Asymptotic behaviour of the number of the Eulerian circuits

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Abstract

We determine the asymptotic behaviour of the number of the Eulerian circuits in undirected simple graphs with large second eigenvalue of the Laplacian matrix (the algebraic connectivity). We also prove some new properties of the Laplacian matrix.

1. Introduction

Let D be a directed graph with vertex set VD and edge set ED. A walk of length m on D is a sequence $v_0, v_1, \dots, v_m$ of vertices of D such that $(v_{i-1}, v_i) \in ED$ for $1 \le i \le n$. This walk is a circuit if $v_m = v_0$ and $(v_{i-1}, v_i) \neq (v_{j-1}, v_j)$ for $1 \le i < j \le m$. The Eulerian circuits in D are circuits of length $|ED|$. Two Eulerian circuits are called equivalent if one is a cyclic permutation of the other. It is clear that the size of such an equivalence class equals the common length of the walks in the class. Let $Eul(D)$ denote the number of equivalence classes of the Eulerian circuits in D.

If G is an undirected graph, we can define the concepts of walk, circuit, the Eulerian circuit and $Eul(G)$ in the same way. The only clarification needed is that a walk $v_0, v_1, \dots, v_m$ is a circuit only if $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$ for $1 \le i < j \le m$; i.e., edges may be traversed in one direction only. Also note that an eulerian circuit and its reverse are counted separately for $n > 1$ in an undirected graph.

Consider the problem of counting the number of the Eulerian circuits in an undirected graph. As the appearance of Euler's name suggests, this problem is one of the oldest in graph theory. In his 1736 paper on the famous Königsberg Bridges Problem, see [4], Euler proved that $Eul(G) = 0$ if there is a vertex $v \in VG$ with odd degree. English translation of this paper can be found in [2].

Even in the case of $G = K_n$ the problem of counting $Eul(G)$ has a long history. The value of $Eul(K_7)$ was calculated for the first time in [9] as the number of legal circular arrangements of the 21 dominoes (doubles excluded) over the set $\{1, \dots, 7\}$. An asymptotic formula for $Eul(K_n)$ as $n \to \infty$ with n odd is given in [8]. (For more detailed history for the case of $G = K_n$ see, for example, [8] and references therein.)

It was shown in [11] that for any graph G the number of the Eulerian circuits can be written as the solution of a triangular set of linear equations. However, the number of equations is extremely large for the dense graphs.

The problem of counting the number of the Eulerian circuits in an undirected graph is complete for the class $\#P$, see [3]. Thus this problem is difficult in terms of complexity theory.

In the present work we continue studies of [8]. We determine asymptotic behaviour of the number of the Eulerian circuits in simple graphs with all vertices of even degree as $n \to \infty$, where n is the number of the vertices. The asymptotic formula for $Eul(K_n)$ and our main result are presented and discussed in Section 2. In Section 3 we prove some basic properties of the Laplacian matrix, which may be of independent interest. In Section 4 we express $Eul(G)$ in terms of an n-dimensional integral using Cauchy's formula. The value of the integral is estimated in Sections 5 and 6, using some Lemmas proved in Section 8. We prove the main result in Section 7.

2. Asymptotic estimates of the number of the Eulerian circuits

In what follows we suppose that undirected graph G has no loops and multiple edges, i.e.

$$G \text{ is a connected simple graph.} \tag{2.1}$$

We also assume that

$$\text{all vertices of } G \text{ have even degrees.} \tag{2.2}$$

Define the $n \times n$ matrix Q by

$$Q_{jk} = \begin{cases} -1, & (v_j, v_k) \in EG, \\ d_j, & j = k, \\ 0, & \text{otherwise} \end{cases}, \tag{2.3}$$

where $n = |VG|$ and d_j is the degree of the vertex $v_j \in VG$. The matrix $Q = Q(G)$ is called the Laplacian matrix of the graph G. The eigenvalues $\lambda_0 \le \lambda_1 \le \ldots \le \lambda_{n-1}$ of the matrix Q are always non-negative real numbers and $\lambda_0 = 0$. The eigenvalue λ_1 is called the algebraic connectivity of the graph G. (For more information about the spectral properties of the Laplace matrix see, for example, [5] and [7].)

According to the Kirchhoff's Matrix-Tree-Theorem, see [6], we have that

$$t(G) = \frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1}, \tag{2.4}$$

where $t(G)$ denotes the number of spanning trees of the graph G.

Let $p \ge 1$ be a real number and $\mathbf{x} \in \mathbb{R}^n$. We use notation

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}. \tag{2.5}$$

For $p = \infty$ we have the maximum norm

$$\|\mathbf{x}\|_\infty = \max_j |x_j|. \tag{2.6}$$

The matrix norm corresponding to the p-norm for vectors is

$$\|A\|_p = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}. \tag{2.7}$$

We denote by $\|A\|_{HS}$ the Hilbert–Schmidt norm of the matrix A.

$$\|A\|_{HS} = \sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n}|A_{jk}|^2} \tag{2.8}$$

If f is bounded both above and below by g asymptotically, we use the notation

$$f(n) = \Theta_{k_1,k_2}\left(g(n)\right), \tag{2.9}$$

which implies as $n \to \infty$, eventually

$$k_1|g(n)| \le |f(n)| \le k_2|g(n)|. \tag{2.10}$$

When functions f and g depend not only on n, but also on other parameters $\boldsymbol{\xi}$, we use notation (2.9) meaning that condition (2.10) holds uniformly for all possible values of $\boldsymbol{\xi}$.

The main result of the present work is the following theorem.

Theorem 2.1. *Let matrix Q be the Laplacian matrix of graph G with n vertices. Let conditions (2.1), (2.2) hold and the algebraic connectivity $\lambda_1 \ge \sigma n$ for some $\sigma > 0$. Then as $n \to \infty$*

$$Eul(G) = \Theta_{k_1,k_2}\left(2^{E-\frac{n-1}{2}}\pi^{-\frac{n-1}{2}}\sqrt{t(G)}\prod_{j=1}^{n}\left(\frac{d_j}{2}-1\right)!\right), \tag{2.11}$$

where $E = |EG|$, d_j is the degree of the vertex v_j, $t(G)$ denotes the number of spanning trees of the graph G and constants $k_1, k_2 > 0$ depend only on σ.

Remark 2.1. We can replace condition $\lambda_1 \ge \sigma n$ for some $\sigma > 0$ in Theorem 2.1 by the condition that for some $\sigma > 1/2$ the degree of each vertex of G at least σn.

For the complete graph K_n one can show that $\lambda_1 = n$ and $t(K_n) = n^{n-2}$. The following theorem is the variation of Theorem 4 of [8].

Theorem 2.2. *As $n \to \infty$ with n odd*

$$Eul(K_n) = 2^{\frac{(n-1)^2}{2}}\pi^{-\frac{n-1}{2}}n^{\frac{n-2}{2}}\left(\left(\frac{n-1}{2}-1\right)!\right)^n\left(1+O(n^{-1/2+\epsilon})\right) \tag{2.12}$$

for any $\epsilon > 0$.

In fact, Theorem 2.2 is stronger than Theorem 2.1 in the case of $G = K_n$. However, the asymptotic estimate of Theorem 2.1 holds for considerably broader class of graphs.

3. Some basic properties of the Laplacian matrix

Consider the graph G such that conditions (2.1), (2.2) hold. The Laplacian matrix Q has the eigenvector $[1, 1, \ldots, 1]^T$, corresponding to the eigenvalue $\lambda_0 = 0$. We use notation $\hat{Q} = Q + J$, where J denotes the matrix with every entry 1. Note that Q and $\hat{Q}$ have the same set of eigenvectors and eigenvalues, except for the eigenvalue corresponding to the eigenvector $[1, 1, \ldots, 1]^T$, which equals 0 for Q and n for $\hat{Q}$.

Since the spectral norm is bounded above by any matrix norm we get that

$$\lambda_{n-1} = ||Q||_2 \le ||\hat{Q}||_2 \le ||\hat{Q}||_1 = \max_j \sum_{k=1}^{n} |\hat{Q}_{jk}| = n. \tag{3.1}$$

We denote by G_r the graph which arises from G by removing vertices $v_1, v_2, \dots, v_r$ and all adjacent edges.

Lemma 3.1. *Let conditions (2.1), (2.2) hold for graph G with n vertices. Then*

$$\lambda_1(G) \le \frac{n}{n-1} \min_j d_j, \tag{3.2}$$

$$\lambda_1(G) \ge 2 \min_j d_j - n + 2, \tag{3.3}$$

$$\lambda_1(G_r) \ge \lambda_1(G) - r, \tag{3.4}$$

where $\lambda_1(G)$ is the algebraic connectivity of G and d_j is the degree of the vertex $v_j \in VG$.

The proof of Lemma 3.1 can be found in [5].

Lemma 3.2. *Let the assumptions of Theorem 2.1 hold. Then there is a constant $c_\infty > 0$ depending only on σ such that*

$$||\hat{Q}^{-1}||_1 = ||\hat{Q}^{-1}||_\infty \le \frac{c_\infty}{n}. \tag{3.5}$$

Proof of Lemma 3.2. We consider $\mathbf{x} \in \mathbb{R}^n$ such that $||\mathbf{x}||_\infty = 1$. For simplicity, we assume that $|x_1| = 1$. We denote by J_σ the set of the indices j such that $|x_j| \ge \sigma/8$.

In the case of $|J_\sigma| \ge \sigma n/4$ we have that

$$||\mathbf{x}||_2 \ge \sqrt{\frac{\sigma^2}{64} \sigma n/4}. \tag{3.6}$$

Since the algebraic connectivity $\lambda_1 \ge \sigma n$, we get that

$$\sqrt{n||\hat{Q}\mathbf{x}||_\infty^2} \ge ||\hat{Q}\mathbf{x}||_2 \ge \lambda_1 ||\mathbf{x}||_2 \ge \sigma n ||\mathbf{x}||_2 \ge \sigma n \sqrt{\frac{\sigma^3 n}{256}} \tag{3.7}$$

In the case of $|J_\sigma| \le \sigma n/4$ we have that

$$\begin{aligned} ||\hat{Q}\mathbf{x}||_\infty &\ge (d_1 + 1)|x_1| - \sum_{j=2}^{n} |x_j| \ge \\ &\ge d_1 + 1 - \sum_{j \in J_\sigma} |x_j| - \sum_{j \notin J_\sigma} |x_j| \ge \\ &\ge d_1 + 1 - \sigma n/4 - n\sigma/8. \end{aligned} \tag{3.8}$$

Using again $\lambda_1 \ge \sigma n$ and (3.2) we get that

$$d_1 \ge \min_j d_j \ge \frac{n-1}{n} \sigma n \ge \sigma n/2. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) we obtain that

$$||\hat{Q}\mathbf{x}||_\infty \geq c_\infty^{-1} n ||\mathbf{x}||_\infty, \tag{3.10}$$

for some constant $c_\infty > 0$ depending only on σ. ∎

The following lemmas will be applied to estimate the determinant of a matrix close to the identity matrix I.

Lemma 3.3. *Let X be an $n \times n$ matrix such that $\|X\|_2 < 1$. Then for fixed $m \geq 2$*

$$\det(I + X) = \exp\left(\sum_{r=1}^{m-1} \frac{(-1)^{r+1}}{r} tr(X^r) + E_m(X)\right), \tag{3.11}$$

where tr is the trace function and

$$|E_m(X)| \leq \frac{n}{m} \frac{\|X\|_2^m}{1 - \|X\|_2}. \tag{3.12}$$

Lemma 3.3 was also formulated and proved in [8].

Lemma 3.4. *Let the assumptions of Lemma 3.3 hold and all eigenvalues of X are non-negative real numbers. Then*

$$\det(I - X) \geq \exp\left(-\frac{tr(X)}{1 - \|X\|_2}\right). \tag{3.13}$$

Proof of Lemma 3.4. Using Lemma 3.3 we get that

$$\det(I - X) = \exp\left(\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} (-1)^r tr(X^r)\right) \tag{3.14}$$

Since all eigenvalues of X are non-negative real numbers

$$0 \leq tr(X^r) \leq tr(X) \|X\|_2^{r-1}. \tag{3.15}$$

Hence

$$\det(I - X) \geq \exp\left(-\sum_{r=1}^{\infty} \frac{tr(X)}{r} \|X\|_2^{r-1}\right) \geq \exp\left(-\frac{tr(X)}{1 - \|X\|_2}\right). \tag{3.16}$$

∎

Lemma 3.5. *Let the assumptions of Theorem 2.1 hold. Then there is a constant $c_1 > 0$ depending only on σ such that*

$$|\det M_{11}| \leq c_1 \frac{\det \hat{Q}}{n}, \tag{3.17}$$

where M_{11} denotes the $(n-1) \times (n-1)$ matrix that results from deleting the first row and the first column of $\hat{Q} = Q + J$.

Proof of Lemma 3.5. Since the algebraic connectivity $\lambda_1 \geq \sigma n$, using (3.2), we get the following estimate for the degree d_k of the vertex $v_k \in VG$.

$$d_k \geq \min_j d_j \geq \frac{n-1}{n}\sigma n \geq \sigma n/2. \tag{3.18}$$

Consider the $n \times n$ matrix X such that

$$X_{jk} = \begin{cases} \frac{1}{d_1+1}, & \text{if } (v_1, v_j) \in EG \text{ and } (v_1, v_k) \in EG, \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

After performing one step of the Gaussian elimination for $\hat{Q} + X$, we obtain that

$$\det(\hat{Q} + X) = (d_1 + 1) \det M_{11}, \tag{3.20}$$

Since the spectral norm is bounded above by any matrix norm, we get that

$$||X||_2 \leq ||X||_1 = 1. \tag{3.21}$$

Since $\lambda_1 \geq \sigma n$, taking into account (3.1), we obtain that

$$||X\hat{Q}^{-1}||_2 \leq ||X||_2 ||\hat{Q}^{-1}||_2 \leq \frac{1}{\lambda_1} \leq \frac{1}{\sigma n} \tag{3.22}$$

Combining Lemma 3.3 with (3.22), we get that as $n \to \infty$

$$\det\left(I + X\hat{Q}^{-1}\right) = \exp\left(tr\left(X\hat{Q}^{-1}\right) + E_2\left(X\hat{Q}^{-1}\right)\right) \leq \exp\left(n\frac{1}{\sigma n} + O(n^{-1})\right). \tag{3.23}$$

From (3.20) and (3.23) we have that as $n \to \infty$

$$(d_1 + 1) \det M_{11} = \det\left(I + X\hat{Q}^{-1}\right) \det \hat{Q} \leq \det \hat{Q} \exp\left(1/\sigma + O(n^{-1})\right). \tag{3.24}$$

Since $\hat{Q}$ is positive definite, using (3.18) in (3.24), we obtain (3.17). ∎

Lemma 3.6. *Let the assumptions of Theorem 2.1 hold. Let G_r be the graph which arises from G by removing vertices $v_1, v_2, \ldots, v_r$ and all adjacent edges. Then there is a constant $c_2 > 0$ depending only on σ such that for any $\epsilon \in (0, 1)$ and $r \leq n^{\epsilon}$*

$$\det \hat{Q}(G_r) \geq \frac{\det \hat{Q}(G)}{(c_2 n)^r}. \tag{3.25}$$

Proof of Lemma 3.6. We give first a proof for the case of $r = 1$. For our purpose it is convenient to use notations $\hat{Q} = \hat{Q}(G)$ and $\hat{Q}_1 = \hat{Q}(G_1)$. Note that the matrix M_{11} that results from deleting the first row and the first column of $\hat{Q}$ coincides with the matrix $\hat{Q}_1$ with the exception of the diagonal elements. In a similar way as (3.20) we get that

$$\det(\hat{Q} + \Omega + X) = (d_1 + 1) \det \hat{Q}_1, \tag{3.26}$$

where X is such that

$$X_{jk} = \begin{cases} \frac{1}{d_1+1}, & \text{if } (v_1, v_j) \in EG \text{ and } (v_1, v_k) \in EG, \\ 0, & \text{otherwise.} \end{cases} \tag{3.27}$$

and Ω is such diagonal matrix that

$$\Omega_{jj} = \begin{cases} 1, & \text{if } (v_1, v_j) \in EG, \\ 0, & \text{otherwise.} \end{cases} \tag{3.28}$$

Taking into account (3.21), we have that

$$||\Omega + X||_2 \leq ||\Omega||_2 + ||X||_2 \leq 2. \tag{3.29}$$

In a similar way as (3.22) we get that

$$||(\Omega + X)\hat{Q}^{-1}||_2 \leq ||\Omega + X||_2 ||\hat{Q}^{-1}||_2 \leq 2\frac{1}{\lambda_1} \leq \frac{2}{\sigma n} \tag{3.30}$$

Combining Lemma 3.3 with (3.30), we get that as $n \to \infty$

$$\begin{aligned} \det\left(I + (\Omega + X)\hat{Q}^{-1}\right) &= \exp\left(tr\left((\Omega + X)\hat{Q}^{-1}\right) + E_2\left((\Omega + X)\hat{Q}^{-1}\right)\right) \geq \\ &\geq \exp\left(-n\frac{2}{\sigma n} + O(n^{-1})\right). \end{aligned} \tag{3.31}$$

From (3.26) and (3.31) we have that as $n \to \infty$

$$(d_1 + 1)\det\hat{Q}_1 = \det\left(I + (\Omega + X)\hat{Q}^{-1}\right)\det\hat{Q} \geq \det\hat{Q}\exp\left(-2/\sigma + O(n^{-1})\right). \tag{3.32}$$

Since $d_1 + 1 \leq n$ we get (3.25) for the case of $r = 1$.

Taking into account (3.4) and using r times (3.32) we get (3.25) for the general case. ■

According to (2.4), we have that

$$t(G) = \frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1} = \frac{\det\hat{Q}}{n^2}, \tag{3.33}$$

where $t(G)$ denotes the number of spanning trees of the graph G.

Lemma 3.7. *Let the assumptions of Theorem 2.1 hold. Then for some $c_3 > 0$ depending only on σ the number of spanning trees of the graph G with maximum degree greater than d is less then $c_3^n \det\hat{Q}/d!$ for all $d \geq 0$.*

Proof of Lemma 3.7. According to Lemma 5 of [8] the number of labelled trees on n vertices with first vertex having degree greater than d is less than $2n^{n-2}/d!$ for all $d \geq 0$. We have that

$$\det\hat{Q} \geq \lambda_1^n \geq (\sigma n)^n. \tag{3.34}$$

To complete proof it remains to note that the number of spanning trees with maximum degree greater than d in G does not exceed the number of such spanning trees in the complete graph with n vertices. ■

Consider a spanning tree T of the graph G. We denote by G_T the graph which arises from G by removing all edges of the tree T.

Lemma 3.8. *Let the assumptions of Theorem 2.1 hold. Let T be the spanning tree of G with the maximum degree at most $\sigma n/4$. Then the algebraic connectivity $\lambda_1(G_T) \geq \sigma n/2$ and*

$$\det \hat{Q}(G_T) \geq c_4 \det \hat{Q}(G) \tag{3.35}$$

for some $c_4 > 0$ depending only on σ.

Proof of Lemma 3.8. Note that

$$\hat{Q}(G_T) = \hat{Q}(G) - Q(T). \tag{3.36}$$

Since the spectral norm is bounded above by any matrix norm and the maximum degree of vertex of T at most $\sigma n/4$ we get that

$$||Q(T)||_2 \leq ||Q(T)||_1 \leq \sigma n/2. \tag{3.37}$$

Therefore, since the algebraic connectivity $\lambda_1(G) \geq \sigma n$, we have that

$$\lambda_1(G_T) \geq \lambda_1(G) - ||Q(T)||_2 \geq \sigma n/2 \tag{3.38}$$

and

$$\det \hat{Q}(G_T) = \det \hat{Q}(G) \det(I - X), \tag{3.39}$$

where $X = Q(T)\hat{Q}(G)^{-1}$. Note that $\hat{Q}(G)$ is the matrix of positive definite quadratic form and $Q(T)$ is the matrix of quadratic form with non-negative eigenvalues. Considering the basis in which both matrices are diagonal, we have that

$$tr(Q(T)\hat{Q}(G)^{-1}) \leq tr(Q(T))||\hat{Q}(G)^{-1}||_2 \tag{3.40}$$

and

$$\text{all eigenvalues of } Q(T)\hat{Q}(G)^{-1} \text{ are non-negative.} \tag{3.41}$$

Using again the fact that the algebraic connectivity $\lambda_1(G) \geq \sigma n$ and (3.37) we get that

$$||X||_2 \leq ||Q(T)||_2||\hat{Q}(G)^{-1}||_2 \leq \frac{\sigma n}{2}\frac{1}{\sigma n} = \frac{1}{2}. \tag{3.42}$$

Since T is the spanning tree $tr(Q(T)) = 2(n-1)$. Using (3.40), we get that

$$tr(X) \leq tr(Q(T))||\hat{Q}(G)^{-1}||_2 \leq 2n\frac{1}{\sigma n} = \frac{2}{\sigma}. \tag{3.43}$$

To complete the proof it remains to combine Lemma 3.4 with (3.39), (3.42) and (3.43). ∎

4. The result expressed as an integral

The reasoning below is similar to the arguments of Section 2 of [8]

An Eulerian orientation of G is an orientation of its edges with the property that for every vertex both the in-degree and the out-degree are equal. Any Eulerian circuit induces an Eulerian orientation by orienting each edge in accordance with its direction of traversal.

A directed tree with root v is a connected directed graph T such that $v \in VT$ has out-degree zero, and each other vertex has out-degree one. Thus, T is a tree which has each edge oriented towards v.

Let D be a directed graph with n vertices, and let $v \in VD$. A directed spanning tree of D with root v is a spanning subgraph of D which is a directed tree with root v.

The following famous theorem, sometimes called the BEST Theorem, is due to de Bruijn, van Aardenne-Ehrenfest, Smith, and Tutte [1, 10].

Theorem 4.1. *Let D be a directed graph with vertices $v_1, v_2, \ldots, v_n$. Suppose that there are numbers $d_1, d_2, \ldots, d_n$ such that, for every vertex v_r, both the in-degree and the out-degree of v_r are equal to d_r. Let $t_r = t_r(D)$ be the number of directed spanning trees of D rooted at v_r. Then t_r is independent of i, and*

$$Eul(D) = t_r \prod_{j=1}^{n} (d_j - 1)!. \tag{4.1}$$

Consider the undirected graph G with n vertices such that conditions (2.1), (2.2) hold. Note that for every spanning tree T of the graph G and any vertex $v_r \in VG$ there is only one orientation of the edges of T such that we obtain a directed tree with root v_r. We denote by $\mathcal{T}_r$ the set of directed trees with root v_r obtained in such a way. For $T \in \bigcup_{r=1}^{n} \mathcal{T}_r$ denote by $EO(T)$ the number of the Eulerian orientations of G that the corresponding graphs contain T.

From Theorem 4.1 in the case of a graph D corresponding to the Eulerian orientation of the graph G we find that

$$Eul(D) = t_r(D) \prod_{j=1}^{n} \left(\frac{d_j}{2} - 1\right)!, \tag{4.2}$$

where d_j is the degree of the vertex $v_j \in VG$. Let denote by $\mathcal{EO}$ the set of all graphs corresponding to the Eulerian orientations of the graph G. Grouping the Eulerian circuits according to the induced orientations, we obtain that

$$Eul(G) = \sum_{D \in \mathcal{EO}} Eul(D) = \prod_{j=1}^{n} \left(\frac{d_j}{2} - 1\right)! \sum_{D \in \mathcal{EO}} t_r(D) \tag{4.3}$$

for any fixed natural number $r \leq n$.

Regrouping the terms of the final summation according to the directed subtrees rooted at v_r, we find that

$$Eul(G) = \prod_{j=1}^{n} \left(\frac{d_j}{2} - 1\right)! \sum_{T \in \mathcal{T}_r} EO(T). \tag{4.4}$$

For $n \geq 1$ and $R \geq 0$ we use notation $U_n(R) = \{(x_1, x_2, \ldots, x_n) \mid |x_i| < R \text{ for all } i\}$. The value of $EO(T)$ is the constant term in

$$\prod_{(v_j, v_k) \in EG} ({x_j}^{-1} x_k + {x_k}^{-1} x_j) \prod_{(v_j, v_k) \in ET} \frac{x_k^{-1} x_j}{({x_j}^{-1} x_k + {x_k}^{-1} x_j)}, \tag{4.5}$$

which we can extract via Cauchy's Theorem using the unit circle as a contour for each variable. Making the substitution $x_j = e^{i\theta_j}$ for each j, we find that

$$Eul(G) = \prod_{j=1}^{n} \big(\frac{d_j}{2} - 1\big)!\, 2^{|EG|-n+1}\pi^{-n} S, \tag{4.6}$$

where

$$S = \int\limits_{U_n(\pi/2)} \prod_{(v_j,v_k)\in EG} \cos\Delta_{jk} \sum_{T\in\mathcal{T}_r} \prod_{(v_j,v_k)\in ET} (1 + i\tan\Delta_{jk})\; d\boldsymbol{\theta}, \tag{4.7}$$

having put $\Delta_{jk} = \theta_j - \theta_k$ and using the fact that the integrand is unchanged by the substitutions $\theta_j \to \theta_j + \pi$ if condition (2.2) holds.

We approach the integral by first estimating it in the region that would turn out to be the asymptotically significant one. Then we bound the integral over the remaining regions and show that it is vanishingly small in comparison with the significant part.

5. The dominant part of the integral

In what follows, we fix some small constant $\epsilon > 0$. Define

$$V_0 = \{\boldsymbol{\theta} \in U_n(\pi/2) :\ |\theta_j - \bar\theta| \le n^{-1/2+\epsilon} \text{ where } \bar\theta = \frac{\theta_1 + \ldots + \theta_n}{n}\}, \tag{5.1}$$

and let S_0 denote the contribution to S of $\boldsymbol{\theta} \in V_0$. Since the integrand is invariant under uniform translation of all the θ_j's mod π, we can fix $\bar\theta = 0$ and multiply it by the ratio of its range π to the length $n^{-1/2}$ of the vector $\frac{1}{n}[1, 1, \ldots, 1]^T$. Thus we have that

$$S_0 = \pi n^{1/2} \int\limits_{L\cap V_0} \prod_{(v_j,v_k)\in EG} \cos\Delta_{jk} \sum_{T\in\mathcal{T}_r} \prod_{(v_j,v_k)\in ET} (1 + i\tan\Delta_{jk})\; dL, \tag{5.2}$$

where L denotes the orthogonal complement to the vector $[1, 1, \ldots, 1]^T$.

The sum over $\mathcal{T}_r$ in the integrand of (5.2) can be expressed as a determinant, according to the following theorem of [12].

Theorem 5.1. *Let w_{jk} $(1 \le j, k \le n,\ j \ne k)$ be arbitrary. Define the $n \times n$ matrix A by*

$$A_{jk} = \begin{cases} -w_{jk}, & \text{if } j \ne k, \\ \sum_{r\ne j} w_{jr}, & \text{if } k = j \end{cases}, \tag{5.3}$$

the sum being over $1 \le r \le n$ with $r \ne j$. For any r with $1 \le r \le n$, let M_r denote the principal minor of A formed by removing row r and column r. Then

$$\det M_r = \sum_{T} \prod_{(v_j,v_k)\in ET} w_{jk}, \tag{5.4}$$

where the sum is over all directed trees T with $VT = \{v_1, v_2, \ldots, v_n\}$ and root v_r.

Lemma 5.1. *Let the assumptions of Theorem 2.1 hold. Let $\hat{Q} = Q + J$, where J denotes the matrix with every entry 1. Then for $\boldsymbol{\theta} \in V_0$ as $n \to \infty$*

$$\sum_{r=1}^{n} \sum_{T \in \mathcal{T}_r} \prod_{(v_j,v_k) \in ET} (1 + i \tan \Delta_{jk}) = \frac{e^{i\boldsymbol{\theta}^T Q \boldsymbol{\alpha} + tr(\Lambda \hat{Q}^{-1})^2}}{n} \det \hat{Q} \Big(1 + O(n^{-1/2+3\epsilon}) \Big), \tag{5.5}$$

where $\boldsymbol{\alpha}$ denotes the vector composed of the diagonal elements of the matrix $\hat{Q}^{-1}$, Λ denotes the diagonal matrix whose diagonal elements are equal to the components of the vector $Q\boldsymbol{\theta}$.

Proof of Lemma 5.1. Define the $n \times n$ matrix B by

$$B_{jk} = \begin{cases} -\tan \Delta_{jk}, & \text{for } (v_j, v_k) \in EG, \\ \sum\limits_{l:(v_j,v_l) \in EG} \tan \Delta_{jl}, & \text{for } k = j, \\ 0 & \text{otherwise .} \end{cases} \tag{5.6}$$

Using Theorem 5.1 with the matrix $A = Q + iB$, we get that

$$\sum_{r=1}^{n} \sum_{T \in \mathcal{T}_r} \prod_{(v_j,v_k) \in ET} (1 + i \tan \Delta_{jk}) = \sum_{r=1}^{n} M_r, \tag{5.7}$$

where M_r denotes the principal minor of A formed by removing row r and column r. Since the vector $[1, 1, \ldots, 1]^T$ is the common eigenvector of the matrices Q and B, corresponding to the eigenvalue 0, we find that

$$\sum_{r=1}^{n} M_r = \frac{\det(\hat{Q} + iB)}{n}. \tag{5.8}$$

Note that for $\boldsymbol{\theta} \in V_0$

$$\Delta_{jk} = (\theta_j - \bar{\theta}) - (\theta_k - \bar{\theta}) \leq 2n^{-1/2+\epsilon}. \tag{5.9}$$

Since the spectral norm is bounded above by any matrix norm we get that

$$||B||_2 \leq ||B||_1 = \max_j \sum_{k=1}^{n} |B_{jk}| = O(n^{1/2+\epsilon}). \tag{5.10}$$

Let $\Phi = B\hat{Q}^{-1}$. Since the algebraic connectivity $\lambda_1 \geq \sigma n$, we get that

$$||\Phi||_2 \leq ||B||_2 ||\hat{Q}^{-1}||_2 \leq \frac{1}{\lambda_1} ||B||_2 = O(n^{-1/2+\epsilon}). \tag{5.11}$$

Using Lemma (3.3) with the matrix $i\Phi$, we find that as $n \to \infty$

$$\det(I + i\Phi) = \exp\left(tr(i\Phi) + \frac{tr(\Phi^2)}{2} + O(n^{-1/2+3\epsilon}) \right). \tag{5.12}$$

Let

$$B = B_{skew} + B_{diag}, \tag{5.13}$$

where B_{skew} is the skew-symmetric matrix and B_{diag} is the diagonal matrix. Since $\hat{Q}$ is the symmetric matrix

$$tr(B_{skew}\hat{Q}^{-1}) = 0. \tag{5.14}$$

Using (5.9), note that as $n \to \infty$

$$||B_{diag} - \Lambda||_2 = O(n^{-1/2+3\epsilon}), \tag{5.15}$$

where Λ denotes the diagonal matrix whose diagonal elements are equal to the components of the vector $Q\boldsymbol{\theta}$. Since the algebraic connectivity $\lambda_1 \geq \sigma n$, we get that as $n \to \infty$

$$\left|tr\left((B_{diag} - \Lambda)\hat{Q}^{-1}\right)\right| \leq n||B_{diag} - \Lambda||_2||\hat{Q}^{-1}||_2 = O(n^{-1/2+3\epsilon}). \tag{5.16}$$

Using (5.14) and (5.16), we obtain that as $n \to \infty$

$$tr(\Phi) = tr(B_{diag}\hat{Q}^{-1}) = tr(\Lambda\hat{Q}^{-1}) + O(n^{-1/2+3\epsilon}) = \boldsymbol{\theta}^T Q\boldsymbol{\alpha} + O(n^{-1/2+3\epsilon}), \tag{5.17}$$

where $\boldsymbol{\alpha}$ denotes the vector composed of the diagonal elements of the matrix $\hat{Q}^{-1}$.

Using the property of the trace function $tr(XY) = tr(YX)$, we have that

$$tr\left(\Phi^2\right) = tr\left((B_{skew}\hat{Q}^{-1})^2\right) + tr\left((B_{diag}\hat{Q}^{-1})^2\right) + 2\,tr\left(B_{skew}\hat{Q}^{-1}B_{diag}\hat{Q}^{-1}\right). \tag{5.18}$$

Since B_{skew} is the skew-symmetric matrix and $\hat{Q}^{-1}B_{diag}\hat{Q}^{-1}$ is the symmetric matrix

$$tr\left(B_{skew}\hat{Q}^{-1}B_{diag}\hat{Q}^{-1}\right) = 0. \tag{5.19}$$

One can show that

$$\begin{gathered} tr\left(X^2\right) \leq ||X||_{HS}^2, \\ ||XY||_{HS} \leq ||X||_{HS}||Y||_2. \end{gathered} \tag{5.20}$$

Therefore we get that

$$\left|tr\left((B_{skew}\hat{Q}^{-1})^2\right)\right| \leq ||B_{skew}\hat{Q}^{-1}||_{HS}^2. \tag{5.21}$$

Since the algebraic connectivity $\lambda_1 \geq \sigma n$, using (5.9), we obtain that as $n \to \infty$

$$||B_{skew}\hat{Q}^{-1}||_{HS} \leq ||\hat{Q}^{-1}||_2||B_{skew}||_{HS} \leq \frac{1}{\lambda_1}||B_{skew}||_{HS} = O(n^{-1/2+\epsilon}). \tag{5.22}$$

Using (5.9) and (5.15), we get that

$$\left|tr\left((B_{diag} - \Lambda)\hat{Q}^{-1}B_{diag}\hat{Q}^{-1}\right)\right| \leq n\frac{1}{\lambda_1^2}||(B_{diag} - \Lambda)||_2||B_{diag}||_2 = O(n^{-1+4\epsilon}) \tag{5.23}$$

and

$$\left|tr\left((B_{diag} - \Lambda)\hat{Q}^{-1}(B_{diag} - \Lambda)\hat{Q}^{-1}\right)\right| \leq n\frac{1}{\lambda_1^2}||(B_{diag} - \Lambda)||_2^2 = O(n^{-2+6\epsilon}). \tag{5.24}$$

Thus

$$tr\left((B_{diag}\hat{Q}^{-1})^2\right) = tr\left((\Lambda\hat{Q}^{-1})^2\right) + O(n^{-1+4\epsilon}). \tag{5.25}$$

Combining (5.18), (5.19), (5.21), (5.22) and (5.25), we obtain that

$$tr\left(\Phi^2\right) = tr\left((\Lambda\hat{Q}^{-1})^2\right) + O(n^{-1+4\epsilon}). \tag{5.26}$$

Using (5.17) and (5.26) in (5.12), we get that

$$\det(I+i\Phi)=\exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}+\frac{tr\left((\Lambda\hat{Q}^{-1})^2\right)}{2}+O(n^{-1/2+3\epsilon})\right). \tag{5.27}$$

Combining (5.7), (5.8) and (5.27), we obtain (5.5). ∎

We denote by $P(\boldsymbol{\theta})$ the orthogonal projection onto the space L.

Lemma 5.2. *Let the assumptions of Theorem 2.1 hold. Let $\hat{Q}=Q+J$, where J denotes the matrix with every entry 1. For positive constants a,b,c,d_1,d_2 let sequence of vectors $\{\boldsymbol{\alpha}_n\}$ and sequence of differentiable functions $\{R_n(\boldsymbol{\theta})\}$ be such that*

$$||\boldsymbol{\alpha}_n||_\infty\le c/n, \tag{5.28}$$

$$|R_n(\boldsymbol{\theta})|\le d_1\frac{\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}{n}, \tag{5.29}$$

$$R_n(\boldsymbol{\theta})=R_n(P(\boldsymbol{\theta})) \tag{5.30}$$

and for $\boldsymbol{\theta}\in U_n(\frac{4}{\sigma}n^{-1/2+\epsilon})$

$$\left\|\frac{\partial R_n(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right\|_\infty\le d_2\,n^{-1/2+\epsilon}. \tag{5.31}$$

For $n\ge 2$ define

$$J_n=\int\limits_{L\cap V_0}\exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}_{\mathbf{n}}-a\sum_{(v_j,v_k)\in EG}\Delta_{ij}^2-b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4+R_n(\boldsymbol{\theta})\right)dL. \tag{5.32}$$

Then as $n\to\infty$

$$J_n=\Theta_{k_1,k_2}\left(\pi^{\frac{n-1}{2}}a^{-\frac{n-1}{2}}n^{1/2}\Big/\sqrt{\det\hat{Q}}\right), \tag{5.33}$$

where constants $k_1,k_2>0$ depend only on a, b, c, d_1, d_2 and σ.

Lemma 5.2 is proved in Section 8.

Lemma 5.3. *Let the assumptions of Theorem 2.1 hold. Then as $n\to\infty$*

$$S_0=\Theta_{k_1,k_2}\left(2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}n^{-1}\sqrt{\det\hat{Q}}\right), \tag{5.34}$$

where constants $k_1,k_2>0$ depend only on σ.

Proof of Lemma 5.3. Using formula (5.2) with $r=1,2\ldots,n$ and summing, we obtain that

$$\begin{aligned}nS_0&=\sum_{r=1}^{n}\pi n^{1/2}\int\limits_{L\cap V_0}\prod_{(v_j,v_k)\in EG}\cos\Delta_{jk}\sum_{T\in\mathcal{T}_r}\prod_{(v_j,v_k)\in ET}(1+i\tan\Delta_{jk})\,dL=\\&=\pi n^{1/2}\int\limits_{L\cap V_0}\prod_{(v_j,v_k)\in EG}\cos\Delta_{jk}\sum_{r=1}^{n}\sum_{T\in\mathcal{T}_r}\prod_{(v_j,v_k)\in ET}(1+i\tan\Delta_{jk})\,dL.\end{aligned} \tag{5.35}$$

By Taylor's theorem we have that for $\boldsymbol{\theta} \in V_0$

$$\prod_{(v_j,v_k)\in EG} \cos \Delta_{jk} = \exp\left(-\frac{1}{2} \sum_{(v_j,v_k)\in EG} \Delta_{ij}^2 - \frac{1}{12} \sum_{(v_j,v_k)\in EG} \Delta_{ij}^4 + O(n^{-1+6\epsilon}) \right). \tag{5.36}$$

Combining (5.35) with (5.36) and Lemma 5.1, we obtain that as $n \to \infty$

$$S_0 \sim \pi \frac{\det \hat{Q}}{n^{3/2}} \int\limits_{L\cap V_0} \exp\left(i\, \boldsymbol{\theta}^T Q \boldsymbol{\alpha} - \frac{1}{2} \sum_{(v_j,v_k)\in EG} \Delta_{ij}^2 - \frac{1}{12} \sum_{(v_j,v_k)\in EG} \Delta_{jk}^4 + R(\boldsymbol{\theta}) \right) dL, \tag{5.37}$$

where $\boldsymbol{\alpha}$ denotes the vector composed of the diagonal elements of the matrix $\hat{Q}^{-1}$ and

$$R(\boldsymbol{\theta}) = tr(\Lambda \hat{Q}^{-1})^2, \tag{5.38}$$

where Λ is the diagonal matrix whose diagonal elements are equal to the components of the vector $Q\boldsymbol{\theta}$. Since vector $[1, 1 \ldots, 1]^T$ is the eigenvector of Q, corresponding to eigenvalue 0, we have that

$$Q\boldsymbol{\theta} = QP(\boldsymbol{\theta}). \tag{5.39}$$

Thus

$$R(\boldsymbol{\theta}) = R(P(\boldsymbol{\theta})). \tag{5.40}$$

Note that Lemma 3.5 implies as $n \to \infty$

$$||\boldsymbol{\alpha}||_\infty \leq c_1/n, \tag{5.41}$$

where $c_1 = c_1(\sigma) > 0$. Since the algebraic connectivity $\lambda_1 \geq \sigma n$, using (5.20), we get that

$$\begin{aligned} |R(\boldsymbol{\theta})| &\leq ||\Lambda \hat{Q}^{-1}||_{HS}^2 \leq ||\Lambda||_{HS}^2 ||\hat{Q}^{-1}||_2^2 = \\ &= ||Q\boldsymbol{\theta}||_2^2 \, ||\hat{Q}^{-1}||_2^2 \leq ||Q||_2^2 \, ||\hat{Q}^{-1}||_2^2 \, ||\boldsymbol{\theta}||_2^2 \leq \\ &\leq \lambda_{n-1}^2 \frac{1}{\lambda_1^2} ||\boldsymbol{\theta}||_2^2 \leq n^2 \frac{1}{\lambda_1^2} \frac{\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}}{\lambda_1} \leq \frac{1}{\sigma^3} \frac{\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}}{n}. \end{aligned} \tag{5.42}$$

Note that for $\boldsymbol{\theta} \in U_n(\frac{4}{\sigma} n^{-1/2+\epsilon})$ and for some $1 \leq k \leq n$

$$||\Lambda||_2 \leq \sum_{j=1}^{n} |\Delta_{jk}| = O\left(n^{1/2+\epsilon}\right). \tag{5.43}$$

For $1 \leq k \leq n$ we denote by $(\hat{Q}^{-1})_k$ the k-th column of the matrix $\hat{Q}^{-1}$. Using again $\lambda_1 \geq \sigma n$, we get that

$$1 = ||\hat{Q}(\hat{Q}^{-1})_k||_2 \geq \lambda_1 ||(\hat{Q}^{-1})_k||_2 \geq \sigma n ||(\hat{Q}^{-1})_k||_2. \tag{5.44}$$

Note that

$$\frac{\partial R}{\partial \theta_k} = 2tr(\frac{\partial \Lambda}{\partial \theta_k} \hat{Q}^{-1} \Lambda \hat{Q}^{-1}) = 2d_k (\hat{Q}^{-1} \Lambda \hat{Q}^{-1})_{kk} + 2tr(\tilde{\Lambda} \hat{Q}^{-1} \Lambda \hat{Q}^{-1}), \tag{5.45}$$

where $(\hat{Q}^{-1}\Lambda\hat{Q}^{-1})_{kk}$ denotes (k,k)-th element of the matrix $\hat{Q}^{-1}\Lambda\hat{Q}^{-1}$ and the matrix $\tilde{\Lambda}$ is such that for any $1 \le j \le n$ the diagonal element $|\Lambda_{jj}| \le 1$. Since the algebraic connectivity $\lambda_1 \ge \sigma n$, using (5.43) (5.44), we get that as $n \to \infty$

$$|tr(\tilde{\Lambda}\hat{Q}^{-1}\Lambda\hat{Q}^{-1})| \le n\,||\tilde{\Lambda}\hat{Q}^{-1}\Lambda\hat{Q}^{-1}||_2 \le n\,||\tilde{\Lambda}||_2\;||\Lambda||_2\;\frac{1}{\lambda_1^2} = O(n^{-1/2+\epsilon}) \tag{5.46}$$

and

$$2d_k(\hat{Q}^{-1}\Lambda\hat{Q}^{-1})_{kk} = 2d_k\Lambda_{kk}||(\hat{Q}^{-1})_k||_2^2 = O(n^{-1/2+\epsilon}). \tag{5.47}$$

Thus for some $d > 0$, depending only on σ

$$\left\| \frac{\partial R_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\|_\infty \le d\,n^{-1/2+\epsilon}. \tag{5.48}$$

Combining (5.37), (5.40), (5.41), (5.42), (5.48) and using Lemma 5.2 we obtain (5.34). ∎

6. The insignificant parts of the integral

In this section we prove that S_0 contributes almost all of S, even though it involves only a tiny part of the region of integration, compare with Section 4 of [8]. We continue to use the same value of ϵ as in the previous section.

Let assumptions of Theorem 2.1 hold. Define $E'T = \{(v_j, v_k), (v_k, v_j) \mid (v_j, v_k) \in ET\}$. We express the integrand of (4.7) as

$$F(\boldsymbol{\theta}) = \sum_{T \in T_r} \prod_{jk \in EG} f_{jk}(T, \boldsymbol{\theta}), \tag{6.1}$$

where

$$f_{jk}(T,\boldsymbol{\theta}) = \begin{cases} \cos\Delta_{jk}(1 + i\tan\Delta_{jk}), & (v_j, v_k) \in ET, \\ \cos\Delta_{jk}(1 - i\tan\Delta_{jk}), & (v_k, v_j) \in ET, \\ \cos\Delta_{jk}, & \text{otherwise.} \end{cases} \tag{6.2}$$

Note that $|f_{jk}(T,\boldsymbol{\theta})| \le 1$ for all values of the parameters. One can show that

$$|\cos(x)| \le \exp(-\frac{1}{2}x^2) \;\text{ for }\; |x| \le \frac{9}{16}\pi. \tag{6.3}$$

Divide the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ mod π into 32 equal intervals $H_0, \ldots, H_{31}$ such that $H_0 = [-\frac{1}{64}\pi, \frac{1}{64}\pi]$. For each j, define the region $W_j \subseteq U_n(\pi/2)$ as the set of points having at least $\frac{1}{32}n$ coordinates in H_j. Clearly, the W_j's cover $U_n(\pi/2)$ and also each W_j can be mapped to W_0 by a uniform translation of the θ_j mod π. This mapping preserves the integrand of (4.7) and also maps V_0 to itself, so we have that

$$\int\limits_{U_n(\pi/2)-V_0} |F(\theta)|d\boldsymbol{\theta} \le 32Z, \tag{6.4}$$

where

$$Z = \int\limits_{W_0-V_0} |F(\theta)|d\boldsymbol{\theta}. \tag{6.5}$$

We proceed by defining integrals $S_1, \dots, S_4$ in such a way that Z is obviously bounded by their sum. We then show that $S_j = o(S_0)$ for $j = 1, 2, 3, 4$ separately. Write

$$F(\boldsymbol{\theta}) = F_a(\boldsymbol{\theta}) + F_b(\boldsymbol{\theta}), \tag{6.6}$$

where $F_a(\theta)$ and $F_b(\theta)$ are defined by restricting the sum to trees with maximum degree greater than $\sigma n/4$ and no more than $\sigma n/4$, respectively. Also define regions V_1 and V_2 as follows.

$$\begin{aligned} V_1 &= \{\boldsymbol{\theta} \in W_0 \mid |\theta_j| \geq \frac{1}{32}\pi \text{ for fewer than } n^{\epsilon} \text{ values of } j\}, \\ V_2 &= \{\boldsymbol{\theta} \in V_1 \mid |\theta_j| \geq \frac{1}{16}\pi \text{ for at least one value of } j\}. \end{aligned} \tag{6.7}$$

Then our four integrals can be defined as

$$\begin{aligned} S_1 &= \int\limits_{W_0 - V_1} |F(\boldsymbol{\theta})| d\boldsymbol{\theta}, \\ S_2 &= \int\limits_{V_1} |F_a(\boldsymbol{\theta})| d\boldsymbol{\theta}, \\ S_3 &= \int\limits_{V_2} |F_b(\boldsymbol{\theta})| d\boldsymbol{\theta}, \\ S_4 &= \int\limits_{V_1 - V_2 - V_0} |F_b(\boldsymbol{\theta})| d\boldsymbol{\theta}. \end{aligned} \tag{6.8}$$

We begin with S_1. If $|\theta_j| \leq \frac{1}{64}\pi$ and $|\theta_k| \geq \frac{1}{32}\pi$ or vice versa, but $(v_j, v_k) \notin E'T$, we have that $|f_{jk}(T, \boldsymbol{\theta})| \leq \cos(\frac{1}{64}\pi)$. Since this includes more than $\frac{1}{32}n^{1+\epsilon} - n$ pairs (j, k), using (3.1) and (3.33), we get that as $n \to \infty$

$$S_1 \leq t(G)(2\pi)^n \left(\cos\frac{\pi}{64}\right)^{n^{1+\epsilon}/32 - n} = O\Big(\exp(-cn^{1+\epsilon})\Big) 2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} n^{-1} \sqrt{\det \hat{Q}} \tag{6.9}$$

for some constant $c > 0$ depending only on σ.

To bound S_2, we first note from Lemma 3.7 that the number of trees with maximum degree greater than $\sigma n/4$ is less than $c_3^n \det \hat{Q}/(\sigma n/4)!$. Using (6.3), we see that

$$|f_{jk}(T, \boldsymbol{\theta})| \leq \exp(-\frac{1}{2}\Delta_{jk}^2) \tag{6.10}$$

except for at most $n^{2\epsilon}$ pairs (j, k) with $|\Delta_{jk}| \geq \frac{1}{16}\pi$ and fewer than n pairs in $E'T$. In those excluded cases the value $\exp(-\frac{1}{2}\Delta_{jk}^2)$ may be high by a factor $\exp(\frac{1}{2}\pi^2)$. Hence, we have that

$$S_2 \leq \frac{c_3^n \det \hat{Q}}{(\sigma n/4)!} \exp\left(\frac{1}{2}\pi^2(n + n^{2\epsilon})\right) \int\limits_{U_n(\pi/2)} \exp\left(-\frac{1}{2} \sum_{(v_j, v_k) \in EG} \Delta_{ij}^2\right) d\boldsymbol{\theta}. \tag{6.11}$$

Lemma 6.1. *Let the assumptions of Theorem 2.1 hold. Then*

$$\int\limits_{U_n(\pi/2)} \exp\left(-\frac{1}{2} \sum_{(v_j, v_k) \in EG} \Delta_{ij}^2\right) d\boldsymbol{\theta} \leq \frac{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} n}{\sqrt{\det \hat{Q}}}. \tag{6.12}$$

Lemma 6.1 is proved in Section 8. Combining (6.11) and (6.12), we obtain that as $n \to \infty$

$$S_2 = O(n^{-cn})\, 2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} n^{-1} \sqrt{\det \hat{Q}} \tag{6.13}$$

for some constant $c > 0$ depending only on σ.

We denote by G_T the graph which arises from G by removing all edges of the tree T. Let $G_{T,\boldsymbol{\theta}}$ be the graph resulting from from G_T by removing vertices, corresponding to those values of j for which $|\theta_j| \geq \frac{1}{16}\pi$.

For $1 \leq r \leq n^{\epsilon}$ let $S_3(r)$ denote the contribution to S_3 of those $\theta \in V_2$ such that $|\theta_j| \geq \frac{1}{16}\pi$ for exactly r values of j. If $|\theta_j| \leq \frac{1}{32}\pi$ and $|\theta_k| \geq \frac{1}{16}\pi$ or vice versa, we have that

$$|f_{jk}(T, \boldsymbol{\theta})| \leq \cos\left(\frac{1}{32}\pi\right) \tag{6.14}$$

unless $(v_j, v_k) \in E'T$. This includes at least $r(\sigma n/2 - \sigma n/4 - n^{\epsilon})$ pairs (j,k), because the degree of any vertex of the graph G is at least $\sigma n/2$, see (3.2). For pairs (j,k) such that $|\theta_j|, |\theta_k| \leq \frac{1}{16}\pi$, but $(v_j, v_k) \notin E'T$, we use (6.10). We put $\boldsymbol{\theta'} = (\theta_1, \ldots, \theta_{n-r})$. Then, allowing n^r for the choice of those values of j for which $|\theta_j| \geq \frac{1}{16}\pi$, we get that

$$S_3(r) \leq \pi^r n^r \left(\cos\frac{\pi}{32}\right)^{r(\sigma n/4 - n^{\epsilon})} \sum_{T} \int\limits_{U_{n-r}(\pi/2)} \exp\left(-\frac{1}{2} \sum_{(v_j,v_k) \in EG_{T,\boldsymbol{\theta}}} \Delta_{ij}^2\right) d\boldsymbol{\theta'}, \tag{6.15}$$

where the first sum is over trees with maximum degree $\sigma n/4$. Using Lemma 3.8 and then Lemma 3.6 for the graph G_T, we obtain that

$$\lambda_1(G_{T,\boldsymbol{\theta}}) \geq \sigma n/2 - n^{\epsilon} \tag{6.16}$$

and

$$\det \hat{Q}(G_{T,\boldsymbol{\theta}}) \geq \frac{\det \hat{Q}}{(c_5 n)^r}, \tag{6.17}$$

where $c_5 = c_5(\sigma) > 0$ and $\hat{Q} = \hat{Q}(G)$. According to Lemma 6.1, we have that

$$\int\limits_{U_{n-r}(\pi/2)} \exp\left(-\frac{1}{2} \sum_{(v_j,v_k) \in EG_{T,\boldsymbol{\theta}}} \Delta_{ij}^2\right) d\boldsymbol{\theta'} \leq \frac{2^{\frac{n-r-1}{2}} \pi^{\frac{n-r+1}{2}} n}{\sqrt{\det \hat{Q}(G_{T,\boldsymbol{\theta}})}}. \tag{6.18}$$

Combining (6.15) with (6.17) and (6.18), we obtain that

$$S_3(r) \leq 2^{\frac{n-r-1}{2}} \pi^{\frac{n+r+1}{2}} n^{r+1} \left(\cos\frac{\pi}{32}\right)^{r(\sigma n/4 - n^{\epsilon})} \frac{t(G)(c_5 n)^{r/2}}{\sqrt{\det \hat{Q}}} \tag{6.19}$$

and, using (3.33), we can calculate that

$$S_3 = \sum_{r=1}^{n^{\epsilon}} S_3(r) = O(c^{-n})\, 2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} n^{-1} \sqrt{\det \hat{Q}} \tag{6.20}$$

for some constant $c > 1$ depending only on σ.

Since $\Delta_{jk} \le \frac{1}{8}\pi$ for $\boldsymbol{\theta} \in V_1 - V_2 - V_0$ and the integrand is invariant under uniform translation of all the θ_j's mod π, we can fix $\bar{\theta} = 0$ and multiply it by the ratio of its range π to the length $n^{-1/2}$ of the vector $\frac{1}{n}[1, 1, \dots, 1]^T$. Thus we get that

$$S_4 \le \pi n^{1/2} \int\limits_{L \cap U_n(\pi/8) - V_0} |F_b(\boldsymbol{\theta})| dL, \tag{6.21}$$

where L denotes the orthogonal complement to the vector $[1, 1, \dots, 1]^T$. In a similar way as (6.15) we find that

$$S_4 \le \pi n^{1/2} \sum_T \int\limits_{L \cap U_n(\pi/8) - V_0} \exp\left(-\frac{1}{2} \sum_{(v_j, v_k) \in EG_T} \Delta_{ij}^2\right) dL, \tag{6.22}$$

where the first sum is over trees with maximum degree $\sigma n/4$.

Lemma 6.2. *Let the assumptions of Theorem 2.1 hold. Then as $n \to \infty$*

$$\int\limits_{L - U_n(n^{-1/2+\epsilon})} \exp\left(-\frac{1}{2} \sum_{(v_j, v_k) \in EG} \Delta_{ij}^2\right) dL = O\left(\exp(-cn^{2\epsilon})\right) \frac{2^{\frac{n-1}{2}} \pi^{\frac{n-1}{2}} n^{1/2}}{\sqrt{\det \hat{Q}}} \tag{6.23}$$

for some $c > 0$ depending only on σ.

Lemma 6.2 is proved in Section 8. Using Lemma 3.8 and combining (6.23), (6.22) and (3.33), we obtain that as $n \to \infty$

$$S_4 = O\left(\exp(-cn^{2\epsilon})\right) t(G) \frac{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} n}{\sqrt{\det \hat{Q}}} = O\left(\exp(-cn^{2\epsilon})\right) 2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} n^{-1} \sqrt{\det \hat{Q}} \tag{6.24}$$

for some $c > 0$ depending only on σ. Combining (6.9), (6.13), (6.20), (6.24) and Lemma 5.3, we obtain the desired result.

Lemma 6.3. *Let the assumptions of Theorem 2.1 hold. Then as $n \to \infty$*

$$S = \left(1 + O\left(\exp(-cn^{2\epsilon})\right)\right) S_0 \tag{6.25}$$

for some $c > 0$ depending only on σ.

7. Proof of Theorem 2.1

According to (4.6) and (4.7)

$$Eul(G) = \prod_{j=1}^{n} \left(\frac{d_j}{2} - 1\right)! \, 2^{|EG| - n + 1} \pi^{-n} S, \tag{7.1}$$

where

$$S = \int\limits_{U_n(\pi/2)} \prod_{(v_j, v_k) \in EG} \cos \Delta_{jk} \sum_{T \in \mathcal{T}_r} \prod_{(v_j, v_k) \in ET} (1 + i \tan \Delta_{jk}) \, d\boldsymbol{\theta}. \tag{7.2}$$

Combining Lemma 5.3 and Lemma 6.3 we get that as $n \to \infty$

$$S = \Theta_{k_1,k_2}\left(2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}n^{-1}\sqrt{\det \hat{Q}}\right), \tag{7.3}$$

where constants $k_1, k_2 > 0$ depend only on σ. Taking into account (3.33) we obtain (2.11). ■

If for some $\sigma > 1/2$ the degree of each vertex of the graph G at least σn, we can use (3.3) and get that

$$\lambda_1(G) \geq 2\min_j d_j - n + 2 > (2\sigma - 1)n. \tag{7.4}$$

8. Proofs of Lemma 5.2, Lemma 6.1 and Lemma 6.2

Let assumptions of Theorem 2.1 hold. We define

$$\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\theta}) = (\phi_1(\boldsymbol{\theta}), \dots \phi_n(\boldsymbol{\theta})) = \hat{Q}\boldsymbol{\theta}. \tag{8.1}$$

We continue to use notation $P(\boldsymbol{\theta})$ for the orthogonal projection onto the space L, where L is the orthogonal complement to the vector $[1, 1, \dots, 1]^T$. For any $a > 0$ we have that

$$\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} = \pi^{n/2}a^{-n/2}\Big/\sqrt{\det \hat{Q}} \tag{8.2}$$

and

$$\int\limits_{L} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}dL = \int\limits_{L} e^{-a\,\boldsymbol{\theta}^TQ\boldsymbol{\theta}}dL = \pi^{\frac{n-1}{2}}a^{-\frac{n-1}{2}}n^{1/2}\Big/\sqrt{\det \hat{Q}}. \tag{8.3}$$

Proof of Lemma 6.1. Note that

$$\sum_{(v_j,v_k)\in EG} \Delta_{ij}^2 = \boldsymbol{\theta}^TQ\boldsymbol{\theta}. \tag{8.4}$$

Since diagonal of $U_n(\pi/2)$ is equal to $\pi n^{1/2}$ and $Q\boldsymbol{\theta} = QP(\boldsymbol{\theta})$ we have that

$$\int\limits_{U_n(\pi/2)} \exp\left(-\frac{1}{2}\sum_{(v_j,v_k)\in EG}\Delta_{ij}^2\right)d\boldsymbol{\theta} \leq \pi n^{1/2}\int\limits_{L} e^{-\frac{1}{2}\,\boldsymbol{\theta}^TQ\boldsymbol{\theta}}dL. \tag{8.5}$$

Using (8.3), we obtain (6.12). ■

Note that for some $g_1(\boldsymbol{\theta}) = g_1(\theta_2, \dots, \theta_n)$

$$\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} = \frac{\phi_1(\boldsymbol{\theta})^2}{d_1+1} + g_1(\boldsymbol{\theta}). \tag{8.6}$$

Using (3.9), we get that as $n \to \infty$

$$\begin{aligned}\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} &= \int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty} e^{-a\,g_1(\theta_2,\dots,\theta_n)}\left(\int\limits_{-\infty}^{+\infty} e^{-a\,\frac{\phi_1(\boldsymbol{\theta})^2}{d_1+1}}d\theta_1\right)d\theta_2\dots d\theta_n \\ &= \left(1 + O\left(\exp(-\tilde{c}n^{2\epsilon})\right)\right)\int\limits_{|\phi_1(\boldsymbol{\theta})|\leq\frac{1}{2}c_\infty^{-1}n^{1/2+\epsilon}} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\end{aligned} \tag{8.7}$$

for some $\tilde{c} > 0$ depending only on σ and a, where c_∞ is the constant of Lemma 3.2. Combining similar expressions for $\phi_1, \phi_2, \dots \phi_n$, we obtain that as $n \to \infty$

$$\int\limits_{||\boldsymbol{\phi}(\boldsymbol{\theta})||_\infty \le \frac{1}{2}c_\infty^{-1} n^{1/2+\epsilon}} e^{-a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta} = \left(1 + O\left(\exp(-cn^{2\epsilon})\right)\right) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta} \tag{8.8}$$

for some $c > 0$ depending only on σ and a. Using Lemma 3.2, we get that as $n \to \infty$

$$\int\limits_{U_n(\frac{1}{2}n^{-1/2+\epsilon})} e^{-a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta} = \left(1 + O\left(\exp(-cn^{2\epsilon})\right)\right) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}. \tag{8.9}$$

Proof of Lemma 6.2. Note that

$$||P(\boldsymbol{\theta})||_\infty = ||\boldsymbol{\theta} - \bar{\theta}[1, 1, \dots, 1]^T||_\infty \le 2||\boldsymbol{\theta}||_\infty, \tag{8.10}$$

where

$$\bar{\theta} = \frac{\theta_1 + \theta_2 + \dots \theta_n}{n}. \tag{8.11}$$

Thus

$$U_n(\frac{1}{2}n^{-1/2+\epsilon}) \subset \left\{\boldsymbol{\theta} \mid P(\boldsymbol{\theta}) \in U_n(n^{-1/2+\epsilon})\right\} \tag{8.12}$$

Since $Q\boldsymbol{\theta} = QP(\boldsymbol{\theta})$, using (8.4) and (8.12), we get that

$$\begin{aligned} &\int\limits_{L \cap U_n(n^{-1/2+\epsilon})} \exp\left(-\frac{1}{2} \sum_{(v_j,v_k) \in EG} \Delta_{ij}^2\right) dL = \int\limits_{L \cap U_n(n^{-1/2+\epsilon})} e^{-\frac{1}{2}\,\boldsymbol{\theta}^T Q \boldsymbol{\theta}} dL \\ &= \int\limits_{P(\boldsymbol{\theta}) \in U_n(n^{-1/2+\epsilon})} e^{-\frac{1}{2}\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta} \Big/ \int\limits_{-\infty}^{+\infty} e^{-\frac{1}{2}nx^2} dx \ge \frac{n^{1/2}}{\sqrt{2\pi}} \int\limits_{U_n(\frac{1}{2}n^{-1/2+\epsilon})} e^{-\frac{1}{2}\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}. \end{aligned} \tag{8.13}$$

Combining (8.2), (8.9) and (8.13) we obtain (6.23). ∎

To prove Lemma 5.2 we separate the integrand in (5.32) into three factors.

- $\exp\left(i\,\boldsymbol{\theta}^T Q \boldsymbol{\alpha}_{\mathbf{n}}\right)$ — the oscillatory factor,
- $\exp\left(a \sum\limits_{(v_j,v_k) \in EG} \Delta_{ij}^2\right)$ — the regular factor,
- $\exp\left(b \sum\limits_{(v_j,v_k) \in EG} \Delta_{jk}^4 + R_n(\boldsymbol{\theta})\right)$ — the residual factor.

The proof consists of the following steps.

1. In Lemma 8.3 we estimate an integral analogous to (5.32) but without an oscillatory factor.

2. Using Lemma 8.5, we get rid of the oscillatory factor in (5.32).

3. Combining Lemma 8.5 and Lemma 8.3, we complete the proof of Lemma 5.2.

At first, we prove two technical statements.

Lemma 8.1. *For any $a > 0$ and sequence of functions $r_n(x)$ such that as $n \to \infty$*

$$\sup_{|x| \le n^{-1/2+\epsilon}} |r_n(x)| = o(1). \tag{8.14}$$

Then as $n \to \infty$

$$\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} nx^2 e^{-anx^2+r_n(x)} dx = \left(\frac{1}{a} + o(1)\right) \int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-anx^2+r_n(x)} dx. \tag{8.15}$$

and

$$\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} n^2x^4 e^{-anx^2+r_n(x)} dx = \left(\frac{3}{4a^2} + o(1)\right) \int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-anx^2+r_n(x)} dx. \tag{8.16}$$

Proof of Lemma 8.1. Note that

$$\int\limits_{-n^{\epsilon}}^{n^{\epsilon}} t^2 e^{-at^2} dt = \left(1 + O\left(\exp(-cn^{2\epsilon})\right)\right) \int\limits_{-\infty}^{+\infty} t^2 e^{-at^2} dt = \left(\frac{1}{a} + o(1)\right) \int\limits_{-n^{\epsilon}}^{n^{\epsilon}} e^{-at^2} dt \tag{8.17}$$

and

$$\int\limits_{-n^{\epsilon}}^{n^{\epsilon}} t^4 e^{-at^2} dt = \left(1 + O\left(\exp(-cn^{2\epsilon})\right)\right) \int\limits_{-\infty}^{+\infty} t^4 e^{-at^2} dt = \left(\frac{3}{4a^2} + o(1)\right) \int\limits_{-n^{\epsilon}}^{n^{\epsilon}} e^{-at^2} dt. \tag{8.18}$$

Using (8.14), we get that as $n \to \infty$

$$\sup_{|x| \le n^{-1/2+\epsilon}} |e^{r_n(x)} - 1| = o(1). \tag{8.19}$$

Making the substitution $t = \sqrt{n}x$ and combining (8.17) and (8.18) with (8.19), we obtain (8.15) and (8.16), respectively. ∎

Lemma 8.2. *Under the assumptions of Lemma 8.1, let $r_n(x)$ be differentiable and as $n \to \infty$*

$$\sup_{|x| \le n^{-1/2+\epsilon}} |r_n'(x)| = O(n^{-1/2+3\epsilon}). \tag{8.20}$$

Then as $n \to \infty$

$$\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} xe^{-anx^2+r_n(x)} dx = O\left(n^{-3/2+4\epsilon}\right) \int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-anx^2+r_n(x)} dx. \tag{8.21}$$

Proof of Lemma 8.2. Note that

$$\int\limits_0^{n^\epsilon} te^{-at^2}dt = \left(1+O\left(\exp(-cn^{2\epsilon})\right)\right)\int\limits_0^{+\infty} te^{-at^2}dt = \frac{1}{2a}+O\left(\exp(-cn^{2\epsilon})\right). \tag{8.22}$$

According to the Mean Value Theorem, we have that for some $|\tilde{x}|\le|x|$

$$|e^{r_n(x)}-e^{r_n(-x)}| = |e^{r_n(\tilde{x})}r_n'(\tilde{x})2x|. \tag{8.23}$$

Using (8.20), we get that as $n\to\infty$

$$\sup_{|x|\le n^{-1/2+\epsilon}}|e^{r_n(x)}-e^{r_n(-x)}| = O\left(n^{-1+4\epsilon}\right). \tag{8.24}$$

We have that

$$\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} xe^{-anx^2+r_n(x)}dx = \int\limits_0^{n^{-1/2+\epsilon}} xe^{-anx^2}\left(e^{r_n(x)}-e^{r_n(-x)}\right)dx. \tag{8.25}$$

Making the substitution $t=\sqrt{n}x$ and combining (8.22) with (8.24), we obtain (8.21). ∎

We use notation

$$\mu_m = \sum_{j=1}^n |\phi_j|^m. \tag{8.26}$$

According to the Generalized Mean Inequality, we have that

$$\mu_1/n \le \left(\mu_4/n\right)^{1/4}. \tag{8.27}$$

Since

$$\phi_k = (d_k+1)\theta_k + \sum_{(v_k,v_j)\in EG}\theta_j, \tag{8.28}$$

and (see (3.18))

$$d_k \ge \sigma n/2 \tag{8.29}$$

we obtain that

$$|\theta_k| \le \frac{2}{\sigma n}\left(|\phi_k|+\sum_{j\ne k}|\theta_j|\right) \tag{8.30}$$

Using Lemma 3.2, we find that

$$\sum_{j\ne k}|\theta_j| \le ||\theta||_1 \le \frac{c_\infty}{n}||\phi||_1 = \frac{c_\infty}{n}\mu_1 \tag{8.31}$$

Combining (8.27), (8.30) and (8.31), we get that

$$\theta_k^4 \le \frac{16}{\sigma^4 n^4}\left(|\phi_k|+\frac{c_\infty}{n}\mu_1\right)^4 \le \frac{16}{\sigma^4 n^4}\left(|\phi_k|+c_\infty\left(\mu_4/n\right)^{1/4}\right)^4. \tag{8.32}$$

Using the inequality $(x+y)^4\le 8(x^4+y^4)$, we obtain that

$$\theta_k^4 \le c_\phi\frac{\phi_k^4}{n^4}+c_\mu\frac{\mu_4}{n^5}, \tag{8.33}$$

where constants $c_\phi, c_\mu>0$ depend only on σ.

Lemma 8.3. *Let assumptions of Theorem 2.1 hold. Let $\{a_n\}$ be sequence of positive numbers having limit $a > 0$. Then for any $b > 0$ as $n \to \infty$*

$$\int_{U_n(n^{-1/2+\epsilon})} \exp\left(-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - b \sum_{(v_j,v_k)\in EG} \Delta_{jk}^4\right) d\boldsymbol{\theta} = \Theta_{k_1,k_2}\left(\int_{\mathbb{R}^n} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}\right), \tag{8.34}$$

where constants $k_1, k_2 > 0$ depend only on a, b and σ.

Proof of Lemma 8.3. Using the inequality $(x+y)^4 \le 8(x^4+y^4)$, we find that

$$\sum_{(v_j,v_k)\in EG} \Delta_{jk}^4 \le 8n \sum_{j=1}^{n} \theta_j^4. \tag{8.35}$$

We define $R_1(\boldsymbol{\theta}) = 8n \sum\limits_{j=1}^{n} \theta_j^4$. Thus we have that

$$\int_{U_n(n^{-1/2+\epsilon})} \exp\left(-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - b \sum_{(v_j,v_k)\in EG} \Delta_{jk}^4\right) d\boldsymbol{\theta} \ge \int_{\mathbb{R}^n} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\boldsymbol{\theta}. \tag{8.36}$$

Using (8.9), we find that as $n \to \infty$

$$\begin{aligned}\int_{U_n(n^{-1/2+\epsilon})} \phi_1^4 e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\boldsymbol{\theta} = \int_{U_n(\frac{4}{\sigma} n^{-1/2+\epsilon})} \phi_1^4 e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\boldsymbol{\theta} + \\ + O\left(\exp(-cn^{2\epsilon})\right) \int_{\mathbb{R}^n} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}\end{aligned} \tag{8.37}$$

for some $c > 0$ depending only on a and σ. It follows that

$$\begin{aligned}&\int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \cdots \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \left(\int_{-\frac{4}{\sigma} n^{-1/2+\epsilon}}^{\frac{4}{\sigma} n^{-1/2+\epsilon}} \phi_1^4 e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\theta_1\right) d\theta_2 \dots d\theta_n = \\ &= \int_{U_n(n^{-1/2+\epsilon})} \phi_1^4 e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\boldsymbol{\theta} + O\left(\exp(-cn^{2\epsilon})\right) \int_{\mathbb{R}^n} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}\end{aligned} \tag{8.38}$$

Using (8.6), we find that

$$\begin{aligned}&\int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \cdots \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \left(\int_{-\frac{4}{\sigma} n^{-1/2+\epsilon}}^{\frac{4}{\sigma} n^{-1/2+\epsilon}} \phi_1^4 e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\theta_1\right) d\theta_2 \dots d\theta_n = \\ &= \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \cdots \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-a_n g_1(\theta_2,\dots,\theta_n) - R_2(\boldsymbol{\theta})} \left(\int_{-\frac{4}{\sigma} n^{-1/2+\epsilon}}^{\frac{4}{\sigma} n^{-1/2+\epsilon}} \phi_1^4 e^{-a_n \frac{\phi_1^2}{d_1+1} - 8bn\theta_1^4} d\theta_1\right) d\theta_2 \dots d\theta_n,\end{aligned} \tag{8.39}$$

where $R_2(\boldsymbol{\theta}) = 8n\sum\limits_{j=2}^{n}\theta_j^4$. Using (8.30), we get that as $n\to\infty$

$$\int\limits_{-\frac{4}{\sigma}n^{-1/2+\epsilon}}^{\frac{4}{\sigma}n^{-1/2+\epsilon}}\phi_1^4 e^{-a_n\frac{\phi_1^2}{d_1+1}-8bn\theta_1^4}d\theta_1 = \left(1+O\left(\exp(-cn^{2\epsilon})\right)\right)\int\limits_{|\phi_1|\le n^{1/2+\epsilon}}\phi_1^4 e^{-a_n\frac{\phi_1^2}{d_1+1}-8bn\theta_1^4}d\theta_1 \quad (8.40)$$

Combining (8.38), (8.39), (8.40) and Lemma 8.1 with $x=\phi_1/n$, we obtain that as $n\to\infty$

$$\begin{aligned}\int_{U_n(n^{-1/2+\epsilon})}\phi_1^4 e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_1(\boldsymbol{\theta})}d\boldsymbol{\theta} \le c'n^3\int_{U_n(n^{-1/2+\epsilon})}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_1(\boldsymbol{\theta})}d\boldsymbol{\theta}+\\ +O\left(\exp(-cn^{2\epsilon})\right)\int_{\mathbb{R}^n}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\end{aligned} \quad (8.41)$$

for some constants $c, c'>0$ depending only on a and σ.

Combining similar to (8.41) inequalities for $\phi_1,\phi_2,\ldots,\phi_n$ and using (8.33), we find that as $n\to\infty$

$$\begin{aligned}\int_{U_n(n^{-1/2+\epsilon})}\theta_1^4 e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_1(\boldsymbol{\theta})}d\boldsymbol{\theta} \le \frac{(c_\phi+c_\mu)c'}{n^2}\int_{U_n(n^{-1/2+\epsilon})}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_1(\boldsymbol{\theta})}d\boldsymbol{\theta}+\\ +O\left(\exp(-cn^{2\epsilon})\right)\int_{\mathbb{R}^n}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}\end{aligned} \quad (8.42)$$

for some $c>0$ depending only on a and σ. Note that as $n\to\infty$

$$\begin{aligned}\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}}\cdots\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}}e^{-a_n g_1(\theta_2,\ldots,\theta_n)-R_2(\boldsymbol{\theta})}\left(\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}}e^{-a_n\frac{\phi_1^2}{d_1+1}-8bn\theta_1^4}d\theta_1\right)d\theta_2\ldots d\theta_n =\\ =\cdots\left(\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}}e^{-a_n\frac{\phi_1^2}{d_1+1}}\left(1-8bn\theta_1^4+O\left(n^{-2+8\epsilon}\right)\right)d\theta_1\right)d\theta_2\ldots d\theta_n\end{aligned} \quad (8.43)$$

Combining (8.42) and (8.43), we get that as $n\to\infty$

$$\begin{aligned}\int_{U_n(n^{-1/2+\epsilon})}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_1(\boldsymbol{\theta})}d\boldsymbol{\theta} \ge \left(1+\frac{\tilde{c}}{n}\right)\int_{U_n(n^{-1/2+\epsilon})}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_2(\boldsymbol{\theta})}d\boldsymbol{\theta}+\\ +O\left(\exp(-cn^{2\epsilon})\right)\int_{\mathbb{R}^n}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta},\end{aligned} \quad (8.44)$$

where $\tilde{c}$ depends only on a, b and σ.

We continue similarly to (8.44)

$$\begin{aligned}\int_{U_n(n^{-1/2+\epsilon})}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_k(\boldsymbol{\theta})}d\boldsymbol{\theta} \ge \left(1+\frac{\tilde{c}}{n}\right)\int_{U_n(n^{-1/2+\epsilon})}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}-R_{k+1}(\boldsymbol{\theta})}d\boldsymbol{\theta}+\\ +O\left(\exp(-cn^{2\epsilon})\right)\int_{\mathbb{R}^n}e^{-a_n\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta},\end{aligned} \quad (8.45)$$

where

$$R_k(\boldsymbol{\theta}) = 8n \sum_{j=k}^{n} \theta_j^4. \tag{8.46}$$

Combining all inequalities of (8.45) for $R_1, R_2, \ldots, R_n$, we get that

$$\begin{aligned}\int_{U_n(n^{-1/2+\epsilon})} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\boldsymbol{\theta} \geq \left(1 + \frac{\tilde{c}}{n}\right)^n \int_{U_n(n^{-1/2+\epsilon})} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta} + \\ + O\left(\exp(-cn^{2\epsilon})\right) \int_{\mathbb{R}^n} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}\end{aligned} \tag{8.47}$$

for some $c > 0$ depending only on a and σ. Note also that

$$\int_{U_n(n^{-1/2+\epsilon})} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} - R_1(\boldsymbol{\theta})} d\boldsymbol{\theta} \leq \int_{U_n(n^{-1/2+\epsilon})} e^{-a_n \boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta} \tag{8.48}$$

Combining (8.9), (8.47) and (8.48), we obtain (8.34). ∎

Lemma 8.4. *Let the assumptions of Theorem 2.1 hold. For positive constants a, b, d_1, d_2 let sequence of differentiable functions $\{R_n(\boldsymbol{\theta})\}$ be such that*

$$Re\left(R_n(\boldsymbol{\theta})\right) \leq d_1 \frac{\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}}{n}, \tag{8.49}$$

and for $\boldsymbol{\theta} \in U_n(\frac{4}{\sigma} n^{-1/2+\epsilon})$

$$\left| \frac{\partial R_n(\boldsymbol{\theta})}{\partial \theta_k} \right| \leq d_2\, n^{-1/2+3\epsilon}. \tag{8.50}$$

Then as $n \to \infty$

$$\begin{aligned}\int\limits_{U_n(n^{-1/2+\epsilon})} \phi_k e^{i\frac{b}{n}\phi_k - a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} = \Theta_{k_1,k_2} \left(\int\limits_{U_n(n^{-3/2+\epsilon})} e^{i\frac{b}{n}\phi_k - a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} \right) + \\ + O(n^{-1/2+4\epsilon}) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta}\end{aligned} \tag{8.51}$$

and

$$\begin{aligned}\int\limits_{U_n(n^{-1/2+\epsilon})} \phi_k^2 e^{i\frac{b}{n}\phi_k - a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} = \Theta_{k_1,k_2} \left(n \int\limits_{U_n(n^{-3/2+\epsilon})} e^{i\frac{b}{n}\phi_k - a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} \right) + \\ + O\left(\exp(-\tilde{c}n^{2\epsilon})\right) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T \hat{Q} \boldsymbol{\theta}} d\boldsymbol{\theta},\end{aligned} \tag{8.52}$$

where constants $k_1, k_2, \tilde{c} > 0$ depend only on a, b, d_1, d_2 and σ .

Proof of Lemma 8.4. For our purpose it is convenient to assume that $k = 1$. Note that

$$\int\limits_{U_n(n^{-1/2+\epsilon})} |e^{i\frac{b}{n}\phi_1 - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})}| d\boldsymbol{\theta} \leq \int\limits_{U_n(n^{-1/2+\epsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + d_1\frac{\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}{n}} d\boldsymbol{\theta}. \tag{8.53}$$

Using (8.2), we get that as $n \to \infty$

$$\int\limits_{U_n(n^{-1/2+\epsilon})} |e^{i\frac{b}{n}\phi_1 - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})}| d\boldsymbol{\theta} = O\left(1\right) \int\limits_{U_n(n^{-1/2+\epsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta}. \tag{8.54}$$

Similar to (8.37), for $m = 1, 2$ we find that as $n \to \infty$

$$\begin{aligned} \int_{U_n(\frac{4}{\sigma}n^{-1/2+\epsilon}) - U_n(n^{-1/2+\epsilon})} \phi_1^m |e^{i\frac{b}{n}\phi_1 - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})}| d\boldsymbol{\theta} = \\ = O\left(\exp(-cn^{2\epsilon})\right) \int_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta} \end{aligned} \tag{8.55}$$

for some $c > 0$ depending only on a, d_1 and σ. It follows that

$$\begin{aligned} &\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \cdots \int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \left(\int\limits_{-\frac{4}{\sigma}n^{-1/2+\epsilon}}^{\frac{4}{\sigma}n^{-1/2+\epsilon}} \phi_1^m e^{i\frac{b}{n}\phi_1 - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\theta_1 \right) d\theta_2 \ldots d\theta_n = \\ &= \int_{U_n(n^{-1/2+\epsilon})} \phi_1^m e^{i\frac{b}{n}\phi_1 - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} + O\left(\exp(-cn^{2\epsilon})\right) \int_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta} \end{aligned} \tag{8.56}$$

We define

$$R_n'(\boldsymbol{\theta}) = R_n(0, \theta_2, \ldots, \theta_n) \tag{8.57}$$

and

$$r_1(\boldsymbol{\theta}) = R_n(\boldsymbol{\theta}) - R_n'(\boldsymbol{\theta}). \tag{8.58}$$

According to Mean Value Theorem, for $\theta \in U_n(\frac{4}{\sigma}n^{-1/2+\epsilon})$ we have that

$$|r_1(\boldsymbol{\theta})| = |R_n(\boldsymbol{\theta}) - R_n'(\boldsymbol{\theta})| = \left| \theta_1 \frac{\partial R_n(\tilde{\boldsymbol{\theta}})}{\partial \theta_k} \right| = O(n^{-1+4\epsilon}). \tag{8.59}$$

Using (8.6), we find that

$$\begin{aligned} &\int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \cdots \int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \left(\int\limits_{-\frac{4}{\sigma}n^{-1/2+\epsilon}}^{\frac{4}{\sigma}n^{-1/2+\epsilon}} \phi_1^m e^{i\frac{b}{n}\phi_1 - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\theta_1 \right) d\theta_2 \ldots d\theta_n = \\ &= \cdots \int\limits_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-ag_1(\theta_2,\ldots,\theta_n) + R_n'(\boldsymbol{\theta})} \left(\int\limits_{-\frac{4}{\sigma}n^{-1/2+\epsilon}}^{\frac{4}{\sigma}n^{-1/2+\epsilon}} \phi_1^m e^{i\frac{b}{n}\phi_1 - a\frac{\phi_1^2}{d_1+1} + r_1(\boldsymbol{\theta})} d\theta_1 \right) d\theta_2 \ldots d\theta_n, \end{aligned} \tag{8.60}$$

Using (8.30), we get that as $n \to \infty$

$$\begin{aligned}\int\limits_{-\frac{4}{\sigma}n^{-1/2+\epsilon}}^{\frac{4}{\sigma}n^{-1/2+\epsilon}} \phi_1^m e^{i\frac{b}{n}\phi_1 - a\frac{\phi_1^2}{d_1+1}+r_1(\boldsymbol{\theta})}d\theta_1 = \\ = \left(1+O\left(\exp(-cn^{2\epsilon})\right)\right) \int\limits_{|\phi_1|\le n^{1/2+\epsilon}} \phi_1^m e^{i\frac{b}{n}\phi_1 - a\frac{\phi_1^2}{d_1+1}+r_1(\boldsymbol{\theta})}d\theta_1\end{aligned} \tag{8.61}$$

Combining (8.56), (8.60), (8.61) with $m = 2$ and (8.15) with $x = \phi_1/n$, we obtain (8.52).

Note that

$$\begin{aligned}\int\limits_{|\phi_1|\le n^{1/2+\epsilon}} \phi_1 e^{i\frac{b}{n}\phi_1 - a\frac{\phi_1^2}{d_1+1}+r_1(\boldsymbol{\theta})}d\theta_1 = \\ = \int\limits_{|\phi_1|\le n^{1/2+\epsilon}} \phi_1\left(1+i\frac{b}{n}\phi_1+O\left(n^{-1+2\epsilon}\right)\right)e^{-a\frac{\phi_1^2}{d_1+1}+r_1(\boldsymbol{\theta})}d\theta_1\end{aligned} \tag{8.62}$$

Since $\partial r_1/\partial\theta_1 = \partial R_n/\partial\theta_1$, using (8.21) with $x = \phi_1/n$, we get that

$$\int\limits_{|\phi_1|\le n^{1/2+\epsilon}} \phi_1 e^{-a\frac{\phi_1^2}{d_1+1}+r_1(\boldsymbol{\theta})}d\theta_1 = O\left(n^{-1/2+4\epsilon}\right) \int\limits_{|\phi_1|\le n^{1/2+\epsilon}} e^{-a\frac{\phi_1^2}{d_1+1}+r_1(\boldsymbol{\theta})}d\theta_1 \tag{8.63}$$

Combining (8.56), (8.60), (8.61) with $m = 1$ and (8.52), (8.54) with $b = 0$ and (8.63), we obtain (8.51). ∎

Lemma 8.5. *Let the assumptions of Theorem 2.1 hold. For positive constants a, b, d_1, d_2 let sequence of vectors $\{\boldsymbol{\beta}_n\}$ and sequence of real differentiable functions $\{R_n(\boldsymbol{\theta})\}$ be such that*

$$||\boldsymbol{\beta}_n||_\infty \le b, \tag{8.64}$$

$$R_n(\boldsymbol{\theta}) \le d_1\frac{\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}{n}, \tag{8.65}$$

and for $\boldsymbol{\theta} \in U_n(\frac{4}{\sigma}n^{-1/2+\epsilon})$

$$\left\|\frac{\partial R_n(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right\|_\infty \le d_2\, n^{-1/2+3\epsilon}. \tag{8.66}$$

Then as $n \to \infty$

$$\begin{aligned}\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,\boldsymbol{\beta}_n^T\boldsymbol{\theta} - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+\tilde{R}_n(\boldsymbol{\theta})}d\boldsymbol{\theta} = \Theta_{\tilde{k}_1,\tilde{k}_2}\left(\int\limits_{U_n(n^{-1/2+\epsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+\tilde{R}_n(\boldsymbol{\theta})}d\boldsymbol{\theta}\right) + \\ +O(n^{-1/2+4\epsilon})\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta},\end{aligned} \tag{8.67}$$

where $\tilde{R}_n = R_n - \frac{1}{2}\sum\limits_{j=1}^n \beta_j^2\theta_j^2$ and constants $\tilde{k}_1, \tilde{k}_2$ depend only on a, b, d_1, d_2 and σ .

Proof of Lemma 8.5. Using (8.54), we get that as $n \to \infty$

$$\begin{aligned}
&\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,\boldsymbol{\beta}_n^T\boldsymbol{\theta}-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})}d\boldsymbol{\theta} = \\
&= \left(1+i\beta_1\theta_1-\frac{\beta_1^2\theta_1^2}{2}+O(n^{-3/2+3\epsilon})\right)\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})}d\boldsymbol{\theta} = \\
&= \int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})-\frac{1}{2}\beta_1^2\theta_1^2}d\boldsymbol{\theta}+O(n^{-3/2+3\epsilon})\int\limits_{U_n(n^{-1/2+\epsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta}+ \\
&+i\beta_1\theta_1\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})}d\boldsymbol{\theta}.
\end{aligned} \tag{8.68}$$

Taking into account (8.28) and using Lemma 8.4, we find that as $n \to \infty$

$$\begin{aligned}
&\int\limits_{U_n(n^{-1/2+\epsilon})} \phi_k e^{i\,(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})}d\boldsymbol{\theta} = \\
&= \Theta_{k_1,k_2}\left(\int\limits_{U_n(n^{-3/2+\epsilon})} e^{i(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})}d\boldsymbol{\theta}\right)+ \\
&\qquad +O(n^{-1/2+4\epsilon})\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta} = \\
&= \Theta_{k_1,k_2}\left(\int\limits_{U_n(n^{-3/2+\epsilon})} e^{i(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})-\frac{1}{2}\beta_1^2\theta_1^2}d\boldsymbol{\theta}\right)+ \\
&\qquad +O(n^{-1/2+4\epsilon})\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta},
\end{aligned} \tag{8.69}$$

where constants k_1, k_2 depend only on a, b, d_1, d_2 and σ.

According to Lemma 3.2, we have that

$$||\hat{Q}^{-1}||_1 = ||\hat{Q}^{-1}||_\infty \le \frac{c_\infty}{n}. \tag{8.70}$$

Thus as $n \to \infty$

$$\begin{aligned}
&\left|\int\limits_{U_n(n^{-1/2+\epsilon})} \theta_k e^{i\,(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})}d\boldsymbol{\theta}\right| \le \\
&\le \frac{\tilde{c}}{n}\left|\int\limits_{U_n(n^{-3/2+\epsilon})} e^{i(\boldsymbol{\beta}_n^T\boldsymbol{\theta}-\beta_1\theta_1)-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}+R_n(\boldsymbol{\theta})-\frac{1}{2}\beta_1^2\theta_1^2}d\boldsymbol{\theta}\right| + O(n^{-3/2+4\epsilon})\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}d\boldsymbol{\theta},
\end{aligned} \tag{8.71}$$

where $\tilde{c} > 0$ depends only on a, b, d_1, d_2 and σ.

Combining (8.68) and (8.71), we get that as $n \to \infty$

$$\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,\boldsymbol{\beta}_n^T\boldsymbol{\theta} - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} =$$
$$= \left(1 + \frac{c^{(1)}}{n}\right) \int\limits_{U_n(n^{-3/2+\epsilon})} e^{i(\boldsymbol{\beta}_n^T\boldsymbol{\theta} - \beta_1\theta_1) - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta}) - \frac{1}{2}\beta_1^2\theta_1^2} d\boldsymbol{\theta} + \tag{8.72}$$
$$+ O(n^{-3/2+4\epsilon}) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta},$$

where $|c^{(1)}| \le \tilde{c}\beta_1 \le \tilde{c}b$.

We define

$$R_n^{(k)} = R_n - \frac{1}{2}\sum_{j=1}^{k} \beta_j^2\theta_j^2. \tag{8.73}$$

We continue similarly to (8.72)

$$\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\left(\boldsymbol{\beta}_n^T\boldsymbol{\theta} - \sum\limits_{j=1}^{k}\beta_j\theta_j\right) - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n^{(k)}(\boldsymbol{\theta})} d\boldsymbol{\theta} =$$
$$= \left(1 + \frac{c^{(k+1)}}{n}\right) \int\limits_{U_n(n^{-3/2+\epsilon})} e^{i\left(\boldsymbol{\beta}_n^T\boldsymbol{\theta} - \sum\limits_{j=1}^{k+1}\beta_j\theta_j\right) - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n^{(k+1)}(\boldsymbol{\theta})} d\boldsymbol{\theta} + \tag{8.74}$$
$$+ O(n^{-3/2+4\epsilon}) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta},$$

where $|c^{(k+1)}| \le \tilde{c}\beta_k \le \tilde{c}b$.

Combining all inequalities of (8.74) for $k = 0, 1, \ldots, n-1$, we get that as $n \to \infty$

$$\int\limits_{U_n(n^{-1/2+\epsilon})} e^{i\,\boldsymbol{\beta}_n^T\boldsymbol{\theta} - a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + R_n(\boldsymbol{\theta})} d\boldsymbol{\theta} =$$
$$= \left(1 + \frac{c^{(1)}}{n}\right) \cdots \left(1 + \frac{c^{(n)}}{n}\right) \int\limits_{U_n(n^{-3/2+\epsilon})} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + \tilde{R}_n(\boldsymbol{\theta})} d\boldsymbol{\theta} + \tag{8.75}$$
$$+ O(n^{-1/2+4\epsilon}) \int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}} d\boldsymbol{\theta}.$$

Since $|c^{(k)}| \le b\tilde{c}$ for $k = 0, 1, \ldots, n-1$, using (8.75), we obtain (8.67).

■

Proof of Lemma 5.2. Note that for $\boldsymbol{\theta} \in U_n(\frac{4}{\sigma}n^{-1/2+\epsilon})$ as $n \to \infty$

$$\left\| \frac{\partial}{\partial \boldsymbol{\theta}} \sum_{(v_j, v_k) \in EG} \Delta_{jk}^4 \right\|_\infty = O(n^{-1/2+3\epsilon}). \tag{8.76}$$

We define $\boldsymbol{\beta}_n = Q\boldsymbol{\alpha}_{\mathbf{n}}$. Using Lemma 8.5, we find that as $n \to \infty$

$$\begin{aligned}\int\limits_{U_n(n^{-1/2+\epsilon})} \exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}_{\mathbf{n}} - a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + R_n(\boldsymbol{\theta})\right) d\boldsymbol{\theta} = \\ = \Theta_{\tilde{k}_1,\tilde{k}_2}\left(\int\limits_{U_n(n^{-1/2+\epsilon})} \exp\left(-a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + \tilde{R}_n(\boldsymbol{\theta})\right) d\boldsymbol{\theta}\right) + \\ + O(n^{-1/2+4\epsilon})\int\limits_{\mathbb{R}^n} e^{-a\,\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}},\end{aligned} \tag{8.77}$$

where $\tilde{R}_n = R_n - \frac{1}{2}\sum\limits_{j=1}^{n}\beta_j^2\theta_j^2$ and constants $\tilde{k}_1, \tilde{k}_2$ depend only on a, b, d_1, d_2 and σ. Note that for some $d_3 > 0$, depending only on c and σ,

$$\frac{1}{2}\sum_{j=1}^{n}\beta_j^2\theta_j^2 \le d_3\frac{\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}{n}. \tag{8.78}$$

Combining (8.2), (8.77) and Lemma 8.3, we find that as $n \to \infty$

$$\begin{aligned}\int\limits_{U_n(n^{-1/2+\epsilon})} \exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}_{\mathbf{n}} - a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + R_n(\boldsymbol{\theta})\right) d\boldsymbol{\theta} = \\ = \Theta_{k_1',k_2'}\left(2^{\frac{n-1}{2}}\pi^{\frac{n-1}{2}}\Big/\sqrt{\det\hat{Q}}\right),\end{aligned} \tag{8.79}$$

where constants k_1', k_2' depend only on a, b, d_1, d_2 and σ. Note that

$$\begin{aligned}\int\limits_{-\infty}^{+\infty} e^{-anx^2}dx \int\limits_{L\cap V_0} \exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}_{\mathbf{n}} - a\sum_{(v_j,v_k)\in EG}\Delta_{ij}^2 - b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + R_n(\boldsymbol{\theta})\right) dL = \\ = \int\limits_{P(\theta)\in L\cap V_0} \exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}_{\mathbf{n}} - a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + R_n(\boldsymbol{\theta})\right) d\boldsymbol{\theta}.\end{aligned} \tag{8.80}$$

We have that

$$\left|\exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha}_{\mathbf{n}} - a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} - b\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + R_n(\boldsymbol{\theta})\right)\right| \le e^{-a\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta} + \frac{d_1}{n}\boldsymbol{\theta}^T\hat{Q}\boldsymbol{\theta}}. \tag{8.81}$$

Thus, combining (8.9), (8.12), (8.79) and (8.80), we obtain (5.33) ∎

Final remarks

In fact, using Lemma 5.1 and Lemma 6.3, the estimation of the number of the Eulerian circuits is reduced (see proof of Lemma 5.3) to estimating the integral

$$\int\limits_{V_0} \exp\left(i\,\boldsymbol{\theta}^T Q\boldsymbol{\alpha} - \frac{1}{2}\sum_{(v_j,v_k)\in EG}\Delta_{ij}^2 - \frac{1}{12}\sum_{(v_j,v_k)\in EG}\Delta_{jk}^4 + tr(\Lambda\hat{Q}^{-1})^2 \right) d\boldsymbol{\theta}, \tag{8.82}$$

where α denotes the vector composed of the diagonal elements of $\hat{Q}^{-1}$, Λ denotes the diagonal matrix whose diagonal elements are equal to components of the vector $Q\boldsymbol{\theta}$. Apparently, it is possible to estimate integral (8.82) more accurately for particular classes of graphs and obtain asymptotic formulas for $Eul(G)$, similar to (2.12).

Finally, we want to note that the following expression

$$2^{|EG|-\frac{n-1}{2}}\pi^{-\frac{n-1}{2}}\sqrt{t(G)}\prod_{j=1}^{n}\left(\frac{d_j}{2}-1\right)! \tag{8.83}$$

gives a surprisingly good estimate for the number of the Eulerian circuits in graphs. Namely, we calculated the exact numbers of the Eulerian circuits for small random graphs and in all cases the values given by (8.83) differ from the exact ones within not more than 30% error.

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