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# GLOBAL WELL POSEDNESS FOR THE NAVIER-STOKES EQUATIONS

ABDELHAFID YOUNSI

ABSTRACT. We investigate general topology properties to show that the set of solutions of the Navier-Stokes is homeomorphic to the set of solutions of regularized Navier-Stokes equations by adding a high-order viscosity term. This result means that the set of solutions is reduced to one solution for each dimension  $d \leq 4$ . We also prove a high regularity of solution Navier-Stokes equations.

## 1. INTRODUCTION

Two of the profound open problems in the theory of three dimensional viscous flow are the unique solvability theorem for all time and the regularity of solutions.

For the 3D Navier-Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [3], only the uniqueness of weak solutions remains as an open problem.

We consider in this paper a “regularized” Navier-Stokes system was proposed by J. L. Lions [4], who added the artificial hyper-viscosity  $\varepsilon(-\Delta)^l$ ,  $l > 1$ , to the Navier-Stokes equation. Here  $\varepsilon > 0$  is the artificial dissipation parameter.

For such a modified problem considered in a bounded domain, J. L. Lions was able to prove (cf. [4, Chap.1, Remarque 6.11]) the existence of a unique regular solution provided ( $l \geq (d+2)/2$  for the  $d$ -dimensional problem). We have shown in our earlier study [9], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter  $\varepsilon$  goes to zero for  $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$  in each  $d \leq 4$ .

A natural question then is to investigate the possibility of establishing such results for the conventional Navier-Stokes equations by a limit process.

We shall denote by  $\mathfrak{S}_\varepsilon = S(f, u_{\varepsilon_0}, \varepsilon)$  the set of solutions  $u_\varepsilon$  of the regularized problem. In order to show the uniqueness of weak solutions of the Navier-Stokes equations in each dimension  $d \leq 4$ , we investigate general topology properties to identify  $\mathfrak{S}_\varepsilon$  to the set of solutions  $u$  of the conventional problem. We show that the set of solutions of the Navier-Stokes equations is homeomorphic to  $\mathfrak{S}_\varepsilon$ . We finish by establishing a high regularity result for the solutions of Navier-Stokes equations.

In this paper we give an answer to the uniqueness question for the Navier-Stokes equations for  $d \leq 4$ . It appears that this method is an interesting way to establish the uniqueness by a limit process in other branches of partial differential equations.

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## 2. PRELIMINARY

We denote by  $H^m(\Omega)$ , the Sobolev space of  $L$ -periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \left( \sum_{|\beta| \leq m} \|D^\beta u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad (2.1)$$

and by  $H^{-m}(\Omega)$  the dual space of  $H^m(\Omega)$ . We define the spaces  $V_s$  as completions of smooth, divergence-free, periodic, zero-average functions with respect to the  $H^s$  norms.  $V'_s$  denote the dual space of  $V_s$  and  $V$  denote the space  $V_0$ .

We denote by  $A$  the Stokes operator  $Au = -\Delta u$  for  $u \in D(A) = \{u \in V_0, Au \in V_0\} = V_2$ . The spectral theory of  $A$  allows us to define the powers  $A^l$  of  $A$  for  $l \geq 1$ ,  $A^l$  is an unbounded self-adjoint operator in  $V_0$ . We set here

$$A^l u = (-\Delta)^l u \text{ for } u \in D(A^l) = V_{2l} \cap V_0. \quad (2.2)$$

Now define the trilinear form  $b(\cdot, \cdot, \cdot)$  associated with the inertial terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \quad (2.3)$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator  $B(u, v); V_2 \times V_2 \rightarrow V'_2$  will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V_2. \quad (2.4)$$

Recall that for  $u$  satisfying  $\nabla \cdot u = 0$  we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.5)$$

Hereafter,  $c_i \in \mathbb{N}$ , will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. We recall some inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{1}{q}}} b^q, \quad a, b, \epsilon > 0, p > 1, q = \frac{p}{p-1}. \quad (2.6)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \quad (2.7)$$

for all  $u \in V$ , where  $\lambda_1$  is the smallest eigenvalue of the Stokes operator  $A$ .

## 3. THE REGULARIZED NAVIER-STOKES SYSTEM

In this paper, we study the uniqueness of weak solutions of the modified Navier-Stokes equations in each dimension  $d \leq 4$ . We regularized the Navier-Stokes system by adding a high order artificial viscosity term to the conventional system

$$\begin{aligned} \frac{du_\epsilon}{dt} + \varepsilon(-\Delta)^l u_\epsilon - \nu \Delta u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon + \nabla p &= f(x), \text{ in } \Omega \times (0, \infty) \\ \operatorname{div} u_\epsilon &= 0, \text{ in } \Omega \times (0, \infty), \\ p(x + Le_i, t) &= p(x, t), \quad u(x + Le_i, t) = u(x, t) \quad i = 1, \dots, d \quad t \in (0, \infty) \\ u_\epsilon(x, 0) &= u_{\varepsilon 0}(x), \text{ in } \Omega, \end{aligned} \quad (3.1)$$

where  $\Omega = (0, L)^d$  with periodic boundary conditions and  $(e_1, \dots, e_d)$  is the natural basis of  $\mathbb{R}^d$ . Here  $\varepsilon > 0$  is the artificial dissipation parameter,  $u^\varepsilon$  is the velocity vector field,  $p$  is the pressure,  $\nu > 0$  is the kinematic viscosity of the fluid and  $f$  is

a given force field. For  $\varepsilon = 0$ , the model is reduced to Navier–Stokes system. For further discussion of theoretical results concerning (3.1), see [1, 4, 9].

Using the operators defined in the previous section, we can write the modified system (3.1) in the evolution form

$$\begin{aligned} \partial_t u_\varepsilon + \varepsilon A^l u_\varepsilon + \nu A u_\varepsilon + B(u_\varepsilon, u_\varepsilon) &= f(x), & \text{in } \Omega \times (0, \infty) \\ u_\varepsilon(x, 0) &= u_{\varepsilon 0}, & \text{in } \Omega. \end{aligned} \quad (3.2)$$

The existence and uniqueness results for initial value problem (3.1) can be found in Lions [4].

The following theorem collects the main result in this work.

**Theorem 3.1.** *For  $l \geq \frac{d+2}{4}$ ,  $d$  is the space dimension, for  $\varepsilon > 0$  fixed,  $f \in L^2(0, T; V_0')$  and  $u_{\varepsilon 0} \in V_0$  be given. There exists a unique weak solution of (3.1) which satisfies*

$$u_\varepsilon \in L^2(0, T; V_l) \cap L^\infty(0, T; V_0), \forall T > 0.$$

Notice that the conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{du}{dt} + \nu A u + B(u, u) &= f(x), & \text{in } \Omega \times (0, \infty) \\ u(0) &= u_0, & \text{in } \Omega. \end{aligned} \quad (3.3)$$

**Theorem 3.2.** *For  $d \leq 4$ , for  $f \in L^2(0; T; V_0)$  and  $u_0 \in V_0$  be given. There exists a weak solution of (3.3) which satisfies  $u \in L^\infty(0; T; V_0) \cap L^2(0; T; V_1)$ , for  $T > 0$ . For  $d = 2$ ,  $u$  is unique (J. Lions [4]).*

In the work [9], we demonstrated the following theorem

**Theorem 3.3.** *For  $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$  and for  $d \leq 4$ , the weak solution  $u_\varepsilon$  of the modified Navier-Stokes equations (3.1) given by Theorem 3.1 converges strongly in  $L^2(0, T; V_0)$  as  $\varepsilon \rightarrow 0$  to  $u$  a weak solution of the system (3.3).*

At the first we give a prior estimate on the solution  $u_\varepsilon(t)$ , we show that the system has a global attractor.

A compact set  $\mathfrak{A} \Subset E$  is said to be a global attractor of a semigroup  $\{S(t), t > 0\}$  acting in a Banach or Hilbert space  $E$  if  $\mathfrak{A}$  is strictly invariant with respect to  $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A} \forall t \geq 0$  and  $\mathfrak{A}$  attracts any bounded set  $B \subset E$ :

$$\text{dist}(S(t)B, \mathfrak{A}) \rightarrow 0 \text{ when } t \rightarrow \infty, \quad (3.4)$$

see for instance Temam [8] or Chepyzhov and Vishik [5].

For  $\varepsilon > 0$  and  $f \in V_0$  a time independent function, let  $S^\varepsilon(t)$  denote the semiflow generated by the weak solutions of the regularized Navier-Stokes equations (3.1). Thus

$$S^\varepsilon(t) = S^\varepsilon(f, t) u_{\varepsilon 0}, \quad (3.5)$$

where  $u_\varepsilon(t) = S^\varepsilon(f, t) u_{\varepsilon 0}$  is the weak solution of (3.1) that satisfies  $u_\varepsilon(0) = u_{\varepsilon 0}$ .

Now, we show that the semigroup  $S^\varepsilon(t)$  has an absorbing ball in  $V_0$  and an absorbing ball in  $V_1$ . Then we show that  $S^\varepsilon(t)$  admits a compact attractor in  $V_0$  for each  $\varepsilon > 0$ .

**Theorem 3.4.** *For  $\varepsilon > 0$ , the dynamical system associated with the regularized Navier Stokes equations (3.1) admits a compact attractor  $\mathfrak{A}_\varepsilon$ , which attracts bounded sets of  $V_0$ .*

*Proof.* We take the inner product of (3.1) with  $u_\varepsilon$ , we obtain

$$\frac{d}{dt} \|u_\varepsilon\|^2 + 2\varepsilon (A^l u_\varepsilon, u_\varepsilon) + 2\nu \|\nabla u_\varepsilon\|^2 = 2(f, u_\varepsilon). \quad (3.6)$$

Here we have used the fact that  $b(u_\varepsilon, u_\varepsilon, u_\varepsilon) = 0$ .

By applying Young's inequality and the Poincaré Lemma, we get

$$\frac{d}{dt} \|u_\varepsilon\|^2 + 2\varepsilon \|A^{\frac{l}{2}} u_\varepsilon\|^2 + \nu \|\nabla u_\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu \lambda_1}, \quad (3.7)$$

we drop the term  $2\varepsilon \|A^{\frac{l}{2}} u_\varepsilon\|^2$ , we obtain

$$\frac{d}{dt} \|u_\varepsilon\|^2 + \nu \lambda_1 \|u_\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu \lambda_1}, \quad (3.8)$$

by integrating the above inequality from 0 to  $t$ , we get

$$\|u_\varepsilon(t)\|^2 \leq \|u_{\varepsilon 0}\|^2 e^{-\nu \lambda_1 t} + \rho_0^2 (1 - e^{-\nu \lambda_1 t}), \quad t > 0, \quad (3.9)$$

where  $\rho_0 = \frac{1}{\nu \lambda_1} \|f\|$ . Hence for any ball  $B_{R_0} = \{u_{\varepsilon 0} \in V_0; \|u_{\varepsilon 0}\| \leq R_0\}$  there is a ball  $B(0, \delta_0)$  in  $V_0$  centered at origin with radius  $\delta_0 > \rho_0$  ( $R_0 > \delta_0$ ) such that

$$S^\varepsilon(t)B_{R_0} \subset B_{r_0} \text{ for } t \geq t_0(B_{R_0}) = \frac{1}{\nu \lambda_1} \log \frac{R_0^2 - \rho_0^2}{\delta_0^2 - \rho_0^2}. \quad (3.10)$$

The ball  $B_{\delta_0}$  is said to be absorbing and invariant under the action of  $S^\varepsilon(t)$ . Taking the limit in (3.9) we get,

$$\limsup_{t \rightarrow \infty} \|u_\varepsilon(t)\| \leq \rho_0. \quad (3.11)$$

We integrate (3.8) from  $t$  to  $t+r$ , we obtain for  $u_{\varepsilon 0} \in B_{R_0}$

$$\int_t^{t+r} \|u_\varepsilon\|_1^2 ds \leq \frac{1}{\nu} \left( \frac{r \|f\|^2}{\nu \lambda_1} + \|u_\varepsilon(t)\|^2 \right), \quad \forall r > 0, \quad \forall t \geq t_0(B_{R_0}). \quad (3.12)$$

With the use of (3.11) we conclude that

$$\limsup_{t \rightarrow \infty} \int_t^{t+r} \|u_\varepsilon\|_1^2 ds \leq \frac{r}{\nu^2 \lambda_1} \|f\|^2 + \frac{\|f\|^2}{\nu^3 \lambda_1^2}. \quad (3.13)$$

To show that the semigroup  $S^\varepsilon(t)$  has an absorbing set in  $V_1$ , we consider the strong solutions and take the inner product of (3.1) with  $Au_\varepsilon$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{\frac{l}{2}} u_\varepsilon\|^2 + \varepsilon (A^l u_\varepsilon, Au_\varepsilon) + \nu \|Au_\varepsilon\|^2 = -b(u_\varepsilon, u_\varepsilon, Au_\varepsilon) + (f, Au_\varepsilon). \quad (3.14)$$

By applying Young's inequality, we get

$$\begin{aligned} (f, Au_\varepsilon) &\leq \|f\| \|Au_\varepsilon\| \\ &\leq \frac{\nu}{4} \|Au_\varepsilon\|^2 + \frac{1}{\nu} \|f\|^2. \end{aligned} \quad (3.15)$$

For the second member, we have by Holder's inequality that

$$|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| \leq \|u_\varepsilon\|_\infty \|u_\varepsilon\|_1 \|Au_\varepsilon\|, \quad (3.16)$$

using (3.11) we get

$$|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| \leq \rho_0 \|u_\varepsilon\|_1 \|Au_\varepsilon\|. \quad (3.17)$$

By applying Young's inequality, we get

$$|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| \leq \frac{\rho_0^2}{\nu} \|u_\varepsilon\|_1^2 + \frac{\nu}{4} \|Au_\varepsilon\|^2. \quad (3.18)$$

Substituting the above result into (3.14), we obtain

$$\frac{d}{dt} \|u_\varepsilon\|_1^2 + 2\varepsilon \|A^{\frac{l+1}{2}} u_\varepsilon\|^2 + \nu \|Au_\varepsilon\|^2 \leq \frac{2}{\nu} \|f\|^2 + \frac{\rho_0^2}{\nu} \|u_\varepsilon\|_1^2. \quad (3.19)$$

If we drop the positive term associated with  $\varepsilon$ , we obtain

$$\frac{d}{dt} \|u_\varepsilon\|_1^2 \leq \frac{2}{\nu} \|f\|^2 + \frac{\rho_0^2}{\nu} \|u_\varepsilon\|_1^2. \quad (3.20)$$

We apply the uniform Gronwall Lemma to (3.20). Thanks to (3.9)-(3.13) we estimate the quantities  $a_1, a_2, a_3$  by

$$a_1 = \frac{\rho_0^2}{\nu}, \quad a_2 = \frac{2}{\nu} \|f\|^2, \quad a_3 = \frac{r \|f\|^2}{\nu^2 \lambda_1} + \frac{\|f\|^2}{\nu^3 \lambda_1^2}. \quad (3.21)$$

Then, we obtain

$$\|u_\varepsilon(t)\|_1^2 \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1) = R_1^2 \text{ for } t \geq t_0, \quad t_0 \text{ as in (3.10)}. \quad (3.22)$$

Hence, for any ball  $B_{R_1}$ , there exists a ball  $B_{\delta_1}$ , in  $V_1$  centered at origin with radius  $R_1 > \delta_1 > \rho_1$  such that

$$S^\varepsilon(t)B_{R_1} \subset B_{\delta_1} \text{ for } t \geq t_1(B_{R_0}) = t_0(B_{R_0}) + 1 + \frac{1}{\nu \lambda_1} \log \frac{R_1^2 - \rho_1^2}{\delta_1^2 - \rho_1^2}. \quad (3.23)$$

The ball  $B_{\delta_1}$  is said to be absorbing and invariant for the semigroup  $S^\varepsilon(t)$ .

Furthermore, if  $B$  is any bounded set of  $V_0$ , then  $S^\varepsilon(t)B \subset B_{\delta_1}$  for  $t \geq t_1(B, R_0)$ , this shows the existence of an absorbing set in  $V_1$ . Since the embedding of  $V_1$  in  $V_0$  is compact, we deduce that  $S^\varepsilon(t)$  maps a bounded set in  $V_0$  into a compact set in  $V_0$ . In addition, the operators  $S^\varepsilon(t)$  are uniformly compact for  $t \geq t_1(B, R_0)$ . That is,

$$\bigcup_{t \geq t_1} S^\varepsilon(t, 0, B_{R_0}) \quad (3.24)$$

is relatively compact in  $V_0$ .

Due to a the standard procedure (cf., for example, Temam [8, Theorem I.1.1] for details), there is a global attractor  $\mathfrak{A}_\varepsilon$  for the operators  $S^\varepsilon(t)$  for  $\varepsilon > 0$ .

Note that the global attractor  $\mathfrak{A}_\varepsilon$  must be contained in the absorbing balls  $V_0$  and  $V_1$

$$\mathfrak{A}_\varepsilon = \bigcap_{t_1 \geq 0} \overline{\bigcup_{t \geq t_1} B_{\delta_1}(t)} \subset B_{\delta_0} \cap B_{\delta_1}. \quad (3.25)$$

□

It is well known (cf. J. L. Lions [3]) that the problem reduces to a functional equation involving only  $u_\varepsilon$ . We recall that  $\mathfrak{S}_\varepsilon = S(\varepsilon, f, u_{\varepsilon_0})$  denotes the set of solutions  $u_\varepsilon$  of the problem (3.1) for all  $\varepsilon > 0$  fixed.

We shall first conclude this subsection by listing some properties of  $\mathfrak{S}_\varepsilon$  which are either well known or are easy consequences of well-known results.

**Theorem 3.5.** *Assume that  $d \leq 4$  and  $\varepsilon > 0$  fixed, let  $S(\varepsilon, f, u_{\varepsilon_0})$  the set of solutions of the modified Navier-Stokes equations (3.1) given by Theorem 3.1. Then*

- i).  $S(\varepsilon, f, u_{\varepsilon_0})$  is not empty (existence of solutions),*
- ii).  $S(\varepsilon, f, u_{\varepsilon_0})$  is closed and bounded in  $V_0$ , compact in  $V_0$ ,*

iii).  $S(\varepsilon, f, u_{\varepsilon_0})$  is reduced to one point (uniqueness of solutions) for all  $\varepsilon > 0$  fixed, if  $l \geq \frac{d+2}{4}$ .

*Proof.* The existence of solutions (point i)) is established in [4, 6.163]. The closedness of the set  $S(\varepsilon, f, u_{\varepsilon_0})$  and its compactness in  $V_0$ , are consequences of the Theorem 3.4 . The proof of (iii)) is also established in (cf. [4, 6.164]).  $\square$

The set of solutions  $u$  of problem (3.3) is denoted by  $\mathfrak{S}_0 = S(0, f, u_0)$ . The set  $\mathfrak{S}_0$  contains all solutions of the system (3.1 ).

**Lemma 3.6.** *Let  $v_n$  and  $u_n$  be two convergent sequences such that  $u_n$  converges to  $u$  and  $v_n$  converges to  $v$  in  $V_0$ , then for all  $\varepsilon > 0$  there exists a positive number  $N \in \mathbb{N}$  such that for all  $n > N$  we have*

1).

$$\|u_n - v_n\| \leq \|u - v\| + \varepsilon \quad (3.26)$$

2).

$$\|u - v\| \leq \|u_n - v_n\| + \varepsilon.$$

*Proof.* We have

$$\|u_n - v_n\| \leq \|u_n - u\| + \|u - v\| + \|v - v_n\|$$

thus, for all  $\varepsilon > 0$  there exists  $N$  such that for all  $n > N$

$$\|u_n - v_n\| \leq \varepsilon/2 + \|u - v\| + \varepsilon/2$$

Whence the result.

The proof of part 2) is similar to that of part 1).

$$\begin{aligned} \|u - v\| &\leq \|u_n - u\| + \|u_n - v_n\| + \|v - v_n\| \\ &\leq \|u_n - v_n\| + \varepsilon, \end{aligned}$$

for all  $\varepsilon > 0$  and for all  $n > N$ .  $\square$

An interesting question is then whether we can construct a continuous operator from  $\mathfrak{S}_\varepsilon$  into  $\mathfrak{S}_0$ .

**Theorem 3.7.** *Assume that  $l \geq \frac{d}{2}$  in each  $d \leq 4$ , then there exists  $\varepsilon' > 0$  such that for all  $\varepsilon \leq \varepsilon'$ , there exists a sequence of continuous functions  $\Phi_\alpha$ , from  $\mathfrak{S}_\varepsilon$  to  $\mathfrak{S}_0$ , for all  $\alpha > 0$ .*

*Proof.* Let  $v_\varepsilon(t)$  and  $u_\varepsilon(t)$  two weak solutions of (3.1 ), we obtain for  $w_\varepsilon(t) = v_\varepsilon(t) - u_\varepsilon(t)$

$$\frac{dw_\varepsilon}{dt} + \varepsilon A^l w_\varepsilon + \nu A w_\varepsilon + B(w_\varepsilon, v_\varepsilon) + B(v_\varepsilon, w_\varepsilon) - B(w_\varepsilon, w_\varepsilon) = 0 \quad (3.27)$$

and  $w_\varepsilon(0) = v_{\varepsilon 0} - u_{\varepsilon 0} = 0$ .

Taking the inner product of the last equation with  $w_\varepsilon$ , we obtain

$$\frac{d}{dt} \|w_\varepsilon\|^2 + 2\varepsilon \|A^{\frac{l}{2}} w_\varepsilon\|^2 + 2\nu \|w_\varepsilon\|_1^2 \leq |2b(w_\varepsilon, v_\varepsilon, w_\varepsilon)|. \quad (3.28)$$

The second member satisfies [4, 6.168]

$$\frac{d}{dt} \|w_\varepsilon\|^2 \leq 2c_1 \|w_\varepsilon\|^2 \|v_\varepsilon\|_1^{\frac{1}{2}}. \quad (3.29)$$

Using (3.22), we obtain the following differential inequality

$$\frac{d}{dt} \|w_\varepsilon\|^2 \leq c_1 R_1^{\frac{1}{d}} \|w_\varepsilon\|^2. \quad (3.30)$$

Integrating from 0 to  $t$  to obtain

$$\|w_\varepsilon\|^2 \leq c_1 R_1^{\frac{1}{d}} \int_0^t \|w_\varepsilon\|^2 ds. \quad (3.31)$$

Integrating again from 0 to  $t$  to obtain

$$\|w_\varepsilon\|_{(L^2(0,T),V_0)}^2 \leq c_1 T R_1^{\frac{1}{d}} \|w_\varepsilon\|_{(L^2(0,T),V_0)}^2. \quad (3.32)$$

Since  $w_\varepsilon$  converges strongly in  $V$  to  $w = u - v$  a weak solution of the conventional Navier stokes [9, Theorem 3.9.] for  $l \geq \frac{d}{2}$ , using 2) in Lemma3.6 we can show that there exists a positive constant  $\varepsilon'$ , such that at the limit we find, there exists a real parameter  $\alpha$ , such that for all  $\alpha > 0$

$$\|w\|_{(L^2(0,T),V_0)}^2 \leq c_1 T R_1^{\frac{1}{d}} \|w_\varepsilon\|_{(L^2(0,T),V_0)}^2 + \alpha \text{ for each } \varepsilon \leq \varepsilon'. \quad (3.33)$$

The inequality (3.33) describes a relationship between solutions of two systems (3.1) and (3.3). We will use this relationship to define a mapping between  $\mathfrak{S}_\varepsilon$  and  $\mathfrak{S}_0$ . Now, let  $\Phi_\alpha$  be a function maps  $\mathfrak{S}_\varepsilon$  into  $\mathfrak{S}_0$ . The function  $\Phi_\alpha$  is defined as follows

$$\Phi_\alpha(u_\varepsilon) = u. \quad (3.34)$$

The inequality (3.33) means that  $\Phi_\alpha$  also satisfies

$$\|\Phi_\alpha(v_\varepsilon) - \Phi_\alpha(u_\varepsilon)\|_{(L^2(0,T),V_0)} \leq k \|v_\varepsilon - u_\varepsilon\|_{(L^2(0,T),V_0)} + \alpha, \quad (3.35)$$

where

$$k = c_1 T R_1^{\frac{1}{d}}. \quad (3.36)$$

For all  $\varepsilon > 0$ , there exists a  $\eta \geq 0$ , such that for all  $v_\varepsilon, u_\varepsilon \in \mathfrak{S}_\varepsilon$  satisfy

$$\|v_\varepsilon - u_\varepsilon\|_{(L^2(0,T),V_0)} \leq \eta \quad (3.37)$$

we get

$$\|\Phi_\alpha(v_\varepsilon) - \Phi_\alpha(u_\varepsilon)\|_{(L^2(0,T),V_0)} \leq \epsilon. \quad (3.38)$$

We have taken  $\alpha = \epsilon/2$  and  $k\eta = \epsilon/2$ . This shows that  $\Phi_\alpha$  is uniformly continuous.  $\square$

We need the following result

**Proposition 3.8.** *For  $l \geq \frac{d}{2}$  in each  $d \leq 4$  and for all  $\varepsilon \leq \varepsilon'$ , the set  $\mathfrak{S}_\varepsilon$  is dense in  $\mathfrak{S}_0$ .*

*Proof.* By subtracting (3.3) from (3.2), we obtain for  $\xi = u_\varepsilon(t) - u(t)$

$$\partial_t \xi + \varepsilon A^l u_\varepsilon + A\xi + B(u_\varepsilon, u_\varepsilon) - B(u, u) = 0. \quad (3.39)$$

By taking inner product with  $\xi$  for the above equation we get

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \varepsilon (A^l u_\varepsilon, \xi) + \nu \|A^{\frac{1}{2}} \xi\|^2 = b(\xi, \xi, u_\varepsilon). \quad (3.40)$$

The second term of left hand side of equation (3.40) can be written as

$$\begin{aligned} \varepsilon (A^l u_\varepsilon, \xi) &= \varepsilon (A^l u_\varepsilon, u_\varepsilon) - \varepsilon (A^l u_\varepsilon, u) \\ &= \varepsilon \|A^{\frac{1}{2}} u_\varepsilon\|^2 - \varepsilon (A^l u_\varepsilon, u). \end{aligned} \quad (3.41)$$



This leads to

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \varepsilon \|A^{\frac{1}{2}} u_\varepsilon\|^2 + \nu \|A^{\frac{1}{2}} \xi\|^2 = \varepsilon (A^l u_\varepsilon, u) + b(\xi, \xi, u_\varepsilon). \quad (3.42)$$

By applying Young's inequality, we have

$$(A^l u_\varepsilon, u) \leq \frac{1}{2\nu} \|A^l u_\varepsilon\|^2 + \frac{\nu}{2} \|u\|^2. \quad (3.43)$$

The trilinear term can be estimated as in [4, 4.167]

$$|b(\xi, \xi, u_\varepsilon)| \leq \|\xi\|^2 \|u_\varepsilon\|_1^{1/\theta}. \quad (3.44)$$

Combining all these inequalities in (3.42), we have

$$\frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq \|\xi\|^2 \|u_\varepsilon\|_1^{\frac{1}{\theta}} + \frac{\varepsilon}{2\nu} \|A^l u_\varepsilon\|^2 + \frac{\varepsilon\nu}{2} \|u\|^2. \quad (3.45)$$

Applying now Gronwall's inequality to (3.45), for  $t \geq 0$  we get

$$\begin{aligned} \|\xi(t)\|^2 &\leq \|\xi(0)\|^2 \exp c_4 \left( \int_0^t \|u_\varepsilon\|_1^{1/\theta} ds \right) \\ &\quad + \frac{\varepsilon}{\nu} \int_0^t \left( \frac{1}{2\nu} \|A^l u_\varepsilon\|^2 + \frac{\nu}{2} \|u\|^2 \right) \exp c_4 \left( \int_0^t \|u_\varepsilon\|_1^{1/\theta} ds \right) dh, \end{aligned} \quad (3.46)$$

using estimate (3.22) we find

$$\begin{aligned} \|\xi(t)\|^2 &\leq \|\xi(0)\|^2 \exp c_4 T R_1^{\frac{1}{\theta}} \\ &\quad + \frac{\varepsilon \exp c_4 T R_1^{\frac{1}{\theta}}}{\nu} \int_0^t \left( \frac{1}{2\nu} \|A^l u_\varepsilon\|^2 + \frac{\nu}{2} \|u\|^2 \right) dh. \end{aligned} \quad (3.47)$$

Since  $\xi(0) = u_\varepsilon(0) - u(0) = 0$ , this means

$$\|\xi(t)\|^2 \leq \varepsilon \frac{\exp c_4 T R_1^{\frac{1}{\theta}}}{\nu} \int_0^t \left( \frac{1}{2\nu} \|A^l u_\varepsilon\|^2 + \frac{\nu}{2} \|u\|^2 \right) dh \quad (3.48)$$

this means that for each  $u \in \mathfrak{S}_0$  there exists a sequence  $u_\varepsilon \in \mathfrak{S}_\varepsilon$  such that  $\|u_\varepsilon - u\| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , it follows that  $\mathfrak{S}_\varepsilon$  is dense in  $\mathfrak{S}_0$ .  $\square$

Since the uniqueness of weak solutions of the Navier–Stokes equations is not known. It may happen that  $\mathfrak{S}_0 = S(0, f, u_0)$  the set of solutions of the conventional Navier stokes equations (3.3) is not reduced to one point. there can exist “ghost solutions” that solve the original problem, but are not limits of the perturbed problem. For this, we consider two subsets  $A$  and  $B$  such that  $\mathfrak{S}_0 = A \cup B$ .

We say that  $u \in A$  if  $u$  is a limit of the perturbed problem and  $u \in B$  if  $u$  is not limit of the perturbed problem. We clearly have  $\overline{A} \cap B = \emptyset$ , it follows that  $d_V(A, B) > 0$  ( $d$  is the Hausdorff distance).

**Proposition 3.9.** *For  $l \geq \frac{d}{2}$  in each  $d \leq 4$  and for all  $\varepsilon \leq \varepsilon'$ , the two sets  $\mathfrak{S}_\varepsilon$  and  $\mathfrak{S}_0$  are homeomorphic.*

*Proof.* Since the limits solutions are in  $A$

$$A = \Phi_\alpha(\mathfrak{S}_\varepsilon). \quad (3.49)$$

$A$  is a compact set. Thus,  $\mathfrak{S}_\varepsilon$  is dense in  $\mathfrak{S}_0$ ,  $\mathfrak{S}_\varepsilon$  is dense in  $A \cup B$ , then for each  $u' \in B$

$$d(u', A) = d(u', u) \leq d(u', u_\varepsilon) + d(u_\varepsilon, u) \leq \varepsilon, \quad (3.50)$$

it follows that  $u' \in \overline{A}$ , but  $A = \overline{A}$ , we conclude that  $A = B$ . Consequently,

$$\Phi_\alpha(\mathfrak{S}_\varepsilon) = \mathfrak{S}_0 = A \text{ for all } \varepsilon \leq \varepsilon' \quad (3.51)$$

and  $\Phi_\alpha$  is surjective from  $\mathfrak{S}_\varepsilon$  into  $\mathfrak{S}_0$ . The injectivity is a consequence of the uniqueness of the solution of the perturbed system. Since  $\mathfrak{S}_\varepsilon$  is a compact set,  $\Phi_\alpha$  is a homeomorphism.  $\square$

Our main result is then

**Theorem 3.10.** *For each dimension  $d \leq 4$ , the Navier-Stokes system (3.3) has a unique weak solution.*

*Proof.* For  $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$  in each  $d \leq 4$  and for fixed  $\varepsilon \leq \varepsilon'$ , the set  $\mathfrak{S}_\varepsilon$  is reduced to one point and From the Proposition 3.9  $\Phi_\alpha$  is a homeomorphism for all  $\alpha > 0$ , that is

$$\mathfrak{S}_0 = \Phi_\alpha(\mathfrak{S}_\varepsilon) = \{u\}. \quad (3.52)$$

Thus, the problem (3.3) has a unique solution.  $\square$

Finally we finish by a regularity theorem

**Theorem 3.11.** *Assume that  $l \geq \frac{d}{2}$  in each  $d \leq 4$ , then there exists a finite time  $T^*$  depending on the data such that for any  $t \leq T^*$  there exists a unique solution of (3.3) satisfying*

$$u \in L^2(0, T^*; V_l) \cap L^\infty(0, T^*; V_0). \quad (3.53)$$

*Proof.* Let  $C$  be a nonempty subset of a Banach space  $X$  and fix a sequence  $\{a_n\}$  in  $[0, \infty)$  with  $a_n \rightarrow 0$ . A mapping  $\Theta : C \rightarrow C$  will be called nearly Lipschitzian with respect to  $\{a_n\}$  if for each  $n \in \mathbb{N}$ , there exists a constant  $k_n \geq 0$  such that

$$\|\Theta^n(u_\varepsilon) - \Theta^n(v_\varepsilon)\| \leq k_n(\|u_\varepsilon - v_\varepsilon\| + a_n) \text{ for all } u_\varepsilon, v_\varepsilon \in C, \quad (3.54)$$

(cf. D. R. Sahu [7]). The infimum of constants  $k_n$  for which (3.54) holds will be called nearly Lipschitz constant. A nearly Lipschitzian mapping  $\Phi_{\alpha_n}$  with sequence  $\{(\alpha_n, k_n)\}$  is said to be nearly contraction if

$$\lim_{n \rightarrow \infty} (k_n)^{\frac{1}{n}} < 1 \text{ for all } n \in \mathbb{N}. \quad (3.55)$$

From (3.55) we find that

$$\|\Phi_{\alpha_n}^n(u_\varepsilon) - \Phi_{\alpha_n}^n(v_\varepsilon)\| \leq k^n \|u_\varepsilon - v_\varepsilon\| + a_n \text{ for all } u_\varepsilon, v_\varepsilon \in \mathfrak{S}_\varepsilon.$$

with

$$a_n = \alpha_n(1 + k + \dots + k^n). \quad (3.56)$$

We assume that  $\alpha_n = k^{2n}$ , then

$$\|\Phi_{\alpha_n}^n(u_\varepsilon) - \Phi_{\alpha_n}^n(v_\varepsilon)\| \leq k^n(\|u_\varepsilon - v_\varepsilon\| + R_n), \quad (3.57)$$

where

$$R_n = k^{n+1} + k^{n+2} + \dots + k^{2n+1}. \quad (3.58)$$

Note that (3.55) implies that there exists a  $T^* < c_2 R_1^{-\frac{1}{2\theta}}$ , such that for any  $t \leq T^*$

$$k < 1. \quad (3.59)$$

Since  $R_n$  is the  $n^{\text{th}}$  remainder of a convergent series, it follows that

$$\lim_{n \rightarrow \infty} R_n = 0. \quad (3.60)$$

Thus  $\Phi_{\alpha_n}$  is a nearly contracting Lipschitz map on  $(L^2(0, T^*), V_0)$ . As a result,  $\Phi_{\alpha_n}$  has a unique fixed point on  $L^2((0, T^*), V_0)$  such that

$$\Phi_\alpha(u) = u_\varepsilon = u. \quad (3.61)$$

This fixed point is the limit of the solution of equations (3.1) as  $\varepsilon$  goes to zero on  $(0, T^*)$ , and because of the contraction property, this limit is uniquely determined. So the theorem is proved.  $\square$

This result gives a new way to the uniqueness and regularity of the Navier stokes equations. This method can also be applied to other nonlinear partial differential equations to study the questions of the uniqueness and regularity of solutions.

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