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Abdelhafid Younsi

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ON THE UNIQUENESS IN THE 3D NAVIER-STOKES EQUATIONS

ABDELHAFID YOUNSI

ABSTRACT. In this paper, we study the Cauchy problem of the 3D incompressible Navier–Stokes equations. We prove an uniqueness result of weak solutions for the 3D Navier-Stokes equations satisfying the energy inequality.

1. INTRODUCTION

Two of the profound open problems in the theory of three dimensional viscous flow are the unique solvability theorem for all time and the regularity of solutions. For the three-dimensional Navier-Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [4], it is not known if the weak solution is unique or what further assumption could make it unique only the uniqueness of weak solutions remains as an open problem. In this paper we use the quotient of Dirichlet to prove the uniqueness, this quantity was used by Constantin [2] and Kukavica [3] to study the backward uniqueness in 2D Navier-Stokes equations.

In this paper, we prove that, if two weak solutions of the 3-dimensional Navier-Stokes are equal in such time t_0 , then they are equal for all $t \geq t_0$. This particular result is interesting not only for mathematical understanding of the equations but also for understanding the phenomenon of turbulence [1].

2. PRELIMINARY

We denote by $H_{per}^m(\Omega)$, the Sobolev space of L -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

We define the spaces V_m as completions of smooth, divergence-free, periodic, zero-average functions with respect to the H_{per}^m norms. V'_m denotes the dual space of V_m and V denotes the space V_0 .

We denote by A the Stokes operator $Au = -\Delta u$ for $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact, $D(A) = V_2$. Now define the trilinear form $b(., ., .)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \quad (2.1)$$

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The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B(u, v); V \times V \rightarrow V'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V. \quad (2.2)$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.3)$$

Hereafter, $c_i \in \mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, p > 1, q = \frac{p}{p-1}. \quad (2.4)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \text{ for all } u \in V_0, \quad (2.5)$$

where λ_1 is the smallest eigenvalue of the Stokes operator A .

3. NAVIER-STOKES EQUATIONS

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{\partial u}{\partial t} + \nu Au + B(u, u) &= f, \quad t > 0, \\ \operatorname{div} u &= 0, \text{ in } \Omega \times (0, \infty) \text{ and } u(x, 0) = u_0, \text{ in } \Omega. \end{aligned} \quad (3.1)$$

We recall that a Leray weak solution of the Navier-Stokes equations is a solution which is bounded and weakly continuous in the space of periodic divergence-free L^2 functions, whose gradient is square-integrable in space and time and which satisfies the energy inequality. The proof of the following theorem is given in [1, 5, 7]

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ and $f \in L^2(0, T; V_1')$, $u_0 \in V_0$ be given. Then there exists a weak solution u of (3.1) which satisfies $u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, $\forall T > 0$, furthermore if $n = 2$, u is unique.*

In this paper we will be especially interested in the case where $n = 3$.

Proposition 3.2. *If we consider $w = u - v$, the difference of two weak solutions of the 3D Navier-Stokes equations, u and v , then we have $w(0) = 0$ implies $w(t) = 0$ for all $t \in [0, T]$.*

Proof. We obtain the equation for $w = u - v$ as

$$\partial_t w + Aw + B(v, v) - B(u, u) = 0, \quad (3.2)$$

with $\operatorname{div} w = 0$. Taking the scalar product of (3.2) with w , we have

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu \|A^{\frac{1}{2}} w\|^2 = b(w, w, u). \quad (3.3)$$

Using the generalised version of Holder's inequality, we have

$$|b(w, w, u)| \leq c_1 \|w\|_{L^4}^2 \left\| A^{\frac{1}{2}} u \right\|_{L^2}. \quad (3.4)$$

A straightforward application of Peetr's Theorem [5, 6]

$$H^{(1-\theta)m}(\Omega) \subset L^{q_\theta}(\Omega), \frac{1}{q_\theta} = \frac{1}{2} - \frac{(1-\theta)m}{n} \quad (3.5)$$

if we consider $m = 1$ then $q_\theta \geq 4$ for $\theta \leq \frac{1}{4}$, inequality (3.4) means that

$$|b(w, w, u)| \leq c_2 \|w\|_1^{2(1-\theta)} \|w\|_{L^2}^{2\theta} \|u\|_1. \quad (3.6)$$

Hence, applying the Poincaré inequality gives

$$|b(w, w, u)| \leq c_3 \|w\|_1^{(2-\theta)} \|w\|_{L^2}^\theta \|u\|_1. \quad (3.7)$$

We define the Dirichlet quotients χ for solutions w of the equation (3.3)

$$\chi = \frac{\|w\|_1}{\|w\|}. \quad (3.8)$$

If we consider the Poincaré inequality (2.5), we obtain from the above considerations that

$$\frac{1}{\chi} \text{ is finite for } \|w(t)\|_1 \neq 0, \forall t \geq 0 \quad (3.9)$$

and

$$\frac{1}{\chi} \leq \frac{1}{\lambda_1^{1/2}}. \quad (3.10)$$

This means to

$$\frac{1}{\chi} |b(w, w, u)| \leq c_3 \|w\|_1^{(1-\theta)} \|w\|_{L^2}^{1+\theta} \|u\|_1. \quad (3.11)$$

Using Young's inequality on the right-hand side, we obtain

$$\frac{1}{\chi} \frac{d}{dt} \|w\|^2 + \frac{\nu}{\chi} \|w\|_1^2 \leq \frac{\nu}{2\chi} \|w\|_1^2 + c_4 \left(\frac{\nu}{2\chi}\right)^{\frac{4}{1-\theta^2}} (\|w\|_{L^2}^{1+\theta} \|u\|_1)^{\frac{2}{1+\theta}}. \quad (3.12)$$

Combining all these inequalities yields

$$\frac{1}{\chi} \frac{d}{dt} \|w\|^2 + \frac{\nu}{2\chi} \|w\|_1^2 \leq c_4 \left(\frac{\nu}{2\chi}\right)^{\frac{4}{1-\theta^2}} \|w\|^2 \|u\|_1^{\frac{2}{\theta+1}}. \quad (3.13)$$

Dropping the positive term $\frac{\nu}{2\chi} \|w\|_1^2$, we get

$$\frac{1}{\chi} \frac{d}{dt} \|w\|^2 \leq c_5 \left(\frac{1}{\chi}\right)^{\frac{4}{1-\theta^2}} \|u\|_1^{\frac{2}{\theta+1}} \|w\|^2. \quad (3.14)$$

Since

$$\frac{4}{1-\theta^2} \leq \frac{4}{1-\theta}, \quad (3.15)$$

this means to

$$\frac{d}{dt} \|w\|^2 \leq c_5 \left(\frac{1}{\lambda_1}\right)^{\frac{3+\theta}{2-2\theta}} \|u\|_1^{\frac{2}{\theta+1}} \|w\|^2. \quad (3.16)$$

Applying Gronwall's inequality on (3.16) and using the fact that

$$\frac{2}{\theta+1} \leq 2 \quad (3.17)$$

yields

$$\|w(t)\|^2 \leq c_6 \|w(0)\|^2 \int_0^t \|u\|_1^2 ds. \quad (3.18)$$

Since $u \in L^2(0, T; V_1)$, the integral expression on the right-hand side is finite, which implies both continuous dependence on initial conditions and uniqueness, this means that $w(x, t) = 0$ for $0 \leq t \leq T$ if $w(0) = 0$. \square

Note that the terms in the right-hand side of (3.18) are independent of χ . As a continuation of the previous work [8, 9] on the Navier-Stokes system, we introduce in this paper a new direction for the study of the uniqueness of weak solutions.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF DJELFA , ALGERIA.
E-mail address: younsihafid@gmail.com