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Dominique Lecomte, Miroslav Zeleny

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Dominique LECOMTE and Miroslav ZELENY[1,2]

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• Université Paris 6, Institut de Mathématiques de Jussieu, Projet Analyse Fonctionnelle
Couloir 16-26, 4ème étage, Case 247, 4, place Jussieu, 75 252 Paris Cedex 05, France
dominique.lecomte@upmc.fr

• Université de Picardie, I.U.T. de l'Oise, site de Creil,
13, allée de la faïencerie, 60 107 Creil, France

•[1] Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis
Sokolovská 83, 186 75 Prague, Czech Republic
zeleny@karlin.mff.cuni.cz

Abstract. The $\mathbb{G}_0$-dichotomy due to Kechris, Solecki and Todorčević characterizes the analytic relations having a Borel-measurable countable coloring. We give a version of the $\mathbb{G}_0$-dichotomy for $\mathbf{\Sigma}^0_\xi$-measurable countable colorings when $\xi \leq 3$. A $\mathbf{\Sigma}^0_\xi$-measurable countable coloring gives a covering of the diagonal consisting of countably many $\mathbf{\Sigma}^0_\xi$ squares. This leads to the study of countable unions of $\mathbf{\Sigma}^0_\xi$ rectangles. We also give a Hurewicz-like dichotomy for such countable unions when $\xi \leq 2$.

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1 Introduction

The reader should see [K] for the standard descriptive set theoretic notation used in this paper. We study a definable coloring problem. We will need some more notation:

Notation. The letters X, Y will refer to some sets. We set $\Delta(X) := \{(x_0, x_1) \in X^2 \mid x_0 = x_1\}$.

Definition 1.1 *(1) Let $A \subseteq X^2$. We say that A is a $digraph$ if A does not meet $\Delta(X)$.*
(2) Let A be a digraph. A $countable\ coloring$ of (X, A) is a map $c : X \to \omega$ such that A does not meet $(c \times c)^{-1}\big(\Delta(\omega)\big)$.

In [K-S-T], the authors characterize the analytic digraphs of having a Borel countable coloring. The characterization is given in terms of the following notion of comparison between relations.

Notation. Let X, Y be Polish spaces, A (resp., B) be a relation on X (resp., Y), and $\mathbf{\Gamma}$ be a class of sets. We set

$$(X, A) \preceq_{\mathbf{\Gamma}} (Y, B) \ \Leftrightarrow\ \exists f : X \to Y \ \ \mathbf{\Gamma}\text{-measurable with } A \subseteq (f \times f)^{-1}(B).$$

In this case, we say that f is a $\mathbf{\Gamma}$-$measurable\ homomorphism$ from (X, A) into (Y, B). This notion essentially makes sense for digraphs (we can take f to be constant if B is not a digraph).

We also have to introduce a minimum digraph without Borel countable coloring:

• Let $\psi : \omega \to 2^{<\omega}$ be a natural bijection. More specifically, $\psi(0) := \emptyset$ is the sequence of length 0, $\psi(1) := 0$, $\psi(2) := 1$ are the sequences of length 1, and so on. Note that $|\psi(n)| \leq n$ if $n \in \omega$. Let $n \in \omega$. As $|\psi(n)| \leq n$, we can define $s_n := \psi(n)0^{n-|\psi(n)|}$. The crucial properties of the sequence $(s_n)_{n\in\omega}$ are the following:

- $(s_n)_{n\in\omega}$ is $dense$ in $2^{<\omega}$. This means that for each $s \in 2^{<\omega}$, there is $n \in \omega$ such that s_n extends s (denoted $s \subseteq s_n$).
- $|s_n| = n$.

• We put $\mathbb{G}_0 := \{(s_n 0\gamma, s_n 1\gamma) \mid n \in \omega \text{ and } \gamma \in 2^\omega\} \subseteq 2^\omega \times 2^\omega$. Note that $\mathbb{G}_0$ is analytic (in fact difference of two closed sets) since the map $(n, \gamma) \mapsto (s_n 0\gamma, s_n 1\gamma)$ is continuous.

The previous definitions were given, when $\mathbf{\Gamma} = \mathbf{\Delta}^1_1$, in [K-S-T], where the following is proved:

Theorem 1.2 *(Kechris, Solecki, Todorčević) Let X be a Polish space, and A be an analytic relation on X. Then exactly one of the following holds:*
(a) There is a Borel countable coloring of (X, A), i.e., $(X, A) \preceq_{\mathbf{\Delta}^1_1} \big(\omega, \neg\Delta(\omega)\big)$,
(b) $(2^\omega, \mathbb{G}_0) \preceq_{\mathbf{\Sigma}^0_1} (X, A)$.

This result had several developments during the last decade. Here is a non-exhaustive list:

- We can characterize the potentially open sets via a Hurewicz-like test, and in finite dimension it is a consequence of the previous result. Let us specify this. The following definition can be found in [Lo2] (see Definition 3.3).

Definition 1.3 *(Louveau) Let X, Y be Polish spaces, A be a Borel subset of $X\times Y$, and $\mathbf{\Gamma}$ be a Borel class. We say that A is $potentially\ in$ $\mathbf{\Gamma}$ $\big($denoted $A\in pot(\mathbf{\Gamma})\big)$ if we can find a finer Polish topology σ (resp., τ) on X (resp., Y) such that $A\in\mathbf{\Gamma}\big((X,\sigma)\times(Y,\tau)\big)$.*

The pot$(\mathbf{\Sigma}^0_1)$ sets are the countable unions of Borel rectangles. A consequence of this is that the Borel hierarchy built on the Borel rectangles is exactly the hierarchy of the classes of the sets potentially in some Borel class. The good notion of comparison to study the pot$(\mathbf{\Gamma})$ sets is as follows (see [L3]). Let X_0, X_1, Y_0, Y_1 be Polish spaces, and $A^\varepsilon_0, A^\varepsilon_1\subseteq X_\varepsilon\times Y_\varepsilon$ be disjoint. We set

$(X_0,Y_0,A^0_0,A^0_1)\leq(X_1,Y_1,A^1_0,A^1_1)\Leftrightarrow$
$\exists f:X_0\to X_1\ \ \exists g:Y_0\to Y_1$ continuous with $A^0_\varepsilon\subseteq(f\times g)^{-1}(A^1_\varepsilon)$ for each $\varepsilon\in 2$.

The following theorem is proved in [L1], and is a consequence of Theorem 1.2:

Theorem 1.4 *Let X, Y be Polish spaces, and A_0, A_1 be disjoint analytic subsets of $X\times Y$. Then exactly one of the following holds:*
(a) The set A_0 can be separated from A_1 by a pot$(\mathbf{\Sigma}^0_1)=(\mathbf{\Delta}^1_1\times\mathbf{\Delta}^1_1)_\sigma$ set (i.e., there is $S\in pot(\mathbf{\Sigma}^0_1)$ with $A_0\subseteq S\subseteq\neg A_1$),
(b) $\big(2^\omega,2^\omega,\Delta(2^\omega),\mathbb{G}_0\big)\leq(X,Y,A_0,A_1)$.

In [L1], it is also proved that we cannot have f one-to-one in Theorem 1.2.(b) in general. It is easy to check that Theorem 1.2 is also an easy consequence of Theorem 1.4. This means that the study of the Borel countable colorings is highly related to the study of countable unions of Borel rectangles.

- We can extend Theorem 1.2 to any finite dimension, and also in infinite dimension if we change the space in which lives the infinite dimensional version of $\mathbb{G}_0$ (see [L2]).

- B. Miller recently developped some techniques to recover many dichotomy results of descriptive set theory, but without using effective descriptive set theory. He replaces it with some versions of Theorem 1.2. In particular, he can prove Theorem 1.2 without effective descriptive set theory.

When A is Borel, it is natural to ask about the relation between the Borel class of A and that of the coloring f when Theorem 1.2.(a) holds. This leads to consider $\mathbf{\Delta}^0_\xi$-measurable countable colorings (or equivalently $\mathbf{\Sigma}^0_\xi$-measurable countable colorings). We have the following conjecture:

Conjecture 1 *Let $1\leq\xi<\omega_1$. Then there are*
- a 0-dimensional Polish space $\mathbb{X}_\xi$,
- an analytic relation $\mathbb{A}_\xi$ on $\mathbb{X}_\xi$
such that for any 0-dimensional Polish space X, and for any analytic relation A on X, exactly one of the following holds:
(a) $(X,A)\preceq_{\mathbf{\Delta}^0_\xi}\big(\omega,\neg\Delta(\omega)\big)$,
(b) $(\mathbb{X}_\xi,\mathbb{A}_\xi)\preceq_{\mathbf{\Sigma}^0_1}(X,A)$.

We will prove it when $1\leq\xi\leq 2$, and in these cases we do not have to assume that A is analytic. We will also prove it when $\xi=3$, which is much more difficult. We should not have to assume that X is 0-dimensional when $\xi\geq 2$, but we have to do it when $\xi=1$.

We saw that the study of the Borel countable colorings is highly related to the study of countable unions of Borel rectangles, and gave some motivation for studying $\mathbf{\Sigma}^0_\xi$-measurable countable colorings. This motivates the study of countable unions of $\mathbf{\Sigma}^0_\xi$ rectangles. Another motivation is that $(X,A) \preceq_{\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$ is equivalent to the fact that $\Delta(X)$ can be separated from A by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set, by the generalized reduction property for the class $\mathbf{\Sigma}^0_\xi$ (see 22.16 in [K]).

Conjecture 2 *Let $1 \leq \xi < \omega_1$. Then there are 0-dimensional Polish spaces $\mathbb{X}^0_\xi, \mathbb{X}^1_\xi$ and disjoint analytic subsets $\mathbb{A}^0_\xi, \mathbb{A}^1_\xi$ of $\mathbb{X}^0_\xi \times \mathbb{X}^1_\xi$ such that for any Polish spaces X, Y, and for any pair A_0, A_1 of disjoint analytic subsets of $X \times Y$, exactly one of the following holds:*

(a) The set A_0 can be separated from A_1 by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set,

(b) $(\mathbb{X}^0_\xi, \mathbb{X}^1_\xi, \mathbb{A}^0_\xi, \mathbb{A}^1_\xi) \leq (X, Y, A_0, A_1)$.

It is trivial to prove this when $\xi = 1$. We will prove that Conjecture 2 holds when $\xi \leq 2$, which is significantly more and more difficult when ξ increases. We use effective descriptive set theory, and give effective strengthenings of our results. The reader should see [M] for basic notions of effective descriptive set theory. In particular, we will see that to test whether an analytic relation has a $\mathbf{\Sigma}^0_\xi$-measurable countable coloring, it is enough to test countably many partitions instead of continuum many. We will use the topology T_ξ generated by the $\Sigma^1_1 \cap \mathbf{\Pi}^0_{<\xi}$ subsets of a recursively presented Polish space (introduced in [Lo1]) when ξ is 2 or 3 (T_1 is just the basic topology). The last result can be strengthened as follows (see [L3]).

Theorem 1.5 *Let $1 \leq \xi \leq 2$. Then there are 0-dimensional Polish spaces $\mathbb{X}^0_\xi, \mathbb{X}^1_\xi$ and disjoint analytic subsets $\mathbb{A}^0_\xi, \mathbb{A}^1_\xi$ of $\mathbb{X}^0_\xi \times \mathbb{X}^1_\xi$ such that for any recursively presented Polish spaces X, Y, and for any pair A_0, A_1 of disjoint Σ^1_1 subsets of $X \times Y$, the following are equivalent:*

(a) The set A_0 cannot be separated from A_1 by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set.

(b) The set A_0 cannot be separated from A_1 by a $\Delta^1_1 \cap (\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set.

(c) The set A_0 cannot be separated from A_1 by a $\mathbf{\Sigma}^0_1(T_\xi \times T_\xi)$ set.

(d) $A_0 \cap \overline{A_1}^{T_\xi \times T_\xi} \neq \emptyset$.

(e) $(\mathbb{X}^0_\xi, \mathbb{X}^1_\xi, \mathbb{A}^0_\xi, \mathbb{A}^1_\xi) \leq (X, Y, A_0, A_1)$.

2 Some general effective facts

One can hope for an effective strengthening of Conjecture 1:

Effective conjecture 1 *Let $1 \leq \xi < \omega_1$. We can find a 0-dimensional Polish space $\mathbb{X}_\xi$ and an analytic relation $\mathbb{A}_\xi$ on $\mathbb{X}_\xi$ such that $(\mathbb{X}_\xi, \mathbb{A}_\xi) \not\preceq_{\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$, and for any $\alpha \in \omega^\omega$ with $1 \leq \xi < \omega_1^\alpha$, for any 0-dimensional recursively in α presented Polish space X, and for any $\Sigma^1_1(\alpha)$ relation A on X, one of the following holds:*

(a) $(X, A) \preceq_{\Delta^1_1(\alpha) \cap \mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$,

(b) $(\mathbb{X}_\xi, \mathbb{A}_\xi) \preceq_{\mathbf{\Sigma}^0_1} (X, A)$.

We will see that this effective conjecture is true when $1 \leq \xi \leq 3$. The following statement is a corollary of this effective conjecture, and is in fact a theorem:

Theorem 2.1 *Let $1 \leq \xi < \omega_1^{CK}$, X be a 0-dimensional recursively presented Polish space, and A be a Σ^1_1 relation on X. We assume that $(X, A) \preceq_{\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$. Then $(X, A) \preceq_{\Delta^1_1 \cap \mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$.*

A consequence of this is that to test whether an analytic relation has a $\mathbf{\Sigma}^0_\xi$-measurable countable coloring, it is enough to test countably many partitions instead of continuum many. Another consequence is the equivalence between Conjecture 1 and the Effective conjecture 1. We have in fact preliminary results that will help us to prove also the equivalence between (a)-(d) in Theorem 1.5, in the general case.

Lemma 2.2 *Let $1 \leq \xi < \omega_1^{CK}$, X, Y be recursively presented Polish spaces, $A \in \Sigma^1_1(X) \cap \mathbf{\Sigma}^0_\xi$, $B \in \Sigma^1_1(Y) \cap \mathbf{\Sigma}^0_\xi$, and $C \in \Sigma^1_1(X \times Y)$ disjoint from $A \times B$. Then there are $A', B' \in \Delta^1_1 \cap \mathbf{\Sigma}^0_\xi$ such that $A' \times B'$ separates $A \times B$ from C. This also holds for $\mathbf{\Pi}^0_\xi$ instead of $\mathbf{\Sigma}^0_\xi$.*

Proof. Note that A and $\{x \in X \mid \exists y \in B \ \ (x, y) \in C\}$ are disjoint Σ^1_1 sets, separable by a $\mathbf{\Sigma}^0_\xi$ subset of X. By Theorems 1.A and 1.B in [Lo1], there is $A' \in \Delta^1_1 \cap \mathbf{\Sigma}^0_\xi$ separating these two sets. Similarly, B and $\{y \in Y \mid \exists x \in A' \ \ (x, y) \in C\}$ are disjoint Σ^1_1 sets, and there is $B' \in \Delta^1_1 \cap \mathbf{\Sigma}^0_\xi$ separating these two sets. The proof for $\mathbf{\Pi}^0_\xi$ is identical to the one for $\mathbf{\Sigma}^0_\xi$. □

Theorem 2.3 *Let $1 \leq \xi < \omega_1^{CK}$, X, Y be recursively presented Polish spaces, and A_0, A_1 be disjoint Σ^1_1 subsets of $X \times Y$. We assume that A_0 is separable from A_1 by a $\big(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi\big)_\sigma$ set. Then A_0 is separable from A_1 by a $\Delta^1_1 \cap \big((\Delta^1_1 \cap \mathbf{\Sigma}^0_\xi) \times (\Delta^1_1 \cap \mathbf{\Sigma}^0_\xi)\big)_\sigma$ set.*

Proof. By Example 2 of Chapter 3 in [Lo2], the family $\big(N(n, X)\big)_{n \in \omega}$ is regular without parameter. By Corollary 2.10 in [Lo2], $\mathbf{\Pi}^0_\xi(X)$, as well as $\mathbf{\Sigma}^0_\xi(X) = \big(\bigcup_{\eta < \xi} \mathbf{\Pi}^0_\eta(X)\big)_\sigma$, are regular without parameter. By Theorem 2.12 in [Lo2], $\mathbf{\Sigma}^0_\xi(X) \times \mathbf{\Sigma}^0_\xi(Y)$ is also regular without parameter. By Theorem 2.8 in [Lo2], the family $\Phi := \big(\mathbf{\Sigma}^0_\xi(X) \times \mathbf{\Sigma}^0_\xi(Y)\big)_\sigma$ is separating, which implies the existence of $S \in \Delta^1_1 \cap \Phi$ separating A_0 from A_1.

With the notation of [Lo2], let n be an integer with $(0^\infty, n) \in W$ and $C_{0^\infty, n} = S$. Then $(0^\infty, n)$ is in W_Φ, which by Theorem 2.8.(ii) in [Lo2] is

$$\left\{ (\alpha, n) \in W \mid \exists \beta \in \Delta^1_1(\alpha) \ \ \forall m \in \omega \ \ \big(\alpha, \beta(m)\big) \in W_{\mathbf{\Sigma}^0_\xi(X) \times \mathbf{\Sigma}^0_\xi(Y)} \text{ and } C_{\alpha, n} = \bigcup_{m \in \omega} C_{\alpha, \beta(m)} \right\}.$$

This implies that $S \in \Delta^1_1 \cap \big(\Delta^1_1 \cap (\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)\big)_\sigma$. It remains to check that

$$\Delta^1_1 \cap (\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi) = (\Delta^1_1 \cap \mathbf{\Sigma}^0_\xi) \times (\Delta^1_1 \cap \mathbf{\Sigma}^0_\xi).$$

The second set is clearly a subset of the first one. So assume that $R = A \times B \in \Delta^1_1 \cap (\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)$. We may assume that R is not empty. Then the projections A, B are Σ^1_1 since $R \in \Delta^1_1$. Lemma 2.2 gives $A', B' \in \Delta^1_1 \cap \mathbf{\Sigma}^0_\xi$ with $A \times B \subseteq A' \times B' \subseteq R = A \times B$. □

Recall that if A is a relation on X and $D \subseteq X$, then D is A-*discrete* if $A \cap D^2 = \emptyset$.

Proof of Theorem 2.1. We apply Theorem 2.3 to $Y := X$, $A_0 := \Delta(X)$ and $A_1 := A$. As

$$(X, A) \preceq_{\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big),$$

$\Delta(X)$ is separable from A by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set. Theorem 2.3 gives $C_n, D_n \in \mathit{\Delta}^1_1 \cap \mathbf{\Sigma}^0_\xi$ such that $S := \bigcup_{n\in\omega} C_n \times D_n \in \mathit{\Delta}^1_1$ separates $\Delta(X)$ from A. As the set of codes for $\mathit{\Delta}^1_1 \cap \mathbf{\Sigma}^0_\xi$ subsets of X is $\mathit{\Pi}^1_1$ (see Proposition 1.4 and Theorem 1.A in [Lo1]), the $\mathit{\Delta}^1_1$-selection theorem and the separation theorem imply that we may assume that the sequences (C_n) and (D_n) are $\mathit{\Delta}^1_1$. Note that $(C_n \cap D_n)$ is a $\mathit{\Delta}^1_1$ covering of X into A-discrete $\mathit{\Delta}^1_1 \cap \mathbf{\Sigma}^0_\xi$ sets. As X is 0-dimensional we can reduce this covering into a $\mathit{\Delta}^1_1$ covering of X consisting of $\mathit{\Delta}^1_1 \cap \mathbf{\Sigma}^0_\xi$ sets, which are in fact $\mathbf{\Delta}^0_\xi$. This gives the desired partition. □

Notation. Following [Lo1], we define the following topologies on a 0-dimensional recursively in α presented Polish space X, for any $\alpha \in \omega^\omega$. Let $T_1(\alpha)$ be the topology of X, and, for $2 \le \xi < \omega_1$, $T_\xi(\alpha)$ be the topology generated by the $\mathit{\Sigma}^1_1(\alpha) \cap \mathbf{\Pi}^0_{<\xi}$ subsets of X. The next proposition gives a reformulation of the inequality $(X, A) \preceq_{\mathit{\Delta}^1_1(\alpha)\cap\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$ of the Effective conjecture 1.

Proposition 2.4 *Let $1 \le \xi < \omega_1^{CK}$, X be a 0-dimensional recursively presented Polish space, and A be a $\mathit{\Sigma}^1_1$ relation on X. Then $(X, A) \preceq_{\mathit{\Delta}^1_1\cap\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$ is equivalent to $\Delta(X) \cap \overline{A}^{T_\xi \times T_\xi} = \emptyset$.*

Proof. Assume first that $(X, A) \preceq_{\mathit{\Delta}^1_1\cap\mathbf{\Delta}^0_\xi} \big(\omega, \neg\Delta(\omega)\big)$. Then there is a partition (B_n) of X into A-discrete $\mathit{\Delta}^1_1 \cap \mathbf{\Delta}^0_\xi$ sets. In particular, Theorem 1.A in [Lo1] implies that B_n is a countable union of $\mathit{\Delta}^1_1 \cap \mathbf{\Pi}^0_{<\xi}$ sets if $\xi \ge 2$. In particular, B_n is T_ξ-open and $\Delta(X)$ is disjoint from $\overline{A}^{T_\xi \times T_\xi}$ (even if $\xi = 1$).

Conversely, assume that $\Delta(X) \cap \overline{A}^{T_\xi \times T_\xi} = \emptyset$. Then each element x of X is contained in a A-discrete $\mathit{\Sigma}^1_1 \cap \mathbf{\Pi}^0_{<\xi}$ set (basic clopen set if $\xi = 1$). Lemma 2.2 implies that each element x of X is in fact contained in a A-discrete $\mathit{\Delta}^1_1 \cap \mathbf{\Pi}^0_{<\xi}$ set if $\xi \ge 2$. It remains to apply Proposition 1.4 in [Lo1] and the $\mathit{\Delta}^1_1$-selection theorem to get the desired partition. □

One can also hope for an effective strengthening of Conjecture 2 generalizing Theorem 1.5:

Effective conjecture 2 *Let $1 \le \xi < \omega_1$. Then there are*
- 0-dimensional Polish spaces $\mathbb{X}^0_\xi, \mathbb{X}^1_\xi$,
- disjoint analytic subsets $\mathbb{A}^0_\xi, \mathbb{A}^1_\xi$ of the space $\mathbb{X}^0_\xi \times \mathbb{X}^1_\xi$, not separable by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set,
such that for any $\alpha \in \omega^\omega$ such that $1 \le \xi < \omega_1^\alpha$, for any recursively in α presented Polish spaces X, Y, and for any pair A_0, A_1 of disjoint $\mathit{\Sigma}^1_1(\alpha)$ subsets of $X \times Y$, the following are equivalent:
(a) The set A_0 cannot be separated from A_1 by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set.
(b) The set A_0 cannot be separated from A_1 by a $\mathit{\Delta}^1_1(\alpha) \cap (\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set.
(c) The set A_0 cannot be separated from A_1 by a $\mathbf{\Sigma}^0_1\big(T_\xi(\alpha) \times T_\xi(\alpha)\big)$ set.
(d) $A_0 \cap \overline{A_1}^{T_\xi(\alpha)\times T_\xi(\alpha)} \ne \emptyset$.
(e) $(\mathbb{X}^0_\xi, \mathbb{X}^1_\xi, \mathbb{A}^0_\xi, \mathbb{A}^1_\xi) \le (X, Y, A_0, A_1)$.

In fact, the statements (a)-(d) are indeed equivalent:

Theorem 2.5 *Let $1 \leq \xi < \omega_1^{CK}$, X, Y be recursively presented Polish spaces, and A_0, A_1 be disjoint Σ^1_1 subsets of $X \times Y$. The following are equivalent:*

(a) The set A_0 cannot be separated from A_1 by a $(\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set.

(b) The set A_0 cannot be separated from A_1 by a $\Delta^1_1 \cap (\mathbf{\Sigma}^0_\xi \times \mathbf{\Sigma}^0_\xi)_\sigma$ set.

(c) The set A_0 cannot be separated from A_1 by a $\mathbf{\Sigma}^0_1(T_\xi \times T_\xi)$ set.

(d) $A_0 \cap \overline{A_1}^{T_\xi \times T_\xi} \neq \emptyset$.

Proof. Theorem 2.3 implies that (a) is indeed equivalent to (b). It also implies, using the proof of Proposition 2.4, that (c) implies (a), and the converse is clear. It is also clear that (c) and (d) are equivalent. □

A consequence of this is that Conjecture 2 and the Effective conjecture 2 are equivalent.

3 The case $\xi = 1$

(A) Continuous colorings

As in [L3], we can separate Conjecture 1 in two parts. We introduce the following notion, that will help us to characterize the relations $\mathbb{A}$ for which there is a continuous homomorphism from $\mathbb{A}$ into any relation without countable continuous coloring:

Definition 3.1 *Let ξ be a countable ordinal, $\mathbf{\Pi}^0_0 := \mathbf{\Delta}^0_1$, and $\mathbb{X}$ be a 0-dimensional Polish space. A family $\mathcal{F}$ of subsets of $\mathbb{X}$ is ξ-$disjoint$ if the elements of $\mathcal{F}$ are $\mathbf{\Pi}^0_\xi$ and pairwise disjoint.*

The first part ensures the existence of complicated examples.

Lemma 3.2 *(a) Assume that $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ is a 0-disjoint family of subsets of the space $\mathbb{X}$ such that $\mathbb{X} \setminus (\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i) \neq \emptyset$ and no clopen set meeting $\mathbb{X} \setminus (\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)$ is $(\bigcup_{i\in\omega} C^0_i \times C^1_i)$-discrete. Then $(\mathbb{X}, \bigcup_{i\in\omega} C^0_i \times C^1_i) \not\leq_{\mathbf{\Delta}^0_1} \big(\omega, \neg\Delta(\omega)\big)$.*

(b) There is a 0-disjoint family $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of 2^ω satisfying the assumption (and thus the conclusion) of (a).

Proof. (a) We argue by contradiction, which gives $f : \mathbb{X} \to \omega$ continuous such that $f(x) \neq f(y)$ if $(x, y) \in \bigcup_{i\in\omega} C^0_i \times C^1_i$. We set $D_k := f^{-1}(\{k\})$, so that $(D_k)_{k\in\omega}$ is a partition of $\mathbb{X}$ into clopen sets discrete for $\bigcup_{i\in\omega} C^0_i \times C^1_i$. Choose $z \in \mathbb{X} \setminus (\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)$, and k with $z \in D_k$. This gives $(x, y) \in (\bigcup_{i\in\omega} C^0_i \times C^1_i) \cap D^2_k$, which is absurd.

(b) We set $C^\varepsilon_i := N_{0^{2i+\varepsilon}1}$, so that $\bigcup_{i\in\omega} C^0_i \times C^1_i = \{(0^{2i}1\alpha, 0^{2i+1}1\beta) \mid i \in \omega \text{ and } \alpha, \beta \in 2^\omega\}$. Note that $\{0^\infty\} = \mathbb{X} \setminus (\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)$. If C is a clopen neighborhood of 0^∞, then $N_{0^i} \subseteq C$ if i is big enough. This gives an integer i with $(0^{2i}1^\infty, 0^{2i+1}1^\infty) \in (\bigcup_{i\in\omega} C^0_i \times C^1_i) \cap C^2$. □

The second part ensures the existence of the continuous homomorphism.

Lemma 3.3 *Let $\mathbb{X}$ be a 0-dimensional Polish space, $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ be a 0-disjoint family of subsets of $\mathbb{X}$, X be a 0-dimensional Polish space, and A be a relation on X. Then one of the following holds:*

(a) $(X,A)\preceq_{\mathbf{\Delta}^0_1}\big(\omega,\neg\Delta(\omega)\big)$,

(b) $(\mathbb{X},\bigcup_{i\in\omega}\ C_i^0\times C_i^1)\preceq_{\mathbf{\Sigma}^0_1}(X,A)$.

Proof. Assume that (a) does not hold. Let us fix a compatible complete metric on X. In the sequel, the diameter will refer to this metric (this will also be the case in all the proofs where diameters are involved to come). We enumerate a basis $\big(N(p,X)\big)_{p\in\omega}$ for the topology of X made of clopen sets.

• We build

- an increasing sequence of integers $(p_i)_{i\in\omega}$,

- a sequence $(x_k)_{k\in\omega}$ of points of X.

We want these objects to satisfy the following conditions:

(1) $(x_{2i},x_{2i+1})\in A\cap N(p_i,X)^2$
(2) $N(p_{i+1},X)\subseteq N(p_i,X)$
(3) $\mathrm{diam}\big(N(p_i,X)\big)\leq 2^{-i}$
(4) There is no covering of $N(p_i,X)$ consisting of A-discrete clopen subsets of X

• Assume that this is done. Then we can define a point x of X by $\{x\}=\bigcap_{i\in\omega}\ N(p_i,X)$. Note that $(x_k)_{k\in\omega}$ tends to x. We define $f:\mathbb{X}\to X$ by $f(z):=x$ if $z\notin\bigcup_{(\varepsilon,i)\in 2\times\omega}\ C_i^\varepsilon$, $f(z):=x_{2i+\varepsilon}$ if $z\in C_i^\varepsilon$. Note that f is continuous. Moreover, $\big(f(y),f(z)\big)=(x_{2i},x_{2i+1})\in A$ if $(y,z)\in C_i^0\times C_i^1$, so that (b) holds.

• Let us prove that the construction is possible. We set $N(p_{-1},X):=X$. Assume that $(p_i)_{i<l}$ and $(x_{2i},x_{2i+1})_{i<l}$ satisfying (1)-(4) have been constructed, which is the case for $l=0$. We choose a covering of $N(p_{l-1},X)$ with basic clopen sets of diameter at most 2^{-l}, contained in $N(p_{l-1},X)$. Then one of these basic sets, say $N(p_l,X)$, satisfies (4). It remains to choose (x_{2l},x_{2l+1}) in the set $A\cap N(p_l,X)^2$. □

We set $\mathbb{X}_1:=2^\omega$ and $\mathbb{A}_1:=\{(0^{2i}1\alpha,0^{2i+1}1\beta)\mid i\in\omega\text{ and }\alpha,\beta\in 2^\omega\}=\bigcup_{i\in\omega}\ C_i^0\times C_i^1$, so that $\mathbb{A}_1$ is a $\mathbf{\Sigma}^0_1$ relation on $\mathbb{X}_1$.

Corollary 3.4 *Let X be a 0-dimensional Polish space, and A be a relation on X. Then exactly one of the following holds:*

(a) $(X,A)\preceq_{\mathbf{\Delta}^0_1}\big(\omega,\neg\Delta(\omega)\big)$,

(b) $(\mathbb{X}_1,\mathbb{A}_1)\preceq_{\mathbf{\Sigma}^0_1}(X,A)$.

Moreover, there are a non 0-dimensional Polish space X, and a closed relation A on X, for which neither (a), nor (b) holds (with this couple $(\mathbb{X}_1,\mathbb{A}_1)$ or any other). There are also a 0-dimensional Polish space X, and a relation A on X (a difference of two closed sets), for which it is not possible to have f one-to-one in (b) (with this couple $(\mathbb{X}_1,\mathbb{A}_1)$ or any other).

Proof. Note first that (a) and (b) cannot hold simultaneously, by Lemma 3.2. Lemma 3.3 implies that (a) or (b) holds.

• Consider now $X := \mathbb{R}$ and $A := \{(0,1)\}$. Then (a) does not hold since $\mathbb{R}$ is connected. If (b) holds, then we must have $f(0^{2i}1\alpha) = 0$ and $f(0^{2i+1}1\beta) = 1$. By continuity of f, we get $f(0^\infty) = 0 = 1$.

This would be the same with any $(\mathbb{X}_1, \mathbb{A}_1)$. Indeed, as $(\mathbb{X}_1, \mathbb{A}_1) \not\preceq_{\mathbf{\Delta}^0_1} (\omega, \neg\Delta(\omega))$, we have $\overline{\Pi_0[\mathbb{A}_1]} \cap \overline{\Pi_1[\mathbb{A}_1]} \neq \emptyset$, since otherwise there would be a clopen subset C of the 0-dimensional space $\mathbb{X}_1$ separating $\overline{\Pi_0[\mathbb{A}_1]}$ from $\overline{\Pi_1[\mathbb{A}_1]}$, and we would have $\Delta(\mathbb{X}_1) \subseteq C^2 \cup (\neg C)^2 \subseteq \neg\mathbb{A}_1$. So we can choose $x \in \overline{\Pi_0[\mathbb{A}_1]} \cap \overline{\Pi_1[\mathbb{A}_1]}$, $x_{2i} \in \Pi_0[\mathbb{A}_1]$ such that (x_{2i}) tends to x, $y_{2i+1} \in \Pi_1[\mathbb{A}_1]$ such that (y_{2i+1}) tends to x, y_{2i} with $(x_{2i}, y_{2i}) \in \mathbb{A}_1$, and x_{2i+1} with $(x_{2i+1}, y_{2i+1}) \in \mathbb{A}_1$. Then $f(x_{2i}) = 0$, $f(y_{2i+1}) = 1$ and we conclude as before.

• Consider $X := 2^\omega$ and $A := \{0^\infty\} \times (2^\omega \setminus \{0^\infty\})$. Then (a) does not hold since if a clopen subset C of 2^ω contains 0^∞, then it contains also some $\alpha \neq 0^\infty$, so that $(0^\infty, \alpha) \in A \cap C^2$. If (b) holds, then $f(0^{2i}1\alpha) = 0^\infty$ for each integer i and f is not one-to-one.

This argument works as soon as $\Pi_0[\mathbb{A}_1]$ has at least two elements. If we argue in the other factor, then we see that an example $(\mathbb{X}_1, \mathbb{A}_1)$ with injectivity must satisfy that $\mathbb{A}_1$ is a singleton $\{(\alpha, \beta)\}$. As $(\mathbb{X}_1, \mathbb{A}_1) \preceq_{\mathbf{\Sigma}^0_1} (2^\omega, \mathbb{G}_0)$, $\alpha \neq \beta$. So take a clopen subset C of $\mathbb{X}_1$ containing α but not β. Then $\Delta(\mathbb{X}_1) \subseteq C^2 \cup (\neg C)^2 \subseteq \neg\mathbb{A}_1$. □

The notion of a 0-disjoint family is essential in the following sense:

Proposition 3.5 *Let $\mathbb{X}$ be a 0-dimensional Polish space, and $\mathbb{A}$ be a relation on $\mathbb{X}$. The following are equivalent:*

(a) For any 0-dimensional Polish space X, and any relation A on X,

$$(X, A) \not\preceq_{\mathbf{\Delta}^0_1} (\omega, \neg\Delta(\omega)) \Rightarrow (\mathbb{X}, \mathbb{A}) \preceq_{\mathbf{\Sigma}^0_1} (X, A).$$

(b) There is a 0-disjoint family $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of $\mathbb{X}$ such that $\mathbb{A} \subseteq \bigcup_{i\in\omega} C^0_i \times C^1_i$.

Proof. (a) ⇒ (b) We set $X := \mathbb{X}_1$ and $A := \mathbb{A}_1$. By Lemma 3.2, we get $f : \mathbb{X} \to 2^\omega$ such that $\mathbb{A} \subseteq (f \times f)^{-1}(\mathbb{A}_1)$. We set $C^\varepsilon_i := f^{-1}(N_{0^{2i+\varepsilon}1})$.

(b) ⇒ (a) By Lemma 3.3 we get $(\mathbb{X}, \bigcup_{i\in\omega} C^0_i \times C^1_i) \preceq_{\mathbf{\Sigma}^0_1} (X, A)$, so that $(\mathbb{X}, \mathbb{A}) \preceq_{\mathbf{\Sigma}^0_1} (X, A)$. □

(B) Countable unions of open rectangles (i.e., open sets)

The content here is completely trivial. It is just the fact that a subset of a metric space is not open exactly when it contains a point that we can approximate by a countable sequence contained in its complement. We give some statements since the situation will be more complicated in the case $\xi = 2$. As in (A) we can characterize the tuples $(\mathbb{X}^0, \mathbb{X}^1, \mathbb{A}^0, \mathbb{A}^1)$ $\leq$-below any tuple (X, Y, A_0, A_1) with A_0 not separable from A_1 by a $(\mathbf{\Sigma}^0_1 \times \mathbf{\Sigma}^0_1)_\sigma$ set.

Lemma 3.6 *(a) Assume that $(C_i^\varepsilon)_{i\in\omega}$ is a 0-disjoint family of subsets of the space $\mathbb{X}^\varepsilon$ such that $\big(\mathbb{X}^0\setminus(\bigcup_{i\in\omega}\ C_i^0)\big)\times\big(\mathbb{X}^1\setminus(\bigcup_{i\in\omega}\ C_i^1)\big)\neq\emptyset$ and no open set meeting $\big(\mathbb{X}^0\setminus(\bigcup_{i\in\omega}\ C_i^0)\big)\times\big(\mathbb{X}^1\setminus(\bigcup_{i\in\omega}\ C_i^1)\big)$ is disjoint from $\bigcup_{i\in\omega}\ C_i^0\times C_i^1$. Then $\big(\mathbb{X}^0\setminus(\bigcup_{i\in\omega}\ C_i^0)\big)\times\big(\mathbb{X}^1\setminus(\bigcup_{i\in\omega}\ C_i^1)\big)$ is not separable from $\bigcup_{i\in\omega}\ C_i^0\times C_i^1$ by a $(\mathbf{\Sigma}^0_1\times\mathbf{\Sigma}^0_1)_\sigma$ set.*

(b) There are 0-disjoint families of subsets of 2^ω satisfying the assumption (and thus the conclusion) of (a).

Proof. (a) is obvious.

(b) We set $C_i^\varepsilon := N_{0^i1}$, so that $\bigcup_{i\in\omega}\ C_i^0\times C_i^1=\{(0^i1\alpha,0^i1\beta)\mid i\in\omega\text{ and }\alpha,\beta\in2^\omega\}$. Note that $\{0^\infty\}=\mathbb{X}^\varepsilon\setminus(\bigcup_{i\in\omega}\ C_i^\varepsilon)$. If O is an open neighborhood of $(0^\infty,0^\infty)$, then $N^2_{0^i}\subseteq O$ if i is big enough. This gives an integer i with $(0^i1^\infty,0^i1^\infty)\in(\bigcup_{i\in\omega}\ C_i^0\times C_i^1)\cap O$. □

Lemma 3.7 *Let $\mathbb{X}^0,\mathbb{X}^1$ be 0-dimensional Polish spaces, $(C_i^\varepsilon)_{i\in\omega}$ be a 0-disjoint family of subsets of $\mathbb{X}^\varepsilon$, X,Y be Polish spaces, and A_0,A_1 be disjoint subsets of $X\times Y$. Then one of the following holds:*

(a) A_0 is separable from A_1 by a $(\mathbf{\Sigma}^0_1\times\mathbf{\Sigma}^0_1)_\sigma$ set,

(b) $\big(\mathbb{X}^0,\mathbb{X}^1,\big(\mathbb{X}^0\setminus(\bigcup_{i\in\omega}\ C_i^0)\big)\times\big(\mathbb{X}^1\setminus(\bigcup_{i\in\omega}\ C_i^1)\big),\bigcup_{i\in\omega}\ C_i^0\times C_i^1\big)\leq(X,Y,A_0,A_1)$.

Proof. Assume that (a) does not hold. Pick $(x,y)\in A_0\cap\overline{A_1}$, and (x_i,y_i) in A_1 tending to (x,y). We define $f:\mathbb{X}^0\to X$ by $f(z):=x$ if $z\notin\bigcup_{i\in\omega}\ C_i^0$, x_i if $z\in C_i^0$. Note that f is continuous. Similarly, we define $g:\mathbb{X}^1\to Y$, so that (b) holds. □

We define $\mathbb{X}_1^\varepsilon:=2^\omega$, $\mathbb{A}_1^0:=\{(0^\infty,0^\infty)\}=\big(\mathbb{X}_1^0\setminus(\bigcup_{i\in\omega}\ C_i^0)\big)\times\big(\mathbb{X}_1^1\setminus(\bigcup_{i\in\omega}\ C_i^1)\big)$ and also $\mathbb{A}_1^1:=\{(0^i1\alpha,0^i1\beta)\mid i\in\omega\text{ and }\alpha,\beta\in2^\omega\}=\bigcup_{i\in\omega}\ C_i^0\times C_i^1$. As in (A) we get the two following consequences:

Corollary 3.8 *Let X,Y be Polish spaces, and A_0,A_1 be disjoint subsets of $X\times Y$. Then exactly one of the following holds:*

(a) A_0 is separable from A_1 by a $(\mathbf{\Sigma}^0_1\times\mathbf{\Sigma}^0_1)_\sigma$ set,

(b) $(\mathbb{X}_1^0,\mathbb{X}_1^1,\mathbb{A}_1^0,\mathbb{A}_1^1)\leq(X,Y,A_0,A_1)$.

Proposition 3.9 *Let $\mathbb{X}^0,\mathbb{X}^1$ be 0-dimensional Polish spaces, and $\mathbb{A}^0,\mathbb{A}^1\subseteq\mathbb{X}^0\times\mathbb{X}^1$ be disjoint. The following are equivalent:*

(a) For any Polish spaces X,Y, and any $A_0,A_1\subseteq X\times Y$ disjoint,

$$A_0\text{ is not separable from }A_1\text{ by a }(\mathbf{\Sigma}^0_1\times\mathbf{\Sigma}^0_1)_\sigma\text{ set }\Rightarrow(\mathbb{X}^0,\mathbb{X}^1,\mathbb{A}^0,\mathbb{A}^1)\leq(X,Y,A_0,A_1).$$

(b) There is a 0-disjoint family $(C_i^\varepsilon)_{i\in\omega}$ of subsets of $\mathbb{X}^\varepsilon$ such that the inclusions $\mathbb{A}^1\subseteq\bigcup_{i\in\omega}\ C_i^0\times C_i^1$ and $\mathbb{A}^0\subseteq\big(\mathbb{X}^0\setminus(\bigcup_{i\in\omega}\ C_i^0)\big)\times\big(\mathbb{X}^1\setminus(\bigcup_{i\in\omega}\ C_i^1)\big)$ hold.

4 The case $\xi = 2$

(A) Baire class one colorings

Lemma 4.1 *(a) Assume that $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is a 1-disjoint family of subsets of $\mathbb{X}$ such that no nonempty clopen subset of $\mathbb{X}$ is $(\bigcup_{i\in\omega} C_i^0\times C_i^1)$-discrete. Then $(\mathbb{X},\bigcup_{i\in\omega} C_i^0\times C_i^1)\preceq_{\mathbf{\Delta}_2^0}(\omega,\neg\Delta(\omega))$.*
(b) There is a 1-disjoint family $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of ω^ω satisfying the assumption (and thus the conclusion) of (a).

Proof. (a) We argue by contradiction, which gives a $\mathbf{\Delta}_2^0$-measurable map $f:\mathbb{X}\to\omega$ with $f(x)\neq f(y)$ if $(x,y)\in\bigcup_{i\in\omega} C_i^0\times C_i^1$. We set $D_k:=f^{-1}(\{k\})$, so that $(D_k)_{k\in\omega}$ is a partition of $\mathbb{X}$ into $\mathbf{\Delta}_2^0$ sets discrete for $\bigcup_{i\in\omega} C_i^0\times C_i^1$. By Baire's theorem, there are an integer k and a nonempty clopen subset C of $\mathbb{X}$ such that D_k contains C. This gives $(x,y)\in(\bigcup_{i\in\omega} C_i^0\times C_i^1)\cap C^2\subseteq(\bigcup_{i\in\omega} C_i^0\times C_i^1)\cap D_k^2$, which is absurd.

(b) Let $b:\omega\to\omega^{<\omega}$ be a bijection. We set $C_i^\varepsilon:=\big\{b(i)\big(2|b(i)|+\varepsilon\big)^\infty\big\}$, so that

$$\bigcup_{i\in\omega} C_i^0\times C_i^1=\big\{\big(u(2|u|)^\infty,u(2|u|+1)^\infty\big)\mid u\in\omega^{<\omega}\big\}.$$

If $\emptyset\neq C\in\mathbf{\Delta}_1^0(\omega^\omega)$, then C contains some basic clopen set N_u, and $\big(u(2|u|)^\infty,u(2|u|+1)^\infty\big)$ is in $(\bigcup_{i\in\omega} C_i^0\times C_i^1)\cap C^2$. □

Remark. There are a 1-disjoint family $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of ω^ω, and a relation A on ω^ω such that $(\omega^\omega,A)\not\preceq_{\mathbf{\Delta}_2^0}(\omega,\neg\Delta(\omega))$ and $(\omega^\omega,\bigcup_{i\in\omega} C_i^0\times C_i^1)\not\preceq_{\mathbf{\Sigma}_1^0}(\omega^\omega,A)$, so that Lemma 3.3 cannot be extended to $\mathbf{\Sigma}_2^0$-measurable countable colorings.

Indeed, we set $C_i^\varepsilon:=\{u(2i+\varepsilon)^\infty\mid u\in\omega^i\}$ and $A:=\big\{\big(u(2|u|)^\infty,u(2|u|+1)^\infty\big)\mid u\in\omega^{<\omega}\big\}$. Then $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is clearly a 1-disjoint family. Lemma 4.1 gives the first assertion. For the second assertion, assume, towards a contradiction, that $f:\omega^\omega\to\omega^\omega$ is continuous and satisfies the inclusion $\bigcup_{i\in\omega} C_i^0\times C_i^1\subseteq(f\times f)^{-1}(A)$. If $i\in\omega$, then there is $u_i\in\omega^{<\omega}$ with

$$f[C_i^0]\times f[C_i^1]\subseteq\big\{\big(u_i(2|u_i|)^\infty,u_i(2|u_i|+1)^\infty\big)\big\}.$$

In particular, for any $\alpha,\beta\in\omega^\omega$ we get

$$\big(f(\alpha),f(\beta)\big)=\lim_{i\to\infty}\Big(f\big((\alpha|i)(2i)^\infty\big),f\big((\beta|i)(2i+1)^\infty\big)\Big)$$

$$=\lim_{i\to\infty}\big(u_i(2|u_i|)^\infty,u_i(2|u_i|+1)^\infty\big).$$

But this implies that f is constant, which is absurd. To fix this, we refine the notion of a ξ-disjoint family.

Definition 4.2 *Let $1\leq\xi<\omega_1$. A ξ-disjoint family $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of a 0-dimensional Polish space $\mathbb{X}$ is said to be $comparing$ if for each integer q there is a partition $(O_q^p)_{p\in\omega}$ of $\mathbb{X}$ into $\mathbf{\Delta}_\xi^0$ sets such that, for each $i\in\omega$,*

(a) if $q<i$, then there is $p_q^i\in\omega$ such that $C_i^0\cup C_i^1\subseteq O_q^{p_q^i}$,
(b) if $q\geq i$ and $\varepsilon\in 2$, then $C_i^\varepsilon\subseteq O_q^{2i+\varepsilon}$.

Lemma 4.3 *There is a comparing* 1*-disjoint family* $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ *of subsets of* ω^ω *satisfying the assumption (and thus the conclusion) of Lemma 4.1.(a).*

Proof. Let $b:\omega\to\omega^{<\omega}$ be a bijection satisfying $b^{-1}(s)\leq b^{-1}(t)$ if $s\subseteq t$. It can be built as follows. Let $(p_q)_{q\in\omega}$ be the sequence of prime numbers, and $I:\omega^{<\omega}\to\omega$ defined by $I(s):=p_0^{s(0)+1}...p_{|s|-1}^{s(|s|-1)+1}$ if $s\neq\emptyset$, and $I(\emptyset):=1$. Note that I is one-to-one, so that there is an increasing bijection $\varphi:I[\omega^{<\omega}]\to\omega$. We set $b:=(\varphi\circ I)^{-1}:\omega\to\omega^{<\omega}$. We define $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ as in the proof of Lemma 4.1.(b), so that $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is a 1-disjoint family. It remains to see that $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is comparing. We set

$$O_q^p:=\begin{cases} N_{b(i)(2|b(i)|+\varepsilon)}^{\max_{l\leq q}\ (|b(l)|+1)-|b(i)|} \text{ if } p=2i+\varepsilon\leq 2q+1,\\ \\ \omega^\omega\setminus(\bigcup_{p'\leq 2q+1}\ O_q^{p'}) \text{ if } p=2q+2,\\ \\ \emptyset \text{ if } p\geq 2q+3,\end{cases}$$

so that $(O_q^p)_{p\in\omega}$ is a partition of ω^ω into $\mathbf{\Delta}_1^0$ sets. Note that (b) is fulfilled. If $q<i$, then there is at most one couple $(j,\varepsilon)\in(q+1)\times 2$ such that $b(j)(2|b(j)|+\varepsilon)^{\max_{l\leq q}\ (|b(l)|+1)-|b(j)|}$ is compatible with $b(i)$. If it exists and if $|b(i)|\geq\max_{l\leq q}\ (|b(l)|+1)$, then $C_i^0\cup C_i^1\subseteq O_q^{2j+\varepsilon}$ and we set $p_q^i:=2j+\varepsilon$. Otherwise, $C_i^0\cup C_i^1\subseteq O_q^{2q+2}$ and we set $p_q^i:=2q+2$. □

We have a stronger result than Conjecture 1, in the sense that we do not need any regularity assumption on A, neither that X is 0-dimensional.

Lemma 4.4 *Let* $\mathbb{X}$ *be a* 0*-dimensional Polish space,* $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ *be a comparing* 1*-disjoint family of subsets of* $\mathbb{X}$*,* X *be a Polish space, and* A *be a relation on* X*. Then one of the following holds:*
(a) $(X,A)\preceq_{\mathbf{\Delta}_2^0}\big(\omega,\neg\Delta(\omega)\big)$,
(b) $(\mathbb{X},\bigcup_{i\in\omega}\ C_i^0\times C_i^1)\preceq_{\mathbf{\Sigma}_1^0}(X,A)$.

Proof. If A is not a digraph, then choose x_0 with $(x_0,x_0)\in A$, and put $f(x):=x_0$. So we may assume that A is a digraph. We set

$$U:=\bigcup\Big\{V\in\mathbf{\Sigma}_1^0(X)\mid\exists D\in\mathbf{\Pi}_1^0(\omega\times X)\ \ V\subseteq\bigcup_{p\in\omega}\ D_p \text{ and } \forall p\in\omega\ \ A\cap D_p^2=\emptyset\Big\}.$$

Case 1. $U=X$.

There is a countable covering of X into A-discrete $\mathbf{\Sigma}_2^0$ sets. We just have to reduce them to get a partition showing that (a) holds.

Case 2. $U\neq X$.

Then $Y:=X\setminus U$ is a nonempty closed subset of X.

Claim *If* $V\in\mathbf{\Sigma}_1^0(X)$ *meets* Y*, then* $V\cap Y$ *is not* A*-discrete.*

We argue by contradiction. As $V\cap U$ can be covered with some $\bigcup_{p\in\omega}\ D_p$'s, so is V. Thus $V\subseteq U$, so that $V\cap Y\subseteq U\setminus U=\emptyset$, which is the desired contradiction. ◇

• We construct a family $(x_u)_{u\in\omega^{<\omega}}$ of points of Y, and a family $(X_u)_{u\in\omega^{<\omega}}$ of open subsets of Y. We want these objects to satisfy the following conditions:

$$\begin{array}{l}(1)\ x_u\in X_u\\ (2)\ \overline{X_{up}}\subseteq X_u\\ (3)\ \mathrm{diam}(X_u)\leq 2^{-|u|}\\ (4)\ (x_{u(2n)},x_{u(2n+1)})\in A \text{ if } u\in\omega^n\\ (5)\ x_{u(2n+\varepsilon)}=x_u \text{ if } u\notin\omega^n \text{ and } \varepsilon\in 2\end{array}$$

• Assume that this is done. We define $f:\mathbb{X}\to Y\subseteq X$ by

$$\{f(x)\}:=\bigcap_{q\in\omega}\overline{X_{p_0\dots p_{q-1}}}=\bigcap_{q\in\omega}X_{p_0\dots p_{q-1}},$$

where p_i satisfies $x\in O_i^{p_i}$ witnessing comparability, so that f is continuous. Note that $f(x)$ is the limit of $x_{p_0\dots p_{q-1}}$, and that $x_{u(2|u|+\varepsilon)}=x_{u(2|u|+\varepsilon)^2}=\dots=x_{u(2|u|+\varepsilon)^{q+1}}$ for each $(u,\varepsilon)\in\omega^{<\omega}\times 2$. Thus $f(x)=\lim_{q\to\infty}x_{u(2|u|+\varepsilon)^{q+1}}=x_{u(2|u|+\varepsilon)}$ if $x\in C_i^\varepsilon$ and $u:=p_0^i\dots p_{i-1}^i$, and

$$\big(f(x),f(y)\big)=(x_{u(2|u|)},x_{u(2|u|+1)})\in A$$

if $(x,y)\in C_i^0\times C_i^1$. So (b) holds.

• Let us prove that the construction is possible. We choose $x_\emptyset\in Y$ and an open neighborhood $X_\emptyset$ of $x_\emptyset$ in Y, of diameter at most 1. Assume that $(x_u)_{u\in\omega^{\leq l}}$ and $(X_u)_{u\in\omega^{\leq l}}$ satisfying (1)-(5) have been constructed, which is the case for $l=0$.

An application of the claim gives $(x_{u(2l)},x_{u(2l+1)})\in A\cap X_u^2$ if $u\in\omega^l$. We satisfy (5), so that the definition of the x_u's is complete. Note that $x_u\in X_{u|l}$ if $u\in\omega^{l+1}$.

We choose an open neighborhood X_u of x_u in Y, of diameter at most 2^{-l-1}, ensuring the inclusion $\overline{X_u}\subseteq X_{u|l}$. □

We set $\mathbb{X}_2:=\omega^\omega$ and $\mathbb{A}_2:=\big\{\big(u(2|u|)^\infty,u(2|u|+1)^\infty\big)\mid u\in\omega^{<\omega}\big\}=\bigcup_{i\in\omega}C_i^0\times C_i^1$, so that $\mathbb{A}_2$ is a $\mathbf{\Sigma}_2^0$ relation on $\mathbb{X}_2$. As in Section 3.(A) we get the two following consequences:

Corollary 4.5 *Let X be a Polish space, and A be a relation on X. Then exactly one of the following holds:*

(a) $(X,A)\preceq_{\mathbf{\Delta}_2^0}\big(\omega,\neg\Delta(\omega)\big)$,

(b) $(\mathbb{X}_2,\mathbb{A}_2)\preceq_{\mathbf{\Sigma}_1^0}(X,A)$.

Proposition 4.6 *Let $\mathbb{X}$ be a 0-dimensional Polish space, and $\mathbb{A}$ be a relation on $\mathbb{X}$. The following are equivalent:*

(a) For any Polish space X, and any relation A on X,

$$(X,A)\not\preceq_{\mathbf{\Delta}_2^0}\big(\omega,\neg\Delta(\omega)\big)\Rightarrow(\mathbb{X},\mathbb{A})\preceq_{\mathbf{\Sigma}_1^0}(X,A).$$

(b) There is a comparing 1-disjoint family $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of $\mathbb{X}$ such that $\mathbb{A}\subseteq\bigcup_{i\in\omega}C_i^0\times C_i^1$.

(B) Countable unions of $\mathbf{\Sigma}^0_2$ rectangles

Lemma 4.7 *(a) Assume that $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ is a 1-disjoint family of meager subsets of $\mathbb{X}$ such that no nonempty clopen subset of $\mathbb{X}$ is $(\bigcup_{i\in\omega} C^0_i\times C^1_i)$-discrete. Then $\Delta\big(\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)\big)$ is not separable from $\bigcup_{i\in\omega} C^0_i\times C^1_i$ by a $\Big(\mathbf{\Sigma}^0_2\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^1_i)\big)\times\mathbf{\Sigma}^0_2\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^0_i)\big)\Big)_\sigma$ set.*

(b) There is a comparing 1-disjoint family $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of ω^ω satisfying the assumption (and thus the conclusion) of (a).

Proof. (a) We argue by contradiction, which gives $C_n\in\mathbf{\Pi}^0_1\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^1_i)\big)$ on one side, and also $D_n\in\mathbf{\Pi}^0_1\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^0_i)\big)$ with $\Delta\big(\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)\big)\subseteq\bigcup_{n\in\omega}(C_n\times D_n)\subseteq\neg(\bigcup_{i\in\omega} C^0_i\times C^1_i)$. In particular, $\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)=\bigcup_{n\in\omega} C_n\cap D_n$, and Baire's theorem gives n and a nonempty clopen subset C of $\mathbb{X}$ such that $C\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)\subseteq C_n\cap D_n$. Note that $C\setminus(\bigcup_{i\in\omega} C^1_i)\subseteq C_n$ and $C\setminus(\bigcup_{i\in\omega} C^0_i)\subseteq D_n$ since the C^ε_i's are meager and $\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)$ is dense in $\mathbb{X}\setminus(\bigcup_{i\in\omega} C^\varepsilon_i)$. The assumption gives $(x,y)\in(\bigcup_{i\in\omega} C^0_i\times C^1_i)\cap C^2$. Then $(x,y)\in(\bigcup_{i\in\omega} C^0_i\times C^1_i)\cap(C_n\times D_n)$, which is absurd.

(b) Let $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ be the family given by Lemmas 4.1.(b) and 4.3. As the C^ε_i's are singletons, they are meager. □

Remark. Note that $\Delta\big(\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C^\varepsilon_i)\big)=\Delta(\mathbb{X})\cap\Big(\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^1_i)\big)\times\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^0_i)\big)\Big)$ is a closed subset of $\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^1_i)\big)\times\big(\mathbb{X}\setminus(\bigcup_{i\in\omega} C^0_i)\big)$. This shows that the spaces $\mathbb{X}^0_2,\mathbb{X}^1_2$ of Conjecture 2 cannot be both compact, which is quite unusual in this kind of dichotomy (even if it was already the case in [L2]). Indeed, our example shows that $\mathbb{A}^0_2,\mathbb{A}^1_2$ must be separable by a closed set C, and $C,\mathbb{A}^1_2$ must have disjoint projections. If $\mathbb{X}^0_2,\mathbb{X}^1_2$ are compact, then C and its projections are compact too. The product of these compact projections is a $(\mathbf{\Sigma}^0_2\times\mathbf{\Sigma}^0_2)_\sigma$ set separating $\mathbb{A}^0_2$ from $\mathbb{A}^1_2$, which cannot be. We will meet an example where $\mathbb{X}=3^\omega$. This fact implies that we cannot extend the continuous maps of Theorem 1.5.(e) to 3^ω in general.

To ensure the possibility of the reduction, we introduce the following notion:

Definition 4.8 *Let $1\leq\xi<\omega_1$. A ξ-disjoint family $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of a 0-dimensional Polish space $\mathbb{X}$ is said to be $very\ comparing$ if for each integer q there is a partition $(O^p_q)_{p\in\omega}$ of $\mathbb{X}$ into $\mathbf{\Delta}^0_\xi$ sets such that, for each $i\in\omega$,*

(a) if $q<i$, then there is $p^i_q\in\omega$ such that $C^0_i\cup C^1_i\subseteq O^{p^i_q}_q$,

(b) if $q\geq i$ and $\varepsilon\in 2$, then $C^\varepsilon_i\subseteq O^{2i+\varepsilon}_q$,

(c) if $(\varepsilon,i)\in 2\times\omega$, then $\bigcup_{r\geq i}\bigcap_{q\geq r} O^{2i+\varepsilon}_q=C^\varepsilon_i$.

Lemma 4.9 *There is a very comparing 1-disjoint family $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of ω^ω satisfying the assumptions (and thus the conclusion) of Lemma 4.7.(a).*

Proof. Let $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ be the family given by Lemmas 4.1.(b), 4.3, and 4.7.(b). It remains to check Condition (c). Note first that the inclusion $\bigcup_{r\geq i}\bigcap_{q\geq r} O^{2i+\varepsilon}_q\supseteq C^\varepsilon_i$ holds for any comparing ξ-disjoint family.

Conversely,

$$\bigcup_{r\geq i}\ \bigcap_{q\geq r}\ O_q^{2i+\varepsilon}=\bigcup_{r\geq i}\ \bigcap_{q\geq r}\ N_{b(i)(2|b(i)|+\varepsilon)}{}^{\max_{l\leq q}\ (|b(l)|+1)-|b(i)|}$$
$$=\{b(i)(2|b(i)|+\varepsilon)^\infty\}=C_i^\varepsilon.$$

This finishes the proof. □

Notation. We now recall some facts about the Gandy-Harrington topology (see [L2]). Let Z be a recursively presented Polish space. The $Gandy\text{-}Harrington\ topology$ GH on Z is generated by the Σ^1_1 subsets of Z. We set $\Omega_Z := \{z\in Z \mid \omega_1^z=\omega_1^{\rm CK}\}$. Then Ω_Z is Σ^1_1, dense in $(Z,{\rm GH})$, and $W\cap\Omega_Z$ is a clopen subset of $(\Omega_Z,{\rm GH})$ for each $W\in\Sigma^1_1(Z)$. Moreover, $(\Omega_Z,{\rm GH})$ is a 0-dimensional Polish space. So we fix a complete compatible metric $d_{\rm GH}$ on $(\Omega_Z,{\rm GH})$.

The following notion is important for the next proof.

Definition 4.10 *Let a be a countable set, $\xi<\omega_1$, and $\mathcal{F}:=(S_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ be a ξ-disjoint family of subsets of a^ω. We say that $s\in a^{<\omega}$ is $\mathcal{F}$-$suitable$ if there is $(\alpha,\beta)\in\bigcup_{i\in\omega}\ S_i^0\times S_i^1$ such that $s=\alpha\wedge\beta$ is the longest common initial segment of α and β.*

Example. In the next proof, we will take $a:=3$, $\xi:=1$, and s will be $suitable$ when s is empty or finishes with 2. If $\theta:\omega\to\{s\in 3^{<\omega}\mid s \text{ is suitable}\}$ is a bijection such that $\big(|\theta(i)|\big)_{i\in\omega}$ is non-decreasing, then we can define a 1-disjoint family $\mathcal{F}$ of subsets of 3^ω by $S_i^\varepsilon:=\{\theta(i)\varepsilon\alpha\mid\alpha\in 2^\omega\}$, and s is suitable exactly when s is $\mathcal{F}$-suitable.

In particular, a non suitable sequence is of the form $s\varepsilon t$, where s is suitable, $\varepsilon\in 2$ and $t\in 2^{<\omega}$ (we will use this notation in the next proof). If $\emptyset\neq s$ is suitable, then we set

$$s^-:=s|\max\{l<|s|\mid s|l \text{ is suitable}\}.$$

Lemma 4.11 *Let $\mathbb{X}$ be a 0-dimensional Polish space, $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ be a very comparing 1-disjoint family of subsets of $\mathbb{X}$, X,Y be Polish spaces, and A_0,A_1 be disjoint analytic subsets of $X\times Y$. Then one of the following holds:*

(a) A_0 is separable from A_1 by a $(\mathbf{\Sigma}^0_2\times\mathbf{\Sigma}^0_2)_\sigma$ set,

(b) $\big(\mathbb{X}\setminus(\bigcup_{i\in\omega}\ C_i^1),\mathbb{X}\setminus(\bigcup_{i\in\omega}\ C_i^0),\Delta\big(\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega}\ C_i^\varepsilon)\big),\bigcup_{i\in\omega}\ C_i^0\times C_i^1\big)\leq(X,Y,A_0,A_1)$.

Proof. We may assume that X,Y are recursively presented and that A_0,A_1 are Σ^1_1. Assume that (a) does not hold. By Theorem 2.5 we get $N:=A_0\cap\overline{A_1}^{T_2\times T_2}\neq\emptyset$. Lemma 2.2 implies that

$$(x,y)\notin\overline{A_1}^{T_2\times T_2}\Leftrightarrow\exists C,D\in\Sigma^1_1\cap\mathbf{\Pi}^0_1\ \ (x,y)\in C\times D\subseteq\neg A_1$$
$$\Leftrightarrow\exists C,D\in\Delta^1_1\cap\mathbf{\Pi}^0_1\ \ (x,y)\in C\times D\subseteq\neg A_1.$$

This and Proposition 1.4 in [Lo1] show that N is Σ^1_1. We construct

- A sequence $(x_u)_{u\in 3^{<\omega}}$ of points of X,
- A sequence $(y_u)_{u\in 3^{<\omega}}$ of points of Y,
- A sequence $(X_u)_{u\in 3^{<\omega}}$ of Σ^0_1 subsets of X,
- A sequence $(Y_u)_{u\in 3^{<\omega}}$ of Σ^0_1 subsets of Y,
- A sequence $(V_s)_{s\in 3^{<\omega}\ \text{suitable}}$ of Σ^1_1 subsets of $X\times Y$.

We want these objects to satisfy the following conditions:

(1) $(x_u, y_u) \in X_u \times Y_u$
(2) $(x_s, y_s) \in V_s \subseteq N \cap \Omega_{X\times Y}$ if s is suitable
(3) $\overline{X_{u\varepsilon}} \subseteq X_u$ if u is suitable or $u = s0t$, and $\overline{X_{s1t2}} \subseteq X_s$
(4) $\overline{Y_{u\varepsilon}} \subseteq Y_u$ if u is suitable or $u = s1t$, and $\overline{Y_{s0t2}} \subseteq Y_s$
(5) $V_s \subseteq V_{s^-}$ if $\emptyset \neq s$ is suitable
(6) $\mathrm{diam}(X_u), \mathrm{diam}(Y_u) \leq 2^{-|u|}$
(7) $\mathrm{diam}_{\mathrm{GH}}(V_s) \leq 2^{-|s|}$ if s is suitable
(8) $(x_{s0}, y_{s1}) \in \big(\overline{\Pi_0[(X_s \times Y_s) \cap V_s]} \times \overline{\Pi_1[(X_s \times Y_s) \cap V_s]}\big) \cap A_1$ if s is suitable
(9) $(x_{s0t}, y_{s1t}) = (x_{s0}, y_{s1})$ if s is suitable and $t \in 2^{<\omega}$

• Assume that this is done. We define $\phi : \omega^{<\omega} \to 3^{<\omega}$ by $\phi(\emptyset) := \emptyset$ and

$$\phi(sn) := \begin{cases} \phi(s)2\varepsilon \text{ if } n = 2|s| + \varepsilon \text{ or } \big(s \neq \emptyset \text{ and } n \neq s(|s|-1)\big), \\ \phi(s)\varepsilon \text{ if } \big(n = 2q+\varepsilon \text{ and } q \neq |s|\big) \text{ and } \Big(s = \emptyset \text{ or } \big(s \neq \emptyset \text{ and } n = s(|s|-1)\big)\Big). \end{cases}$$

This map allows us to define $\Phi : \omega^\omega \to 3^\omega$ by $\Phi(\gamma)(p) := \phi\big(\gamma|(p+1)\big)(p)$, and Φ is continuous.

As $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is very comparing, there are some witnesses $(O_q^p)_{p\in\omega}$. Let $x \in \mathbb{X}$. As in the proof of Lemma 4.4, we associate the sequence $(p_q)_{q\in\omega} \in \omega^\omega$ defined by $x \in O_q^{p_q}$. As $(C_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is very comparing, $(p_q)_{q\in\omega}$ is not eventually constant if $x \notin \bigcup_{(\varepsilon,i)\in 2\times\omega} C_i^\varepsilon$. Thus $\Phi\big((p_q)_{q\in\omega}\big)$ has infinitely many 2's in this case. If $x \in C_i^\varepsilon$, then

$$\Phi\big((p_q)_{q\in\omega}\big) = \Phi\big(p_0^i \ldots p_{i-1}^i (2i+\varepsilon)^\infty\big) = \phi(p_0^i \ldots p_{i-1}^i) 2\varepsilon^\infty.$$

If $x \in \mathbb{X} \setminus (\bigcup_{i\in\omega} C_i^1)$, then the increasing sequence $(n_k^0)_{k\in\omega}$ of integers such that $\Phi\big((p_q)_{q\in\omega}\big)|n_k^0$ is suitable or of the form $s0t$ is infinite. Condition (3) implies that $(\overline{X_{\Phi((p_q)_{q\in\omega})|n_k^0}})_{k\in\omega}$ is non-increasing. Moreover, $(\overline{X_{\Phi((p_q)_{q\in\omega})|n_k^0}})_{k\in\omega}$ is a sequence of nonempty closed subsets of X whose diameters tend to 0, so that we can define $\{f(x)\} := \bigcap_{k\in\omega} \overline{X_{\Phi((p_q)_{q\in\omega})|n_k^0}} = \bigcap_{k\in\omega} X_{\Phi((p_q)_{q\in\omega})|n_k^0}$. This defines a continuous map $f : \mathbb{X} \setminus (\bigcup_{i\in\omega} C_i^1) \to X$ with $f(x) = \lim_{k\to\infty} x_{\Phi((p_q)_{q\in\omega})|n_k^0}$. Similarly, we define $g : \mathbb{X} \setminus (\bigcup_{i\in\omega} C_i^0) \to Y$ continuous with $g(x) = \lim_{k\to\infty} y_{\Phi((p_q)_{q\in\omega})|n_k^1}$.

If $x \notin \bigcup_{(\varepsilon,i)\in 2\times\omega} C_i^\varepsilon$, then the sequence $(k_j)_j$ of integers such that $\Phi\big((p_q)_{q\in\omega}\big)|k_j$ is suitable is an infinite subsequence of both $(n_k^0)_{k\in\omega}$ and $(n_k^1)_{k\in\omega}$. Note that $(V_{\Phi((p_q)_{q\in\omega})|k_j})_{j\in\omega}$ is a non-increasing sequence of nonempty closed subsets of $\Omega_{X\times Y}$ whose GH-diameters tend to 0, so that we can define $F(x)$ by $\{F(x)\} := \bigcap_{j\in\omega} V_{\Phi((p_q)_{q\in\omega})|k_j} \subseteq N \subseteq A_0$. As $F(x)$ is the limit (in $(X \times Y, \mathrm{GH})$, and thus in $X \times Y$) of $(x_{\Phi((p_q)_{q\in\omega})|k_j}, y_{\Phi((p_q)_{q\in\omega})|k_j})_{j\in\omega}$, we get $F(x) = \big(f(x), g(x)\big)$. Therefore $\Delta\big(\mathbb{X} \setminus (\bigcup_{(\varepsilon,i)\in 2\times\omega} C_i^\varepsilon)\big) \subseteq (f \times g)^{-1}(A_0)$.

Note that $x_{s0} = x_{s0^2} = \ldots = x_{s0^{q+1}}$ for each s suitable. Thus

$$f(x) = \lim_{q\to\infty} x_{\phi(p_0^i \ldots p_{i-1}^i)20^{q+1}} = x_{\phi(p_0^i \ldots p_{i-1}^i)20}$$

if $x \in C_i^0$. Similarly, $g(y) = y_{\phi(p_0^i \ldots p_{i-1}^i)21}$ if $y \in C_i^1$ and $\bigcup_{i\in\omega} C_i^0 \times C_i^1 \subseteq (f \times g)^{-1}(A_1)$.

• Let us prove that the construction is possible. As $N\neq\emptyset$, we can choose $(x_\emptyset,y_\emptyset)\in N\cap\Omega_{X\times Y}$, a Σ^1_1 subset $V_\emptyset$ of $X\times Y$ with $(x_\emptyset,y_\emptyset)\in V_\emptyset\subseteq N\cap\Omega_{X\times Y}$ of GH-diameter at most 1, and a Σ^0_1 neighborhood $X_\emptyset$ (resp., $Y_\emptyset$) of $x_\emptyset$ (resp., $y_\emptyset$) of diameter at most 1. Assume that $(x_u)_{u\in 3^{\leq l}}$, $(y_u)_{u\in 3^{\leq l}}$, $(X_u)_{u\in 3^{\leq l}}$, $(Y_u)_{u\in 3^{\leq l}}$ and $(V_s)_{s\in 3^{\leq l}}$ suitable satisfying (1)-(9) have been constructed, which is the case for $l=0$.

Let $s\in 3^{<\omega}$ be suitable. Note that $(x_s,y_s)\in(X_s\times Y_s)\cap V_s\subseteq\overline{A_1}^{T_2\times T_2}$. We choose $X',Y'\in\Sigma^0_1$ with $(x_s,y_s)\in X'\times Y'\subseteq\overline{X'}\times\overline{Y'}\subseteq X_s\times Y_s$. As $\Pi_\varepsilon[(X'\times Y')\cap V_s]$ is Σ^1_1, $\overline{\Pi_\varepsilon[(X'\times Y')\cap V_s]}$ is $\Sigma^1_1\cap\mathbf{\Pi}^0_1$. In particular, $\overline{\Pi_\varepsilon[(X'\times Y')\cap V_s]}$ is T_2-open. This shows the existence of

$$(x_{s0},y_{s1})\in\big(\overline{\Pi_0[(X'\times Y')\cap V_s]}\times\overline{\Pi_1[(X'\times Y')\cap V_s]}\big)\cap A_1.$$

Note that $(x_{s0},y_{s1})\in\overline{X'}\times\overline{Y'}\subseteq X_s\times Y_s$. We set $x_{s1}:=x_s$, $y_{s0}:=y_s$. We defined x_u,y_u when $u\in 3^{l+1}$ is not suitable but $u|l$ is suitable.

Assume now that $u\in 3^{l+1}$ is suitable, but not $u|l$. This gives (s,ε,t) such that $u=s\varepsilon t2$. Assume first that $\varepsilon=0$. Note that $x_{s0t}=x_{s0}\in X_{s0t}\cap\overline{\Pi_0[(X_s\times Y_s)\cap V_s]}$. This gives

$$x_u\in X_{s0t}\cap\Pi_0[(X_s\times Y_s)\cap V_s],$$

and also y_u with $(x_u,y_u)\in\big((X_{s0t}\cap X_s)\times Y_s\big)\cap V_s=(X_{s0t}\times Y_s)\cap V_s$. If $\varepsilon=1$, then similarly we get $(x_u,y_u)\in(X_s\times Y_{s1t})\cap V_s$.

If u and $u|l$ are both suitable, or both non suitable, then we set $(x_u,y_u):=(x_{u|l},y_{u|l})$. So we defined x_u,y_u in any case. Note that Conditions (8) and (9) are fulfilled, and that $(x_s,y_s)\in V_{s^-}$ if s is suitable. Moreover, $x_u\in X_{u|l}$ if $u|l$ is suitable or $u|l=s0t$, and $x_u\in X_s$ if $u=s1t2$, and similarly in Y. We choose Σ^0_1 sets X_u,Y_u of diameter at most 2^{-l-1} with

$$(x_u,y_u)\in X_u\times Y_u\subseteq\overline{X_u}\times\overline{Y_u}\subseteq\begin{cases}X_{u|l}\times Y_{u|l}\text{ if }u\text{ is not suitable or }u|l\text{ is suitable,}\\ X_{u|l}\times Y_s\text{ if }u=s0t2,\\ X_s\times Y_{u|l}\text{ if }u=s1t2.\end{cases}$$

It remains to choose, when s is suitable, $V_s\in\Sigma^1_1(X\times Y)$ of GH-diameter at most 2^{-l-1} such that $(x_s,y_s)\in V_s\subseteq V_{s^-}$. □

We set $\mathbb{X}^\varepsilon_2:=\omega^\omega\setminus\{u(2|u|+1-\varepsilon)^\infty\mid u\in\omega^{<\omega}\}=\mathbb{X}\setminus(\bigcup_{i\in\omega}C_i^{1-\varepsilon})$,

$$\mathbb{A}^0_2:=\Delta(\omega^\omega\setminus\{u(2|u|+\varepsilon)^\infty\mid(u,\varepsilon)\in\omega^{<\omega}\times 2\})=\Delta\big(\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega}C_i^\varepsilon)\big),$$

and $\mathbb{A}^1_2:=\big\{\big(u(2|u|)^\infty,u(2|u|+1)^\infty\big)\mid u\in\omega^{<\omega}\big\}=\bigcup_{i\in\omega}C_i^0\times C_i^1$.

Corollary 4.12 *Let X,Y be Polish spaces, and A_0,A_1 be disjoint analytic subsets of $X\times Y$. Then exactly one of the following holds:*

(a) A_0 is separable from A_1 by a $(\mathbf{\Sigma}^0_2\times\mathbf{\Sigma}^0_2)_\sigma$ set,

(b) $(\mathbb{X}^0_2,\mathbb{X}^1_2,\mathbb{A}^0_2,\mathbb{A}^1_2)\leq(X,Y,A_0,A_1)$.

Remark. In the remark after Lemma 4.7 we announced an example with $\mathbb{X}:=3^\omega$. In fact, we already met it after Definition 4.10. Recall that the formula $S_i^\varepsilon := \{\theta(i)\varepsilon\alpha \mid \alpha \in 2^\omega\}$ defines a 1-disjoint family of subsets of 3^ω, which are clearly meager. It is also clear that no nonempty clopen subset of 3^ω is $(\bigcup_{i\in\omega} S_i^0\times S_i^1)$-discrete. One can check that the formula

$$O_q^p := \begin{cases} \bigcup_{t\in 2^{|\theta(q)|-|\theta(i)|}} N_{\theta(i)\varepsilon t} \text{ if } p=2i+\varepsilon\leq 2q+1, \\ 3^\omega\setminus(\bigcup_{p'\leq 2q+1} O_q^{p'}) \text{ if } p=2q+2, \\ \emptyset \text{ if } p\geq 2q+3, \end{cases}$$

defines witnesses for the fact that $(S_i^\varepsilon)_{(\varepsilon,i)\in 2\times\omega}$ is very comparing.

To close this section, we notice that the notion of a very comparing 1-disjoint family gives only a sufficient condition, and not a characterization like in 3.5, 3.9 or 4.6:

Proposition 4.13 *Let $\mathbb{X}$ be a 0-dimensional Polish space, $(C_i^\varepsilon)_{i\in\omega}$ be a very comparing 1-disjoint family of subsets of $\mathbb{X}$, $\mathbb{X}^\varepsilon\subseteq\mathbb{X}\setminus(\bigcup_{i\in\omega} C_i^{1-\varepsilon})$, $\mathbb{A}^0\subseteq\Delta\big(\mathbb{X}\setminus(\bigcup_{(\varepsilon,i)\in 2\times\omega} C_i^\varepsilon)\big)$, and $\mathbb{A}^1\subseteq\bigcup_{i\in\omega} C_i^0\times C_i^1$ be as in the definition of $\leq$. Then for any Polish spaces X,Y, and any disjoint analytic subsets A_0, A_1 of $X\times Y$,*

$$A_0 \text{ is not separable from } A_1 \text{ by a } (\mathbf{\Sigma}_1^0\times\mathbf{\Sigma}_1^0)_\sigma \text{ set } \Rightarrow (\mathbb{X}^0,\mathbb{X}^1,\mathbb{A}^0,\mathbb{A}^1)\leq(X,Y,A_0,A_1).$$

5 The case $\xi=3$: Baire class two colorings

Remark. Unlike when $\xi\in\{1,2\}$, we cannot have $\mathbb{A}_3$ of the form $\bigcup_{n\in\omega} C_n^0\times C_n^1$, where $(C_n^\varepsilon)_{(\varepsilon,n)\in 2\times\omega}$ is a 2-disjoint family. Indeed, we will see that there is a Borel graph $\mathbb{G}\subseteq 2^\omega\times 2^\omega$ of a partial injection such that $(2^\omega,\mathbb{G})\not\preceq_{\mathbf{\Delta}_3^0}\big(\omega,\neg\Delta(\omega)\big)$. We would get $f:\mathbb{X}_3\to 2^\omega$ continuous such that $\mathbb{A}_3\subseteq(f\times f)^{-1}(\mathbb{G})$, and $f[C_n^0]\times f[C_n^1]$ would be a singleton. The set $(f\times f)[\mathbb{A}_3]$ would be countable, and $(2^\omega,(f\times f)[\mathbb{A}_3])\preceq_{\mathbf{\Delta}_3^0}\big(\omega,\neg\Delta(\omega)\big)$, $(\mathbb{X}_3,\mathbb{A}_3)\preceq_{\mathbf{\Delta}_3^0}\big(\omega,\neg\Delta(\omega)\big)$ would hold, which is absurd. However, the following result holds.

Theorem 5.1 *There are a 0-dimensional Polish space $\mathbb{X}_3$ and an analytic relation $\mathbb{A}_3$ on $\mathbb{X}_3$ such that for any Polish space X, and for any analytic relation A on X, exactly one of the following holds:*
(a) $(X,A)\preceq_{\mathbf{\Delta}_3^0}\big(\omega,\neg\Delta(\omega)\big)$,
(b) $(\mathbb{X}_3,\mathbb{A}_3)\preceq_{\mathbf{\Sigma}_1^0}(X,A)$.

We can take $\mathbb{X}_3=\omega^\omega$, but this is not the most natural thing to do. Note that we can replace $\mathbb{X}_3$ with any copy of it. Our space $\mathbb{X}_3$ will be a dense G_δ subset of 2^ω, in fact a copy of ω^ω. This G_δ subset is not necessary to see that $(\mathbb{X}_3,\mathbb{A}_3)$ satisfies the "exactly" part of Theorem 5.1 (i.e., that $(\mathbb{X}_3,\mathbb{A}_3)\not\preceq_{\mathbf{\Delta}_3^0}\big(\omega,\neg\Delta(\omega)\big)$), but it is useful to build and ensure the continuity of the homomorphism of Statement (b). The definition of $\mathbb{X}_3$ and $\mathbb{A}_3$ is based on the construction of the following basic objects.

Recall the sequence $(s_n)_{n\in\omega}$ defined in the introduction. In $\mathbb{G}_0$, we put s_n, i.e., a finite sequence of elements of 2, before the changed coordinate. In $\mathbb{A}_3$, we will put a finite sequence of elements of $2^{<\omega}$, together with a way to recover them after concatenation, before the changed coordinate. In order to do that, we identify ω with ω^2.

Notation. Let $<.,.>:\omega^2\to\omega$ be a natural bijection. More precisely, $<n,p>:=(\Sigma_{k\leq n+p}\ k)+p$. Note that the inverse bijection $q\mapsto\big((q)_0,(q)_1\big)$ is build as follows. We set, for $q\in\omega$,

$$M(q):=\max\{m\in\omega\mid \Sigma_{k\leq m}\ k\leq q\}.$$

Then we define $\big((q)_0,(q)_1\big):=\big(M(q)-q+(\Sigma_{k\leq M(q)}\ k),q-(\Sigma_{k\leq M(q)}\ k)\big)$. More concretely,

$$\omega=\{<0,0>,<1,0>,<0,1>,\dots,<M(q),0>,<M(q)-1,1>,...,<0,M(q)>,...\}.$$

If $u\in 2^{\leq\omega}$ and $n\in\omega$, then we define $(u)_n\in 2^{\leq\omega}$ by $(u)_n(p):=u(<n,p>)$ if $<n,p><|u|$. Here also we define $<\alpha_0,\alpha_1,...>\in 2^\omega$ by $<\alpha_0,\alpha_1,...>(<n,p>):=\alpha_n(p)$, for any sequence $(\alpha_n)_{n\in\omega}$ of elements of 2^ω. In particular, $\alpha\mapsto\big((\alpha)_n\big)_{n\in\omega}$ and $(\alpha_n)_{n\in\omega}\mapsto<\alpha_0,\alpha_1,...>$ are inverse bijections.

Lemma 5.2 *Let $u,v\in 2^{<\omega}$.*
(a) $u\subseteq v$ implies that $(u)_n\subseteq(v)_n$ for each $n\in\omega$.
(b) $|(u)_0|\leq|u|$.
(c) $|(u)_n|\leq|u|+1-n$ if $n\leq|u|+1$.

Proof. (a) If $<n,p><|u|$, then $(u)_n(p)=u(<n,p>)=v(<n,p>)=(v)_n(p)$ because of the inequality $<n,p><|v|$, so that $(u)_n\subseteq(v)_n$.

(b) We set, for $n,q\in\omega$, $c^n_q:=\text{Card}(\{p\in\omega\mid<n,p><q\})$. As $<.,.>$ is a bijection, we get $c^n_{q+1}\leq c^n_q+1$. As $c^n_0=0$, $c^n_q\leq q$. We are done since $|(u)_n|=c^n_{|u|}$.

(c) Note first that $<n,p>=(\Sigma_{k\leq n+p}\ k)+p<(\Sigma_{k\leq n'+p'}\ k)+p'=<n',p'>$ if $n+p<n'+p'$, and that $(q)_0+(q)_1=M(q)\leq q<q+1$. This implies that $q=<(q)_0,(q)_1><<n,q+1-n>$ if $n\leq q+1$. It remains to apply this to $q:=|u|$ since $|(u)_n|=c^n_{|u|}$. □

We can view $\mathbb{G}_0$ as the countable union $\bigcup_{n\in\omega}\text{Gr}(\varphi_n)$, where φ_n is the homeomorphism defined on the basic clopen set N_{s_n0} onto the clopen set N_{s_n1} defined by $\varphi_n(s_n0\gamma):=s_n1\gamma$. The set $\mathbb{A}_3$ will also be the countable union of the graphs of some homeomorphisms, indexed by $\omega^{<\omega}$ instead of ω. Their domain and range will be G_δ subsets of 2^ω instead of clopen sets. We first define the closures of these G_δ's. They will be copies of 2^ω. In fact, our homeomorphims will also be defined on the closure of these final domains. We will fix the coordinates whose number is in one of the verticals before that of the number of the changed coordinate. This leads to the following notation.

Notation. If $t\in\omega^{<\omega}$ and $k\leq|t|$, then we set $\Sigma^t_k:=\Sigma_{j<k}\ \big(t(j)+2\big)$, and

$$\Sigma_t:=<\Sigma_{j<|t|}\ \big(t(j)+2\big),0>=<\Sigma^t_{|t|},0>$$

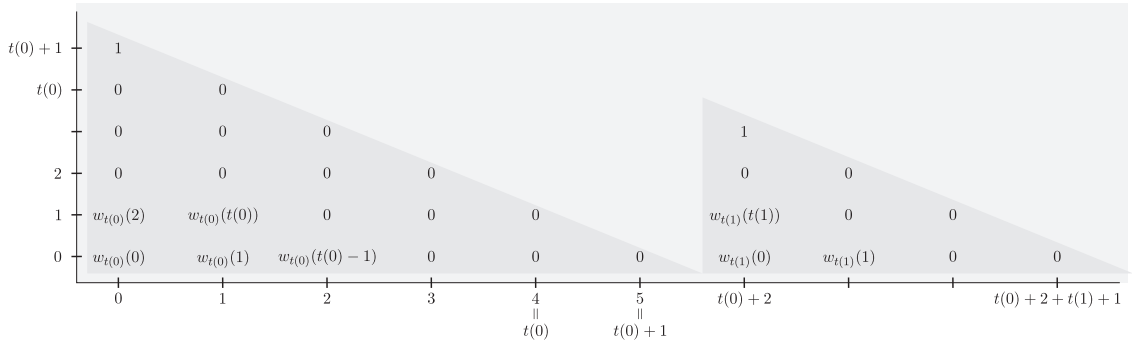
(Σ_t will be the number of the unique changed coordinate). We set $w_n:=s_n0$, so that $|w_n|=n+1$ and $(w_n)_{n\in\omega}$ is dense (we want w_n to be nonempty).

We define the following objects for $t \in \omega^{<\omega}$.

• We first define a copy K_t of 2^ω by

$$K_t := \big\{\alpha \in 2^\omega \mid \forall k < |t| \;\; (\alpha)_{\Sigma^t_k} = (w_{t(k)})_0 0^{t(k)+1-|(w_{t(k)})_0|} 10^\infty \text{ and } \forall 0 < i < t(k)+2 \;\; (\alpha)_{\Sigma^t_k + i} = \big(w_{t(k)}\big)_i 0^\infty\big\}.$$

This is well defined since $|w_{t(k)}| = t(k)+1$, so that we can apply Lemma 5.2.(b) to $u := w_{t(k)}$ and $t(k)+1-|(w_{t(k)})_0| \geq 0$. In particular, the last 1 in $(\alpha)_{\Sigma^t_k}$ is at the position $t(k)+1$. Here is the picture of K_t when $t = (4,2)$:



• We define a non-trivial partition (K^0_t, K^1_t) of K_t into clopen sets by $K^\varepsilon_t := \{\alpha \in K_t \mid \alpha(\Sigma_t) = \varepsilon\}$.

• We define a homeomorphism $\varphi_t : K^0_t \to K^1_t$ by $\varphi_t(\alpha)(m) := 1$ if $m = \Sigma_t$, $\alpha(m)$ otherwise.

We can view the construction of K_t, K^ε_t and φ_t inductively. Indeed, $K_\emptyset = 2^\omega$, $K^\varepsilon_\emptyset$ is the basic clopen set N_ε, and $\varphi_\emptyset(\alpha)(m)$ is 1 when $m = 0$, $\alpha(m)$ otherwise. Then

$$K_{tn} := \{\alpha \in K_t^{w_n(0)} \mid (\alpha)_{\Sigma^t_{|t|}} = (w_n)_0 0^{n+1-|(w_n)_0|} 10^\infty \text{ and } \forall 0 < i < n+2 \;\; (\alpha)_{\Sigma^t_{|t|}+i} = (w_n)_i 0^\infty\},$$

$K^\varepsilon_{tn} := \{\alpha \in K_{tn} \mid \alpha(< \Sigma^t_{|t|}+n+2, 0 >) = \varepsilon\}$, and $\varphi_{tn}(\alpha)(m)$ is 1 when m is equal to $< \Sigma^t_{|t|}+n+2, 0 >$, $\alpha(m)$ otherwise.

The set $\mathbb{S}_3 := \{\alpha \in 2^\omega \mid \exists m \in \omega \;\; \forall n \in \omega \;\; \exists p \geq n \;\; (p)_0 = m \text{ and } \alpha(p) = 1\}$ is a standard $\mathbf{\Sigma}^0_3$-complete set (see 23.A in [K]). We will more or less recover this example, but the 1's have to be well placed. This leads to the following technical but crucial notion.

Definition 5.3 *We say that $u \in 2^{<\omega}$ is placed if $u \neq \emptyset$ and there is $t \in \omega^{<\omega}$ such that $N_u \cap K_t \neq \emptyset$, $(|u|-1)_0 = \Sigma^t_{|t|}$, and $u(|u|-1) = 1$ if $(|u|-1)_1 > 0$. We also say that t is a $witness$ for the fact that u is placed.*

This means that the last coordinate of u has a number on the vertical $\Sigma^t_{|t|}$, on which the coordinates of the elements of K_t are left free by t, and which is the first vertical with this property. The coordinates of u whose number is on one of the verticals before the previous one are determined by t. Finally, the last coordinate of u is 1, except maybe if this coordinate has the number Σ_t, which is at the bottom of the vertical $\Sigma^t_{|t|}$.

Examples. Let $\alpha \in K_{nj} = K_{(n,j)}$. Then

• $\alpha|1$, $\alpha|(<0, n+1>+1)$ are placed with witness $\emptyset$.

• $\alpha|(<n+2, 0>+1)$, $\alpha|(<n+2, j+1>+1)$ are placed with witness (n).

• $\alpha|(< n+2+j+2, 0 > +1)$ is placed with witness (n, j). If $\alpha(< n+2+j+2, q >) = 1$, then $\alpha|(<n+2+j+2, q>+1)$ is placed with witness (n, j).

We are now ready to define $\mathbb{X}_3$ and $\mathbb{A}_3$.

Notation. We set $\mathbb{X}_3 := \big\{\alpha \in 2^\omega \mid \forall n \in \omega \ \ \exists p \geq n \ \ \alpha|p \text{ is placed}\big\}$. Let $t \in \omega^{<\omega}$. We set

$$H_t := \{\alpha \in K_t^0 \mid \forall n \in \omega \ \ \exists p \geq n \ \ (p)_0 = \Sigma^t_{|t|} \text{ and } \alpha(p) = 1\},$$

and $\mathbb{A}_3 := \bigcup_{t \in \omega^{<\omega}} \ \mathrm{Gr}(\varphi_{t|H_t})$. In this sense, we recover $\mathbb{S}_3$. More concretely,

$$\mathbb{A}_3 = \bigcup_{t \in \omega^{<\omega}} \Big\{(u0\gamma, u1\gamma) \mid |u| = \Sigma_t \text{ and } u0\gamma \in K_t \text{ and } \forall n \in \omega \ \ \exists p \geq n \ \ (u0\gamma)(< \Sigma^t_{|t|}, p >) = 1\Big\}.$$

Lemma 5.4 *Let $t \in \omega^{<\omega}$ and $\varepsilon \in 2$.*

(a) $(K_{tn})_{n \in \omega, w_n(0) = \varepsilon}$ is a sequence of pairwise disjoint meager subsets of K_t^ε.

(b) Any nonempty open subset of K_t^ε contains one of the K_{tn}'s.

Proof. (a) This comes from the fact that the last 1 in $(\alpha)_{\Sigma^t_{|t|}}$ is at the position $n+1$ if $\alpha \in K_{tn}$.

(b) A nonempty open subset of K_t^ε contains a basic clopen set C of the form

$$\{\alpha \in K_t^\varepsilon \mid \varepsilon u \subseteq < (\alpha)_{\Sigma^t_{|t|}}, (\alpha)_{\Sigma^t_{|t|}+1}, ... >\},$$

where $u \in 2^{<\omega}$. We choose $n \in \omega$ such that $\varepsilon u \subseteq w_n$. It remains to see that $K_{tn} \subseteq C$. So let $m =< i, p >\leq |u|$. Note first that $M(q) \leq \min\big(q, M(q+1)\big)$. Thus, as $|u| \leq |w_n| = n+1$,

$$i = (m)_0 \leq (m)_0 + (m)_1 = M(m) \leq M(|u|) \leq M(n+1) \leq n+1 < n+2.$$

Lemma 5.2.(a) allows us to write

$$\begin{aligned}(\varepsilon u)(m) &= (\varepsilon u)(< i, p >) = (\varepsilon u)_i(p) = (w_n)_i(p) = (\alpha)_{\Sigma^t_{|t|}+i}(p) \\ &= (< (\alpha)_{\Sigma^t_{|t|}}, (\alpha)_{\Sigma^t_{|t|}+1}, ... >)_i(p) = (< (\alpha)_{\Sigma^t_{|t|}}, (\alpha)_{\Sigma^t_{|t|}+1}, ... >)(m).\end{aligned}$$

This finishes the proof. □

We now start to prove the required properties of $\mathbb{X}_3$ and $\mathbb{A}_3$.

Lemma 5.5 *(a) The set $\mathbb{X}_3$ is a dense $\mathbf{\Pi}^0_2$ subset of 2^ω. In particular, $\mathbb{X}_3$ is a 0-dimensional Polish space.*

(b) Let $t \in \omega^{<\omega}$. The set H_t is a dense $\mathbf{\Pi}^0_2$ subset of K_t^0.

(c) The set $\mathbb{A}_3$ is a $\mathbf{\Sigma}^0_3$ subset of $\mathbb{X}_3^2$. In particular, $\mathbb{A}_3$ is an analytic relation on $\mathbb{X}_3$.

(d) Let $\beta \in \omega^\omega$. Then $\bigcap_{n \in \omega} \ K_{\beta|(n+1)} \subseteq \mathbb{X}_3$.

(e) $(\mathbb{X}_3, \mathbb{A}_3) \not\leq_{\mathbf{\Delta}^0_3} \big(\omega, \neg\Delta(\omega)\big)$.

Proof. (a) $\mathbb{X}_3$ is clearly a $\mathbf{\Pi}^0_2$ subset of 2^ω. So let us prove its density. We just have to prove that $\big\{\alpha\in 2^\omega \mid \exists p\geq n \ \ \alpha|p \text{ is placed}\big\}$ is dense in 2^ω for each integer n. So let $\emptyset\neq w\in 2^{<\omega}$. Note that $\alpha:=w1^\infty\in N_{w(0)}=K^{w(0)}_\emptyset$. Let $p\geq\max(n,|w|)$ with $(p)_0=0$. Then $\alpha|(p+1)$ is placed with witness $t:=\emptyset$.

(b) H_t is clearly a $\mathbf{\Pi}^0_2$ subset of K^0_t. So let us prove its density. We just have to prove the density in K^0_t of the set $\big\{\alpha\in K^0_t \mid \exists p\geq n \ \ (p)_0=\Sigma^t_{|t|} \text{ and } \alpha(p)=1\big\}$, for each integer n. If $|t|=0$, then $K^0_t=K^0_\emptyset$. As in the proof of (a), we see (with $w(0):=0$) that $\alpha:=w1^\infty\in N_w\cap K^0_t$ and $p\geq\max(n,|w|)$ with $(p)_0=0$ are suitable. If $|t|\geq 1$, then we argue similarly. We put again $w1^\infty$, in the coordinates not determined by t.

(c) By (b), $\mathbb{A}_3$ is a $\mathbf{\Sigma}^0_3$ subset of $2^\omega\times 2^\omega$. So we just have to see that $\mathbb{A}_3\subseteq\mathbb{X}^2_3$, which is clear.

(d) Let $\alpha\in\bigcap_{n\in\omega} K_{\beta|(n+1)}$. Note that the sequence $\big(\Sigma^{\beta|(n+1)}_{n+1}\big)_{n\in\omega}$ is strictly increasing. In particular, $p:=<\Sigma^{\beta|(n+1)}_{n+1},\beta(n+1)+1>+1\geq<\Sigma^{\beta|(n+1)}_{n+1},0>+1\geq\Sigma^{\beta|(n+1)}_{n+1}+1\geq n$ and $t:=\beta|(n+1)$ are witnesses for the fact fact $\alpha\in\mathbb{X}_3$.

(e) We argue by contradiction, which gives a partition $(C_n)_{n\in\omega}$ of $\mathbb{X}_3$ into $\mathbb{A}_3$-discrete $\mathbf{\Pi}^0_2$ sets. Fix $n\in\omega$ and $t\in\omega^{<\omega}$. Let us prove that there is $i\in\omega$ such that $C_n\cap K_{ti}=\emptyset$. We argue by contradiction. By Lemma 5.4, $C_n\cap K^\varepsilon_t$ is dense in K^ε_t for each $\varepsilon\in 2$. As $C_n\cap K^\varepsilon_t$ is $\mathbf{\Pi}^0_2$, it is comeager in K^ε_t. By (b), H_t is also comeager in K^0_t, so that this is also the case of $C_n\cap H_t$. In particular, $\varphi_t[C_n\cap H_t]$ is comeager in K^1_t, and $C_n\cap\varphi_t[C_n\cap H_t]$ too. In particular, $C_n\cap\varphi_t[C_n\cap H_t]$ is not empty, which contradicts the $\mathbb{A}_3$-discreteness of C_n.

Applying this inductively, we construct $\beta\in\omega^\omega$ such that $C_n\cap K_{\beta|(n+1)}=\emptyset$ for each $n\in\omega$. By compactness, there is $\alpha\in\bigcap_{n\in\omega} K_{\beta|(n+1)}$, and $\alpha\notin\bigcup_{n\in\omega} C_n=\mathbb{X}_3$. But this contradicts (d). □

The following uniqueness properties will be important in the sequel.

Lemma 5.6 *Let $t\in\omega^{<\omega}$ and $\alpha\in K_t$.*
(a) Assume that $u\in 2^{<\omega}$ is placed with witness t. Then $\Sigma_t<|u|$, the last 1 in u strictly before the position $<\Sigma^t_{k+1},0>$ is at the position $<\Sigma^t_k,t(k)+1>$ for each $k<|t|$, and t is unique.
(b) Let $p>\Sigma_t$ be such that $\alpha|p$ is placed with witness t'. Then $t\subseteq t'$.

Proof. (a) As $(|u|-1)_0=\Sigma^t_{|t|}$, we may assume that $|t|\geq 1$. Let $\alpha\in N_u\cap K_t$. Then

$$(\alpha)_{\Sigma^t_k}=(w_{t(k)})_0 0^{t(k)+1-|(w_{t(k)})_0|}10^\infty$$

for each $k<|t|$.

In particular, $\alpha(<\Sigma^t_k,t(k)+1>)=1$. As $<n+p+1,0>><n,p>$ and $(|u|-1)_0=\Sigma^t_{|t|}$, we get $|u|>|u|-1\geq<\Sigma^t_{|t|},0>=\Sigma_t\geq<\Sigma^t_{k+1},0>><\Sigma^t_k,t(k)+1>$. Thus $u(<\Sigma^t_k,t(k)+1>)=1$.

• Let us prove that the last 1 in u strictly before the position $<\Sigma^t_{k+1},0>$ is at the position $<\Sigma^t_k,t(k)+1>$. The consecutive integers between the values $<\Sigma^t_k,t(k)+1>$ and $<\Sigma^t_{k+1},0>$ are $<\Sigma^t_k,t(k)+1>$, $<\Sigma^t_k-1,t(k)+2>$, ..., $<0,\Sigma^t_k+t(k)+1>$ and $<\Sigma^t_{k+1},0>$. So we have to see that $u(<\Sigma^t_k-j,t(k)+1+j>)=0$ if $0<j\leq\Sigma^t_k$.

There are $k'<k$ and $i<t(k')+2$ such that $\Sigma^t_k-j=\Sigma^t_{k'}+i$. In particular,

$$t(k)+1+j=\Sigma^t_{k+1}-1-\Sigma^t_{k'}-i\geq t(k')+2-i.$$

Lemma 5.2.(c) applied to $w_{t(k')}$ implies that $|(w_{t(k')})_i|\leq t(k')+2-i$. Thus

$$\alpha(<\Sigma^t_{k'}+i, t(k)+1+j>)=0,$$

and we are done.

• As the last 1 in u strictly before the position $<\Sigma^t_{|t|},0>$ is at the position $<\Sigma^t_{|t|-1}, t(|t|-1)+1>$, $t(|t|-1)$ is determined. It remains to iterate this argument to see the uniqueness of t.

(b) We argue by induction on $l:=|t|$, and we may assume that our property is proved for l. So let $t\in\omega^{l+1}$, $\alpha\in K_t$, and $p>\Sigma_t$ be such that $\alpha|p$ is placed with witness t'. Note that $\alpha\in K_{t|l}$ and $p>\Sigma_{t|l}$. By the induction assumption, we get $t|l\subseteq t'$.

Let us prove that $t|l\neq t'$. We argue by contradiction, so that $(p-1)_0=\Sigma^{t'}_l=\Sigma^t_l$. Note that $<\Sigma^t_l, t(l)+1><\Sigma_t<p$. Thus $(p-1)_1>t(l)+1>0$ and $\alpha(p-1)=1=\alpha(<\Sigma^t_l,(p-1)_1>)$. As $\alpha\in K_t$, the last 1 of $(\alpha)_{\Sigma^t_l}$ is at the position $t(l)+1$, which is absurd.

This shows the existence of $t'(l)$. Let $\beta\in N_{\alpha|p}\cap K_{t'}$. The last 1 of $(\beta)_{\Sigma^t_l}$ is at the position $t'(l)+1$. As $<\Sigma^t_l, t'(l)+1><p-1=<\Sigma^{t'}_{|t'|},(p-1)_1>$, it is also the last 1 of $(\beta|p)_{\Sigma^t_l}=(\alpha|p)_{\Sigma^t_l}$. But the last 1 of $(\alpha|p)_{\Sigma^t_l}$ is at the position $t(l)+1$, so that $t'(l)=t(l)$ and $t\subseteq t'$. □

Definition 5.7 *Let $u\in 2^{<\omega}$ and $l\in\omega$.*

(a) If u is placed, then let $t\in\omega^{<\omega}$ be the unique witness given by Lemma 5.6.(b). We will consider

• *the length $l(u):=|t|$*

• *the sequence $u^{l(u)}\in 2^{|u|}\setminus\{u\}$ defined by $u^{l(u)}(m):=1-u(m)$ if $m=\Sigma_t$, $u(m)$ otherwise. Note that $u^{l(u)}$ is placed with witness t, so that $l(u^{l(u)})=l(u)$ and $(u^{l(u)})^{l(u)}=u$*

• *the digit $\varepsilon(u):=u(\Sigma_t)$. Note that $\varepsilon(u^{l(u)})=1-\varepsilon(u)$.*

(b) We say that u is l-placed if u is placed and $l(u)=l$. We say that u is $(\leq l)$-placed (resp., $(<l)$-placed, $(>l)$-placed) if there is $l'\leq l$ (resp., $l'<l$, $l'>l$) such that u is l'-placed.

The following lemma will be crucial in the construction of the homomorphism. We construct some finite approximations of the homomorphism. The lemma says that these finite approximations can be constructed independently.

Lemma 5.8 *Let $u\neq v\in 2^{<\omega}$ be placed with $\varepsilon(u)=\varepsilon(v)$. Then $\{u,u^{l(u)}\}\cap\{v,v^{l(v)}\}=\emptyset$.*

Proof. Note first that $\varepsilon(u^{l(u)})=1-\varepsilon(u)=1-\varepsilon(v)=\varepsilon(v^{l(v)})$. Thus $u\notin\{v,v^{l(v)}\}$ and $u^{l(u)}\neq v$. If $u^{l(u)}=v^{l(v)}$, then $u=(u^{l(u)})^{l(u^{l(u)})}=(v^{l(v)})^{l(v^{l(v)})}=v$, which is absurd. □

When we consider the finite approximations of an element of $\mathbb{A}_3$, we have to guess the finite sequence t. We usually make some mistakes. In this case, we have to be able to come back to an earlier position. This is the role of the following predecessors.

Notation. Let $u \in 2^{<\omega}$. Note that $< \varepsilon >$ is 0-placed. This allows us to define

$$u^- := \begin{cases} \emptyset \text{ if } |u| \leq 1, \\ u|\max\{n < |u| \mid u|n \text{ is placed}\} \text{ if } |u| \geq 2. \end{cases}$$

and, for $l \in \omega$,

$$u^{-l} := \begin{cases} \emptyset \text{ if } |u| \leq 1, \\ u|\max\{n < |u| \mid u|n \text{ is } (\leq l)\text{-placed}\} \text{ if } |u| \geq 2, \end{cases}$$

Before proving our main theorem, we study the relation between these predecessors and the placed sequences.

Lemma 5.9 *Let $l \in \omega$ and $u \in 2^{<\omega}$ be l-placed with $|u| \geq 2$.*
(a) Assume that u^- is l-placed. Then $\varepsilon(u^-) = \varepsilon(u)$. If moreover $(u^l)^-$ is l-placed, then $(u^l)^- = (u^-)^l$.
(b) u^{-l} is l-placed if and only if $(u^l)^{-l}$ is l-placed. In this case, $\varepsilon(u^{-l}) = \varepsilon(u)$ and $(u^l)^{-l} = (u^{-l})^l$.
(c) Assume that u^- or $(u^l)^-$ is $(<l)$-placed. Then $u^- = u^{-l} = (u^l)^- = (u^l)^{-l}$ is $(l-1)$-placed.
(d) Assume that u^- or $(u^l)^-$ is $(>l)$-placed. Then exactly one of those two sequences is $(>l)$-placed, and the other one is l-placed. If u^- (resp., $(u^l)^-$) is $(>l)$-placed, then $u^{-l} = \big((u^l)^-\big)^l$ (resp., $u^{-l} = u^-$) and $\varepsilon(u^{-l}) = \varepsilon(u)$ (resp., $\varepsilon\big((u^l)^{-l}\big) = \varepsilon(u^l)$).
(e) $l(u^{-l}) \in \{l-1, l\}$.

Proof. Let $t \in \omega^l$ (resp. $t' \in \omega^{<\omega}$) be a witness for the fact that u (resp., u^-) is placed, and $\alpha \in N_u \cap K_t$.

Claim. *Assume that $(|u|-1)_1 = 0$. Then $u^- = u^{-l} = (u^l)^- = (u^l)^{-l}$ is $(l-1)$-placed.*

Proof. Note that $l \geq 1$ since $|u| \geq 2$. The consecutive integers between the values $< \Sigma^t_{l-1}, t(l-1)+1 >$ and Σ_t are $< \Sigma^t_{l-1}, t(l-1)+1 >, < \Sigma^t_{l-1}-1, t(l-1)+2 >, ..., < 0, \Sigma^t_{l-1}+t(l-1)+1 >$ and Σ_t. By Lemma 5.6.(b), $\Sigma_t < |u|$ and the last 1 in u strictly before the position Σ_t is at the position $< \Sigma^t_{l-1}, t(l-1)+1 >$. This shows that $u|(\Sigma_t+1)$ and $u|(< \Sigma^t_{l-1}, t(l-1)+1 > +1)$ are placed and $\big(u|(\Sigma_t+1)\big)^- = u|(< \Sigma^t_{l-1}, t(l-1)+1 > +1)$ since $t(l-1)+j > 0$ if $1 \leq j \leq \Sigma^t_{l-1}+1$. As $(|u|-1)_1 = 0$, $|u| = \Sigma_t + 1$ and the sequence $u^- = u|(< \Sigma^t_{l-1}, t(l-1)+1 > +1)$ is $(l-1)$-placed, which implies that $u^- = u^{-l} = (u^l)^- = (u^l)^{-l}$. $\diamond$

(a) By the claim, $(|u|-1)_1 > 0$. Thus $u|(\Sigma_t+1) \subsetneqq u$ is l-placed, $u|(\Sigma_t+1) \subseteq u^-$ and $\Sigma_t < |u^-|$. As $u^- \subseteq \alpha$, we can apply Lemma 5.6.(c) and $t \subseteq t'$. Thus $t = t'$ since $|t| = |t'| = l$, and the equalities $\varepsilon(u^-) = (u^-)(\Sigma_{t'}) = u(\Sigma_t) = \varepsilon(u)$ hold.

Assume now that $(u^l)^-$ is l-placed. As u^l is l-placed with witness t, there is some $\beta \in N_{u^l} \cap K_t$. As $u|(\Sigma_t+1) \subseteq u^- \subsetneqq u$, we get $\big(u|(\Sigma_t+1)\big)^l \subseteq (u^l)^- \subset \beta$. Thus $|(u^l)^-| > \Sigma_t$. Lemma 5.6.(c) implies that t is the witness for the fact that $(u^l)^-$ is l-placed. If $u^- = u|(< \Sigma^t_l, j_0 > +1)$, then there is no $j_0 < j < (|u|-1)_1$ with $u(< \Sigma^t_l, j >) = 1$, and $(u^l)^- = u^l|(< \Sigma^t_l, j_0 > +1) = (u^-)^l$.

(b) Assume that u^{-l} is l-placed. As in (a) we get $(|u|-1)_1>0$ and j_1 with $u^{-l}=u|(<\Sigma_l^t,j_1>+1)$, and $(u^l)^{-l}=u^l|(<\Sigma_l^t,j_1>+1)=(u^{-l})^l$ is l-placed. The equivalence comes from the fact that $(u^l)^l=u$. We argue as in (a) to see that $\varepsilon(u^{-l})=\varepsilon(u)$ if u^{-l} is l-placed.

(c) Assume first that u^- is $(<l)$-placed. The proof of (a) shows that $|t'|\geq l$ if $(|u|-1)_1>0$. Thus $(|u|-1)_1=0$ and the claim gives the result. If $(u^l)^-$ is $(<l)$-placed, then we apply this to u^l, using the facts that u^l is l-placed and $(u^l)^l=u$.

(d) Assume first that u^- is $(>l)$-placed. As in (a) we get $t\subsetneq t'$. In particular, the last 1 in $(u^-)_{\Sigma_l^t}$ is at the position $t'(l)+1$. Let us prove that $u^{-l}=u|(<\Sigma_l^t,t'(l)+1>+1)$. Note that $u|(<\Sigma_l^t,t'(l)+1>+1)$ is l-placed, so that $u|(<\Sigma_l^t,t'(l)+1>+1)\subseteq u^{-l}\subseteq u^-$. Lemma 5.6.(c) shows that u^{-l} is l-placed with witness t. As the last 1 in $(u^-)_{\Sigma_l^t}$ is at the position $t'(l)+1$, we are done.

Note that $u^l|(<\Sigma_l^t,t'(l)+1>+1)\subseteq(u^l)^-$. We argue by contradiction to see that $(u^l)^-$ is not $(>l)$-placed. This gives a witness t'', which is a strict extension of t by Lemma 5.6.(c). We saw that the last 1 in $\big((u^l)^-\big)_{\Sigma_l^t}$ is at the position $t'(l)+1$. But it is also at the position $t''(l)+1$, which shows that $t''(l)=t'(l)$. Thus $u^l(\Sigma_t)=w_{t''(l)}(0)=w_{t'(l)}(0)$. But $u^-(\Sigma_t)=w_{t'(l)}(0)$. This implies that $u^l(\Sigma_t)=1-w_{t'(l)}(0)$, which is absurd. This shows that $(u^l)^-=u^l|(<\Sigma_l^t,t'(l)+1>+1)=(u^{-l})^l$ is l-placed, so that $u^{-l}=\big((u^l)^-\big)^l$. Moreover, $\varepsilon(u^{-l})=(u^{-l})(\Sigma_t)=u(\Sigma_t)=\varepsilon(u)$.

Assume now that $(u^l)^-$ is $(>l)$-placed. As u^l is l-placed and $(u^l)^l=l$, the previous arguments show that u^- is l-placed. In particular, $u^{-l}=u^-$.

(e) If u^- is l-placed, then $u^{-l}=u^-$ is l-placed. If u^- is $(<l)$-placed, then by (c) u^{-l} is $(l-1)$-placed. If u^- is $(>l)$-placed, then by (d) $(u^l)^-$ is l-placed and $u^{-l}=\big((u^l)^-\big)^l$ is l-placed too. □

Proof of Theorem 5.1. $\mathbb{X}_3$ and $\mathbb{A}_3$ have been defined before. The "exactly" part comes from Lemma 5.5.(e). So we just have to prove that (a) or (b) holds. We may assume that X is recursively presented and A is a Σ_1^1 relation. We set

$$U:=\bigcup\Big\{V\in\Sigma_1^1(X)\cap\mathbf{\Pi}_1^0\mid\exists D\in\Delta_1^1\cap\mathbf{\Pi}_2^0(\omega\times X)\ \ V\subseteq\bigcup_{p\in\omega}D_p\text{ and }\forall p\in\omega\ \ A\cap D_p^2=\emptyset\Big\}.$$

Case 1. $U=X$.

There is a countable covering of X into A-discrete $\mathbf{\Sigma}_3^0$ sets. We just have to reduce them to get a partition showing that (a) holds.

Case 2. $U\neq X$.

Note that if V is as in the definition of U, then V and $\neg\bigcup_{p\in\omega}D_p$ are disjoint Σ_1^1 sets, separable by a $\mathbf{\Pi}_1^0$ set. By Theorems 1.A and 1.B in [Lo1], there is $W\in\Delta_1^1\cap\mathbf{\Pi}_1^0$ separating these two sets. This shows that we can replace the condition "$V\in\Sigma_1^1(X)\cap\mathbf{\Pi}_1^0$" with "$V\in\Delta_1^1(X)\cap\mathbf{\Pi}_1^0$" in the definition of U. Thus U is $\Pi_1^1(X)\cap\mathbf{\Sigma}_2^0$ since the set of codes for $\Delta_1^1\cap\mathbf{\Pi}_\xi^0$ sets is Π_1^1 if $\xi<\omega_1^{\mathrm{CK}}$ (see [Lo1]). This shows that $Y:=X\setminus U$ is a nonempty $\Sigma_1^1\cap\mathbf{\Pi}_2^0$ subset of X.

Claim *If $V \in \Sigma^1_1(X) \cap \mathbf{\Pi}^0_1$ meets Y, then $V \cap Y$ is not A-discrete.*

We argue by contradiction. Note that $V \cap Y \in \Sigma^1_1 \cap \mathbf{\Pi}^0_2$. Lemma 2.2 gives a $\Delta^1_1(X) \cap \mathbf{\Pi}^0_2$ set containing $V \cap Y$ and A-discrete. As $V \cap U$ can be covered with some $\bigcup_{p\in\omega} D_p$'s, so is V. Thus $V \subseteq U$, by Δ^1_1-selection. Therefore $V \cap Y \subseteq U \setminus U = \emptyset$, which is absurd. $\diamond$

• We construct, when u is placed, some points x_u of Y, some Σ^0_1 subsets X_u of X, and some Σ^1_1 subsets U_u of X^2. We want these objects to satisfy the following conditions:

(1) $x_u \in X_u$ and $\begin{cases} (x_u, x_{u^{l(u)}}) \in U_u \text{ if } \varepsilon(u) = 0 \\ (x_{u^{l(u)}}, x_u) \in U_u \text{ if } \varepsilon(u) = 1 \end{cases}$
(2) $\overline{X_u} \subseteq X_{u^-}$ if $|u| \geq 2$
(3) $U_u = U_{u^{l(u)}} \subseteq A \cap Y^2 \cap \Omega_{X^2}$, and $U_u \subseteq U_{u^{-l}}$ if u and u^{-l} are l-placed
(4) $\mathrm{diam}(X_u) \leq 2^{-|u|}$ and $\mathrm{diam}_{\mathrm{GH}}(U_u) \leq 2^{-|u|}$
(5) $U_u \subseteq \begin{cases} \overline{\Pi_0[(X_{u^{-l}} \times X_{(u^{-l})^{l(u^{-l})}}) \cap U_{u^{-l}}]}^2 \text{ if } \varepsilon(u^{-l}) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \text{ is } l\text{-placed and } u^{-l} \text{ is not} \\ \overline{\Pi_1[(X_{(u^{-l})^{l(u^{-l})}} \times X_{u^{-l}}) \cap U_{u^{-l}}]}^2 \text{ if } \varepsilon(u^{-l}) = 1 \end{cases}$

As we will see, Conditions (1)-(4) are sufficient to get the required objects. Condition (5) is used to prove that the construction is possible. The idea is the following. When we extend some $u \in 2^{<\omega}$, some new links may appear. But we may also break some links, and preserve only an initial segment of them. In this case, to ensure Condition (3), we have to be able to come back to the last preserved link. This is possible if we use iteratively Conditions (3) and (5).

• Assume that this is done. Let $\alpha \in \mathbb{X}_3$ and $(p^\alpha_k)_{k\in\omega}$ be the infinite strictly increasing sequence of integers $p^\alpha_k \geq 1$ such that $\alpha|p^\alpha_k$ is placed. Note that $\overline{X_{\alpha|p^\alpha_{k+1}}} \subseteq X_{\alpha|p^\alpha_k}$, by Condition (2). This shows that $(\overline{X_{\alpha|p^\alpha_k}})_{k\in\omega}$ is a non-increasing sequence of nonempty closed subsets of X whose diameters tend to 0, and we define $\{f(\alpha)\} := \bigcap_{k\in\omega} \overline{X_{\alpha|p^\alpha_k}} = \bigcap_{k\in\omega} X_{\alpha|p^\alpha_k}$, so that $f : \mathbb{X}_3 \to X$ is continuous and $f(\alpha) = \lim_{k\to\infty} x_{\alpha|p^\alpha_k}$.

Now let $(\alpha, \beta) \in \mathbb{A}_3$. If $(\alpha, \beta) \in \bigcup_{|t|=l} \mathrm{Gr}(\varphi_{t|H_t})$, then let $(p_j)_{j\in\omega}$ be the infinite strictly increasing sequence of integers $p_j \geq 1$ such that $(p_j - 1)_0 = \Sigma^t_{|t|}$, $(p_j - 1)_1 > 0$ and $\alpha(p_j - 1) = 1$. In particular, $\alpha|p_j$ is l-placed and $\varepsilon(\alpha|p_j) = 0$. Note that $(p_j)_{j\in\omega}$ is also the infinite strictly increasing sequence of integers $p_j \geq 1$ such that $(p_j - 1)_0 = \Sigma^t_{|t|}$, $(p_j - 1)_1 > 0$ and $\beta(p_j - 1) = 1$ on one side, and a subsequence of both $(p^\alpha_k)_{k\in\omega}$ and $(p^\beta_k)_{k\in\omega}$ on the other side.

If moreover $p \geq p_0$ and $\alpha|p$ is placed, then the witness is an extension of t and $l(\alpha|p) \geq l$, by Lemma 5.6.(c). In particular, if $p \geq p_0$ and $\alpha|p$ is l-placed, then the witness is t. This proves that $(p_j)_{j\in\omega}$ is the infinite strictly increasing sequence of integers $p_j \geq p_0$ such that $\alpha|p_j$ is l-placed. Therefore $(\alpha|p_{j+1})^{-l} = \alpha|p_j$. By Condition (3), $(U_{\alpha|p_j})_{j\in\omega}$ is a non-increasing sequence of nonempty clopen subsets of $A \cap \Omega_{X^2}$ whose GH-diameter tend to 0. So we can define $F(\alpha, \beta) \in A$ by $\{F(\alpha, \beta)\} := \bigcap_{j\in\omega} U_{\alpha|p_j}$. Note that $F(\alpha, \beta) = \lim_{j\to\infty} (x_{\alpha|p_j}, x_{\beta|p_j}) = \big(f(\alpha), f(\beta)\big) \in A$, so that $\mathbb{A}_3 \subseteq (f \times f)^{-1}(A)$.

• Let us prove that the construction is possible. We do it by induction on the length k of u.

Subcase 1. $k=0$

We are done since $\emptyset$ is not placed.

Subcase 2. $k=1$

The claim gives $(x_0,x_1)\in A\cap Y^2\cap\Omega_{X^2}$. We choose a Σ_1^0 neighborhood $X_{\varepsilon'}$ of $x_{\varepsilon'}$ with diameter at most 2^{-1}, as well as a Σ_1^1 subset $U_0=U_1$ of X^2 with GH-diameter at most 2^{-1} such that (x_0,x_1) is in $U_0\subseteq A\cap Y^2\cap\Omega_{X^2}$. We are done since $<\varepsilon'>$ is 0-placed and $\varepsilon(<\varepsilon'>)=\varepsilon'$.

Subcase 3. $k\geq 2$

If there is no placed sequence in 2^k, then there is nothing to do. If $u\in 2^k$ is l-placed, then $u^l\in 2^k$ is l-placed and $\varepsilon(u^l)=1-\varepsilon(u)$. Assume for example that $\varepsilon(u)=0$. Lemma 5.8 ensures that we just have to define x_u,x_{u^l},X_u,X_{u^l} and $U_u=U_{u^l}$, independently from the other sequences in 2^k.

- If u^- and $(u^l)^-$ are l-placed, then $\big(u^-,(u^l)^-\big)=\big(u^{-l},(u^l)^{-l}\big)$. Moreover, $\varepsilon(u^-)=\varepsilon(u)=0$, $(u^l)^-=(u^-)^l$ and $(u^l)^{-l}=(u^{-l})^l$, by Lemma 5.9. We set $(x_u,x_{u^l}):=(x_{u^-},x_{(u^-)^l})$, we choose Σ_1^0 sets X_u,X_{u^l} with diameter at most 2^{-k} such that $(x_u,x_{u^l})\in X_u\times X_{u^l}\subseteq\overline{X_u}\times\overline{X_{u^l}}\subseteq X_{u^-}\times X_{(u^-)^l}$, as well as a Σ_1^1 subset U_u of X^2 with GH-diameter at most 2^{-k} such that $(x_u,x_{u^l})\in U_u\subseteq U_{u^-}=U_{u^{-l}}$. We are done since $U_{u^l}=U_u\subseteq U_{u^-}=U_{(u^-)^l}=U_{(u^{-l})^l}=U_{(u^l)^{-l}}$.

- If u^- or $(u^l)^-$ is $(<l)$-placed, then $u^-=u^{-l}=(u^l)^-=(u^l)^{-l}$ is $(l-1)$-placed, by Lemma 5.9.(c). Let W be a Σ_1^0 neighborhood of $x_{u^-}=x_{(u^l)^-}$ with $\overline{W}\subseteq X_{u^-}$. Note that

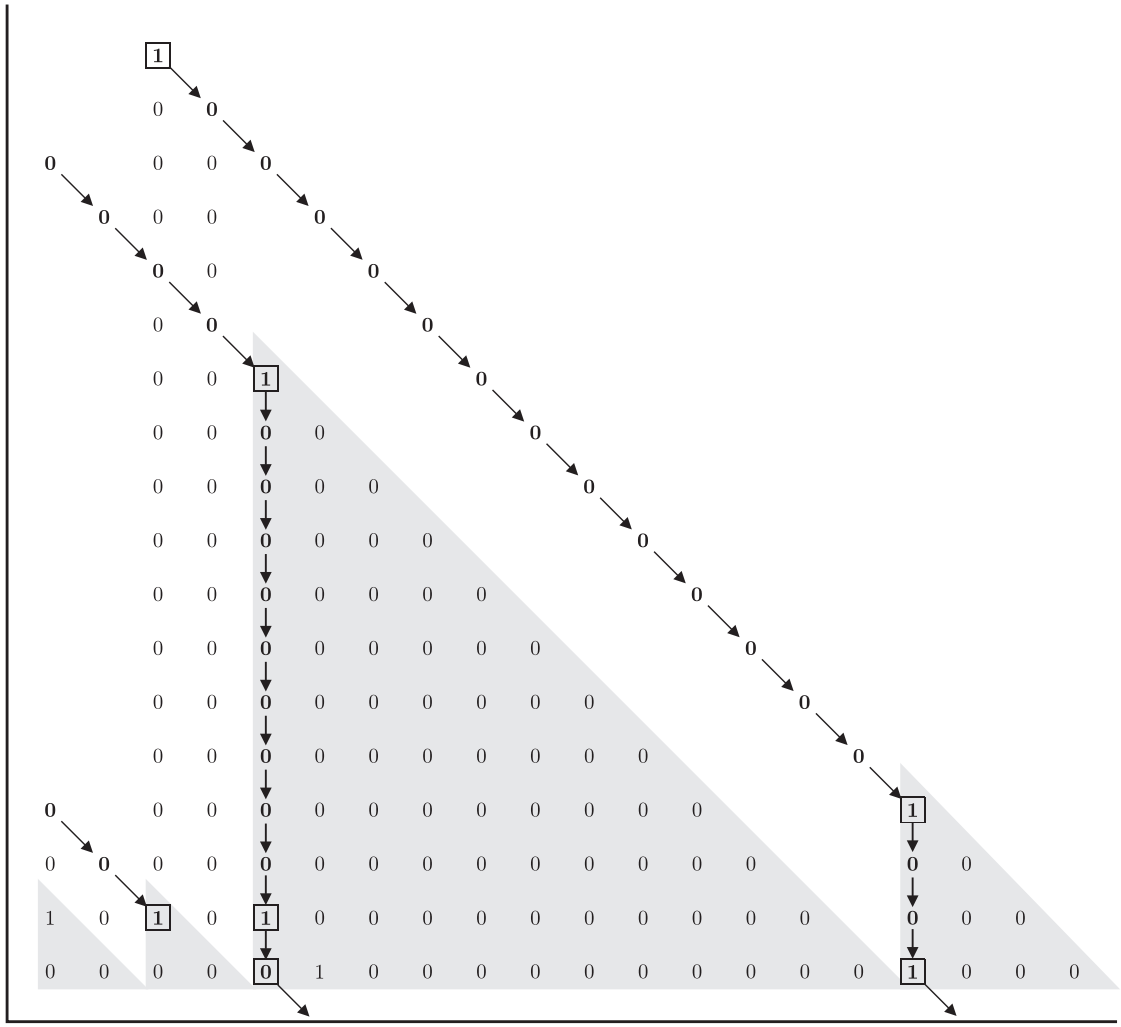
$$x_{u^-}\in\begin{cases}\overline{\Pi_0[(X_{u^-}\times X_{(u^-)^{l-1}})\cap U_{u^-}]}\text{ if }\varepsilon(u^-)=0,\\[2ex]\overline{\Pi_1[(X_{(u^-)^{l-1}}\times X_{u^-})\cap U_{u^-}]}\text{ if }\varepsilon(u^-)=1.\end{cases}$$

Assume for example that we are in the second case. Then

$$x_{u^-}\in\overline{W}\cap\overline{\Pi_1[(X_{(u^-)^{l-1}}\times X_{u^-})\cap U_{u^-}]}\cap Y\neq\emptyset.$$

The claim gives a couple $(x_u,x_{u^l})\in A\cap(\overline{W}\cap\overline{\Pi_1[(X_{(u^-)^{l-1}}\times X_{u^-})\cap U_{u^-}]}\cap Y)^2\cap\Omega_{X^2}$ since the set $\overline{W}\cap\overline{\Pi_1[(X_{(u^-)^{l-1}}\times X_{u^-})\cap U_{u^-}]}$ is $\Sigma_1^1\cap\mathbf{\Pi}_1^0$. We choose Σ_1^0 sets X_u,X_{u^l} with diameter at most 2^{-k} such that $(x_u,x_{u^l})\in X_u\times X_{u^l}\subseteq\overline{X_u}\times\overline{X_{u^l}}\subseteq X_{u^-}\times X_{(u^l)^-}$, as well as a Σ_1^1 subset U_u of X^2 with GH-diameter at most 2^{-k} such that $(x_u,x_{u^l})\in U_u\subseteq A\cap\overline{\Pi_1[(X_{(u^-)^{l-1}}\times X_{u^-})\cap U_{u^-}]}^2\cap Y^2\cap\Omega_{X^2}$.

- If u^- or $(u^l)^-$ is $(>l)$-placed, then by Lemma 5.9.(d) exactly one of those two sequences is $(>l)$-placed, and the other one is l-placed. If u^- (resp., $(u^l)^-$) is $(>l)$-placed, then $u^{-l}=\big((u^l)^-\big)^l$ (resp., $u^{-l}=u^-$). So assume first that u^- is $(>l)$-placed, so that $(u^l)^-=(u^l)^{-l}=(u^{-l})^l$ and u^{-l} is l-placed. Here is an illustration of what is going on in this case.



We define $(u_n)_{n\leq L}$ by $u_0 := u^-$, $u_L := u^{-l}$ and $u_{n+1} := u_n^{-l(u_n)}$ if $n < L$. This can be done, by Lemma 5.9.(e). Note that u_n is placed. We enumerate injectively the sequence $\big(l(u_n)\big)_{n\leq L}$ by the non-increasing sequence $(l_k)_{k\leq K}$. More concretely, $K = l_0 - l \geq 1$, $l(u_0) = ... = l(u_{N_0-1}) = l_0$, $l(u_{N_0}) = ... = l(u_{N_0+N_1-1}) = l_1 = l_0 - 1$, ...,

$$l(u_{N_0+...+N_{K-2}}) = ... = l(u_{N_0+...+N_{K-1}-1}) = l_{K-1} = l_{K-2} - 1 = l + 1$$

and $l(u_{N_0+...+N_{K-1}}) = l(u_L) = l_K = l$, with $N_0, ..., N_{K-1} \geq 1$.

Note that $u_1 = u_0^{-l_0}$ is l_0-placed if $N_0 \geq 2$. By Condition (3), we get $U_{u_0} \subseteq U_{u_1}$. We can iterate this argument, so that the inclusion $U_{u_0} \subseteq U_{u_{N_0-1}}$ holds, even if $N_0 = 1$. By Condition (5), x_{u^-} is in

$$X_{u^-} \cap \begin{cases} \overline{\Pi_0[(X_{u_{N_0}} \times X_{(u_{N_0})^{l_1}}) \cap U_{u_{N_0}}]} \text{ if } \varepsilon(u_{N_0}) = 0, \\ \\ \overline{\Pi_1[(X_{(u_{N_0})^{l_1}} \times X_{u_{N_0}}) \cap U_{u_{N_0}}]} \text{ if } \varepsilon(u_{N_0}) = 1. \end{cases}$$

Similarly, $X_{u^-} \subseteq X_{u_{N_0}}$.

This gives

$$(x_0^0, x_1^0) \in \begin{cases} (X_{u^-} \times X_{(u_{N_0})^{l_1}}) \cap U_{u_{N_0}} \text{ if } \varepsilon(u_{N_0}) = 0, \\ \\ (X_{(u_{N_0})^{l_1}} \times X_{u^-}) \cap U_{u_{N_0}} \text{ if } \varepsilon(u_{N_0}) = 1. \end{cases}$$

If we iterate the previous argument, then we get

$$(x_u, x_{u^l}) := (x_0^{K-1}, x_1^{K-1}) \in (X_{u^-} \times X_{(u_{N_0+\ldots+N_{K-1}})^{l_K}}) \cap U_{u_{N_0+\ldots+N_{K-1}}} = (X_{u^-} \times X_{(u^{-l})^l}) \cap U_{u^{-l}}$$

since $\varepsilon(u^{-l}) = \varepsilon(u) = 0$. We choose Σ_1^0 sets X_u, X_{u^l} with diameter at most 2^{-k} such that

$$(x_u, x_{u^l}) \in X_u \times X_{u^l} \subseteq \overline{X_u} \times \overline{X_{u^l}} \subseteq X_{u^-} \times X_{(u^{-l})^l} = X_{u^-} \times X_{(u^l)^-},$$

as well as a Σ_1^1 subset U_u of X^2 with GH-diameter at most 2^{-k} such that $(x_u, x_{u^l}) \in U_u \subseteq U_{u^{-l}}$.

If now $(u^l)^-$ is $(>l)$-placed, then we argue similarly, using the fact that

$$\varepsilon\big((u^l)^{-l}\big) = \varepsilon(u^l) = 1 - \varepsilon(u) = 1.$$

This finishes the proof. □

At the beginning of the section, we mentioned the fact that it is not necessary to use the dense G_δ subset $\mathbb{X}_3$ of 2^ω to find a relation $\mathbb{G}$ on 2^ω satisfying $(2^\omega, \mathbb{G}) \not\leq_{\mathbf{\Delta}_3^0} \big(\omega, \neg\Delta(\omega)\big)$. We now specify this.

Notation. We set, for $t \in \omega^{<\omega}$, $\tilde{H}_t := K_t^0 \setminus \big(\bigcup_{n\in\omega, w_n(0)=0} K_{tn} \cup \bigcup_{n\in\omega, w_n(0)=1} \varphi_t^{-1}(K_{tn})\big)$. Note that $\tilde{H}_t$ is a $\mathbf{\Pi}_2^0$ subset of 2^ω and $\tilde{H}_t \cap \varphi_t[\tilde{H}_t] = \emptyset$.

Lemma 5.10 *(a) The $\tilde{H}_t \cup \varphi_t[\tilde{H}_t]$'s are pairwise disjoint.*
(b) The set H_t is a subset of $\tilde{H}_t$, and thus satisfies the previous disjointness properties.

Proof. (a) Note first that $K_{tn} \subseteq K_t^{w_n(0)} \subseteq K_t$ and $(K_{tn})_n$ is a sequence of pairwise disjoint sets. This implies that $K_t \cap K_{t'} = \emptyset$ if t, t' are incompatible. In particular, as $\tilde{H}_t \subseteq K_t^0 \setminus (\bigcup_{n\in\omega, w_n(0)=0} K_{tn}) \subseteq K_t$ and $\varphi_t[\tilde{H}_t] \subseteq K_t^1 \setminus (\bigcup_{n\in\omega, w_n(0)=1} K_{tn}) \subseteq K_t$, we also get $(\tilde{H}_t \cup \varphi_t[\tilde{H}_t]) \cap (\tilde{H}_{t'} \cup \varphi_{t'}[\tilde{H}_{t'}]) = \emptyset$ if t and t' are incompatible. Now

$$\begin{aligned} \tilde{H}_t \cap \tilde{H}_{tt'n} &\subseteq \tilde{H}_t \cap K_{t[(t'n)|1]} \\ &\subseteq \begin{cases} K_t^0 \cap K_t^1 \text{ if } s_{(t'n)(0)}(0) = 1, \\ \neg K_{t[(t'n)|1]} \cap K_{t[(t'n)|1]} \text{ if } s_{(t'n)(0)}(0) = 0, \end{cases} \end{aligned}$$

so that $\tilde{H}_t \cap \tilde{H}_{t'} = \emptyset$ if $t \neq t'$. Similarly, $(\tilde{H}_t \cup \varphi_t[\tilde{H}_t]) \cap (\tilde{H}_{t'} \cup \varphi_{t'}[\tilde{H}_{t'}]) = \emptyset$ if $t \neq t'$.

(b) If $\alpha \in K_{tn} \cup \varphi_t^{-1}(K_{tn})$, then $(\alpha)_{\Sigma^t_{|t|}}$ has finitely many 1's. □

Remarks. (a) We set $\mathbb{G} := \bigcup_{t\in\omega^{<\omega}} \mathrm{Gr}(\varphi_{t|\tilde{H}_t})$, so that $(2^\omega, \mathbb{G}) \not\leq_{\mathbf{\Delta}_3^0} \big(\omega, \neg\Delta(\omega)\big)$, by the proof of Lemma 5.5. By Lemma 5.10, $\mathbb{G}$ is the Borel graph of a partial injection, as announced at the beginning of the section.

(b) Note that $(\mathbb{X}_3,\mathbb{A}_3)\preceq_{\mathbf{\Delta}^0_4}\big(\omega,\neg\Delta(\omega)\big)$, as we can see with the following partition of $\mathbb{X}_3$:

$$\mathbb{X}_3=\bigcup_{t\in\omega^{<\omega}}H_t\cup\big(\mathbb{X}_3\setminus(\bigcup_{t\in\omega^{<\omega}}H_t)\big),$$

with $H_t\in\mathbf{\Pi}^0_2$ and $\mathbb{A}_3$-discrete by Lemma 5.10.(b), $\mathbb{X}_3\setminus(\bigcup_{t\in\omega^{<\omega}}\ H_t)\in\mathbf{\Pi}^0_3$ and $\mathbb{A}_3$-discrete.

(c) There are a comparing 2-disjoint family $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ of subsets of $\mathbb{X}_3$, and also homeomorphisms $\varphi_i:C^0_i\to C^1_i$ such that $\mathbb{A}_3=\bigcup_{i\in\omega}\ \mathrm{Gr}(\varphi_i)$. Indeed, we choose a bijection $b:\omega\to\omega^{<\omega}$ with $b^{-1}(s)\leq b^{-1}(t)$ if $s\subseteq t$, as in the proof of Lemma 4.3, and set $C^0_i:=H_{b(i)}$, $C^1_i:=\varphi_{b(i)}[H_{b(i)}]$, $\varphi_i:=\varphi_{b(i)_{|H_{b(i)}}}$, so that $\mathbb{A}_3=\bigcup_{i\in\omega}\ \mathrm{Gr}(\varphi_i)$. It remains to see that $(C^\varepsilon_i)_{(\varepsilon,i)\in 2\times\omega}$ is comparing. We set

$$O^p_q:=\begin{cases}K^\varepsilon_{b(i)}\setminus(\bigcup_{l\leq q,b(i)n\subseteq b(l),w_n(0)=\varepsilon}\ K_{b(i)n})\text{ if }p=2i+\varepsilon\leq 2q+1,\\ \mathbb{X}_3\setminus(\bigcup_{p'\leq 2q+1}\ O^{p'}_q)\text{ if }p=2q+2,\\ \emptyset\text{ if }p\geq 2q+3,\end{cases}$$

so that $(O^p_q)_{p\in\omega}$ is a partition of $\mathbb{X}_3$ into $\mathbf{\Delta}^0_2$ sets since $K_{tn}\subseteq K^{w_n(0)}_t\subseteq K_t$, $K^0_t\cap K^1_t=\emptyset$ and $K_t\cap K_{t'}=\emptyset$ if t and t' are incompatible.

As $\tilde H_t\subseteq K^0_t\setminus(\bigcup_{n\in\omega,w_n(0)=0}\ K_{tn})$, $\varphi_t[\tilde H_t]\subseteq K^1_t\setminus(\bigcup_{n\in\omega,w_n(0)=1}\ K_{tn})$ and $H_t\subseteq\tilde H_t$, (b) in Definition 4.2 is fulfilled. If $q<i$, then

- either there is no $j\leq q$ such that $b(i)$ is compatible with $b(j)$. $C^0_i\cup C^1_i\subseteq K_{b(i)}\subseteq O^{2q+2}_q$ and we set $p^i_q:=2q+2$.

- or there are $j\leq q$ and n such that $b(j)n\subseteq b(i)$, in which case $K_{b(i)}\subseteq K^{w_n(0)}_{b(j)}\cap K_{b(j)n}$. In particular, $K_{b(i)}$ is disjoint from or included in each difference $K^{w_{n'}(0)}_{b(j')}\setminus K_{b(j')n'}$. Thus $K_{b(i)}$ is disjoint from or included in $O^{2j+\varepsilon}_q$. By disjointness, there is at most one couple (j,ε) such that $K_{b(i)}\subseteq O^{2j+\varepsilon}_q$. If it exists, then we set $p^i_q:=2j+\varepsilon$. If it does not exist, then we set $p^i_q:=2q+2$.

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