{g^{-1}H_1g} \mapsto [x, b]_{H_2}$$

and

$$[g] : \text{Sh}_{H_1}(G/Z, X) \to \text{Sh}_{g^{-1}H_1g}(G/Z, X)$$

by

$$[x, b]_{H_1} \mapsto [x, bg]_{g^{-1}H_1g}.$$ Let $\text{pr}_g : \text{Sh}_{H_1}(G/Z, X) \to \text{Sh}_{H_2}(G/Z, X)$ be the composition of $\text{pr}$ with $[g]$. Choose $\theta_g \in \mathbb{Q}$. Set

$$\omega_\varphi := \sum_{g \in \hat{B}^\times} \theta_g \text{pr}_g^*(\omega_\varphi),$$

s.t. $g^{-1}H_1g \subset H_2$

$$(M'_1)^{\text{dR}} = \left( \sum_g \theta_g \text{pr}_g^* \right) (M'_2)^{\text{dR}}$$

and

$$(M'_1)^{\text{B}} = \left( \sum_g \theta_g \text{pr}_g^* \right) (M'_2)^{\text{B}}.$$ 

Proposition 3.3.4. If $\omega_\varphi \neq 0$, then the map

$$\sum_{g \in \hat{B}^\times} \theta_g \text{pr}_g^* \text{ is injective on } \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Cl}_0^\sigma(\omega_\varphi \otimes 1)$$

and $\omega_\varphi \in F'^{H'_{\text{dR}}}(\text{Sh}_H(G/Z, X)/F')^{(E)}$ is rational.

Proof. Assume that $\omega = \sum_{\sigma \in \{\pm 1\}^r} \lambda_\sigma t_\sigma^\omega \omega_\varphi \in \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Cl}_0^\sigma(\omega_\varphi \otimes 1)$ (where $\lambda_\sigma \in \mathbb{C}$) is such that $\sum_g \theta_g \text{pr}_g^*(\omega) = 0$. We have the following equalities:

$$\sum_g \theta_g \text{pr}_g^* \omega = \sum_g \theta_g \text{pr}_g^* \sum_\sigma \lambda_\sigma t_\sigma^\omega \omega_\varphi = \sum_\sigma \lambda_\sigma t_\sigma^\omega \sum_g \theta_g \text{pr}_g^* \omega_\varphi_1.$$

Thus

$$\sum_\sigma \lambda_\sigma t_\sigma^\omega \omega_\varphi_1 = 0 \in \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Cl}_0^\sigma(\omega_\varphi_1),$$

and

$$\forall \sigma \in \{\pm 1\}^r \quad \lambda_\sigma t_\sigma^\omega \omega_\varphi_1 = 0.$$ Hence $\forall \sigma \in \{\pm 1\}^r \quad \lambda_\sigma t_\sigma^\omega \omega_\varphi_1 = 0$. The map $\sum_{g \in \hat{B}^\times} \text{pr}_g^*$ s.t. $g^{-1}H_1g \subset H_2$ $\theta_g \text{pr}_g^*$ commutes with $T_v, v \notin S$ and is an isomorphism $\bigoplus_{\sigma \in \{\pm 1\}^r} \text{Cl}_0^\sigma(\omega_\varphi_1) \to \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Cl}_0^\sigma(\omega_\varphi_1)$. Hence $\omega_\varphi_1 \in \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Cl}_0^\sigma(\omega_\varphi_1 \otimes 1) \cap F'^{H'_{\text{dR}}}(\text{Sh}_H(G/Z, X)/F')^{(E)}$ is rational. 

$\square$
4. Toric orbits

Let $K/F$ be a quadratic extension satisfying the following properties:

1. The places $\tau_1, \ldots, \tau_r$ of $F$ are split in $K$.
2. The places $\tau_1, \tau_{r+1}, \ldots, \tau_d$ are ramified in $K$.

Thanks to the Skolem-Noether theorem, there exists an $F$-embedding $q: K \hookrightarrow B$, unique up to conjugacy. We will denote by $q_j$ (resp. $q$) the induced embedding $K \hookrightarrow B_{\tau_j}$ (resp. $K \hookrightarrow B$, $K_\Lambda \hookrightarrow B_\Lambda$). For each place $v$ of $F$, set $K_v = K \otimes_F F_v$.

4.1. Cycles on $X$. Let $T = \text{Res}_{K/Q}(G_m)/\text{Res}_{F/Q}(G_m)$. Thanks to Hilbert's Theorem 90 we have

$$T(A) = (K \otimes Q A)^\times / (F \otimes Q A)^\times$$

for every $Q$-algebra $A$.

Fix an embedding $q: T \hookrightarrow G/Z(G)$. The group $T(R)$ is identified with $\prod_{j=1}^d K_{\tau_j}^\times / F_{\tau_j}^\times$ which allows us to define $q_j : K_{\tau_j}^\times / F_{\tau_j}^\times \rightarrow G_{\tau_j}$.

Let $\pi_0(T(R))$ be the set of connected components of $T(R)$ and denote by $T(R)^\circ$ the component of the identity. Fix a multi-orientation on $T(R)^\circ = \prod_{j=1}^d (K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ$ (i.e. an orientation of each factor $(K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ$) and remark that

$$\pi_0(T(R)) = T(R)/T(R)^\circ \simeq \prod_{j=2}^r \{\pm 1\}.$$ 

We will focus on the orbits in $X$ under the action of $q(T(R)^\circ)$ by conjugation.

**Proposition 4.1.1.** Let $\mathcal{T}^\circ$ be an orbit of $q(T(R)^\circ)$ in $X$. Then $\mathcal{T}^\circ$ decomposes into a product of orbits in $X_j$ under $q_j(T(R)^\circ)$ and is multi-oriented.

**Proof.** The first part of this assertion follows from the natural decomposition $X = X_1 \times \cdots X_r$. The orbit $\mathcal{T}^\circ$ decomposes into orbits under $q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ)$. For $j = 1$, $q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ) \simeq S^1$ or a point and the orientation does not change. For $j \in \{2, \ldots, r\}$, $q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ) \simeq R_+^\times$. The action of $R_+^\times$ on itself by multiplication does not change the orientation. Hence the multi-orientation induced on $\mathcal{T}^\circ$ by $T(R)^\circ$ is well-defined.

□

In the following sections we shall fix some $q(T(R)^\circ)$-orbit $\mathcal{T}^\circ$ whose projection on $X_1$ is a point.

**Proposition 4.1.2.** $\mathcal{T}^\circ$ is a connected multi-oriented submanifold of real dimension $r - 1$.

**Proof.** Recall that $\mathcal{T}^\circ$ is decomposed as $\mathcal{T}^\circ = \{z_1\} \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_r$. Fix $x \in X$ such that $\mathcal{T}^\circ = q(T(R)^\circ \cdot x$. Then for $j \in \{2, \ldots, r\}$ we have $\mathcal{T}_j = q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ) \cdot \text{pr}_j(x)$. The group $q_j((K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ)$ is naturally identified with $R_+^\times$ and $\mathcal{T}_j$ is a connected oriented manifold of real dimension one.

□

As a corollary, we have the following decomposition:

$$\mathcal{T}^\circ = \{z_1\} \times \gamma_2 \times \cdots \times \gamma_r,$$

when $z_1$ is one of the two fixed points in the action of $q_1(T(R)^\circ)$ on $X_1$ and $\gamma_j$ is an oriented connected submanifold of real dimension one in $X_j$.

When we use the identification of $X$ with $(C \setminus \mathbb{R})^r$, the action of $T(R)$ on $X$ by conjugation is an action of $\text{PGL}_2(R)$ on $(C \setminus \mathbb{R})^r$ by homography. Let $z \in K \setminus F$. For $j \in \{2, \ldots, r\}$ the matrix $q_j(z)$ is hyperbolic with exactly two fixed points in $P^1(R)$, $z_j$ and $z_j'$. The manifold $\gamma_j$ is then a circle arc in the Poincaré upper half-plane joining $z_j$ to $z_j'$ (or a line if $z_j' = \infty$). Figure 1 gives some examples of what could the $\gamma_j$s be in the case of circle arcs.
4. Tori on $Sh_H(G/Z,X)(C)$. Let $b \in \hat{B}^\times$. We will denote by $\mathcal{T}_b^\circ$ the following subset of $Sh_H(G/Z,X)(C)$

$$\mathcal{T}_b^\circ = \{[x,b]_{\hat{H}^\times}, x \in \mathcal{T}^\circ\}.$$  

**Proposition 4.2.1.** $\mathcal{T}_b^\circ$ is an oriented torus of real dimension $r - 1$.

**Proof.** Let $x, x' \in \mathcal{T}^\circ$ and $b \in \hat{B}^\times$; we know that

$$[x,b]_{\hat{H}^\times} = [x',b]_{\hat{H}^\times} \iff \exists k \in B^\times \text{ and } h \in H_{\hat{F}^\times} (kx', kbh) = (x,b) \iff \exists k \in B^\times \cap bH_{\hat{F}^\times}^x b^{-1}$$

Since the projection of $\mathcal{T}^\circ$ on $X_1$ is a point, we have $k \in B \cap q_1(K_{\tau}) = q_1(K)$ and $k \in (q(K^\times) \cap bH_{\hat{F}^\times}^x b^{-1}$. Thus the stabilizer $W$ of $\mathcal{T}_b^\circ$ under the action of $q(K^\times)$ is

$$W = q(K^\times) \cap (bH_{\hat{F}^\times}^x b^{-1})$$

which is commensurable with $O_{K^+,+}^\times / O_F^\times$. This quotient has rank $r - 1$ over $\mathbb{Z}$ as a consequence of Dirichlet’s units theorem:

$$O_{K^+,+}^\times / O_F^\times \simeq \text{torsion} \times \mathbb{Z}^{r-1},$$

and the torsion is finite. The action of $T(R)$ on $\mathcal{T}^\circ$ is given by $\prod_{j=2}^r (K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ$ and there is an isomorphism

$$\prod_{j=2}^r (K_{\tau_j}^\times / F_{\tau_j}^\times)^\circ \simeq R^{r-1}.$$  

The image $\tilde{O}$ of $O_{K^+,+}^\times / O_F^\times$ in $R^{r-1}$ is isomorphic to $\mathbb{Z}^s$ with $s \leq r - 1$. Denote by $\tilde{O}_K^\times$ the image of $O_K^\times$ in $(K \otimes R)^\times$. As

$$\prod_{j \notin \{2,\ldots,r\}} K_{\tau_j}^\times / F_{\tau_j}^\times$$

are compact, $R^{r-1} / \tilde{O}$ is compact. Thus, the image of $O_{K^+,+}^\times / O_F^\times$ in $R^{r-1}$ is a lattice.

The set $\mathcal{T}_b^\circ$ is a principal homogeneous space under

$$q(K^\times) / W \simeq (R / \mathbb{Z})^{r-1}.$$  

It is a real torus in $Sh_H(G/Z,X)(C)$ of dimension $r - 1$, which is oriented by the fixed multiorientation on $\mathcal{T}^\circ$.

For each $u \in \pi_0(T(R))$ and $b \in \hat{B}^\times$ let

$$\mathcal{T}_b^u = \{[q(u) \cdot x, b]_{\hat{H}^\times}, x \in \mathcal{T}^\circ\}.$$  

It is a real oriented torus of dimension $r - 1$. 

**Figure 1.** Case of circle arcs.
Proposition 4.2.2. The set
\[ \{ \mathcal{T}_b^u \mid b \in \hat{B}_x,\ u \in \pi_0(T(R)) \} \]
does not depend on the choice of \( q : K \hookrightarrow B \).

Proof. Let \( \hat{q} : K \hookrightarrow B \) be another embedding. Thanks to the Skolem-Noether theorem there exists \( \alpha \in B^\times \) such that
\[ \forall k \in K \quad \hat{q}(k) = \alpha q(k) \alpha^{-1}. \]

Let \( x_0 \in X \), and assume that \( \mathcal{T}^\circ = q(T(R)^\circ) \cdot x_0 \). We have \( \mathcal{T}^\circ := \hat{q}(T(R)^\circ) \cdot x_0 = \mathcal{T}^\circ \) and for each \( u \in \pi_0(T(R)) \)
\[ \alpha \cdot q(u) \cdot \mathcal{T}^\circ = \hat{q}(u^T(R)^\circ) \cdot \alpha \cdot x_0. \]

Let \( b \in \hat{B}_x \). As \( \alpha \in B^\times \) we have
\[ \mathcal{T}_b^u := \left[ \hat{q}(u) \mathcal{T}^\circ, b \right]_{\hat{F}x} = \left[ \alpha \cdot q(u) \cdot \mathcal{T}^\circ, b \right]_{\hat{F}x} = \left[ q(u) \cdot \mathcal{T}^\circ, \alpha^{-1} \cdot b \right]_{\hat{F}x} = \mathcal{T}_b^{u \cdot \alpha^{-1} \cdot b}. \]
The map \( b \mapsto \alpha^{-1} b \) is a bijection. Thus
\[ \{ \mathcal{T}_b^u, \ b \in \hat{B}_x, \ u \in \pi_0(T(R)) \} = \{ \mathcal{T}_b^u, \ b \in \hat{B}_x, \ u \in \pi_0(T(R)) \}. \]

Action of \( \text{Gal}(K^{ab}/K) \). Let us denote by \( K^{ab} \) the maximal abelian extension of \( K \) and by \( \text{rec}_K : K^{\times}_A/K^{\times} \to \text{Gal}(K^{ab}/K) \) the reciprocity map normalized by letting uniformizers correspond to geometric Frobenius elements.

The group \( K^{\times}_A \) acts on \( \{ \mathcal{T}_b^u \mid b \in \hat{B}_x, \ u \in \pi_0(T(R)) \} \) by
\[ \forall a = (a_\infty, a_f) \in K^{\times}_A = K^{\times}_x \times \hat{K}_x \quad \forall b \in \hat{B}_x \quad \alpha \cdot \mathcal{T}_b^u = \mathcal{T}_{\hat{F} \cdot b}^{u a}. \]

The action of \( k \in K^{\times} \) is trivial; as \( q(k) \in B^\times \), the definition of \( \text{Sh}_G(G/Z, X)(C) \) gives:
\[ k \cdot \mathcal{T}_b^u = [q(k) q(u) \mathcal{T}^\circ, \hat{q}(k) b]_{\hat{F}x} = [q(u) \mathcal{T}^\circ, b]_{\hat{F}x} = \mathcal{T}_b^u. \]

The action of \( F_x^\times \) is trivial. For \( a = (a_\infty, a_f) \in F_x^\times \), and \( b \in \hat{B}_x \), \( \hat{q}(a_f) b = b \hat{q}(a_f) \) and \( q(a_\infty) q(u) \mathcal{T}^\circ = q(u) \mathcal{T}^\circ \) hence
\[ a \cdot \mathcal{T}_b^u = [q(a_\infty) q(u) \mathcal{T}^\circ, q(a_f) b]_{\hat{F}x} = [q(u) \mathcal{T}^\circ, b]_{\hat{F}x} = \mathcal{T}_b^u. \]

4.3. Special cycles on \( \text{Sh}_G(G/Z, X)(C) \). In this section we construct some \( \gamma \)-chain on \( \text{Sh}_G(G/Z, X)(C) \).

Proposition 4.3.1. The homology class \( [\mathcal{T}_b^\circ] \in H_{r-1}(\text{Sh}_G(G/Z, X)(C), \mathbb{Z}) \) of \( \mathcal{T}_b^\circ \) is torsion.

Proof. Let us denote by \( \text{pr} \) the map
\[ \text{pr} : X \times \{ b \} \to \text{Sh}_G(G/Z, X)(C). \]
\( \mathcal{T}_b^\circ \) is in the image of \( \text{pr} \) and
\[ \text{pr}^{-1}(\mathcal{T}_b^\circ) = \{ (z_1) \times \gamma_2 \times \cdots \times \gamma_r \} \times \{ b \}. \]

Let \( \omega \in H^{r-1}(\text{Sh}_G(G/Z, X)(C), C) \). Thanks to the Matsushima-Shimura theorem, \( \omega = \omega_{\text{univ}} + \omega_{\text{cusp}}. \) As \( r \neq 1 \) we know that \( \omega = \omega_{\text{univ}}. \)

- If \( r = 1 \) is odd, then \( H^{r-1}(\text{Sh}_G(G/Z, X)(C), C) = \{ 0 \} \).
- If \( r = 1 = 2s \) is even, \( \omega \) is the pull-back of \( \bigwedge_{j=2}^r \omega^{\langle 1 \rangle} \), where
\[ \omega^{\langle 1 \rangle} = 1 \quad \text{or} \quad \frac{dx_i \wedge dy_i}{y_j^2}. \]

With the notations of the proof of Proposition 4.2.1, \( \mathcal{T}_b^\circ \) is a principal homogeneous space under \( \mathcal{W} \). Fix a fundamental domain \( \mathcal{W}' \) of \( \mathcal{W} \) in \( \gamma_2 \times \cdots \times \gamma_r. \) The incompatibility of degrees gives
\[ \int_{\mathcal{T}_b^\circ} \omega = \int_{\mathcal{W}'} \omega^{(2)} \wedge \cdots \wedge \omega^{(r)} = 0. \]
∀ω ∈ \( H^{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C}) \)
\[ \int_{\mathcal{T}_b^\circ} \omega = 0. \]
This proves that \([\mathcal{T}_b^\circ] = 0 \in H_r(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C})\) and \([\mathcal{T}_b^\circ] \in H_r(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z})\) is torsion.

\( \square \)

**Definition 4.3.2.** Let \( n \in \mathbb{Z}_{>0} \) be the exponent of \( H^{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z})_{\text{tors}} \). Then
\[ n[\mathcal{T}_b^\circ] = \partial \Delta_b^\circ \]
for some piece-wise differentiable \( r \)-chain \( \Delta_b^\circ \).

Proposition 3.2.1 proves that the value of
\[ \left( \frac{1}{\Omega} \xi \int_{\Delta_b^\circ} \omega_{\phi}^\beta \right) \in \mathbb{C} \]
modulo \( \Lambda_1 \) does not depend on the particular choice of \( \Delta_b^\circ \). If \( T(\mathbb{R})^\circ \) is fixed, then we have the following proposition.

**Proposition 4.3.3.** Let \( \mathcal{T}^\circ \) and \( \mathcal{T}'^\circ \) be two special cycles such that \( \text{pr}_1(\mathcal{T}^\circ) = \text{pr}_1(\mathcal{T}'^\circ) = \{z_1\} \). Assume that \( \text{pr}_j(\mathcal{T}^\circ) \) and \( \text{pr}_j(\mathcal{T}'^\circ) \) lie in the same connected component of \( X_j \) for each \( j \in \{2, \ldots, r\} \). Let \( n \) be the exponent of \( H_{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z})_{\text{tors}} \) and let \( \Delta_b^\circ \) and \( \Delta_b'^\circ \) satisfy
\[ n[\mathcal{T}_b^\circ] = \partial \Delta_b^\circ \quad \text{and} \quad n[\mathcal{T}_b'^\circ] = \partial \Delta_b'^\circ. \]
Then we have
\[ \int_{\Delta_b^\circ} \omega_{\phi}^\beta = \int_{\Delta_b'^\circ} \omega_{\phi}^\beta \pmod{\xi^{-1} \alpha^{-1} \Omega^\beta \Lambda_1}. \]

**Proof.** Our hypothesis allows us to decompose \( \Delta_b^\circ - \Delta_b'^\circ \) into
\[ \Delta_b^\circ - \Delta_b'^\circ = \text{pr}(\{z_1\} \times \mathcal{C}) + \mathcal{D}, \]
where \( \mathcal{D} \) is a cycle with \( \partial \mathcal{D} = 0 \) and \( \text{pr} \) is the map
\[
\text{pr}: \begin{cases} X & \to \text{Sh}_H(G/Z, X)(\mathbb{C}) \\ x & \mapsto [x, b]_{H_{\tilde{F}}^r} \end{cases}
\]
Let us show that \( \int_{\Delta_b^\circ - \Delta_b'^\circ} \omega_{\phi}^\beta \in \xi^{-1} \alpha^{-1} \Omega^\beta \Lambda_1 \).
We have
\[
\omega_{\phi}^\beta = \sum_{\varepsilon} \omega_{\varepsilon} \in \bigoplus_{\varepsilon: (\tau_1, \ldots, \tau_r) \to \{\pm 1\}^r} \Gamma(\text{Sh}_H(G/Z, X)(\mathbb{C}), (\Omega_{H_1}^\beta)^r),
\]
Each $\omega_{\varepsilon} \in \Gamma(\text{Sh}_H(G/Z, X)(\mathbb{C}), (\Omega^\text{an}_H)^\varepsilon)$ satisfies

$$\text{pr}^*(\omega_{\varepsilon}) = dz_1 \wedge \omega'_{\varepsilon}$$

We have

$$\int_{\text{pr}(\{z_1\} \times C)} \omega_{\varepsilon} = \int_{\{z_1\} \times C} dz_1 \wedge \omega'_{\varepsilon} = 0,$$

thus

$$\int_{\{z_1\} \times C} \omega'_{\varepsilon} = 0.$$

Thanks to Proposition 3.2.1 we have

$$\int_{D} \omega_{\varepsilon} \in \xi^{-1} \alpha^{-1} \Omega^\beta \Lambda^1$$

and the result follows.

\[ \square \]

**Corollary 4.3.4.** The value modulo $\Lambda_1$ of

$$\left( \frac{1}{\Omega^\beta} \xi \int_{\Delta_b^u} \omega_{\varphi}^\beta \right) \in \mathbb{C}$$

depends neither on the choice of $\mathcal{F}^o$ whose projection on $X_1$ is $\{z_1\}$ nor on $\Delta_b^u$ satisfying $n[\mathcal{F}^o_b] = \partial \Delta_b^u$.

**Definition 4.3.5.** We set $J_{b}^\beta = \frac{1}{\Omega^\beta} \xi \int_{\Delta_b^u} \omega_{\varphi}^\beta \mod \Lambda_1 \in \mathbb{C}/\Lambda_1$, the image of $\mathcal{F}^o_b$ by an exotic Abel-Jacobi map.

**Properties of $J_{b}^\beta$.** For each $u \in \pi_0(T(\mathbb{R}))$ let $\Delta_b^u$ be some piece-wise differentiable chain satisfying

$$n \left[ q(u) \cdot \mathcal{F}^o, b \right]_{H^*} = \partial \Delta_b^u.$$

**Proposition 4.3.6.** We have

$$J_{b}^\beta = \frac{1}{\Omega^\beta} \xi \sum_{u \in \pi_0(T(\mathbb{R}))} \beta(u) \int_{\Delta_b^u} \omega_{\varphi} \mod \Lambda_1.$$

**Proof.** Let us identify $\pi_0(T(\mathbb{R}))$ with $\prod_{j=2}^r \{1, -1\}$ and assume that the image of $T(\mathbb{R})^o$ is $(1, \ldots, 1)$. Then

$$\omega_{\varphi}^\beta = \sum_{u \in \pi_0(T(\mathbb{R}))} \beta(u) t^*_u(\omega_{\varphi}).$$

The chains $t_u \Delta_b^u$ and $\Delta_b^u$ are in the same connected component. Thus using 4.3.3, we have

$$\int_{t_u \Delta_b^u} \omega_{\varphi} = \int_{\Delta_b^u} \omega_{\varphi}$$

and the result follows.

\[ \square \]

Recall that $z_1 \in X_1$ is fixed by $q(K^\times_1)$.

**Proposition 4.3.7.** Let $\mathcal{F}^o$ and $\mathcal{F}'^o$ be two $q(T(\mathbb{R})^o)$-orbits such that $\text{pr}_1(\mathcal{F}^o) = \text{pr}_1(\mathcal{F}'^o) = \{z_1\}$. There exists a unique $u \in \pi_0(T(\mathbb{R}))$ such that, for all $j \in \{2, \ldots, r\}$,

$$\text{pr}_j(\mathcal{F}^o)$$

and $\text{pr}_j(q(u) \cdot \mathcal{F}^o)$

are in the same connected component of $X_j$.

If $J_{b}^\beta \in \mathbb{C}/\Lambda_1$ denotes the value obtained from $\mathcal{F}'^o$, we have

$$J_{b}^\beta = \beta(u) J_{b}^\beta.$$
Proof. Let \( x, x' \in X \) be such that \( \mathcal{T}^\circ = q(T(\mathbb{R})^\circ) \cdot x \) (resp. \( \mathcal{T}'^\circ = q(T(\mathbb{R})^\circ) \cdot x' \)). There exists \( u \in \pi_0(T(\mathbb{R})) \) such that for all \( j \in \{1, \ldots, r\} \), \( \text{pr}_j(q(u) \cdot x) \) and \( \text{pr}_j(x') \) are in the same connected component of \( X_j \). As \( \mathcal{T}'^\circ = q(u) \cdot \mathcal{T}^\circ \), the chain \( \Delta_b^\circ \) whose boundary up to torsion is \([\mathcal{T}'^\circ, b]_{H_F^s}\), equals \( \Delta_b^u \). Thus

\[
\sum_{u' \in \pi_0(T(\mathbb{R}))} \beta(u') \int_{\Delta_b^u} \omega_\varphi = \sum_{u' \in \pi_0(T(\mathbb{R}))} \beta(u') \int_{\Delta_b^u} \omega_\varphi = \beta(u) \sum_{u'' \in \pi_0(T(\mathbb{R}))} \beta(u'') \int_{\Delta_b^{u''}} \omega_\varphi.
\]

\( \square \)

Let \( q, q' : K \hookrightarrow B \) be two embeddings and \( x \in X, \mathcal{T}^\circ = q(T(\mathbb{R})^\circ) \cdot x \) (resp. \( \mathcal{T}'^\circ = q'(T(\mathbb{R})^\circ) \cdot x' \)). There exists \( a \in B^\times \) such that

\[ q' = aqa^{-1} \]

thanks to the Skolem-Noether theorem. For each \( j \in \{1, \ldots, r\} \), \( \text{pr}_j(\mathcal{T}^\circ) \) and \( \text{pr}_j(\mathcal{T}'^\circ) \) are in the same connected component of \( X_j \) if and only if \( \tau_j(\text{nr}(a)) > 0 \).

Using 4.3.7 we obtain

**Proposition 4.3.8.** If

\[ \alpha = (\text{sgn} \circ \tau_j(\text{nr}(a)))_{j \in \{1, \ldots, r\}} \in \{\pm 1\}^{r-1}, \]

then

\[ J_b^\beta = \beta(a)J_b^\beta. \]

Let \( N_{B^\times}(K^\times) \) be the normalizer of \( K^\times \) in \( B^\times \). Let \( a \in N_{B^\times}(K^\times) \setminus K^\times \). After multiplying \( a \) by an element in \( K^\times \) we may assume

\[ \forall j \in \{2, \ldots, r\} \quad \tau_j(\text{nr}(a)) > 0. \]

We have

\[ \text{pr}_1(q(a) \cdot \mathcal{T}^\circ) = t_1(z_1) \]

and

\[ \forall j \in \{2, \ldots, r\} \quad \text{pr}_j(q(a) \cdot \mathcal{T}^\circ) = \text{pr}_j(\mathcal{T}^\circ) \]

but the orientations of \( \text{pr}_j(q(a) \cdot \mathcal{T}^\circ) \) and \( \text{pr}_j(\mathcal{T}^\circ) \) are not the same.

Thus

\[ [t_1, \mathcal{T}^\circ, b]_{H_F^s} = [q(a) \cdot \mathcal{T}^\circ, b]_{H_F^s} = [\mathcal{T}^\circ, \tilde{q}(a)^{-1}b]_{H_F^s}, \]

but the orientations differ by \((-1)^{r-1}\). Hence

**Proposition 4.3.9.** The tori \( \mathcal{T}^\circ_b \) and \( t_1 \mathcal{T}^\circ_{\tilde{q}(a)b} \) are the same up to orientation.

5. Generalized Darmon’s points

5.1. The main conjecture. Let \( \Phi_1 : C/\Delta_1 \xrightarrow{\sim} E_1(C) \) be the Weierstrass uniformization; i.e. the inverse of \( \Phi_1 \) is the Abel-Jacobi map for the differential \( \eta_1 \). For each \( a_\infty \in K_\infty^\times \), fix some \( r \)-chain \( q(a_\infty) \cdot \Delta_b^\beta \) satisfying \( n[q(a_\infty) \cdot \mathcal{T}^\circ_b] = q(a_\infty) \cdot \Delta_b^\beta \) and denote by \( \beta(a_\infty) \) the following sign

\[ \beta(a_\infty) = \prod_{j=2}^r \beta \left( \text{sgn} \left( \prod_{w|\tau_j} a_\infty, w \right) \right). \]

**Conjecture 5.1.1.** The point

\[ P_b^\beta = \Phi_1 \left( \frac{1}{\Omega_b^\beta} \xi_\alpha \int_{\Delta_b^\beta} \omega_\varphi \right) = \Phi_1(J_b^\beta) \in E_1(C) \]

lies in \( E(K^{ab}) \) and

\[ \forall a = (a_\infty, a_f) \in K_\Delta^\times \quad \text{re}_{K}(a) P_b^\beta = \Phi_1 \left( \frac{\xi_\alpha}{\Omega_{q(a_f)}^\beta} \int_{q(a_\infty)} \Delta_b^\beta_{q(a_f)b} \omega_\varphi \right) = \beta(a_\infty) P_b^\beta_{q(a_f)b}. \]
Remark 5.1.2. The choice of \( z_1 \in X_{\lambda_1}^{p_1(K_{11})} \) fixes a morphism \( h_1 : S \to G_{1,1} \), hence a morphism \( C^\times = S(R) \to G_{1,1}(R) = B_{11}^\times = (B \otimes_{F, \tau_1} R)^\times \) satisfying \( h_1(C^\times) = q_1(K_{11}) \). This fixes an embedding \( \tau_{1,K} : K \to C \) such that the following diagram

\[
\begin{array}{c}
C^\times \\
\downarrow \tau_{1,K} \\
(K \otimes_{F, \tau_1} R)^\times
\end{array} \xrightarrow{h_1} \begin{array}{c}
(B \otimes_{F, \tau_1} R)^\times \\
\downarrow q_1
\end{array}
\]

commutes. We may fix \( \tilde{\tau}_1 : K_{ab} \to C \) above \( \tau_{1,K} \), such that

\[
\begin{array}{c}
F \\
\downarrow \tau_1 \\
K \\
\downarrow \tau_{1,K} \\
K_{ab}
\end{array} \xrightarrow{\tilde{\tau}_1} \begin{array}{c}
R \\
\downarrow \tau_1 \\
C \\
\downarrow \tau_{1,K} \\
K_{ab}
\end{array}
\]

commutes. Moreover the isomorphism

\[
\begin{array}{c}
\text{Gal}(K_{ab}/K) \\
\text{Gal}(\tilde{\tau}_1(K_{ab})/\tau_{1,K}(K))
\end{array}
\]

\[
\begin{array}{c}
\sigma \\
\tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1}
\end{array}
\]

does not depend on the choice of \( \tilde{\tau}_1 \). If \( \tilde{\tau}_1' \) is another embedding above \( \tau_{1,K} \), then \( \tilde{\tau}_1' = \tilde{\tau}_1 \circ \sigma' \) with \( \sigma' \in \text{Gal}(K_{ab}/K) \) and

\[
\forall \sigma \in \text{Gal}(K_{ab}/K) \quad \tilde{\tau}_1' \circ \sigma \circ \tilde{\tau}_1^{-1} = \tilde{\tau}_1 \circ \sigma' \circ \tilde{\tau}_1^{-1} = \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1}
\]

because \( \text{Gal}(K_{ab}/K) \) is commutative. Hence the Galois action of 5.1.1 does not depend on the particular choice of \( \tilde{\tau}_1 \).

Remark 5.1.3. Using conjecture 5.1.1, we obtain

\[
\begin{array}{c}
\forall a_{\infty} \in K_{\infty}^\times \\
\text{rec}_K(a_{\infty})P_b^\beta = \beta(a_{\infty})P_b^\beta.
\end{array}
\]

\[
\begin{array}{c}
\forall a \in F_{\infty}^\times \\
\text{rec}_K(a)P_b^\beta = P_b^\beta.
\end{array}
\]

5.2. Field of definition. Let \( B_{+}^\times = \{ b \in B^\times \mid \forall j \in \{2, \ldots, r\}, \tau_j(\text{nr}(b)) > 0 \} \). It is diagonally embedded in \( (B \otimes R)^\times \). Set

\[
K_{b}^+ = (K_b)^{\text{rec}_{K}(q_{\lambda_{1}}(bHF_{\times}B_{+}^\times))} \quad \text{and} \quad K_{b} := (K_b)^{\text{rec}_{K}(q_{\lambda_{1}}(bHF_{\times}B_{+}^\times))} \subset K_{b}^+.
\]

Note that \( K_b \) and \( K_{b}^+ \) depend on the choice of \( q : K \to B \).

Proposition 5.2.1. The point \( P_b^\beta \) is defined over \( K_{b}^+ : P_b^\beta \in E(K_{b}^+) \).

Proof. Let \( a = (1_{\infty}, bhf_{b-1})(a_{\infty}, 1_f) \in q_{\lambda_{1}}(bHF_{\times}b^{-1}B_{+}^\times) \) with \( f \in F_{\times}^\times \) and \( h \in H \). We have

\[
\text{rec}_{K}(a)P_b^\beta = \text{rec}_{K}(q_{\lambda_{1}}((1_{\infty}, bhf_{b-1})))P_b^\beta = P_{bhf_{b-1}}^\beta = P_{bhf_{b}}^\beta = P_b^\beta
\]

\[\square\]

Remark that \( \text{rec}_{K} \) induces a surjection

\[
\mathcal{R} : \pi_0(T(R)) = \frac{(K \otimes_{Q} R)^\times}{(F \otimes_{Q} R)^\times} \simeq \prod_{j=2}^{r} \{ \pm 1 \} \to \text{Gal}(K_{b}^+/K_b).
\]

Thus, we have

Proposition 5.2.2. The points \( P_b^\beta \) lie in \( K_{b}^\beta = (K_{b}^+)^{\mathcal{R}(\text{Ker} \beta)} \).

Remark 5.2.3. As \( \text{Ker} \beta \) has index 2 in \( \prod_{j=2}^{r} \{ \pm 1 \} \), the field \( K_{b}^\beta \) has degree 1 or 2 over \( K_b \).
Assume that the conductor $N$ of $E$ decomposes as $N = N_+N_-$ with $N = p_1\ldots p_t$, $p_i$ distinct prime ideals of $O_F$ and $t \equiv d - r \mod 2$. If $Rm(B) = \{\tau_1, \ldots, \tau_d\} \cup \{p_1, \ldots, p_t\}$ and $H = (R \otimes \widehat{Z})^\times$ where $R \subset B$ is an Eichler order of level $N_+$, then $K_B$ is a ring class field of conductor $f_\ell$ and $K_B^+$ a ring class field of conductor $f_\ell f_\infty$, where $f_\infty = \prod_{j=2}^\infty \tau_j$.

5.3. **Local invariants of** $B$. Let $\pi$ be the irreducible automorphic representation of $B_A^\times$ generated by $\varphi$ and

$$\eta_K = \eta_{K/F} : F_h^\times/F^\times \text{Gal}(K/F) \to \{\pm 1\}$$

the quadratic character of $K/F$. For each place $v$ of $F$ let $\text{inv}_v(B_v) = \{(\pm 1)\}$ be the invariant of $B$:

$$\text{inv}_v(B_v) = 1 \text{ if and only if } B_v \simeq M_2(F_v).$$

Fix $b \in \widehat{B}^\times$ and a character

$$\chi : \text{Gal}(K_0^+/K) \to \mathbb{C}^\times,$$

which will be identified with

$$K_A^\times \text{rec} \rightarrow \text{Gal}(K_{ab}/K) \to \text{Gal}(K_0^+/K) \to \mathbb{C}^\times.$$ 

Let $L(\pi \times \chi, s)$ be the Rankin-Selberg $L$ function, see [Jac72] page 132 and [JL70] section 12. This function admits, since $\pi$ has trivial central character, a holomorphic extension to $\mathbb{C}$ satisfying

$$L(\pi \times \chi, s) = \varepsilon(\pi \times \chi, s)L(\pi \times \chi, 1 - s).$$

In this section, we prove the following

**Proposition 5.3.1.** Let $b \in \widehat{B}^\times$ and assume conjecture 5.1.1. If

$$\varepsilon(\pi, v) = \sum_{\sigma \in \text{Gal}(K_{b}^+/K)} \chi(\sigma) \otimes P^b_\sigma \in E(K_{b}^+/K) \otimes \mathbb{Z}[\chi]$$

is not torsion, then $\beta = \chi_\infty$,

$$\forall v \neq \tau_1 \quad \eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v) \quad \text{and} \quad \varepsilon(\pi \times \chi, \frac{1}{2}) = -1.$$ 

We shall use the following theorem ([Tun83] and [Sai93]).

**Theorem 5.3.2.** The equality $\eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v)$ holds if and only if there exists a non-zero invariant linear form

$$\ell_v : \pi_v \times \chi_v \to \mathbb{C}$$

unique up to a scalar satisfying

$$\forall a \in K_v^\times \forall u \in \pi_v \quad \ell_v(q_v(a)u) = \chi_v(a)^{-1}\ell_v(u)$$

i.e. $\ell_v$ is $q(K_v^\times)$-invariant.

**Proof.** (of Proposition 5.3.1) We follow the proof of [AN10], Proposition 2.6.2.

Let $S'$ be a finite set of finite places of $F$ containing the places where $B$, $\pi$ or $K_B^+/F$ ramify, and such that the map $r = (r_v : K_v^\times \to \text{Gal}(K_{b}^+/K))_{v \in S'}$ obtained by composition

$$r : \prod_{v \in S'} K_v^\times \to K_A^\times \text{rec} \rightarrow \text{Gal}(K_{ab}/K) \to \text{Gal}(K_B^+/K)$$

is surjective.

For each $v \in S'$ let

$$j_v : \begin{cases} K_v \to B_v \mapsto b_v^{-1}q_v(k)b_v \end{cases}$$

and

$$j = (j_v)_{v \in S'} : \prod_{v \in S'} K_v \hookrightarrow \prod_{v \in S'} B_v.$$ 

As $S'$ does not contain any archimedean place of $F$,

$$\forall a \in \prod_{v \in S'} K_v^\times \quad [\mathcal{T}, \check{q}(a)b]_{HF^\times} = [\mathcal{T}, b_{jv}(a)]_{HF^\times}$$

and

$$\forall a \in \prod_{v \in S'} K_v^\times \forall b \in \widehat{B}^\times \quad \text{rec}_K(a)P^b_\sigma = P^b_{\check{q}(a)b} = P^b_{bjv(a)},$$

where $\mathcal{T}$ is the space of test functions (in $\mathcal{T}$).

**Remark.** It is convenient to see $\mathcal{T}$ as the space of test functions on $\mathcal{T}$.

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Let \((K_v^\times)^{\circ} \subset K_v^\times\) be the inverse image of \((K_v^\times / \mathcal{O}_{K,v}^\times)^{\text{Gal}(K/F)} \subset K_v^\times / \mathcal{O}_{K,v}^\times\).
We have
\[
K_v^\times / \mathcal{O}_{K,v}^\times F_v^\times \xrightarrow{\sim} \begin{cases} 
0 & \text{if } v \text{ is inert in } K/F \\
\mathbb{Z}/2\mathbb{Z} & \text{if } v \text{ ramifies in } K/F \\
\mathbb{Z} & \text{if } v \text{ splits in } K/F,
\end{cases}
\]
the quotient \((K_v^\times)^{\circ} / F_v^\times\) is compact and
\[
D_v := K_v^\times / (K_v^\times)^{\circ} \xrightarrow{\sim} \begin{cases} 
\mathbb{Z} & \text{if } v \text{ splits in } K/F \\
0 & \text{otherwise},
\end{cases}
\]
\[
(K_v^\times)^{\circ} / \mathcal{O}_{K,v}^\times F_v^\times \xrightarrow{\sim} \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & \text{if } v \text{ ramifies in } K/F \\
0 & \text{otherwise}.
\end{cases}
\]
For each \(v \in S', C_v = \mathcal{O}_{K,v}^\times \cap \text{Ker}(\tau_v)\) is an open subgroup of \(\mathcal{O}_{K,v}^\times\) and \(V_v^\circ = (K_v^\times)^{\circ} / F_v^\times C_v\) is finite.
Let \(V_v\) be the following subset of \(K_v^\times / F_v^\times C_v\):

- if \(v\) does not split in \(K/F\), \(V_v^\circ = K_v^\times / F_v^\times C_v\) and \(V_v := V_v^\circ\).
- If \(v\) splits in \(K/F\), we fix some section of \(K_v^\times \twoheadrightarrow K_v^\times / (K_v^\times)^{\circ} \xrightarrow{\sim} \mathbb{Z}\). Hence \(K_v^\times = (K_v^\times)^{\circ} \times D_v\) and there exists \(n_v \geq 1\) such that \(\text{Ker}(\tau_v|_{D_v}) = n_v D_v\).

Fix a set of representatives \(D'_v \subset D_v\) of \(D_v / n_v D_v\) and set \(V_v = V_v^\circ D'_v \subset K_v^\times / F_v^\times C_v\).

Let \(V = \prod_{v \in S'} V_v \subset \prod_{v \in S'} K_v^\times / F_v^\times C_v\) which is stable under multiplication by the abelian group \(V^\circ = \prod_{v \in S'} V_v^\circ\) and such that \(V \hookrightarrow \prod_{v \in S'} K_v^\times / F_v^\times C_v \twoheadrightarrow \text{Gal}(K^+_v / K)\) is surjective with fibers of cardinality \(\frac{|V|}{|\text{Gal}(K^+_v / K)|}\).

Fix some open-compact subgroup \(H_1 \subset \bigcap_{a \in V} j(a)Hj(a)^{-1}\). Using the maps
\[
\text{Sh}_{H_1}(G/Z, X) \xrightarrow{[j(a)]} \text{Sh}_{j(a)^{-1}H_1j(a)}(G/Z, X) \xrightarrow{\text{pr}} \text{Sh}_H(G/Z, X),
\]
we have
\[
\sum_{a \in V} \chi(a) \int_{\Delta_b^a(a)} \omega^\beta = \sum_{a \in V} \chi(a) \int_{\Delta_b^a} [j(a)]^* \omega^\beta
= \int_{\Delta_b^a} \sum_{a \in V} \chi(a) [j(a)]^* \omega^\beta
= \int_{\Delta_b^a} \omega^\beta',
\]
where
\[
\omega^\beta := \sum_{a \in V} \chi(a) [j(a)]^* \omega^\beta_a.
\]
Whenever \(\frac{|V|}{|\text{Gal}(K^+_v / K)|} \chi(P_b^\beta) = \sum_{a \in V} \chi(a) \otimes P_b^\beta_{j(a)} \in \mathbb{Z}[\chi] \otimes_{\mathbb{Z} E(K_b^+)} \mathbb{Z} / \mathcal{O}_1\) is not torsion, there exists \(\sigma : \mathbb{Z}[\chi] \hookrightarrow C / \Lambda_1\) such that
\[
\frac{\xi \alpha}{\Omega} \int_{\Delta_b^a} \sum_{a \in V} \sigma \chi(a) [j(a)]^* \omega^\beta_a \notin \mathbb{Q}[\chi] \cdot \Lambda_1,
\]
where \(\sigma \chi = \sigma \circ \chi\). The vector
\[
\omega_1 = \sum_{a \in V} \sigma \chi(a) [j(a)]^* \omega_\phi \in \pi^{H_1} \cap \Gamma(\text{Sh}_{H_1}(G/Z, X), \Omega_{H_1})
\]
is non-zero and invariant under \(j(\prod_{v \in S'} (K_v^\times)^{\circ})\). Moreover,
\[
\forall a \in \prod_{v \in S'} (K_v^\times)^{\circ} \quad j(a) \omega_1 = \sigma \chi^{-1}(a) \omega_1.
\]
Let
\[ \sigma_{\ell_{S'}} : \bigotimes_{v \in S'} \sigma_{\pi_v} = \bigotimes_{v \in S'} \pi_v \rightarrow \mathbb{C}(\sigma_{\chi^{-1}}) \]
be the \( j(\prod_{v \in S'} (K_v^x)^{\circ}) \)-invariant projection on \( \mathbb{C}(\omega_1) \).

Assume that \( v \in S' \) does not split in \( K \). In this case \( (K_v^x)^{\circ} = K_v^x \) and \( \sigma_{\ell_{S'}} \) induces a \( q_v(K_v^x) \)-invariant linear form \( \sigma_{\ell_v} : \pi_v \rightarrow \mathbb{C}(\sigma_{\chi_v^{-1}}) \). We have \( \sigma_{\ell_v}(\omega_1) \neq 0 \), where
\[ \omega_{1,v} = \sum_{a_v \in V_v} \sigma_{\chi_v} \circ r_v(a_v) [j_v(a_v)]^* \omega_v. \]

As \( \varepsilon_v(\pi_v \times \sigma_{\chi_v}, \frac{1}{2}) \) is independent of \( \sigma \), \( \mathbb{Z}[\chi] \rightarrow \mathbb{C} \), Theorem 5.3.2 shows that
\[ \eta_{K,v}(-1) \varepsilon_v(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v). \]
When \( v \in S' \) splits in \( K \) or \( v \notin S' \cup S_{\infty} \), the equality
\[ \eta_{K,v}(-1) \varepsilon_v(\pi_v \times \chi_v, \frac{1}{2}) = 1 = \text{inv}_v(B_v) \]
follows from calculations which may be found for example in [Nek06] Proposition 12.6.2.4.

Global sign. If \( v = \tau_j \) is an archimedean place, then \( \varepsilon_v(\pi_v \times \chi_v, \frac{1}{2}) = 1 \). Moreover \( \eta_{K,v}(-1) = 1 \) if and only if \( j \in \{2, \ldots, r\} \) and \( \text{inv}_v(B_v) = 1 \) if and only if \( j \in \{1, \ldots, r\} \). Thus
\[ \eta_{K,v}(-1) \text{inv}_v(B_v) = \begin{cases} -1 \times 1 & \text{if } j = 1 \\ 1 \times 1 & \text{if } j \in \{2, \ldots, r\} \\ -1 \times -1 & \text{if } j > 1. \end{cases} \]

Hence
\[ \varepsilon_v(\pi, \chi_v, \frac{1}{2}) = - \prod_v \eta_{K,v}(-1) \text{inv}_v(B_v) = -1. \]

\[ \square \]

5.4. Global invariant linear form and a conjectural Gross-Zagier formula. For any open subgroup \( H' \subset H, b \in \hat{B}^x \) and \( \nu \in \pi_0(T(R)) \) fix \( \Delta^n_{H', b} \in C'(\text{Sh}_{H}(G/Z,X)(\mathbb{C}), Q) \) such that
\[ \partial \Delta^n_{H', b} = [\mathcal{F}^n_{H', b}], \text{where } \mathcal{F}^n_{H', b} = \{[g(u)x, b]_{H', \mathcal{F}^x}, x \in \mathcal{F}^x \}. \]
Recall that
\[ \forall \nu' \in \pi_0(T(R)) \quad t_{\nu'} \Delta^n_{H', b} = \Delta^n_{b^{\nu'}}. \]

Let \( \pi_{\infty} \) be the archimedean part of \( \pi \). Fix \( \varphi_{\infty} \in \pi_{\infty} \) a lowest weight vector of weight \( (2, \ldots, 2, 0, \ldots, 0) \) of \( \pi_{\infty} \) and \( \omega_{\varphi} \) such that \( \omega_{\varphi} = \varphi_{\infty} \otimes \varphi_f \in \pi_{\infty} \otimes \pi_f \subset S_2(B_{\chi}). \)

Let us denote by \( Q \pi_f \) the sub \( Q \mathcal{F}^x \)-module of \( \pi_f \) generated by \( \varphi_f \).

Proposition 5.4.1. The space \( Q \pi_f \) is a \( Q \)-vector space and \( Q \pi_f \otimes Q \mathbb{C} \rightarrow \pi_f \) is surjective.

Proof. The space \( \text{Im}(Q \pi_f \otimes Q \mathbb{C} \rightarrow \pi_f) \) is a zero subvector space of \( \pi_f \) invariant under \( B_{\chi}^x \). As \( \pi_f \) is irreducible, we have \( \text{Im}(Q \pi_f \otimes Q \mathbb{C} \rightarrow \pi_f) = \pi_f \) and \( Q \pi_f \otimes Q \mathbb{C} \rightarrow \pi_f \) is surjective. \( \square \)

Fix \( \eta \neq 0 \in H^0(E, \Omega_{E/F}) \). There exists \( \alpha \in F^x \) such that
\[ \mathcal{F}(\alpha \omega_{\varphi}) = \eta. \]

Fix a continuous character of finite order \( \chi : K_{\chi}/K_{\chi} F_{\chi}^x \rightarrow \mathbb{Z}[\chi]^x \). Let \( H' \subset H \) be any open compact subgroup of \( \hat{B}^x \) satisfying \( \chi(q_{\chi}^{-1}(H'F_{\chi})) = 1 \). Assume that there exists \( b_0 \in \hat{B}^x \) such that \( b_0^{-1}H'b_0 \subset H \). Let \( \text{pr}_{b_0} \) be the map \( \text{Sh}_{H}(G/Z,X) \rightarrow \text{Sh}_{H}(G/Z,X) \) defined on complex points by
\[ [x, b]_{H', \mathcal{F}^x} \mapsto [x, b b_0]_{H', \mathcal{F}^x}. \]
**Proposition 5.4.2.** If $b_0^{-1}H'b_0 < H$ for some $b_0 \in \tilde{B}^\times$, then
\[ \forall Z' \in C^r(Sh_H(G/Z,X)(C),Z) \quad \int_{Z'} \text{pr}^*_b(\omega_q^{\infty}) \in \mathbb{Q}\alpha^{-1}\Omega^{\infty} \Lambda_1. \]

**Proof.** Let $Z = \text{pr}^*_b(Z') \in C^r(Sh_H(G/Z,X)(C),Z)$. We have
\[ \int_{Z'} \text{pr}^*_b(\omega_q^{\infty}) = \deg(\text{pr}_b : Z' \to Z) \int_Z \omega_q^{\infty}. \]
Thanks to Proposition 3.3.3, we have $\int_Z \omega_q^{\infty} \in \mathbb{Q}\alpha^{-1}\Omega^{\infty} \Lambda_1$ hence $\int_{Z'} \text{pr}^*_b(\omega_q^{\infty}) \in \mathbb{Q}\alpha^{-1}\Omega^{\infty} \Lambda_1$. \(\square\)

Denote by $\text{pr} : Sh_H(G/Z,X) \to Sh_H(G/Z,X)$ the natural projection, and by $(K \otimes R)^\times_+$ the set of elements of $(K \otimes R)^\times$ whose norm to $F$ is positive at each place of $F$. We have $\pi_0(T(R)) = \langle (K \otimes R)^\times_+ \rangle$.

The following formula
\[ \ell_{\chi}(\omega') = \frac{1}{[H : H']\deg(\mathcal{T}_b, pr : \mathcal{T}_b)} \sum_{a \in \mathcal{K}_a^\times} \chi(a) \int_{\mathcal{H}^\times_\infty} \omega' \mod \mathbb{Q}(\chi) \otimes \mathbb{Q}\alpha^{-1}\Omega^{\infty} \Lambda_1, \]
where $\partial \Delta_{H',q(a)}^{q(q_{\infty})_{H',q(a)}}$ is independent of the specific choice of $\Delta_{H',q(a)}^{q(q_{\infty})}$: we can assume that $\omega' = \text{pr}^*_b(\omega_q)$ for some $b_0 \in \tilde{B}^\times$; decompose each $a \in K^\times_\mathcal{A}/q_{\mathcal{A}}^1(H'F^\times_\mathcal{A})(K \otimes R)^\times$ as $a = (a_f, 1) = (1f, a_\infty)$. Remark that
\[ K^\times_\mathcal{A}/q_{\mathcal{A}}^1(H'F^\times_\mathcal{A})(K \otimes R)^\times = \tilde{R}^\times /q^{-1}(H'\tilde{F}^\times) \times (K \otimes R)^\times / (K \otimes R)^\times_+, \]
hence $a_f \in \tilde{R}^\times /q^{-1}(H'\tilde{F}^\times)$ and $a_\infty \in (K \otimes R)^\times / (K \otimes R)^\times_+$.

Thanks to Proposition 5.4.2, the following formula
\[ \sum_{a_\infty \in K^\times_\infty} \chi(a_\infty) \int_{\mathcal{H}^\times_\infty} \omega' = \sum_{a_\infty \in K^\times_\infty} \chi(a_\infty) \int_{H^\times_\infty} \text{pr}^*_b(\omega_q) \]
\[ = \int_{H^\times_\infty} \omega_q^{\infty} \mod \mathbb{Q}\alpha^{-1}\Omega^{\infty} \Lambda_1 \]
does not depend on the specific choice of $\Delta_{H',q(a)}^{q(q_{\infty})_{H',q(a)}}$.

Thus, the expression of $\ell_{\chi}(\omega')$ above defines a linear form
\[ \ell_{\chi} : S_{H'}^\times \cap \mathbb{Q}[\tilde{B}^\times] \omega_q \to \mathbb{Q}(\chi) \otimes \mathbb{Q}(\mathcal{C}/\mathbb{Q}\alpha^{-1}\Omega^{\infty} \Lambda_1). \]

To simplify the notations, let
\[ \delta_{H',H} = \deg(\mathcal{T}_b, pr : \mathcal{T}_b) \quad \text{and} \quad W_{H'} = K^\times_\mathcal{A}/q_{\mathcal{A}}^1(H'F^\times_\mathcal{A})(K \otimes R)^\times. \]

Thus
\[ \ell_{\chi}(\omega') = \frac{1}{[H : H']\delta_{H',H}} \sum_{a \in W_{H'}} \chi(a) \int_{H^\times_\infty} \omega'. \]

**Proposition 5.4.3.**
(1) Let $H'' \subset H \subset H$ be open compact subgroups such that $\chi(q_{\mathcal{A}}^{-1}(H'F^\times_\mathcal{A})) = 1$ and $\text{pr}^*$ the map $\text{pr}^* : S_{H'}^\times(B^\times_\mathcal{A}) \to S_{H''}^\times(B^\times_\mathcal{A})$.

If $\omega' \in S_{H''}^\times(B^\times_\mathcal{A}) \cap \mathbb{Q}[\tilde{B}^\times] \omega_q$, then $\ell_{\chi}(\omega') = \ell_{\chi}(\text{pr}^*(\omega'))$ and $\ell_{\chi}$ defines a linear form on $\mathbb{Q}[\tilde{B}^\times] \omega_q$.

(2) We have
\[ \forall a \in \tilde{R}^\times \forall \omega \in \mathbb{Q}[\tilde{B}^\times] \omega_q \quad \ell_{\chi}([\hat{q}(a)]^\infty \omega) = \chi(f(a))^{-1}\ell_{\chi}(\omega). \]

(3) If $\chi$ factors through $\text{Gal}(K^\times_b / K)$ and if $P_b^\beta = \Phi_1 \left( \int_{H^\times_\infty} \omega_{\phi}^\beta \right) \otimes 1 \in \mathbb{C}/\mathbb{Q} \Lambda_1$, then
\[ \alpha_\mathcal{T}(P^\infty_{b}) = \sum_{\text{Gal}(K^\times_b / K)} \chi(\sigma) \otimes \sigma(P^\infty_{b}) \in \mathbb{Q}(\chi) \otimes \mathbb{Q}(K^\times_b) \subset \mathbb{Q}(\chi) \otimes \mathbb{Q}(\mathbb{C}/\mathbb{Q} \Lambda_1) \]
equals $\Phi_1(\ell_1([b]^{*}\omega_{\varphi}))$, up to a non-zero rational factor.

Proof. **Proof of 1.** Let $a \in \hat{R}^\times$. We have $\text{pr}(\Delta_{H''\hat{\varrho}(a)}) = \Delta_{H''\hat{\varrho}(a)}$ and

$$\int_{\Delta_{H''\hat{\varrho}}} \text{pr}^{*}\omega' = \deg(\mathcal{F}_{H''\hat{\varrho}} \to \mathcal{F}_{H',\hat{\varrho}}) \int_{\Delta_{H''\hat{\varrho}}} \omega' = \delta_{H''H'} \int_{\Delta_{H''\hat{\varrho}}} \omega'. $$

As $\chi(q^{-1}_{A}(H'F_{A}^{\times})) = 1$, we have (thanks to Proposition 5.4.2)

$$\ell_1(\text{pr}^{*}\omega') = \frac{1}{[H' : H''][\delta_{H''H'}]} \sum_{a \in W_{H''}} \chi(a) \otimes \int_{\Delta_{H''[\hat{\varrho}(a)]}} \text{pr}^{*}\omega' \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \frac{\delta_{H''[\hat{\varrho}]} H'}{H'[\delta_{H''H'}]} \sum_{a \in W_{H''}} [H' : H''] \chi(a) \otimes \int_{\Delta_{H''[\hat{\varrho}(a)]}} \omega' \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \frac{[H' : H'']}{[H' : H'][\delta_{H''H'}]} \sum_{a \in W_{H''}} \chi(a) \otimes \int_{\Delta_{H''[\hat{\varrho}(a)]}} \omega' \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \ell_1(\omega').$$

**Proof of 2.** Assume $H''$ is sufficiently small such that $[\hat{q}(a_{f})]^{*}\text{pr}^{*}\omega \in S_{2}^{H''}$. We have

$$\ell_1([\hat{q}(a_{f})]^{*}\omega) = \ell_1([\hat{q}(a_{f})]^{*}\text{pr}^{*}\omega)$$

$$= \frac{1}{[H' : H''][\delta_{H''H'}]} \sum_{a' \in W_{H''}} \chi(a') \otimes \int_{\Delta_{H''[\hat{\varrho}(a')]}^{\prime}} [\hat{q}(a_{f})]^{*}\text{pr}^{*}\omega \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \frac{1}{[H' : H''][\delta_{H''H'}]} \sum_{a' \in W_{H''}} \chi(a') \otimes \int_{\Delta_{H''[\hat{\varrho}(a')]}^{\prime}} \text{pr}^{*}\omega \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \frac{1}{[H' : H''][\delta_{H''H'}]} \sum_{a'' \in W_{H''}} \chi(a'' a^{-1}) \otimes \int_{\Delta_{H''[\hat{\varrho}(a'')]^{\prime}}} \text{pr}^{*}\omega \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \chi_f(a)^{-1} \frac{1}{[H' : H''][\delta_{H''H'}]} \sum_{a'' \in W_{H''}} \chi(a'') \otimes \int_{\Delta_{H''[\hat{\varrho}(a'')]^{\prime}}} \text{pr}^{*}\omega \pmod{Q(\chi) \otimes Q \alpha^{-1}\Omega^{\times} A_1}$$

$$= \ell_1(\omega')$$

**Proof of 3.** As $\omega_{\varphi} \in S_{2}(B_{A}^{\times}) = \bigcup_{H} S_{2}^{H}(B_{A}^{\times})$, there exists $H'$ sufficiently small such that $\omega_{\varphi} \in S_{2}^{H'}$ and $[b]^{*}\omega_{\varphi} \in S_{2}^{H'}$.

Let $m = [K_{A}^{\times}/q_{A}^{-1}(H'F_{A}^{\times})(K \otimes R)_{A}^{\times} : \text{Gal}(K_{b}^{\times}/K)]$ and $\nu = \frac{1}{[H' : H'][\deg(\mathcal{F}_{H'} \to \mathcal{F}_{a})]}$. We have:
Let us consider the Néron-Tate height $h_{NT} : E(K^{ab}) \times E(K^{ab}) \to \mathbb{R}$ extended to an hermitian form

$$h_{NT} : E(K^{ab}) \otimes \mathbb{C} \times E(K^{ab}) \otimes \mathbb{C} \to \mathbb{C}.$$ 

Recall the condition

$$\forall \nu \neq \tau_1 \quad \varepsilon(\pi_\nu \times \chi_\nu, \frac{1}{2})\eta_{K, \nu}(-1) = \operatorname{inv}_\nu(B)$$

from Proposition 5.3.2: if 2 fails, then $P^{\infty}_{\nu} \in E(K^{ab})$ is torsion.

In general, there should be some $k(b, \omega_\nu) \in \mathbb{C}$ such that

$$\forall \sigma : Q(\chi) \hookrightarrow \mathbb{C} \quad h_{NT}(\varepsilon(\sigma(P^{\infty}_{\nu}))) = k(b, \omega_\nu)L'(\pi \times \sigma_{\chi, \frac{1}{2}}),$$

as in Gross-Zagier, Zhang and Yuan-Zhang-Zhang [GZ86, Zha01, YZZ09].

This formula explains the following conjecture:

**Conjecture 5.4.4.** Let $K_X = (K^{ab})^{\operatorname{Ker}(\chi)}$ be the extension of $K$ trivializing $\chi$. If

$$\forall \nu \neq \tau_1 \quad \varepsilon(\pi_\nu \times \chi_\nu, \frac{1}{2})\eta_{K, \nu}(-1) = \operatorname{inv}_\nu(B),$$

then there exists $b \in \tilde{B}^\times$ such that $k(b, \omega_\nu) \neq 0$ and we have the following equivalences:

$$\ell_\chi \neq 0 \iff \exists b \in B_\mathcal{A}^+ \text{ such that } K_X \subset K_b^+ \text{ and } \sigma(\alpha_{\chi, \mathcal{A}}^{\infty}P^{\infty}_{\nu}) \in \mathbb{Z}[\chi] \otimes E(K_b^+) \text{ is not torsion}$$

$$\iff \exists \sigma : Q(\chi) \hookrightarrow \mathbb{C} \quad L'(\pi \times \sigma_{\chi, \frac{1}{2}}) \neq 0$$

$$\iff \forall \sigma : Q(\chi) \hookrightarrow \mathbb{C} \quad L'(\pi \times \sigma_{\chi, \frac{1}{2}}) \neq 0.$$ 

6. A relation to Kudla’s program

The theorem of Gross-Kohnen-Zagier asserts that the positions of the traces to $\mathbb{Q}$ of classical Heegner points are given by the Fourier coefficients of some Jacobi form. The geometric proof of Zagier explained for example in [Zag85] has been recently generalized by Yuan, Zhang and Zhang in [YZZ09] using a result of Kudla-Millson [KM90]. In this section we establish a relation between Darmon’s construction and Kudla’s program. This is a first step in an attempt to apply the arguments of Zagier [Zag85] and Yuan-Zhang and Zhang’s [YZZ09] to Darmon’s points.
6.1. Some computations. Let us fix a modular elliptic curve $E/F$ of conductor $N = N_+N_-$. Assume that $\operatorname{Ram}(B) = \{\tau_{r+1}, \ldots, \tau_d\} \cup \{v \mid N_-\}$ and that the quadratic extension $K/F$ satisfies the following hypothesis

$$\forall v \mid N_+ \text{ splits in } K \quad \forall v \mid N_- \text{ is inert in } K.$$ 

In particular, the relative discriminant $d_{K/F}$ is prime to $N$. Let $R$ be an Eichler order of $B$ of level $N_+$. Identify $K$ with its image in $B$ by $q$ and assume $K \cap R = \mathcal{O}_K$, $H = \hat{R}^\times$ (which implies that $\dim H = 1$).

Recall that $h_{\mathfrak{a}}$ defines an embedding $\tau_{1,K} : K \hookrightarrow \mathbb{C}$ and denote by $c$ the non-trivial element of $\operatorname{Gal}(K/F)$. Assume that Conjecture 5.1.1 is true for $\beta = 1$ and let $P = \operatorname{Tr}_{K_1^+} P_1 \in E(K)$.

**Proposition 6.1.1.** If $\varepsilon$ is the global sign of $E/F$, i.e. $\Lambda(E/F,s) = \varepsilon \Lambda(E/F,2-s)$, where $\Lambda$ is the completed $L$-function of $E/F$, then $c(P) = -\varepsilon P$.

**Proof.** Assume that $K = F(i)$ and $B = K(j)$, with $i^2 = a \in F^\times$, $j^2 = b \in F^\times$ and $ij = -ji$. Recall that

$$\mathcal{O}_1 = [\mathcal{O}, 1]_{H^F}$$

with $\mathcal{O} = \{z_1 \times \gamma_2 \times \cdots \times \gamma_r\}$. Thus

$$c(\mathcal{O}) = \{[t_1 \gamma_1] \times \gamma_2 \times \cdots \times \gamma_r, 1\}_{H^F} = (-1)^{r-1}[j^{-1}(\mathcal{O}), 1]_{H^F}$$

and

$$c(\mathcal{O}) = (-1)^{r-1}[\mathcal{O}, j]_{H^F}$$

since $j \in B^\times$. This shows that $c(P_j) = (-1)^{r-1}P_j$. We will write $P_j$ using only $P_1$. We will make the following abuse of language. For each place $v$ of $F$, $j_v$ shall denote the element $(1, \ldots, 1, j_v, \ldots) \in B^\times_A$ and we will use the following lemma

**Lemma 6.1.2.** Let $b \in \hat{B}^\times$ and $v$ a place of $F$. When $v \mid N_+$, set $k_v \in K_v^\times$ corresponding to

$$
\begin{pmatrix}
1 & 0 \\
0 & \pi_v \text{ord}_v (N_+) \\
\end{pmatrix}
$$

where $\pi_v$ is an unramified $K_v$-extension of $K_v$. If $b_v = 1$, then

$$P_{bj_v} = \begin{cases} 
-\varepsilon_v p_b & \text{if } v \mid N_-
\
\varepsilon_v \text{rec}_K(k_v^{-1})b_v & \text{if } v \mid N_+
\
p_b & \text{if } v \nmid N
\end{cases}$$

**Proof.** (of the lemma)

For each $v$ inert in $K/F$ we have

$$\text{inv}_v(B) = 1 \iff B_v \simeq M_2(F_v)$$

$$\iff b \in N_{K_v/F_v}(K_v^\times) = \mathcal{O}_{F_v}^\times F_v^2$$

$$\iff 2 \mid \text{ord}_v(b)$$

As $\overline{\mathcal{O}} = -j$, we have $\text{nr}(j) = -j^2 = -b$ and

$$\text{inv}_v(B) = 1 \iff 2 \mid \text{ord}_v(\text{nr}(j_v)).$$

If $v \nmid N_-$ then $H_v = \mathcal{O}_{B_v}^\times$, where $\mathcal{O}_{B_v}$ is the unique maximal order in $B_v$ hence $H_v \triangleleft B_v^\times$ and $B_v^\times / H_v^\times \simeq \mathbb{Z}$ by choosing some uniformizer. As $H_v$ is normal in $B_v^\times$, the map

$$[j_v] : \text{Sh}_H(G/Z, X)(\mathbb{C}) \to \text{Sh}_{j_v^{-1}H_j}(G/Z, X)(\mathbb{C})$$

is well-defined on $\text{Sh}_H(G/Z, X)(\mathbb{C})$. Thus $[\mathcal{O}, b_{j_v}]_{H^F} = [j_v][\mathcal{O}, b]_{H^F}$ and

$$\int_{\Delta_{j_v}} \omega_{\varphi} = \int_{\Delta_{\mathcal{O}}} [j_v]^* \omega_{\varphi} = \int_{\Delta_{\mathcal{O}}} \pi_v(j_v) \omega_{\varphi}.$$ 

Decompose $\pi = \pi(\varphi) = \otimes_v \pi_v$. We have

$$\pi_v : B_v^\times \overset{\text{nr}_v}{\longrightarrow} F_v^\times \overset{\text{ord}_v}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \sim \{\pm 1\}.$$ 

Let us denote by $\alpha$ the following unramified character

$$\alpha : F_v^\times \overset{\text{ord}_v}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \sim \{\pm 1\}$$
satisfying $\pi_v = \alpha \circ \text{nr}$.

As $v \mid N_-$, $E$ has multiplicative reduction in $v$. The character $\alpha$ is trivial if and only if $E$ has split multiplicative reduction in $v$, i.e. $\varepsilon_v = -1$.

Hence

$$[j_v]^* \omega_v = \alpha(\text{nr}(j_v)) \omega_v = \left\{ \begin{array}{ll} \omega_v & \text{if } \alpha = 1 \\ (-1)^{\text{ord}_v(\text{nr}(j))} \omega_v & \text{otherwise.} \end{array} \right.$$  

As $v \mid N_-$, $v \in \text{Ram}(B)$ is inert in $K/F$ and $\text{inv}_v(B) = -1$, thus $2 \nmid \text{ord}_v(\text{nr}(j))$. Hence

$$[j_v]^* \omega_v = \alpha(\text{nr}(j_v)) \omega_v = \left\{ \begin{array}{ll} \omega_v & \text{if } \alpha = 1 \\ -\varepsilon_v \omega_v & \text{if } \alpha = -1 \\ -\varepsilon_v \omega_v & \text{otherwise.} \end{array} \right.$$  

and $P_{b_{j_v}} = -\varepsilon_v P_b$.

If $v \mid N_+$, then we fix some uniformizer $\varpi_v$, and an isomorphism $B_v \simeq M_2(F_v)$ which identifies $K_v$ with the set of diagonal matrices and $R_v$ with $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(O_{F,v}) \mid \frac{\omega_v}{\text{ord}_v(N_+)} v \right\}$. As $\text{inv}_v(B_v) = 1$, $j_v$ is a local norm. There exists $k \in K_v$ such that $j_v = v K_v/F_v(k_v)$. We may assume that $j_v^v = 1$. Moreover $j_v$ is in the normalizer of $K_v^\times$ in $B_v^\times$ we thus identify $j_v$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Set

$$W_v = \left( \begin{pmatrix} \frac{1}{\text{ord}_v(N_+)} & 1 \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = k_v j_v.$$  

This matrix is in the normalizer of $R_v$ in $B_v$. As $W_v$ normalize $H_v$, $[\mathcal{F}^\circ, b_{j_v}]_{H_{\mathcal{F}}^\times} = [\mathcal{F}^\circ, b_{k_v^{-1}} W_v]_{H_{\mathcal{F}}^\times} = [W_v][\mathcal{F}^\circ, b_{k_v^{-1}}]_{H_{\mathcal{F}}^\times}$.

Decompose $\omega_v = \bigotimes_{v \mid N_+} \omega_v \otimes \omega'$, where $\omega_v$ satisfies $[W_v]^* \omega_v = \varepsilon_v \omega_v$; then

$$\int_{\Delta_{\mathcal{F}}^v} \omega_v = \varepsilon_v \int_{\Delta_{\mathcal{F}}^v} \omega_v.$$  

As $b_v = 1$,

$$P_{b_{j_v'}} = \varepsilon_v \text{rec}_K(k_v^{-1}) P_b.$$  

If $v \nmid N$, then by a similar calculation we obtain

$$P_{b_{j_v}} = \text{rec}_K(k_v^{-1}) P_b.$$  

End of the proof of Proposition 6.1.1. Lemma 6.1.2 implies that

$$c(P_1) = (-1)^{r-1} \prod_{v \mid N_-} (-\varepsilon_v) \prod_{v \mid N_+} \varepsilon_v \text{rec}_K(k_v^{-1}) P_1$$  

and

$$\forall a \in K_A^\times \quad c(\text{rec}_K(a) P_1) = (-1)^{r-1} \prod_{v \mid N_-} (-\varepsilon_v) \prod_{v \mid N_+} \varepsilon_v \text{rec}_K(k_v^{-1}) \text{rec}_K(a) P_1.$$  

As $P \in E(K)$, we know that $\text{rec}_K(k^{-1}) P = P$. Thus

$$c(P) = (-1)^{r-1} \prod_{v \mid N_-} (-\varepsilon_v) \prod_{v \mid N_+} \varepsilon_v P = (-1)^{r-1} (-1)^{\#(\{v \mid N_\})} \prod_{\varepsilon_v} \varepsilon_v P.$$  

We have to show that $(-1)^{r-1} \prod_{v \mid N_-} (-\varepsilon_v) \prod_{v \mid N_+} \varepsilon_v = -\varepsilon$. For each $v \mid \infty$ we have $\varepsilon_v = -1$.

Since $\prod_{v \mid \infty} = (-1)^d$, the sign in equation (3) is

$$(-1)^d \prod_{v \mid \infty} \varepsilon_v (-1)^{r-1} (-1)^{\#(\{v \mid N_\})}.$$  

Recall that $\{v \mid N_-\} = \text{Ram}(B) \cap S_f$. As $|\text{Ram}(B)|$ is even, we have

$$(-1)^{\#(\{v \mid N_\})} = (-1)^{|\text{Ram}(B) \cap S_f|} = (-1)^{d-r}.$$  

Hence

$$c(P) = (-1)^d \varepsilon (-1)^{r-1} (-1)^{\#(\{v \mid N_\})} P = -\varepsilon P.$$  

$$\Box$$
Remark 6.1.3. The above computations are a particular case of a result of Prasad, [Pra96] Theorem 4, which asserts that if $\text{Hom}_{K^\times} \neq \emptyset$, then the non trivial element in $N_{B^\times} \langle K^\times \rangle \backslash K^\times$ acts on $\text{Hom}_{K^\times} \langle \pi_v, 1 \rangle$ by multiplication by $\text{inv}_v(B) = \text{inv}_v(B) \pi_v, 1/2 \in \{ \pm 1 \}$.

6.2. Orthogonal Shimura manifolds. Until the end of this paper we shall assume $h_F = 1$.

Let us recall some definitions used by Kudla [Kud97] in the particular case $r = 1$. Let $n \in \mathbb{Z}_{\geq 0}$ and let $(V, Q)$ be a quadratic space over $F$ of dimension $n + 2$. We assume that the signature of $V \otimes F, \tau R$ is

$$(n, 2) \times (n + 1, 1)^{\tau R} \times (n + 2, 0)^{\tau R}$$

Denote by $D$ the symmetric space of $G = \text{Res}_F/Q \text{GSpin}(V)$. $D$ is the product of the oriented symmetric spaces of $V_j = V \otimes_{\tau_j, F} R$. Thus $D = D_1 \times \cdots D_d$, where $D_j$ is the set of oriented positive subspaces in $V_j$ of maximal dimension. For each $x \in V$ let $x_j$ be the image of $x$ in $V_j$. Assume that $Q(x)$ is totally positive. Set $V_x = x^\perp$, $G_x = \text{Res}_F/Q \text{GSpin}(V_x)$ and for each $j \in \{1, \ldots, d\}$

$$D_{x_j} = \{ z \in D_j \mid z \perp x_j \}.$$ 

We shall focus on the following real cycle on the Shimura manifold $G(Q) \backslash D \times G(\hat{Q})/H$.

Definition 6.2.1. Let $H$ be an open compact subgroup in $G(\hat{Q})$ and $g \in G(\hat{Q})$. The cycle $Z(x; g; H)$ is defined to be the image of the map

$$Z(x; g; H) : \begin{cases} 
G_x(Q) \backslash D_x \times G_x(\hat{Q})/H^g_x & \rightarrow & G(Q) \backslash D \times G(\hat{Q})/H \\
G_x(Q)(y, u)H^g_x & \rightarrow & G(Q)(y, ug)H\hat{F}^x,
\end{cases}$$

where $H^g_x$ denotes $G_x(\hat{Q}) \cap gHg^{-1}$.

Example (including Proposition 6.2.2) : Fix $D_0 \in F$ satisfying

$$\begin{cases} 
\tau_j(D_0) > 0 & \text{if } j \in \{1, r + 1, \ldots, d\} \\
\tau_j(D_0) < 0 & \text{if } j \in \{2, \ldots, r\}
\end{cases}$$

Set

$$(V, Q) = (B^{\tau = 0}, D_0 \cdot \tau_0).$$

$(V \otimes F, \tau, \tau \circ D_0 \cdot \tau)R$ has signature

$$\begin{cases} 
(1, 2) & \text{if } j = 1 \\
(2, 1) & \text{if } j \in \{2, \ldots, r\} \\
(3, 0) & \text{if } j \in \{r + 1, \ldots, d\}.
\end{cases}$$

Let $G = \text{Res}_F/Q \text{GSpin}(V)$. The action of $B^\times$ on $V$ by conjugation induces an isomorphism

$$B^\times \xrightarrow{\sim} \text{GSpin}(V)$$

$$b \mapsto v \mapsto vb^{-1},$$

thus $G \simeq \text{Res}_F/Q(B^\times)$.

Let $x \in V$ such that $Q(x) \gg 0$, and denote by $x_1$ its image in $V \otimes F, \tau R$. Denote by $K$ the quadratic extension $F + F x$ and $T = \text{Res}_K/Q(G_m)/\text{Res}_F/Q(G_m)$ as above. Let $q$ be the inclusion $K \hookrightarrow V \hookrightarrow B$.

Proposition 6.2.2. The set

$$D_x = D_{x_1} \times \cdots D_{x_r}$$

is a $q(T(\tau R))\circ$-orbit in $D$ whose projection on $D_1$ is a point.

Proof. As $x \in V$, $\text{Tr}(x) = 0$ and $x^2 = -\tau_0(x) = -\frac{Q(x)}{D_0} \in F^\times$. Let $j \in \{1, \ldots, r\}$. We have $\tau_j(Q(x)) > 0$ hence $\tau_j(x^2)\tau_j(D_0) < 0$. Thus $\tau_1$ ramifies in $K$ and $\tau_2, \ldots, \tau_r$ are split. Moreover $q_t(K^\times)$ fixes $x_1$ by definition of $K$. 

□
Let us focus on the general case when \( V \) has dimension \( n \). Fix \( t \in F \) satisfying \( \forall j \in \{1, \ldots, r\} \tau_j(t) > 0 \). \( G(\hat{Q}) \) acts on \( \Omega = \{ x \in V(F) \mid Q(x) = t \} \) by conjugation.

Let \( \varphi \) be a Schwartz function on \( V(\hat{F}) \). Assume \( \Omega \neq \emptyset \) and fix \( x \in \Omega \). Denote by \( Z(y, \varphi; H) \) the following sum
\[
Z(t, \varphi; H) = \sum_{g \in G_\chi(\hat{Q}) \backslash G(\hat{Q})/H} \varphi(g^{-1} \cdot x)Z(x, g; H).
\]

Proposition 4.3.1 showed that for \( n = 1 \) \( [Z(x, g; H)] = 0 \in H_{r-1}(SH_G(G/Z, X)(\mathbb{C}), \mathbb{C}) \). A natural invariant to consider is the refined class
\[
\{Z(t, \varphi; H)\} = \omega \mapsto J^g_\omega \in \text{Im}(H_r(SH_G(G/Z, X)(\mathbb{C}), \mathbb{Z}) \to \text{Harm}(SH_G(G/Z, X)(\mathbb{C}))^*)
\]
where \( \text{Harm}(SH_G(G/Z, X)(\mathbb{C})) \) is the set of harmonic differential forms on \( SH_G(G/Z, X)(\mathbb{C}) \).

In order to adapt the work of Yuan, Zhang and Zhang, we need the following conjecture

**Conjecture 6.2.3.** In the situation of the above example \((V, Q) = (B^{Tr=0}, D_0 \cdot nr)\), the sum
\[
\sum_{t \in \Omega_F \atop t \geq 0} \{Z(t, \varphi; H)\} q^t
\]
is a Hilbert modular form of weight \( 3/2 \).

In [YZZ09], the authors work by induction. To apply their method we would need to prove that the refined classes \( \{Z(t, \varphi; H)\} \) are compatible with the tower of varieties attached to quadratic spaces \( V_v \hookrightarrow V \) of signature \((n-2) \times (n+1, 1)^{r-1} \times (n+2, 0)^{d-r}\) (in which case a generalization of [KM90] should imply that \( \sum_{t \in \Omega_F \atop t \geq 0} \{Z(t, \varphi; H)\} q^t \) is a Hilbert modular form of weight \( 2 \frac{d}{2} + 1 \) with coefficients in \( H^{r+1}(SH_G(G/Z, X)(\mathbb{C}), \mathbb{C}) \).

6.3. A Gross-Kohnen-Zagier-type conjecture.

The Bruhat-Tits tree. In this section we recall some basic facts about the Bruhat-Tits tree (see [Cl] and [Vig80]).

Let \( v \) be a finite place of \( F \). The vertices of the Bruhat-Tits tree of \( \text{PGL}_2(F_v) \) are the maximal orders of \( \mathbb{M}_2(F_v) \). Such maximal orders are endomorphism rings of lattices in \( F_v^2 \) ([Vig80], lemma 2.1). There is an oriented edge between two vertices \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) if and only if there exist \( L_1, L_2 \) lattices in \( F_v^2 \) such that \( \mathcal{O}_1 = \text{End}(L_1), L_2 \subset L_1 \) and \( L_1/L_2 \isom \mathcal{O}_{F_v} / \varpi_v \mathcal{O}_{F_v} \). The intersection of the source and the target of paths of length \( n \) correspond to level \( n\) Eichler orders.

Fix some quadratic extension \( K/F \). This data allow us to organize the Bruhat-Tits tree. Let \( \Psi : K_v \hookrightarrow \mathbb{M}_2(F_v) \) be a \( F_v \)-embedding of \( K_v \). Let \( M_0(N) \) be the set of matrices in \( \mathbb{M}_2(F_v) \) which are upper triangular modulo \( N \). If
\[
\Psi(\mathcal{O}_{K_v}) = \Psi(K_v) \cap M_0(N),
\]
we say that \( \Psi \) has level \( N \). We can organize the vertices of the tree in “levels”, by privileging a direction. Each level corresponds to a level of embedding relatively to \( \mathcal{O}_{K_v} \), i.e. to orders which are in the same orbit under \( K_v^\times \). The maximal orders in \( \text{PGL}_2(F_v) \) which are maximally embedded are on the bottom of the tree.

Figures 2, 3 and 4 illustrate the dependence on the ramification type of \( v \) in \( K \).

Darmon’s points, Kudla’s program and a Gross-Kohnen-Zagier-type theorem. Recall that \( H = (R \otimes_{\mathbb{Z}} \mathbb{Z})^\times \), where \( R \) is an Eichler order of \( B \) of level \( N_v \) and that \( K = F + Fx \) satisfies the following Heegner hypothesis.

**Hypothesis 6.3.1.** Each prime \( p \mid N_v \) splits in \( K \) and each prime \( p \mid N_\times \) is inert in \( K \).

The group \( G_\chi \) is isomorphic to \( K^\times \) and \( Z(x, 1; H) \) is the image of \( K^\times \backslash D_x \times \hat{K}^\times / H \) in \( SH_G(G, X)(\mathbb{C}) \). Note that
\[
Z(x, 1; H) = \mathcal{J}_1^1 + t_1(\mathcal{J}_1^1),
\]
where \( \mathcal{J}_1^1 = \{ \cup_{u \in \pi_0(\mathcal{J}(R))} \eta(u) \cdot \mathcal{J}_1, 1 \} \text{ if } H \neq \emptyset \).

Let \( \varphi = \chi_{R^{Tr=0}} \). We are able to prove an analogue of Proposition A.I.1 of [Kud04] when \( N = 1 \), \( B = \mathbb{M}_2(F), R = \mathbb{M}_2(\mathcal{O}_F), t = Q(x) = D_{\text{nr}}(x) \in F \) and \( K = F + Fx \) is such that \( K \cap R = \mathcal{O}_K \) and \( \mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_Fx \). Set \( c_1(\mathcal{J}_1^1) = \{ [t_1(x), b]_{H \neq \emptyset}, b \in \hat{B}^\times \} \).

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Proposition 6.3.2. If $N = 1$, $r = d$, $B = M_2(F)$, $H = \hat{R}^\times$ with $R = M_2(\mathcal{O}_F)$ and if $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F x$, then $Z(t, \varphi; H)$ is equal to

$$Z(x, 1; H) = \mathcal{J}_1^1 + c_1(\mathcal{J}_1^1) = \mathcal{J}_1^1 - \varepsilon \mathcal{J}_1^1.$$ 

Remark 6.3.3. Under the strong hypotheses above, $\varepsilon = (-1)^d$ and the cycle obtained is zero when $d$ is even.

Proof. By definition

$$Z(t, \varphi; H) = \sum_{g \in \hat{K}^\times \setminus \hat{B}^\times / \hat{R}^\times} 1_{\hat{R}^{\text{tr}=0}}(g^{-1} \cdot x) Z(x, g; H).$$

We have to determine $g \in \hat{K}^\times \setminus \hat{B}^\times / \hat{R}^\times$ satisfying $g^{-1} x g \in \hat{R}^{\text{tr}=0}$, i.e. $x \in g \hat{R}^{\text{tr}=0} g^{-1}$. As $F^\times \subset K^\times$,

$$\hat{K}^\times \setminus \hat{B}^\times / \hat{R}^\times = \prod_v \hat{K}_v^\times \setminus \hat{B}_v^\times / \hat{R}_v^\times = \prod_v \hat{K}_v^\times \setminus B_v^\times / F_v^\times R_v^\times.$$
This allows us to work locally with $K_v^x \backslash B_v^x / F_v^x R_v^x$, which is identified to the $K_v^x$-orbits of maximal orders of $\text{PGL}_2(F_v)$. This gives the following condition, $x_v \in g_v R_v g_v^{-1}$.

First let us consider those $g_v \in B_v^x / R_v^x F_v^x$ satisfying $x_v \in g_v R_v g_v^{-1}$. The ring $g_v R_v g_v^{-1}$ is a maximal order containing $x_v$. Using the fact that $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_x$, we have

$$x_v \in g_v R_v g_v^{-1} \iff g_v R_v g_v^{-1} \cap K_v = \mathcal{O}_K.$$  

Hence the maximal order $g_v R_v g_v^{-1}$ is maximally embedded in $K_v$. It is identified to a vertex at the lowest level of the Bruhat-Tits tree. As each vertex at the same level is in the same $K_v^x$-orbit, we have

$$\forall v \quad g_v = 1 \in K_v^x \backslash B_v^x / F_v^x R_v^x.$$  

Thus $Z(t, \varphi; H) = Z(x, 1; H)$ and as $D_{x_1}$ is a set of two points, $Z(x, 1; H)$ is identified with $\mathcal{T}_1 + c_1(\mathcal{T}_1) = \mathcal{T}_1 - \mathcal{T}_1$, thanks to Proposition 6.1.1.

We now consider the case when $N = N_+ N_- \neq 1$ is prime to $d_{K/F}$. The following proposition is true even if $B \neq M_2(F)$ but we still assume that $R$ is an Eichler order of level $N_+$ and $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_x$.

**Proposition 6.3.4.** Let $N$ be the conductor of $E$. If $N$ is prime to $d_{K/F}$, then

$$Z(t, \varphi; H) = \prod_{v \mid N}(1 + \text{inv}_v(B) \varepsilon_v) Z(x, 1; H).$$  

**Proof.** The proof is analogous to the proof of Proposition 6.3.2. Let us first compute the number of terms in $Z(t, \varphi; H)$. We need to determine for each $v$ the number of $K_v^x$-orbits of oriented paths of length $\text{ord}_v(N_+)$ in the Bruhat-Tits tree; this is equal to the number of $g_v$ such that $x_v \in g_v R_v g_v^{-1}$.

- If $v \nmid N$ then the same argument as in Proposition 6.3.2 shows that there is only one orbit.
- If $v \mid N_+$, $B_v$ is ramified and $v$ is inert in $K$. Hence $K_v^x \backslash B_v^x / R_v^x F_v^x = \{1, \pi_v\}$ where $\pi_v \in B_v^x$ is an element whose reduced norm has order 1 at $v$; $\pi_v$ corresponds to the Atkin-Lehner involution.
- If $v \mid N_+$, $v$ splits in $K$. Denote by $v^S$ the level of the order $R_v$. Each Eichler order of level $v^S$ is the intersection of the origin and the target of an oriented path of length $\delta$. By hypothesis those orders are maximally embedded in $K_v$ and the path corresponding to $g_v R_v g_v^{-1}$ is contained in the lowest level of the tree. As $K_v^x$ acts by translations on this level, there are exactly two $K_v^x$-orbits corresponding to $g_v$ depending on the orientation. We have $g_v^+ \in g_v^-$ which are exchanged by the Atkin-Lehner involution corresponding to $\left( \begin{array}{cc} 0 & \pi_v \\ 1 & 0 \end{array} \right)$.

Let $n$ be the number of prime ideals in the decomposition of $N$. The sum $Z(t, \varphi; H)$ has $2^n$ factors. Let $W$ be the sets of these factors. By definition $Z(x, g; H) = [g]Z(x, 1; H)$. Using Proposition 6.1.1 we obtain

$$Z(t, \varphi; H) = \sum_{g \in W} [g]Z(x, 1; H) = \prod_{v \mid N}(1 + \text{inv}_v(B) \varepsilon_v) Z(x, 1; H).$$  

Let us conclude this paper by another conjecture. Assume that $E(F)$ has rank 1. Denote by $P_0$ some generator of $E(F)$ modulo torsion. For each $t \in \mathcal{O}_F$ totally positive such that $(t)$ is square free and prime to $d_{K/F}$, denote by $K[t]$ the quadratic extension

$$K[t] = F(\sqrt{\text{det} t}),$$  

which satisfies the hypothesis used to build Darmon’s points. Let $P_{t,1}$ be Darmon’s point obtained for $K[t]$ and $b = 1$, and set

$$P_t = \text{Tr}_{K[t]/F} P_{t,1}.$$  

The point $P_t$ lies in $E(F)$ and there exists an integer $[P_t] \in \mathbb{Z}$ such that

$$P_t = [P_t] P_0$$  

modulo torsion.

Proposition 6.3.4 together with Conjecture 6.2.3 suggest the following (as in Conjecture 5.3 of [DT08]).
Conjecture 6.3.5. There exists some Hilbert modular form $g$ of level $3/2$ such that the $[P_1]$s are proportional to some Fourier coefficients of $g$.

Remark 6.3.6. Using the analogy with the Gross-Kohnen-Zagier theorem, the integers $[P_1]$ should be (proportional to) square roots of $L(E-D_0,1)$, where $E-D_0$ is the twist of $E$ by $-D_0$.

Let us end this paper with two open questions.

Question 6.3.7. Does Bruinier’s generalization of Borcherds products [Bru] give anything interesting in this situation?

It is natural to expect that results of Cornut and Vatsal [CV07, CV05] hold also for Darmon’s points.

Question 6.3.8. Would it be possible to deduce such a result from suitable equidistribution properties for the real tori $\mathcal{F}_p^\circ$?

References


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