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Jean-Luc Fouquet, Henri Thuillier, Jean-Marie Vanherpe, Adam Pawel Wojda. On  $(K_{-q}, k)$  vertex stable with minimum size. *Discrete Mathematics*, 2012, 312 (14), pp.2109:2118. 10.1016/j.disc.2011.04.017 . hal-00586145

**HAL Id: hal-00586145**

**<https://hal.science/hal-00586145v1>**

Submitted on 15 Apr 2011

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# On $(K_q, k)$ vertex stable graphs with minimum size

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## Abstract

A graph  $G$  is a  $(K_q, k)$  vertex stable graph if it contains a  $K_q$  after deleting any subset of  $k$  vertices. We give a characterization of  $(K_q, k)$  vertex stable graphs with minimum size for  $q = 3, 4, 5$ .

*Keywords:* Stable graphs

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## 1. Introduction

For terms not defined here we refer to [1]. As usual, the *order* of a graph  $G$  is the number of its vertices (denoted by  $|G|$ ) and the *size* of  $G$  is the number of its edges (denoted by  $e(G)$ ). A complete subgraph of order  $q$  of  $G$  is called a  $q$ -*clique* of  $G$ . The complete graph of order  $q$  is denoted by  $K_q$ . When a graph  $G$  contains a  $q$ -clique as subgraph, we say “ $G$  contains a  $K_q$ ”. The union of  $p$  mutually disjoint copies of  $K_q$  is denoted by  $pK_q$ . When  $A$  is a set of vertices we denote by  $G - A$  the subgraph induced by  $V(G) - A$ .

In [7, 8] Horváth and G.Y Katona consider the notion of  $(H, k)$  *stable* graph: given a simple graph  $H$ , an integer  $k$  and a graph  $G$  containing  $H$  as subgraph,  $G$  is a  $(H, k)$  *stable graph* whenever the deletion of any set of  $k$  edges does not lead to a  $H$ -free graph. These authors consider  $(P_n, k)$  stable graphs and prove a conjecture stated in [6] on the minimum size of a  $(P_4, k)$  stable graph. In [2], Dudek, Szymański and Zwonek are interested in a vertex version of this notion and introduce the  $(H, k)$  vertex stable graphs.

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<sup>1</sup>The research of APW was partially sponsored by polish Ministry of Science and Higher Education.

**Definition 1.1.** [2] Let  $H$  be a graph. A graph is a  $(H, k)$  vertex stable graph if it contains a graph isomorphic to  $H$  after deleting any subset of  $k$  vertices. By  $Q(H, k)$  we denote the minimum size of a  $(H, k)$  vertex stable graph. If  $G$  is  $(H, k)$  vertex stable of size  $Q(H, k)$  we call it *minimum*  $(H, k)$  vertex stable.

In this paper, we are only interested by  $(H, k)$  vertex stable graphs and, since no confusion will be possible, a  $(H, k)$  vertex stable shall be simply called a  $(H, k)$  *stable graph*.

In [2], the authors give values of  $Q(H, k)$  when  $H$  is isomorphic to  $C_3$ ,  $C_4$  or  $K_4$  and provide upper bounds for some other cases while in [3, 4] the bipartite case is considered.

It must be pointed out that in some cases the value of  $Q(H, k)$  can be obtained without the description of extremal graphs, that is  $(H, k)$  vertex stable graphs whose size is precisely  $Q(H, k)$ . In this paper we describe the extremal  $(H, k)$  stable graphs when  $H$  is isomorphic to  $K_q$ , for  $q \in \{3, 4, 5\}$  while in [5] we describe the extremal  $(K_q, k)$  stable graphs when  $k$  is small with respect to  $q$ .

By considering  $(H, k)$  stable graph with minimum size, it must be clear that we can add some isolated vertices, the resulting graph remains to be a  $(H, k)$  stable graph with minimum size. From now on, the graphs considered have no isolated vertices.

**Proposition 1.2.** [2] *If  $G$  is a  $(H, k)$  stable graph with minimum size then every vertex as well as every edge is contained in a subgraph isomorphic to  $H$ .*

*Remark 1.3.* Proposition 1.2 implies, in particular, that when  $H \equiv K_q$  then the minimum degree of a  $(H, k)$  stable graph with minimum size is at least  $q - 1$ .

**Lemma 1.4.** [2] *Let  $k \geq 1$ . If  $G$  is  $(H, k)$  stable then for any vertex  $v$ ,  $G - \{v\}$  is  $(H, k - 1)$  stable.*

**Definition 1.5.** Let  $H$  be a non complete graph on  $q + p + 1$  ( $p \geq 0$ ) vertices and  $u$  be one vertex. Let  $N$  be the neighbourhood of  $u$  and  $R = V(H) - u - N$ . We shall say that  $H$  is a *near complete graph*  $(R, N, u)$  on  $q + p + 1$  vertices (see Figure 1) when

- $H - \{u\}$  is complete,
- $d_H(u) = q + \epsilon$  ( $\epsilon \in \{-1, 0, 1\}$ ).

Note that the set  $R$  is not empty since  $H$  is not complete. Hence,  $|R| = p - \epsilon$ , and since  $H$  is not complete we must have  $p \geq 2$  when  $d_H(u) = q + 1$  and  $p \geq 1$  when  $d_H(u) = q$ .

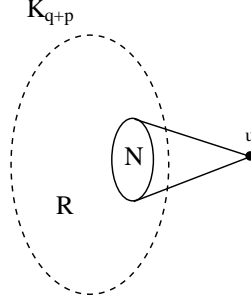


Figure 1: A near complete graph  $(R, N, u)$  on  $q + p + 1$  vertices

## 2. Preliminary results

**Proposition 2.1.** *If  $G$  is a  $(K_q, k)$  stable graph with minimum size ( $q \geq 3$ ) then  $G$  has no component isomorphic to a near complete graph  $(R, N, u)$  on  $q + p + 1$  vertices.*

**Proof.** Suppose, by contradiction, that there exists such a component  $H = (R, N, u)$  on  $q + p + 1$  vertices with  $d_H(u) = q + \epsilon$  ( $\epsilon \in \{-1, 0, 1\}$ ). Since  $G$  is a  $(K_q, k)$ -stable graph with minimum size,  $G - \{u\}$  is not  $(K_q, k)$ -stable. There exists a set  $S$  with at most  $k$  vertices such that  $S$  intersects every  $K_q$  of  $G - \{u\}$ . There exists a  $K_q$  in  $G - S$  and clearly such a  $K_q$  contains  $u$ . Since  $N$  is a  $K_{q+\epsilon}$  and  $N - S$  contains no  $K_q$ ,  $|S \cap N| \geq \epsilon + 1$  (trivial for  $\epsilon = -1$ ). If  $|S \cap N| \geq \epsilon + 2$  then  $|N - S| \leq q - 2$ , and hence  $S$  intersects every  $K_q$  containing  $u$ , a contradiction. Thus,  $|S \cap N| = \epsilon + 1$  and  $|N - S| = q - 1$ . If there exists  $v$  in  $R - S$  then  $(N - S) + \{v\}$  is a  $K_q$  in  $G - \{u\}$ , a contradiction. Thus,  $R \subset S$ . Let  $a \in R$  and  $b \in N - S$ , and set  $S' = S - \{a\} + \{b\}$ . We have  $|S'| \leq k$  and  $G - S'$  contains no  $K_q$ , a contradiction.

□

**Lemma 2.2.** *Let  $q \geq 3$  and  $k \geq 1$  and let  $G$  be a minimum  $(K_q, k)$  stable graph. If  $u$  is a vertex of degree  $q - 1$  then one of the following statements is true*

- $\forall v \in N(u) \quad d(v) \geq q + 1,$
- $Q(K_q, k - 1) + 3(q - 2) \leq Q(K_q, k).$

**Proof.** Since  $d(u) = q - 1$ ,  $\{u\} + N(u)$  induces a complete graph on  $q$  vertices. Assume that some vertex  $w \in N(u)$  has degree  $q + a$  ( $a = -1$  or  $a = 0$ ) and let  $v \in N(u)$  distinct from  $w$ . Then  $G - v$  is a  $(K_q, k - 1)$  stable graph (Lemma 1.4). Since the degree of  $u$  in  $G - \{v\}$  is  $q - 2$ , no edge incident with  $u$  can be contained in a  $K_q$ . We can thus delete these  $q - 2$  edges and the resulting graph (say  $G'$ ) is still a  $(K_q, k - 1)$  stable graph. In  $G'$ , the degree of  $w$  is now  $q + a - 2$ . Hence, no edge incident with  $w$  in  $G'$  can be contained in a  $K_q$ . Deleting these  $q + a - 2$  edges from  $G'$  leads to a graph  $G''$  which remains to be a  $(K_q, k - 1)$  stable graph.

By deleting  $v$ , we have  $e(G - \{v\}) \leq e(G) - (q - 1)$  and hence

$$e(G') \leq e(G) - (q - 1) - (q - 2)$$

We get thus

$$Q(K_q, k - 1) \leq e(G'') \leq e(G) - (q - 1) - (q - 2) - (q + a - 2)$$

Since  $e(G) = Q(K_q, k)$ , the result follows.  $\square$

**Proposition 2.3.** *If  $G$  is a minimum  $(K_q, 1)$  stable graph ( $q \geq 4$ ) then it is isomorphic to  $K_{q+1}$ .*

**Proof.** Let  $G$  be a minimum  $(K_q, 1)$  stable graph. Since  $K_{q+1}$  is  $(K_q, 1)$  stable, clearly  $e(G) \leq \binom{q+1}{2}$ . We can assume that  $G$  is connected. Otherwise, each component contains a  $K_q$ , but  $\binom{q+1}{2} < 2\binom{q}{2}$  as soon as  $q \geq 4$ , a contradiction. Let  $u$  be a vertex of  $G$  and  $Q$  be a subgraph of  $G - \{u\}$  isomorphic to  $K_q$ . Assume that there exists a vertex  $v$  outside  $Q$  and distinct from  $u$ . Note that  $v$  can be a neighbour of  $u$ . Since  $d(u) \geq q - 1$  and  $d(v) \geq q - 1$ ,  $e(G) \geq e(Q) + 2(q - 1) - 1 = \binom{q}{2} + 2q - 3 = \binom{q+1}{2} + q - 3$ . Thus,  $e(G) > e(K_{q+1})$ , a contradiction. Hence,  $V(G) = V(Q) \cup \{u\}$  with  $d(u) \geq q - 1$ . Since for any edge  $e \in K_{q+1} - \{e\}$  is not  $(K_q, 1)$  stable, we see that  $d(u) = q$ , that is  $G$  is isomorphic to  $K_{q+1}$ .  $\square$

*Remark 2.4.* It is easy to see that the minimum  $(K_3, 1)$  stable graphs are  $2K_3$  and  $K_4$ .

**Proposition 2.5.** *If  $G$  is a minimum  $(K_q, 2)$  stable graph ( $q \geq 4$ ) then it is isomorphic to  $K_{q+2}$ .*

**Proof.** Since  $K_{q+2}$  is a  $(K_q, 2)$  stable graph, we can suppose that  $G$  has at most  $\binom{q+2}{2}$  edges. We can suppose, moreover, that  $G$  is not complete, otherwise  $G$  is obviously reduced to  $K_{q+2}$ . Let  $u$  be a vertex of minimum degree (recall that the minimum degree is at least  $q - 1$ ) and let  $v$  be one of its neighbours.

Assume that  $d_G(u) = q - 1$ .  $G - \{v\}$  is a  $(K_q, 1)$  stable graph, but it is not minimum, since none of the remaining edge incident with  $u$  can be contained in a complete graph on  $q$  vertices. By deleting the  $q - 2$  edges incident with  $u$ , we get thus a  $(K_q, 1)$  stable graph.

If  $d(v) \geq q + 1$ , this graph has at most  $\binom{q+2}{2} - (2q - 1)$  edges. Since this number of edges must be greater than  $\binom{q+1}{2}$  by Proposition 2.3, we have

$$(q + 2)(q + 1) - 4q + 2 \geq (q + 1)q$$

That leads to  $q \leq 2$ , a contradiction. If  $d(v) \leq q$ , by Lemma 2.2, we have  $Q(K_q, 1) + 3q - 6 \leq Q(K_q, 2)$  and hence

$$q(q + 1) + 6q - 12 \leq (q + 1)(q + 2)$$

Which gives  $q \leq 3$ , a contradiction.

We can thus assume that the minimum degree of  $G$  is at least  $q$ . Let  $u$  and  $v$  be two non adjacent vertices of  $G$ . Since  $G - \{u, v\}$  contains a  $K_q$  (say  $Q$ ), let  $a$  and  $b$  be two distinct vertices of  $Q$ . Since  $G - \{a, b\}$  must contain also a  $K_q$ , there is certainly a vertex  $w$  distinct from  $v$  and  $u$ , outside  $Q$ , inducing with  $q - 1$  other vertices of  $G - \{a, b\}$  a  $K_q$ . Hence  $G$  contains three vertices ( $u, v$  and  $w$ ) at least in  $G - Q$  and we have:

$$\binom{q + 2}{2} \geq e(G) \geq \binom{q}{2} + 3q - 2$$

Which gives  $q < 3$ , a contradiction. Hence  $G$  is complete and the proposition follows.  $\square$

**Lemma 2.6.** *Let  $G$  be a minimum  $(K_q, 3)$  stable graph,  $q \geq 5$ . Let  $u$  be a vertex of minimum degree in  $G$  and suppose that  $d_G(u) = q + l$ , where  $-1 \leq l \leq 1$ . Then for every neighbour  $v$  of  $u$  we have  $d_G(v) \geq q + l + 2$ .*

**Proof.** Suppose, contrary to our claim, that  $d_G(v) \leq q + l + 1$  for a neighbour  $v$  of  $u$ . Since, by Proposition 1.2, the edge  $uv$  is contained in a clique of order  $q$  and  $q \geq 5$ , there is a set  $A$  of vertices of  $G$  such that  $|A| = l + 2$  and the vertices of the set  $A \cup \{u, v\}$  are mutually adjacent. The graph  $G' = G - A$  is  $(K_q, 3 - (l + 2))$  stable. We have  $d_{G'}(u) = q + l - (l + 2) = q - 2$ , hence also  $G'' = G' - \{u\}$  is  $(K_q, 1 - l)$  stable. But in  $G''$  the degree of the vertex  $v$  is at most  $q - 2$  and therefore  $G''' = G'' - \{v\}$  is  $(K_q, 1 - l)$  stable. Since every vertex of the set  $A \cup \{u, v\}$  has at least  $q - 3$  neighbours outside this set, we have

$$\binom{q + 1 - l}{2} \leq e(G''') \leq \binom{q + 3}{2} - (l + 4)(q - 3) - \binom{l + 4}{2}$$

which contradicts  $q \geq 5$ . □

**Proposition 2.7.** *If  $G$  is a minimum  $(K_q, 3)$  stable graph ( $q \geq 5$ ) then it is isomorphic to  $K_{q+3}$ .*

**Proof.** Note first that to prove the proposition it is sufficient to prove that every vertex of  $G$  has the degree at least  $q + 2$ .

Let  $u$  be a vertex of the minimum degree in  $G$  and suppose, contrary to our claim, that  $d_G(u) \leq q + l$ , where  $-1 \leq l \leq 1$ .

Let  $v_1, v_2, \dots, v_{l+2}$  be such vertices of  $G$  that the set  $\{u, v_1, v_2, \dots, v_{l+2}\}$  induce a clique in  $G$  (such vertices exist since  $u$  is contained in a clique of order  $q$  by Proposition 1.2 and  $q \geq 5$ ). By Lemma 2.6, we have  $d_G(v_i) \geq q + l + 2$  for  $i = 1, 2, \dots, l + 2$ . Set  $G' = G - \{v_1, v_2, \dots, v_{l+2}\}$ . The graph  $G'$  is clearly  $(K_q, 1 - l)$  stable. Moreover, since  $d_{G'}(u) = q - 2$ , the graph  $G'' = G' - \{u\}$  is also  $(K_q, 1 - l)$  stable and we have

$$\binom{q + 1 - l}{2} \leq e(G'') \leq \binom{q + 3}{2} - (l + 2)(q - 1) - (q - 2) - \binom{l + 3}{2}$$

which contradicts  $q \geq 5$ . □

### 3. A characterization of $(K_3, k)$ stable graph with minimum size

Dudek, Szymański and Zwonek in [2] have shown that  $Q(K_3, k) = 3k + 3$  for every nonnegative integer  $k$ . In this section we characterize all that  $(K_3, k)$  stable graphs with minimum size.

Clearly,  $K_3$  is the unique minimum  $(K_3, 0)$  stable graph.

The following theorem characterize all graphs which are  $(K_3, k)$  stable with minimum size.

**Theorem 3.1.** *Let  $G = (V, E)$  be a  $(K_3, k)$  stable graph with minimum size.  $G$  is isomorphic to  $pK_4 + qK_3$ , where  $p$  and  $q$  are such nonnegative integers that  $2p + q = k + 1$ .*

**Proof.** By Remark 2.4,  $K_3$  is the unique minimum  $(K_3, 0)$  stable graph, and the minimum  $(K_3, 1)$  stable graphs are  $2K_3$  and  $K_4$ . Clearly, the graph  $(k + 1)K_3$  is a  $(K_3, k)$  stable graph and has  $3k + 3$  edges. Let  $k_0 \geq 1$  and suppose that for every  $k < k_0$  every minimum  $(K_3, k)$  stable graph is a union of  $p$  copies of  $K_4$  and  $q$  copies  $K_3$  with  $2p + q = k + 1$ .

Let  $G$  be a  $(K_3, k_0)$  stable graph of minimum size. Since  $G - \{v\}$  is  $(K_3, k_0 - 1)$  stable for every vertex  $v$ , we have  $3k_0 \leq e(G - \{v\}) \leq e(G) - d_G(v) \leq 3k_0 + 3 - d_G(v)$ , that is  $d_G(v) \leq 3$ . If every vertex of  $G$  has degree equal to 2, then  $G$  is a union of  $k_0 + 1$  copies of  $K_3$ , and the theorem is proved. So we may suppose that there is a vertex  $v_0$  of degree 3. But then  $G - \{v_0\}$  is  $(K_3, k_0 - 1)$  stable and  $e(G - \{v_0\}) = 3k_0$ , that is  $G - \{v_0\}$  is minimum  $(K_3, k_0 - 1)$  stable. By the induction hypothesis,  $G - \{v_0\}$  is isomorphic to  $p'K_4 + q'K_3$ , where  $2p' + q' = k_0$ . It is clear that all the neighbours of  $v_0$  are in the same component of  $G$ , (otherwise one of the edges incident with  $v_0$  is not contained in any triangle, contrary to Proposition 1.2). Now it is easy to see that  $G$  is isomorphic to  $(p' + 1)K_4 + (q' - 1)K_3$  and  $2(p' + 1) + (q' - 1) = k_0 + 1$  (otherwise there is a set  $A$  of cardinality  $k_0$  which is transversal of all cliques of order 3 in  $G$ ).  $\square$

### 4. A characterization of $(K_4, k)$ stable graph with minimum size

In [2] the minimum number of edges of a  $(K_4, k)$  stable graph is given.



**Theorem 4.1.** [2] *If  $G$  is a  $(K_4, k)$  stable graph with minimum size ( $k \geq 1$ ) then*

- $Q(K_4, 0) = 6$ ,
- $Q(K_4, k) = 5k + 5$  when  $k \geq 1$ .

**Proposition 4.2.** *If  $G$  is a  $(K_4, k)$  stable graph with minimum size ( $k \geq 1$ ) then it has no connected component isomorphic to  $K_4$ .*

**Proof.** Let us consider  $k \geq 2$ . Assume that some component  $H$  of  $G$  is isomorphic to a  $K_4$  with the vertices of  $H$  being  $a, b, c, d$ . Then  $G - H$  has  $5k - 1$  edges. Since  $G - H$  is not a  $(K_4, k - 1)$  stable graph, there is a set  $S$  with at most  $k - 1$  vertices intersecting each  $K_4$  of  $G - H$ . Then  $S + \{a\}$  intersects each  $K_4$  of  $G$  while  $S$  has at most  $k - 1$  vertices, a contradiction.

When  $k = 1$ ,  $G$  must have 10 edges by Theorem 4.1. Since for each vertex  $v$  the graph  $G - v$  contains a  $K_4$ ,  $v$  is joined to this  $K_4$  by 4 edges. Hence  $G$  is a  $K_5$  and the result holds.  $\square$

**Proposition 4.3.** *If  $G$  is a  $(K_4, k)$  stable graph with minimum size ( $k \geq 1$ ) then every vertex of  $G$  has degree 3, 4 or 5.*

**Proof.** By Proposition 1.2 every vertex is contained in a  $K_4$ , hence its degree is at least 3. Assume that  $G$  has a vertex  $v$  with  $d(v) \geq 6$ . Then, by Lemma 1.4,  $G - v$  is a  $(K_4, k - 1)$  stable graph and therefore has at least  $5k$  edges, which is impossible since  $G$  has exactly  $5k + 5$  edges, by Theorem 4.1.  $\square$

**Proposition 4.4.** *Let  $G = (V, E)$  be a  $(K_4, k)$  stable graph with minimum size ( $k \geq 1$ ). If  $H$  is a component containing no vertex of degree 5, then each vertex of  $H$  has degree 4.*

**Proof.** By Proposition 4.3 the vertices of  $G$  have degree 3 or 4. Assume to the contrary that  $H$  contains some vertex  $v$  with degree 3. Let  $N(v) = \{u_1, u_2, u_3\}$  be its neighbourhood. By Proposition 1.2,  $N(v)$  is complete. Since  $H$  is not isomorphic to  $K_4$  by Proposition 4.2, assume that, without loss of generality,  $u_1$  is joined to some new

vertex  $w$ . Since  $u_1w$  must be contained in a  $K_4$  by Proposition 1.2, the vertices of this  $K_4$  are in  $\{u_1, u_2, u_3, w, v\}$ , thus  $w$  must be adjacent to  $u_2$  and  $u_3$ . By Proposition 2.1,  $H$  is not isomorphic to a  $K_5$  minus one edge, hence there must exist some new vertex  $w'$  adjacent to  $w$ . Since each vertex in  $\{u_1, u_2, u_3, w\}$  has degree 4, we cannot find a  $K_4$  using the edge  $ww'$ , a contradiction with Proposition 1.2.  $\square$

**Theorem 4.5.** *If  $G$  is  $(K_4, k)$  stable ( $k \geq 1$ ) with minimum size then it is isomorphic to  $pK_5 + qK_6$ , where  $p$  and  $q$  are nonnegative integers such that  $2p + 3q = k + 1$ .*

**Proof.** The proof is by induction on  $k$ . By Proposition 2.3, the only minimum  $(K_4, 1)$  stable graph with minimum size is  $K_5$ . Let  $k_0 \geq 2$  and suppose that for every integer  $k$ , such that  $1 \leq k < k_0$  every  $(K_4, k)$  stable graph with minimum size is isomorphic to  $pK_5 + qK_6$ , where  $p$  and  $q$  are nonnegative integers such that  $2p + 3q = k + 1$ .

Let  $G$  be a  $(K_4, k_0)$  stable graph with minimum size. By Theorem 4.1 we have  $e(G) = 5k_0 + 5$ . Note that it is sufficient to prove that every component of  $G$  is isomorphic either to  $K_5$  or to  $K_6$ .

By Proposition 4.3, we have  $3 \leq d_G(v) \leq 5$  for every vertex  $v$  of  $G$ . Since by Proposition 1.2, every edge of  $G$  is contained in a  $K_4$ , all the neighbours of a vertex  $v$  are in the same component of  $G - \{v\}$ .

Suppose first that there is a vertex  $v$  in  $G$  such that  $d_G(v) = 5$ . Then  $G - \{v\}$  is  $(K_4, k_0 - 1)$  stable and moreover, since  $e(G - v) = 5k_0$ ,  $G - \{v\}$  is minimum  $(K_4, k_0 - 1)$  stable. Hence every component of  $G - \{v\}$  is either isomorphic to  $K_5$  or to  $K_6$ . If  $v$  is connected in  $G$  to a  $K_6$ , then the component of  $G$  which contains  $v$  is a near complete graph, contradicting Proposition 2.1. So  $v$  is connected to a  $K_5$  and  $G$  is a union of graphs isomorphic to  $K_5$  or  $K_6$ , as desired.

Assume now that no component has a vertex of degree 5. Then, by Proposition 4.4, each component is a 4-regular subgraph.

Let  $v$  be any vertex and let  $N(v) = \{u_1, u_2, u_3, u_4\}$  be its neighbourhood. Since  $v$  is contained in a  $K_4$  by Proposition 1.2, we can suppose, without restriction of generality, that  $u_1u_2$ ,  $u_1u_3$  and  $u_2u_3$  are edges of  $G$ . Since  $vu_4$  must be contained in a  $K_4$  by Proposition 1.2,  $u_4$  must be adjacent to at least 2 vertices of  $N$  (say, without loss of generality,  $u_2$  and  $u_3$ ).

**case 1:**  $u_1u_4 \in E(G)$ . Then the component containing  $v$  is a  $K_5$ .

**case 2:**  $u_1u_4 \notin E(G)$ . Let  $w$  be a new vertex adjacent to  $u_1$  (this new vertex must exist since the component of  $v$  is 4-regular). Then  $u_1w$  cannot be contained in a  $K_4$ , a contradiction.

□

## 5. A characterization of $(K_5, k)$ stable graph with minimum size

In this section we provide the value of  $Q(K_5, k)$  for  $k \geq 5$ , as well as a description of the corresponding minimum stable graphs.

**Lemma 5.1.** *Let  $G$  be a  $(K_5, k)$  stable graph containing a component isomorphic to  $K_p$  with  $p \geq 9$ . Then the graph  $G'$  obtained from  $G$  by deleting two vertices  $v$  and  $v'$  in this  $K_p$  and adding a disjoint  $K_6$  is a  $(K_5, k)$  stable graph such that*

- if  $p \geq 10$  then  $e(G') < e(G)$ ,
- if  $p = 9$  then  $e(G') = e(G)$ .

**Proof.** Let  $A$  be the set of vertices created by the adjunction of the new  $K_6$ . Let  $S$  be a set of vertices with  $|S| \leq k$  in  $G'$ . If  $|S \cap A| \leq 1$ ,  $G - S$  obviously contains a  $K_5$ . If  $|S \cap A| \geq 2$  then  $S' = S - A + \{v, v'\}$  is a subset of  $G$  with at most  $k$  vertices. Hence  $G - S'$  contains a  $K_5$  which still exists in  $G' - S$ .

If  $p \geq 10$  then at least 17 edges are deleted and 15 edges are created, thus  $e(G') < e(G)$ . If  $p = 9$ , 15 edges are deleted while 15 edges are created so  $e(G) = e(G')$ . □

**Lemma 5.2.** *Let  $G$  be a  $(K_5, k)$  stable graph with minimum size. Then  $G$  does not contain 2 components isomorphic to a  $K_p$  with  $5 \leq p \leq 6$ .*

**Proof.** If we have two components (say  $K$  and  $L$ ) isomorphic to a complete graph with 5 vertices then the graph  $G'$  obtained from  $G$  by deleting these two components and adding a complete graph on 6 vertices is still a  $(K_5, k)$  stable graph. Indeed, let  $S'$  be any subset of  $V(G')$  with  $|S'| \leq k$ . If  $G' - S'$  does not contain any  $K_5$  then  $S'$  must

contain at least 2 vertices  $v$  and  $w$  of the new  $K_6$ . Let  $S = S' - \{v, w\} + \{a, b\}$ , where  $a \in K$  and  $b \in L$ , then  $G - S$  does not contain any  $K_5$ , a contradiction.

When we have a  $K_5$  and a  $K_6$ , we get the same kind of contradiction when replacing these two complete graphs with a  $K_7$  as well as when we have two  $K_6$ s replaced by a  $K_8$ .  
 $\square$

**Lemma 5.3.** *Let  $k \geq 5$  and let  $G$  be a  $(K_5, k)$  stable graph with minimum size which is the vertex disjoint union of complete graphs. Then each component is a  $K_7$  or a  $K_8$ .*

**Proof.** By Lemma 5.1, we can consider that each component is a  $K_p$  with  $5 \leq p \leq 9$ . By Lemma 5.2, at most one component is a  $K_5$  or a  $K_6$ . If some component is isomorphic to a  $K_9$  then let us replace this component by a  $K_6$  and a  $K_7$ . By Lemma 5.1 the resulting graph is still a  $(K_5, k)$  stable graph with minimum size. It is clear that no component is isomorphic to a  $K_9$  now. Indeed, applying once more the operation described above leads to a  $(K_5, k)$  stable graph with minimum size having two  $K_6$ s, a contradiction with Lemma 5.2.

Therefore, we have to consider only the case when  $G$  is the vertex disjoint union of complete graphs isomorphic to  $K_7$  or  $K_8$  and at most one  $K_5$  or one  $K_6$ . Replacing a  $K_5$  and a  $K_7$  by one  $K_8$  leads to a  $(K_5, k)$  stable graph with a number of edges less than the number of edges of  $G$ , a contradiction. Replacing a  $K_6$  and a  $K_8$  by two  $K_7$  leads to a  $(K_5, k)$  stable graph with a number of edges less than the number of edges of  $G$ , a contradiction.

It remains to consider the case where the components are all isomorphic to a  $K_7$  with the exception of one  $K_6$  or all isomorphic to a  $K_8$  with the exception of one  $K_5$ . When we have at least two  $K_7$  and a  $K_6$ , these three complete graphs can be replaced by two  $K_8$ , the resulting graph is still a  $(K_5, k)$  stable graph, but the number of edges is less than the number of edges of  $G$ , a contradiction. When we have at least two  $K_8$  and a  $K_5$ , these three complete graphs can be replaced by three  $K_7$ , the resulting graph is still a  $(K_5, k)$  stable graph, but the number of edges is less than the number of edges of  $G$ , a contradiction.

When  $G$  is reduced to a  $K_8$  and a  $K_5$  or to a  $K_7$  and a  $K_6$ , we must have  $k \leq 4$ , which is impossible.

□

**Lemma 5.4.** *Let  $G$  be a  $(K_5, k)$  stable graph with minimum size and maximum degree 6. Assume that some component contains a  $K_6$ . Then either the component is equal to this  $K_6$  or to  $K_7$ .*

**Proof.** Let  $A = \{v_1 \dots v_6\}$  be the set of vertices of the  $K_6$ . If  $d(v_i) = 5$  for each vertex in  $A$  the proof is complete. Assume that the vertex  $v_1$  has degree 6 and let  $w$  be its neighbour outside  $A$ . Since  $v_1 w$  must be contained in a  $K_5$  by Proposition 1.2,  $w$  must be adjacent to 3 other vertices in  $V$  (say  $v_2, v_3$  and  $v_4$ ). In the same way, if  $v_5$  or  $v_6$  has a neighbour outside  $A$ , this vertex must be adjacent to 4 vertices of  $A$ , which is impossible if this vertex is distinct from  $w$ .

Let  $w' \notin A$  be a neighbour of  $w$  (if any). Since  $ww'$  must be contained in a  $K_5$  by Proposition 1.2,  $w'$  must have at least 3 neighbours in  $A$ , which is impossible. Hence the connected component containing the  $K_6$  contains at most one vertex more (the vertex  $w$ ). If  $w$  is not adjacent to at least one of  $v_5$  or  $v_6$  (say  $v_5$ ) then this component is a near complete graph  $(R, N, u)$  on 7 vertices (with  $u = w$ ,  $N = A$  or  $N = A + \{v_5\}$ ,  $R = \{v_5, v_6\}$  or  $R = \{v_5\}$  respectively), which is impossible by Proposition 2.1. If  $w$  is adjacent to  $v_4$  and  $v_5$ , the component containing the  $K_6$  is a  $K_7$  as claimed. □

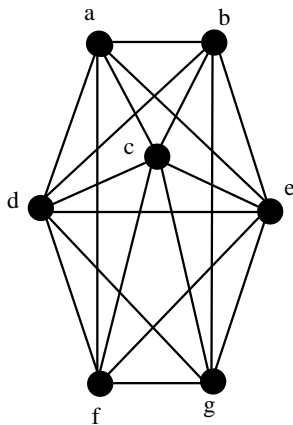


Figure 2: Forbidden component of a  $(K_5, k)$  stable graph with minimum size.

**Lemma 5.5.** *Let  $G$  be a  $(K_5, k)$  stable graph with minimum size. Then no component of  $G$  is isomorphic to the subgraph depicted in Figure 2.*

**Proof.** Since  $G - \{a\}$  is not a  $(K_5, k)$  stable graph, there exists a set  $S$  with  $|S| \leq k$  which intersects each  $K_5$  in  $G - \{a\}$ . If  $S$  contains one of the vertices in  $\{c, d, e\}$ , then  $S$  intersects each  $K_5$  in  $G$ , which is impossible. Since  $\{c, d, e, f, g\}$  induces a  $K_5$ ,  $S$  contains at least one vertex in  $\{f, g\}$ . When  $g \in S$ ,  $S$  intersects each  $K_5$  in  $G$ , which is impossible. Assume that  $f \in S$  then  $S' = S - \{f\} + \{c\}$  intersects each  $K_5$  in  $G$ , a contradiction since  $|S'| \leq k$ .  $\square$

**Lemma 5.6.** *Let  $G$  be a  $(K_5, k)$  stable graph with minimum size. Assume that some component contains vertices with degree 5 or 6 only. Then this component is a complete graph with at least 5 vertices.*

**Proof.** Let  $H$  be a connected component containing vertices of degree 5 or 6 only. By Proposition 1.2, every edge is contained in a  $K_5$ . Let  $U = \{u_1, u_2, u_3, u_4, u_5\}$  be a set of vertices inducing a  $K_5$  in  $H$ .

**case 1 :**  $\exists i \ 1 \leq i \leq 5 \ d_H(u_i) = 6$ .

Without loss of generality we may assume that  $i = 1$ . Let  $w$  and  $w'$  the two neighbours of  $u_1$  outside  $U$ . Since  $u_1w$  must be contained in a  $K_5$  and since this  $K_5$  contains 4 neighbours of  $u_1$ ,  $w$  must be adjacent to at least two vertices in  $U - \{u_1\}$ . Without loss of generality, assume that  $wu_2 \in E(G)$  and  $wu_3 \in E(G)$ . Let us remark that  $w$  is not joined to the two vertices  $u_4$  and  $u_5$ , otherwise,  $H$  contains a  $K_6$  and  $H$  is thus isomorphic to a complete graph by Lemma 5.4. For the same reason,  $w'$  is not joined to all the vertices in  $U$ .

**subcase 1.1 :** If  $w$  or  $w'$  has no other neighbour in  $U$ , say  $w$ , we must have  $ww' \in E(G)$ ,  $w'u_2 \in E(G)$  and  $w'u_3 \in E(G)$ . One of  $u_4$  or  $u_5$ , say  $u_4$ , is not adjacent to  $w'$ , and there must be a vertex  $w''$  adjacent to  $u_4$  ( $d_H(u_4) \geq 5$ ), but the edge  $u_4w''$  cannot be on any  $K_5$ , which is impossible.

**subcase 1.2 :** If  $w$  has an other neighbour (say  $u_5$ ) in  $U$ . When  $w'$  is not adjacent to  $w$ ,  $w'$  must be adjacent to precisely 3 vertices in  $\{u_2, u_3, u_4, u_5\}$ . If  $u_4w'$  is an edge, there must be an edge incident with  $w'$  ( $d_H(w') \geq 5$ ), but this edge cannot be contained

in any  $K_5$ , a contradiction. If  $u_4w'$  is not an edge, there must be an edge incident with  $u_4$  and this edge cannot be contained in any  $K_5$ , which is impossible. Thus,  $w$  and  $w'$  are adjacent and there must be 2 vertices in  $\{u_2, u_3, u_5\}$  adjacent to  $w'$ , say  $u_2$  and  $u_3$ . But now, there is an additional edge incident with  $u_4$  and this edge is  $u_4w'$  otherwise it is not contained in any  $K_5$ . It is a routine matter to check that there is no additional vertex nor edge in  $H$ . Hence  $H$  is isomorphic to the graph depicted in Figure 2, a contradiction with Lemma 5.5.

**case 2 :**  $\forall i \quad 1 \leq i \leq 5 \quad d_H(u_i) = 5$ .

Let  $w$  be the last neighbour of  $u_1$  outside  $U$ . Since  $wu_1$  must be contained in a  $K_5$ ,  $w$  must be adjacent to  $u_2, u_3$  and  $u_4$ , without loss of generality. Hence,  $wu_5 \notin E(G)$  or  $H$  is complete. Since  $d_H(u_5) = 5$ , let  $w' \neq w$  be the last neighbour of  $u_5$  outside  $U$ . Then  $u_5w'$  is not contained in a  $K_5$ , which is impossible.

□

**Lemma 5.7.**  $Q(K_5, 4) = 36$ .

**Proof.** Since  $K_9$  and  $K_6 + K_7$  are  $(K_5, 4)$  stable graphs, we certainly have  $Q(K_5, 4) \leq 36$ .

Assume that some graph  $G$  with  $e(G) \leq 35$  is a  $(K_5, 4)$  stable graph with minimum size. Let  $v$  be a vertex with maximum degree. If  $d(v) \geq 8$  then  $G - v$  is a  $(K_5, 3)$  stable graph with at most 27 edges, a contradiction with Proposition 2.7. If  $d(v) = 7$  then  $G - \{v\}$  is a  $(K_5, 3)$  stable graph with at most 28 edges. Hence we must have  $e(G - \{v\}) = 28$  and  $G$  is a  $(K_5, 3)$  stable graph with minimum size. By Proposition 2.7,  $G - \{v\}$  is a  $K_8$  and  $G$  is a  $K_9$  minus one edge, a contradiction with Proposition 2.1.

We can thus assume that the maximum degree of  $G$  is at most 6. If some vertex  $u$  has degree 4, let  $v$  be one of its neighbours. We know, by Lemma 2.6 that  $d(v) = 6$ . By deleting  $v$ , we get a graph  $G - v$  which is a  $(K_5, 3)$  stable graph. In that graph, the edges incident with  $u$  are not contained in a  $K_5$  since the degree of  $u$  is now 3. We can thus delete these edges and we obtain a  $(K_5, 3)$  stable graph with at most 27 edges, a contradiction with Proposition 2.7.

Hence every vertex must have degree 5 or 6. By Lemma 5.6, the components of  $G$  are complete graphs. It can be easily checked that the only convenient graphs are  $K_9$

and  $K_6 + K_7$ , a contradiction with  $e(G) \leq 35$ . □

**Lemma 5.8.**  *$K_6 + K_7$  and  $K_9$  are the unique  $(K_5, 4)$  stable graph with minimum size.*

**Proof.** By Lemma 5.7, let  $G$  be a  $(K_5, 4)$  stable graph with 36 edges.

If  $G$  has a vertex of degree at least 8 then  $G - \{v\}$  is a  $(K_5, 4)$  stable graph with at most 28 edges. Hence  $G - \{v\}$  must have exactly 28 edges and  $d(v) = 8$ . Since, by Proposition 2.7  $G - \{v\}$  is a  $K_8$ ,  $G$  itself is a  $K_9$ .

We can thus assume that the maximum degree of  $G$  is at most 7. If some vertex  $u$  has degree 4, let  $v$  be one of its neighbours. We know, by Lemma 2.2 that  $d(v) \geq 6$ . By deleting  $v$ , we get a graph  $G - \{v\}$  which is a  $(K_5, 3)$  stable graph. In that graph, the edges incident with  $u$  are not contained in a  $K_5$  since the degree of  $u$  is now 3. We can thus delete these edges and we obtain a  $(K_5, 3)$  stable graph with 27 edges, a contradiction with Proposition 2.7.

Hence the degree of each vertex is 5, 6 or 7.

In the following Claims  $Q_1$  and  $Q_2$  denote any two induced  $K_5$  of  $G$ .

**Claim 5.8.1.**  $|V(Q_1) \cap V(Q_2)| \neq 1$ .

**Proof** Assume that  $|V(Q_1) \cap V(Q_2)| = 1$  then the vertex in the intersection must have degree at least 8, which is impossible. □

**Claim 5.8.2.** *Assume that  $Q_1$  and  $Q_2$  are vertex disjoint and let  $xy \in E(G)$  (if any) such that  $x \in V(Q_1)$  and  $y \in V(Q_2)$ . Then we can find a vertex  $x' \in V(Q_1)$  and a vertex  $y' \in V(Q_2)$  such that  $\{x, x', y, y'\}$  is contained in an induced  $K_5$  of  $G$ . Moreover the 5<sup>th</sup> vertex of this  $K_5$  must be contained in  $V(Q_1) \cup V(Q_2)$ .*

**Proof**

Since  $G$  is a minimum  $(K_5, 4)$  stable graph, the edge  $xy$  must be contained in a  $K_5$  (say  $Q$ ). By Claim 5.8.1  $Q$  contains at least one vertex more in  $Q_1$  (say  $x'$ ) and one vertex more in  $Q_2$  (say  $y'$ ). Let  $a$  be the 5<sup>th</sup> vertex of  $Q$  and assume that  $a \notin V(Q_1) \cup V(Q_2)$ .  $G - \{a\}$  is a  $(K_5, 3)$  stable graph but it is not minimum since the edges between  $\{x, x'\}$  and  $\{y, y'\}$  cannot be contained in a  $K_5$ . By deleting these 4 edges in  $G - \{a\}$  we get a



$(K_5, 3)$  stable with 28 edges. By Proposition 2.7,  $G - \{a\}$  is isomorphic to  $K_8$ , which is impossible.  $\square$

**Claim 5.8.3.**  $|V(Q_1) \cap V(Q_2)| \neq 2$ .

**Proof** Assume that  $V(Q_1) \cap V(Q_2) = \{x, y\}$ . Let us remark that these two vertices have degree 7. Let  $\{u_1, u_2, u_3\}$  and  $\{v_1, v_2, v_3\}$  be the sets of remaining vertices of  $Q_1$  and  $Q_2$  respectively.

Assume that some edge is missing between  $\{u_1, u_2, u_3\}$  and  $\{v_1, v_2, v_3\}$  (say  $u_1v_1 \notin E(G)$ ). Then  $G_1 = G - \{u_2, v_2, v_3\}$  is a  $(K_5, 1)$  stable graph in which the vertices  $x$  and  $y$  are not contained in any  $K_5$ . Hence  $G_2 = G_1 - \{x, y\}$  is a  $(K_5, 1)$  stable graph. Since  $d_G(v_1) \leq 7$ , the degree of  $v_1$  in  $G_2$  is at most 3. Hence  $v_1$  is not contained in any  $K_5$  and  $G_3 = G_2 - \{v_1\}$  is  $(K_5, 1)$  stable graph.

**case 1 :** *The edge  $u_1u_3$  is not contained in a  $K_5$ .*

Then  $G_4 = G_3 \setminus \{u_1, u_3\}$  is a  $(K_5, 1)$  stable graph. By Proposition 2.3,  $G_4$  contains at least 15 edges. Since  $V(Q_1) \cup V(Q_2)$  contains 19 edges, we need to find two more edges. By Claim 5.8.2 no edge can connect  $V(Q_1) \cup V(Q_2)$  to  $G_4$ . Whatever is the place of these edges,  $G - \{x, y\}$  is a  $(K_5, 2)$  stable graph, where no vertex in  $\{u_1, u_2, u_3\}$  nor in  $\{v_1, v_2, v_3\}$  can be contained in a  $K_5$ . Hence  $G - (V(Q_1) \cup V(Q_2))$  is a  $(K_5, 2)$  stable graph and must contain at least 21 edges by Proposition 2.5. That is  $G$  must contain at least 40 edges, a contradiction.

**case 2 :** *The edge  $u_1u_3$  is contained in a  $K_5$ .*

That means that  $u_1$  and  $u_3$  have 3 neighbours outside  $V(Q_1) \cup V(Q_2)$ . In the same way, we can consider that  $u_2$  has also three such neighbours (take  $G_1 = G - \{u_3, v_2, v_3\}$ ) as well as  $v_1, v_2$  and  $v_3$  by symmetry. Hence  $G_3$  contains the 19 edges of  $V(Q_1) \cup V(Q_2)$  and 18 edges connecting  $\{u_1, u_2, u_3\}$  and  $\{v_1, v_2, v_3\}$  to the vertices outside, a contradiction.

We can thus suppose that every vertex in  $\{u_1, u_2, u_3\}$  is joined to every vertex in  $\{v_1, v_2, v_3\}$ . That means that  $V(Q_1) \cup V(Q_2)$  is a connected component of  $G$  and induces a  $K_8$ . No connected component distinct from this  $K_8$  can contain a  $K_5$ , which is impossible.  $\square$

**Claim 5.8.4.**  $|V(Q_1) \cap V(Q_2)| \neq 3$  or  $G$  is isomorphic to  $K_7 + K_6$ .

**Proof** Assume that  $|V(Q_1) \cap V(Q_2)| = \{x, y, z\}$ . Let  $\{u_1, u_2\}$  and  $\{v_1, v_2\}$  be the sets of remaining vertices of  $Q_1$  and  $Q_2$  respectively.

Then  $G_1 = G - \{x, y, z\}$  is a  $(K_5, 1)$  stable graph in which the vertices  $u_1, u_2, v_1, v_2$  are not contained in any  $K_5$  by Claims 5.8.1 and 5.8.3. That means that  $G_2 = G - (V(Q_1) \cup V(Q_2))$  is a  $(K_5, 1)$  stable graph. If  $w \in V(Q_1) \cup V(Q_2)$  is adjacent to some vertex  $w'$  in  $G_2$  then a  $K_5$  using that edge forces 4 more edges more between these two subgraphs, a contradiction since  $G$  would have at least 37 edges (by Proposition 2.3  $G_2$  has at least 15 edges).

If some edge is missing between  $\{u_1, u_2\}$  and  $\{v_1, v_2\}$  (say  $u_1v_1 \notin E(G)$ ), then  $G_3 = G - \{u_2, v_2\}$  is a  $(K_5, 2)$  stable graph where  $x, y, z, u_1, v_1$  are not contained in any  $K_5$ . The graph  $G_3$  is still  $(K_5, 2)$  stable. Hence, by Proposition 2.5  $G$  must have at least 38 edges, a contradiction.

We can thus suppose that  $V(Q_1) \cup V(Q_2)$  induces a  $K_7$ . The remaining part of  $G$  is the  $(K_5, 1)$  stable graph  $G_2$  described above. This graph must have exactly 15 edges. Hence,  $G_2$  is isomorphic to  $K_6$  by Proposition 2.3. That means that  $G$  is isomorphic to  $K_7 + K_6$ .  $\square$

**Claim 5.8.5.**  $|V(Q_1) \cap V(Q_2)| \neq 4$  or  $G$  is isomorphic to  $K_7 + K_6$ .

**Proof** Assume on the contrary that  $|V(Q_1) \cap V(Q_2)| = \{x, y, z, t\}$  and  $G$  is not isomorphic to  $K_7 + K_6$ . Let  $u$  and  $v$  be the remaining vertices of  $Q_1$  and  $Q_2$  respectively.

Let  $r$  be a neighbour of  $u$ , if any, outside  $V(Q_1) \cup V(Q_2)$ . Let  $Q_3$  be a  $K_5$  containing the edge  $ur$ . Then  $V(Q_1) \cap V(Q_3)$  contains 4 vertices (Claims 5.8.1 and 5.8.3) but  $V(Q_2) \cap V(Q_3)$  contains 3 vertices, a contradiction.

Since  $d(u) \geq 5$ , we must have  $uv \in E(G)$  (and, moreover,  $d(u) = d(v) = 5$ ).

**case 1 :** *There are neighbours of  $\{x, y, z, t\}$  outside  $V(Q_1) \cup V(Q_2)$ .*

Let  $s$  be such a neighbour of  $x$ . The edge  $xs$  being contained in a  $K_5$ , this  $K_5$  must have 4 common vertices with  $Q_1$  and 4 common vertices with  $Q_2$  (Claims 5.8.1, 5.8.3 and 5.8.4). Hence,  $s$  must be adjacent to the 4 vertices of  $V(Q_1) \cap V(Q_2)$  and  $\{x, y, z, t, s\}$  induces a  $K_5$  with 4 common vertices with  $Q_1$  and 4 common vertices with  $Q_2$ . By the

above remark, we have  $us \in E(G)$  as well as  $vs \in E(G)$  and  $V(Q_1) \cup V(Q_2)$  induces a  $K_7$ . By deleting 3 vertices of this component, the resulting graph is  $(K_5, 1)$  stable with 15 edges, and hence is isomorphic to  $K_6$ .

**case 2 :** *There are no neighbours of  $\{x, y, z, t\}$  outside  $V(Q_1) \cup V(Q_2)$ .*

Hence,  $V(Q_1) \cup V(Q_2)$  is a connected component of  $G$  inducing a  $K_6$ . By deleting 2 vertices in this component, the resulting graph is  $(K_5, 2)$  stable. Since the remaining vertices of  $V(Q_1) \cup V(Q_2)$  in this graph are not contained in any  $K_5$ , we can delete them and the  $(K_5, 2)$  stable graph obtained in this way must have 21 edges exactly. This component is a  $K_7$  by Proposition 2.5, a contradiction.  $\square$

To end our proof, it is sufficient to say that any two induced  $K_5$  of  $G$  must be disjoint by Claims 5.8.1, 5.8.3, 5.8.4 and 5.8.5. That means that each component of  $G$  is a  $K_5$ , which is impossible since  $G$  must have 36 edges.  $\square$

**Lemma 5.9.**  $Q(K_5, 5) = 42$ .

**Proof.** Since  $K_7 + K_7$  is a  $(K_5, 5)$  stable graphs, we certainly have  $Q(K_5, 5) \leq 42$ . Let  $G$  be a  $(K_5, 5)$  stable graph with minimum size and assume that  $e(G) \leq 41$ . Let us remark that the size of  $G$  is certainly greater than  $Q(K_5, 4)$ .

If  $G$  has a vertex of degree at least 6 then  $G - v$  is a  $(K_5, 4)$  stable graph with at most 35 edges, a contradiction with Lemma 5.7. If  $G$  has a vertex of degree 4 then, since the degree of every neighbour is at most 5, we must have, by Lemma 2.2,  $Q(K_5, 5) \geq Q(K_5, 4) + 9$ , a contradiction.

Hence, every vertex must have degree 5 and by Lemma 5.6, the connected component of  $G$  are isomorphic to  $K_6$ . It is easy to see that no such graph can exist.  $\square$

**Lemma 5.10.**  $K_7 + K_7$  is the unique  $(K_5, 5)$  stable graph with minimum size.

**Proof.** By Lemma 5.9, let  $G$  be a  $(K_5, 5)$  stable graph with 42 edges.

If  $G$  has a vertex of degree at least 7 then  $G - v$  is a  $(K_5, 4)$  stable graph with at most 35 edges, a contradiction with Lemma 5.7.

If  $G$  has a vertex  $u$  of degree 4, let  $v$  be one of its neighbours. By deleting  $v$  we get a  $(K_5, 4)$  stable graph where the edges incident with the vertex  $u$  are not contained in any  $K_5$  since the degree of  $u$  in that graph is 3. By deleting these edges we get a  $(K_5, 4)$  stable graph with at most 35 edges, a contradiction with Lemma 5.7.

Hence every vertex has degree 5 or 6. By Lemma 5.6, the connected components of  $G$  are complete. It is an easy task to see that the only convenient graph  $G$  is isomorphic to  $K_7 + K_7$ , as claimed.  $\square$

**Theorem 5.11.** *If  $G$  is  $(K_5, k)$  stable ( $k \geq 5$ ) with minimum size then  $|E(G)| = 7k + 7$ .*

**Proof.** We can check that the property holds for  $k = 5$  ( $G$  is the vertex disjoint union of two  $K_7$ s by Lemma 5.10). Assume that the property holds for any  $k$  ( $5 \leq k < k_0$ ) and let us consider a  $(K_5, k_0)$  stable graph  $G$  with minimum size. Assume that  $G$  has at most  $7k_0 + 6$  edges and let  $v$  be a vertex of maximum degree. Since  $G - v$  is a  $(K_5, k_0 - 1)$  stable graph, it must have  $7k_0$  edges, which means that  $d(v) \leq 6$ . Moreover, by Proposition 1.2, we certainly have  $d(v) \geq 4$ .

Let  $z$  be a vertex of degree 4 in some component of  $G$ . If  $z$  has a neighbour  $v$  whose degree is 6 then  $G - v$  has exactly  $7k_0$  edges. Hence  $G - v$  is a  $(K_5, k_0 - 1)$  stable graph with minimum size. Since the degree of  $z$  is 3 in  $G - v$ , any edge incident with  $z$  in  $G - v$  is not contained in a  $K_5$ , a contradiction.

If  $z$  has a neighbour  $v$  whose degree is 5 then  $G - v$  has exactly  $7k_0 + 1$  edges.  $G - v$  is a  $(K_5, k_0 - 1)$  stable graph. This graph does not have minimum size since the 3 remaining edges incident with  $z$  are not contained in a  $K_5$ . If we delete these 3 edges, we still have a  $(K_5, k_0 - 1)$  stable graph, but the number of edges is then  $7k_0 - 2$ , which is impossible by the induction hypothesis.

Hence the neighbours of  $z$  have also degree 4, that means that the component containing a vertex of degree 4 is a 4 regular graph containing a  $K_5$ . That is, this component is a  $K_5$ .

Since each component containing only vertices of degree 5 or 6 are complete by Lemma 5.6, we have thus that all the connected components of  $G$  are complete. By Lemma 5.3, each component has 7 vertices or 8 vertices (recall that  $k_0 \geq 5$ ). Assume that we have  $p$  components isomorphic to a  $K_7$  and  $q$  isomorphic to a  $K_8$ , then  $k_0 \leq 3p + 4q - 1$  and  $G$

has  $21p + 28q$  edges. If  $k_0 = 3p + 4q - 1$ , we have  $21p + 28q = 7k_0 + 7$ , a contradiction. If  $k_0 < 3p + 4q - 1$  then deleting one vertex in some component leaves the graph  $(K_5, k_0)$  stable, which is impossible.  $\square$

Dudek, Szymański and Zwonek proposed the following conjecture.

**Conjecture 5.12.** [2] *For every integer  $q \geq 5$  there is an integer  $k(q)$  such that  $Q(K_q, k) = (2q - 3)(k + 1)$  for  $k \geq k(q)$ .*

Theorem 5.11 proves this conjecture for  $q = 5$  with  $k(q) = 5$ .

**Theorem 5.13.** *If  $G$  is  $(K_5, k)$  stable ( $k \geq 5$ ) with minimum size then*

- $|E(G)| = 7k + 7$ ,
- *each connected component is isomorphic to a complete graph with 7 or 8 vertices,*
- *there are  $p$  components isomorphic to  $K_7$  and  $q$  components isomorphic to  $K_8$  for any choice of  $p$  and  $q$  with  $3p + 4q = k + 1$ .*

**Proof.** By Theorem 5.11, the first claim is true. We can check that the property of the second claim holds for  $k = 5$  ( $G$  is the vertex disjoint union of two  $K_7$ s). Assume that the property holds for any  $k$  ( $5 \leq k < k_0$ ) and let us consider a  $(K_5, k_0)$  stable graph  $G$  with minimum size.

If  $G$  has a vertex  $v$  of degree at least 8, then  $G - v$  has at most  $7k_0 - 1$  edges and cannot be a  $(K_5, k_0 - 1)$  stable graph, a contradiction. Thus the maximum degree of  $G$  is at most 7.

**case 1 :**  $\exists v \in V(G) \quad d_H(v) = 7$ .

In that case,  $G - v$  is  $(K_5, k_0 - 1)$  stable graph with minimum size. By the induction hypothesis, each connected component of  $G - v$  is isomorphic to a complete graph with 7 or 8 vertices. Going back to  $G$  by adding the vertex  $v$  leads to join  $v$  to a whole connected component of  $G - v$ , otherwise, some edge incident with  $v$  cannot be contained in a  $K_5$ , a contradiction with Proposition 1.2. The vertex  $v$  cannot be connected to 7 vertices of a  $K_8$ , otherwise we would have a near complete graph, a contradiction. Hence  $v$  is joined to the 7 vertices of a  $K_7$  and the connected component of  $G$  containing  $v$  is a  $K_8$ .

**case 2 :** If some connected component of  $G$  contains vertices of degree 5 or 6 only, then, by Lemma 5.6, this component is a complete graph on at least 7 or 8 vertices (Lemma 5.3), since  $k_0 > 5$ .

**case 3 :** If some connected component of  $G$  contains a vertex  $v$  of degree 4 then, no neighbour  $w$  of  $v$  may have a degree at least 5. Otherwise,  $G - w$  is a  $(K_5, k_0 - 1)$  stable graph with at most  $7k_0 + 2$  edges. Since the degree of  $v$  is 3 in  $G - w$ , the 3 edges incident with  $v$  are not contained in any  $K_5$ . We can thus delete these 3 edges from  $G - w$ , getting a  $(K_5, k_0 - 1)$  stable graph with at most  $7k_0 - 1$  edges, which is impossible by Theorem 5.11. Hence this component is 4-regular. That is, this component is reduced to a  $K_5$ , a contradiction with Lemma 5.3 since  $k_0 > 5$ .

It is now a routine matter to check that the third claim holds. □

**Acknowledgement.** APW wishes to thank *Laboratoire d'Informatique Fondamentale de l'Université d'Orléans*, where a part of this paper was written, for the invitation and hospitality.

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