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# DIVIDED DIFFERENCES & RESTRICTION OPERATOR ON PALEY-WIENER SPACES $PW_\tau^p$ FOR $N$ -CARLESON SEQUENCES

FRÉDÉRIC GAUNARD

ABSTRACT. For a sequence of complex numbers  $\Lambda$  we consider the restriction operator  $R_\Lambda$  defined on Paley-Wiener spaces  $PW_\tau^p$  ( $1 < p < \infty$ ). Lyubarskii and Seip gave necessary and sufficient conditions on  $\Lambda$  for  $R_\Lambda$  to be an isomorphism between  $PW_\tau^p$  and a certain weighted  $l^p$  space. The Carleson condition appears to be necessary. We extend their result to  $N$ -Carleson sequences (finite unions of  $N$  disjoint Carleson sequences). More precisely, we give necessary and sufficient conditions for  $R_\Lambda$  to be an isomorphism between  $PW_\tau^p$  and an appropriate sequence space involving divided differences.

## 1. INTRODUCTION

Let  $X$  be a Banach space of analytic functions defined on a domain  $\Omega$  of the complex plane and  $\Lambda$  a sequence of points lying in  $\Omega$ . The *restriction operator*  $R_\Lambda$  associated to  $\Lambda$  is defined on  $X$  by

$$R_\Lambda : X \ni f \mapsto (f(\lambda))_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda.$$

Our aim is to describe the range of  $R_\Lambda$ , denoted by  $X|_\Lambda$ , as well as the injectivity of  $R_\Lambda$ . This problem is related to interpolation problems in  $X$  and to geometrical properties of reproducing kernels in  $X^*$ . See [10], [16, Part D] or [18].

In the late 1950s and early 1960s, Carleson [4] ( $p = \infty$ ) and Shapiro and Shields [19] ( $1 \leq p < \infty$ ) showed that  $R_\Lambda$  is surjective from the Hardy space onto a suitable weighted  $l^p$  space if and only if  $\Lambda$  satisfies a certain separation condition, the so-called *Carleson condition* (more precise definitions below). Notice that, in Hardy spaces, as soon as the sequence satisfies the Blaschke condition,  $R_\Lambda$  cannot be injective.

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The results of Carleson and Shapiro-Shields have been generalized to finite unions of Carleson sequences (which are called  $N$ -Carleson sequences) by Vasyunin [21] ( $p = \infty$ ) and Hartmann [8] ( $1 < p < \infty$ ). A similar result has been obtained by Bruna, Nicolau and Øyma [3]. In this more general situation the description of the range of  $R_\Lambda$  involves divided differences.

Many authors like Hrushev, Nikolskii, Pavlov [10] or Minkin [13], have investigated interpolation problems in Paley-Wiener spaces using tools from operator theory (for instance invertibility criteria for a suitable Toeplitz operator) since the 1970s. Note that these spaces can be considered as special cases of backward shift invariant subspaces in Hardy spaces. More recently, Lyubarskii and Seip [12] have characterized the sequences  $\Lambda$  for which the associated restriction operator is an isomorphism between the Paley-Wiener space and an appropriate weighted  $l^p$  space. Their proof is in a sense more elementary and allows to consider sequences defined on the whole complex plane while the methods of Hrushev, Nikolskii, Pavlov intrinsically restrict the problem to sequences in a half-plane.

Here we investigate a generalization of Lyubarskii and Seip's result to  $N$ -Carleson sequences, in the spirit of Hartmann. Observe first that the Carleson condition turns out to be necessary for the classical interpolation problem in the Paley-Wiener space. Now, starting from an  $N$ -Carleson sequence  $\Lambda$ , we want to find necessary and sufficient conditions on  $\Lambda$  for  $R_\Lambda$  to be an isomorphism between the Paley-Wiener space and an appropriate sequence space involving now divided differences.

Let us fix the notation and the results we mentioned above. We first recall the definition of the *Hardy space*, for  $1 \leq p < \infty$ ,

$$H^p(\mathbb{C}_a^\pm) := \left\{ f \in \text{Hol}(\mathbb{C}_a^\pm) : \sup_{y \geq a} \int_{\mathbb{R}} |f(x + iy)|^p dx < \infty \right\}$$

on the half-plane

$$\mathbb{C}_a^\pm := \{z \in \mathbb{C} : \text{Im}(z) \gtrless a\}, \quad (a \in \mathbb{R}).$$

For  $p = \infty$ ,

$$H^\infty(\mathbb{C}_a^\pm) := \left\{ f \in \text{Hol}(\mathbb{C}_a^\pm) : \sup_{z \in \mathbb{C}_a^\pm} |f(z)| < \infty \right\}.$$

For short we will write  $\mathbb{C}^\pm := \mathbb{C}_0^\pm$  and  $H_\pm^p := H^p(\mathbb{C}^\pm)$ . A function  $I \in H^\infty(\mathbb{C}_a^\pm)$  satisfying  $|I(x + ia)| = 1$  a.e.  $x \in \mathbb{R}$  is called an *inner function*.

As previously mentioned, Carleson [4], Shapiro and Shields [19] solved the interpolation problem in the Hardy space. Their results were obtained in the unit disk, but translate clearly to any half-plane. Setting

$$l^p(|\operatorname{Im}(\lambda_n) - a|) := \left\{ u = (u_n)_{n \geq 1} : \sum_{n \geq 1} |\operatorname{Im}(\lambda_n) - a| |u_n|^p < \infty \right\},$$

we can state their result as follows. If  $\Lambda = \{\lambda_n : n \geq 1\} \subset \mathbb{C}_a^\pm$ , then

$$H^p(\mathbb{C}_a^\pm) | \Lambda = l^p(|\operatorname{Im}(\lambda_n) - a|)$$

if and only if  $\Lambda$  satisfies the *Carleson* condition

$$(1.1) \quad \inf_{\lambda \in \Lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left| \frac{\lambda - \mu}{\lambda - \bar{\mu} - 2ia} \right| > 0.$$

Such sequences will be simply called *Carleson sequences*.

We consider now the *Paley-Wiener space*  $PW_\tau^p$  (for  $1 \leq p < \infty$ ) which consists of all entire functions of exponential type at most  $\tau$  satisfying

$$\|f\|_p^p = \int_{\mathbb{R}} |f(x)|^p dx < \infty.$$

It is well-known (see e.g. [11]) that in the case  $p = 2$ , the Fourier transform is an isometric isomorphism between  $PW_\tau^2$  and  $L^2(-\tau, \tau)$  which allows to reformulate the problem in terms of geometrical properties of exponentials in  $L^2$  (we still refer to [10]). From the Plancherel-Polyà inequality (see Proposition 20 below), it follows that  $PW_\tau^p = e^{-i\tau \cdot} K_{I_\tau}^p$ , where

$$K_{I_\tau}^p := H_+^p \cap \overline{I_\tau} H_-^p$$

is the backward shift invariant subspace associated with the inner function  $I_\tau(z) := \exp(2i\tau z)$ ,  $z \in \mathbb{C}^+$ . In particular, the Paley-Wiener space can be considered as a subspace of the Hardy space.

Luybarskii and Seip [12] gave necessary and sufficient conditions for  $R_\Lambda$  to be an isomorphism from  $PW_\tau^p$  onto the weighted sequence space  $l^p(e^{-p\tau|\operatorname{Im}(\lambda_n)|} (1 + |\operatorname{Im}(\lambda_n)|))$ . Their proof is based on the boundedness of the Hilbert transform in certain weighted Hardy space.

Recall that the *Hilbert transform*  $\mathcal{H}$  is defined by

$$(1.2) \quad \mathcal{H}f(z) = \int_{-\infty}^{+\infty} \frac{f(t)}{t - z} dt,$$

where the integral has to be understood as a principle value integral for real  $z$ . It is known (see e.g [9] and [5]) that, if  $w > 0$ ,  $\mathcal{H}$  is bounded

from the weighted space

$$L^p(w) := \left\{ f \text{ meas. on } \mathbb{R} : \int_{\mathbb{R}} |f|^p w dm < \infty \right\}$$

into itself, if and only if  $w$  satisfies the *Muckenhoupt*  $(A_p)$  condition

$$(A_p) \quad \sup_I \left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{\frac{p}{p-1}} < \infty,$$

where the supremum is taken over all intervals of finite length. In [12], the authors also introduce the *discrete Hilbert transform* as follows. For fixed  $\epsilon > 0$  and two sequences  $\Gamma := \{\gamma_n\}_n$  and  $\Sigma := \{\sigma_n\}_n$  satisfying  $|\gamma_n - \sigma_n| = \epsilon$ , and  $a = (a_n)_n$ ,

$$(\mathcal{H}_{\Gamma, \Sigma}(a))_n := \sum_j \frac{a_j}{\gamma_j - \sigma_n}.$$

According to [12, Lemma 1]),  $\mathcal{H}_{\Gamma, \Sigma}$  is bounded from  $l^p(w_n)$  into itself if and only if  $(w_n)_n$  satisfies the *discrete Muckenhoupt condition*

$$(\mathfrak{A}_p) \quad \sup_{\substack{k \in \mathbb{Z} \\ n > 0}} \left( \frac{1}{n} \sum_{j=k+1}^{k+n} w_j \right) \left( \frac{1}{n} \sum_{j=k+1}^{k+n} w_j^{-1/(p-1)} \right)^{p-1} < \infty.$$

**Definition 1.** A sequence  $\Lambda \subset \mathbb{C}$  satisfies the condition  $(LS)_{\tau, p}$  for  $\tau > 0$  and  $1 < p < \infty$ , if the following set of conditions hold:

- (i)  $\forall a \in \mathbb{R}$ ,  $\Lambda \cap \mathbb{C}_a^{\pm}$  satisfies the Carleson condition (1.1);
- (ii) The sequence is *relatively dense*:  $\exists r > 0$ ,  $\forall x \in \mathbb{R}$ ,

$$d(x, \Lambda) := \inf_{\lambda \in \Lambda} |x - \lambda| < r;$$

- (iii) The limit

$$S(z) = \lim_{R \rightarrow \infty} \prod_{|\lambda| < R} \left( 1 - \frac{z}{\lambda} \right)$$

exists and defines an entire function of exponential type  $\tau$ ;

- (iv) The function  $x \mapsto \left( \frac{|S(x)|}{d(x, \Lambda)} \right)^p$  satisfies  $(A_p)$ .

Note that if  $0 \in \Lambda$ , then the corresponding factor in (iii) reduces to  $z$ . In order to not complicate the notation we shall assume in all what follows that  $0 \notin \Lambda$  which we can do without loss of generality (for instance, by shifting the sequence). We are now in a position to state the Lyubarski-Seip theorem [12, Theorem 1].

**Theorem 2.** (*Lyubarskii-Seip*). *Let  $\Lambda \subset \mathbb{C}$ ,  $\tau > 0$  and  $1 < p < \infty$ . The following assertions are equivalent.*

- (1)  $R_{\Lambda}$  is an isomorphism from  $PW_{\tau}^p$  onto  $l^p(e^{-p\tau|\operatorname{Im}(\lambda)|} (1 + |\operatorname{Im}(\lambda)|))$ ;
- (2)  $\Lambda$  satisfies  $(LS)_{\tau, p}$ .

*Remark 3.* The condition (iv) can be replaced by the condition (iv)'

- (iv)' There is a relatively dense subsequence  $\Gamma = (\gamma_n)_n \subset \Lambda$  such that the sequence  $(|S'(\gamma_n)|^p)_n$  satisfies the discrete Muckenhoupt condition  $(\mathfrak{A}_p)$ .

The aim of this paper is to generalize the Lyubarskii-Seip result to finite unions of Carleson sequences. In the case of Hardy spaces, this problem has been solved by Vasyunin [21] and Hartmann [8] and involves divided differences.

As mentioned previously, in the case  $p = 2$  the Fourier transform allows to express our main result Theorem 17 in terms of bases of exponentials in  $L^2$  thereby generalizing a result by Avdonin and Ivanov [2, Theorem 3].

This paper is organized as follows. The next section will be devoted to divided differences. Section 3 deals with  $N$ -Carleson sequences. We will state our main result after some technical constructions in the fourth section. For an easier reading, we have postponed the proofs of Section 4 to the fifth section. Finally, in the last section we will discuss the necessity of the  $N$ -Carleson condition with an appropriate definition of the trace  $PW_\tau^p|_\Lambda$ .

A final word on notation. If  $\delta$  is a metric on  $\Omega$ , we will denote by  $D_\delta(x, \eta)$  the ball (relatively to  $\delta$ ) with center  $x \in \Omega$  and radius  $\eta > 0$ , and  $\text{diam}_\delta(E)$  the  $\delta$ -diameter of  $E$ . We shortly write  $\text{diam}(E)$  and  $D(x, \eta)$  when  $\delta$  is the Euclidian distance. If  $\omega = (\omega_n)_{n \geq 1}$  is a sequence of strictly positive numbers and  $1 \leq p < \infty$ , we denote by  $l^p(\omega)$  or  $l^p(\omega_n)$  the space

$$l^p(\omega) := \left\{ a = (a_n)_{n \geq 1} : \sum_{n \geq 1} |a_n|^p \omega_n < \infty \right\}.$$

## 2. DIVIDED DIFFERENCES

Divided differences appear in many results about interpolation or bases of exponentials (see e.g. [21], [8], [3] or [2]). Here we will give the definitions and some properties that we will need later on. We recall that the (non-normalized) Blaschke factors in a half-plane  $\mathbb{C}_a^\pm$  are given by

$$b_\mu^{\pm, a}(z) = \frac{z - \mu}{z - \bar{\mu} - 2ia}.$$

(The formula is actually the same for the upper and the lower half-plane). The associated *pseudohyperbolic distance* will be denoted by

$$\rho_{\pm,a}(z, \mu) := |b_{\mu}^{\pm,a}(z)|.$$

For  $\mathbb{C}^+$ , we will write  $b_{\mu} = b_{\mu}^{+,0}$  and use  $\rho$  for  $\rho_{+,0}$  and  $\rho_{-,0}$ .

The definitions and properties below are stated and proved in  $\mathbb{C}^+$  but are obviously valid for any half-plane  $\mathbb{C}_a^{\pm}$ .

**Definition 4.** Let  $\Gamma := \{\mu_i : 1 \leq i \leq |\Gamma| < \infty\} \subset \mathbb{C}^+$ . For a finite set  $a = \{a_i\}_{1 \leq i \leq |\Gamma|}$ , we define the sequence of (*pseudohyperbolic*) *divided differences* of  $a$  relatively to  $\Gamma$  as follows

$$\Delta_{\Gamma}^0(a_i) := a_i, \quad \Delta_{\Gamma}^1(a_i, a_j) := \frac{a_j - a_i}{b_{\mu_i}(\mu_j)},$$

and

$$\Delta_{\Gamma}^k(a_{i_1}, \dots, a_{i_{k+1}}) := \frac{\Delta_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}) - \Delta_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_k})}{b_{\mu_{i_k}}(\mu_{i_{k+1}})}.$$

We will need to estimate the divided differences when  $\Gamma$  lies in a compact set  $K \subset \mathbb{C}^+$  and  $a = \{f(\mu) : \mu \in \Gamma\}$  for  $f$  an analytic function bounded in  $K$ . Here  $K$  is supposed to be the closure of a non empty open connected set. By  $f \in H^{\infty}(K)$  we mean that  $f$  is holomorphic in the interior of  $K$  and

$$\|f\|_{\infty, K} := \sup_{z \in K} |f(z)| < \infty.$$

**Lemma 5.** *Suppose that  $\Gamma$  lies in a compact set  $K$  with the properties mentioned above, and assume that there exists  $\eta > 0$  such that  $\rho(\Gamma, \partial K) \geq \eta$ . Then, for each function  $f \in H^{\infty}(K)$ , we have*

$$|\Delta_{\Gamma}^j(f(\mu^{(j+1)}))| \leq \left(\frac{2}{\eta}\right)^j \prod_{k=0}^j \left(\frac{1}{1 - \frac{k}{2M}}\right) \|f\|_{\infty, K}$$

where

$$\mu^{(j+1)} = (\mu_1, \dots, \mu_{j+1}) \text{ and } f(\mu^{(j+1)}) = (f(\mu_1), \dots, f(\mu_{j+1})).$$

*Proof.* Set

$$A_j := \left\{ z \in K : \rho(z, \partial K) \geq \frac{j}{2N} \eta \right\}, \quad 0 \leq j \leq N-1.$$

We show by induction over  $j$  that for every  $z \in A_j$ ,

$$|\Delta_{\Gamma}^j(f(\mu^{(j)}, z))| \leq c_j \|f\|_{\infty, K}$$

with

$$c_j = \left(\frac{2}{\eta}\right)^j \prod_{k=0}^j \left(\frac{1}{1 - \frac{k}{2M}}\right).$$

Since  $\Gamma \subset A_{N-1} \subset \dots \subset A_1 \subset A_0$ , the result will follow. The claim is obviously true for  $j = 0$ . Now, the function

$$z \mapsto \Delta_{\Gamma}^{j+1} (f(\mu^{(j+1)}, z))$$

is holomorphic on  $A_{j+1}$  and by the maximum principle and the definition of divided differences, we have for  $z \in A_{j+1}$ ,

$$(2.1) \quad |\Delta_{\Gamma}^{j+1} (f(\mu^{(j+1)}, z))| \leq \sup_{\xi \in \partial A_{j+1}} \left| \frac{\Delta_{\Gamma}^j (f(\mu^{(j)}, \xi)) - \Delta_{\Gamma}^j (f(\mu^{(j+1)}))}{\rho(\xi, \mu_{j+1})} \right|.$$

Let  $\xi \in \partial A_{j+1}$ . It is possible to find a point  $\zeta \in \partial K$  such that

$$\rho(\zeta, \xi) = \left(\frac{j+1}{2N}\right) \eta$$

and so, since  $\mu_{j+1} \in \Gamma$  and  $\rho(\Gamma, \partial K) \geq \eta$ , we have, by the triangle inequality,

$$(2.2) \quad \rho(\xi, \mu_{j+1}) \geq \rho(\zeta, \mu_{j+1}) - \rho(\xi, \zeta) \geq \eta \left(1 - \frac{j+1}{2N}\right).$$

From (2.1), (2.2) and the induction hypothesis, we finally obtain

$$|\Delta_{\Gamma}^{j+1} (f(\mu^{(j+1)}, \xi))| \leq \frac{2}{\eta} \left(\frac{1}{1 - \frac{j+1}{2N}}\right) c_j \|f\|_{\infty, K}$$

which gives the required estimate.  $\square$

The next lemma will be important in the sequel; we can define a rational Newton type interpolating function which interpolates the values  $\{a(\mu) : \mu \in \Gamma\}$  on  $\Gamma$ .

**Lemma 6.** *The holomorphic function*

$$P_{\Gamma, a}(z) := \sum_{k=1}^{|\Gamma|} \Delta_{\Gamma}^{k-1} (a(\mu^{(k)})) \prod_{l=1}^{k-1} b_{\mu_l}(z)$$

satisfies

$$P_{\Gamma, a}(\mu) = a(\mu), \quad \mu \in \Gamma.$$

The proof is quite straightforward (see also [7, p.80]).



*Remark 7.* Divided differences with respect to pseudohyperbolic metric can be found in [3, 8, 21]. We will also need *euclidian divided differences*:

$$\square_{\Gamma}^0 := a_i, \quad \square_{\Gamma}^1(a_i, a_j) := \frac{a_j - a_i}{\mu_j - \mu_i},$$

and

$$\square_{\Gamma}^k(a_{i_1}, \dots, a_{i_{k+1}}) := \frac{\square_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}) - \square_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_k})}{\mu_{k+1} - \mu_k}.$$

### 3. $N$ -CARLESON SEQUENCES

**Definition 8.** Let  $N \geq 1$  be a natural number. A sequence  $\Lambda \subset \mathbb{C}_a^{\pm}$  is called a  $N$ -Carleson sequence if it is possible to find a partition

$$\Lambda = \bigcup_{i=1}^N \Lambda^i$$

such that, for every  $i = 1, \dots, N$ , the sequence  $\Lambda^i$  satisfies the Carleson (1.1) condition in  $\mathbb{C}_a^{\pm}$ .

Note that the number  $N$  is not uniquely defined.

Let us make a link between the  $N$ -Carleson condition and the *Generalized Carleson condition*, also called Carleson-Vasyunin condition (see e.g. [14] and references therein). The following result has originally been stated in  $\mathbb{D}$  (see [8, Proposition 3.1]) but can easily be translated to any half-plane  $\mathbb{C}_a^{\pm}$ .

**Proposition 9.** Let  $\Lambda$  be a sequence of complex numbers, lying in  $\mathbb{C}_a^{\pm}$ . The following assertions are equivalent

- (i)  $\Lambda$  is  $N$ -Carleson in  $\mathbb{C}_a^{\pm}$ ;
- (ii) There exists  $\delta > 0$  and a sequence of Blaschke products  $(B_n)_{n \geq 1}$  such that  $\sup_n \deg B_n \leq N$ ,  $\Lambda = \bigcup_n \sigma_n$ , with  $\sigma_n := \{\lambda \in \mathbb{C}_a^{\pm} : B_n(\lambda) = 0\}$  and  $(B_n)_{n \geq 1}$  satisfies the Generalized Carleson condition

$$(3.1) \quad |B(z)| > \delta \inf_{n \geq 1} |B_n(z)|, \quad z \in \mathbb{C}_a^{\pm},$$

where  $B$  denotes the Blaschke product associated to  $\Lambda$ .

Observe that if  $\Lambda$  satisfies (ii), then, for  $(\lambda, \mu) \in \sigma_n \times \sigma_m$  ( $n \neq m$ ), we have  $\rho(\sigma_n, \sigma_m) \geq \delta$  and thus

$$\inf_{n \neq m} \rho(\sigma_n, \sigma_m) \geq \delta > 0.$$

*Remark 10.* The subsets  $\sigma_n$  can for instance be obtained as intersections  $\tau_n^\epsilon \cap \Lambda$  where  $\tau_n^\epsilon$  are the connected components of  $L(B, \epsilon) := \{z : |B(z)| < \epsilon\}$  and  $\epsilon$  is small enough. Moreover, choosing  $\epsilon$  in a suitable way, it is possible to assume that the pseudohyperbolic diameter of  $\sigma_n$  is arbitrarily small.

**Proposition 11.** *Let  $\Lambda = \{\lambda_n : n \geq 1\}$  be an  $N$ -Carleson sequence in  $\mathbb{C}_a^\pm$ . There exists  $\eta > 0$  such that every connected component of  $\bigcup_{n \geq 1} D_\rho(\lambda_n, \eta)$  admits at most  $N$  elements.*

*Remark 12.* We can deduce from the previous proposition that if  $\Lambda$  is  $N$ -Carleson in  $\mathbb{C}_a^\pm$  (or equivalently satisfies condition (ii) of Proposition 9), it is possible to construct a sequence of rectangles of  $\mathbb{C}_a^\pm$  defined by

$$R_n = \text{Rect}(z_n, L_n, l_n) = \left\{ x + iy \in \mathbb{C}_a^\pm : |x - x_n| \leq \frac{L_n}{2}, |y - y_n| \leq \frac{l_n}{2} \right\}$$

with  $L_n, l_n > 0$  and  $z_n = x_n + iy_n$ . These rectangles satisfy the following properties:

$$(3.2) \quad \sigma_n \subset R_n, \quad n \geq 1;$$

$$(3.3) \quad L_n \asymp l_n \asymp |y_n - a| \asymp d(\partial R_n, \mathbb{R} + ia), \quad n \geq 1;$$

$$(3.4) \quad 0 < \inf_{n \geq 1} \rho(\sigma_n, \partial R_n) \leq \sup_{\substack{n \geq 1 \\ \lambda \in \sigma_n}} \rho(\lambda, \partial R_n) < \infty;$$

and finally, since the diameter of  $\sigma_n$  can be chosen arbitrarily small by Remark 10, we can suppose the  $R_n$  disjoint and even

$$(3.5) \quad \inf_{n \neq k} \rho(R_n, R_k) > 0.$$

Let  $\Lambda$  be  $N$ -Carleson in  $\mathbb{C}_a^\pm$  and  $1 < p < \infty$ . From Proposition 9, we can write

$$\Lambda = \bigcup_{n \geq 1} \sigma_n,$$

with in particular  $|\sigma_n| \leq N$ . We will construct divided differences relatively to  $\sigma_n$ . We set

$$\sigma_n = \{\lambda_{n,k} : 1 \leq k \leq |\sigma_n|\} \quad \text{and} \quad \lambda_n^{(k)} = (\lambda_{n,1}, \dots, \lambda_{n,k}).$$

We choose, in an arbitrarily way,  $\lambda_{n,0}$  in  $\sigma_n$  and introduce, for  $a = (a(\lambda))_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda$ ,

$$\|a\|_{X_{\pm a}^p(\Lambda)} := \left( \sum_{n \geq 1} |\operatorname{Im}(\lambda_{n,0}) - a| \left| \sum_{k=1}^{|\sigma_n|} |\Delta_{\sigma_n}^{k-1}(a(\lambda_n^{(k)}))|^p \right)^{\frac{1}{p}}$$

and the space

$$X_{\pm a}^p(\Lambda) := \left\{ a \in \mathbb{C}^\Lambda : \|a\|_{X_{\pm a}^p(\Lambda)} < \infty \right\}.$$

Observe that for every  $\lambda \in \sigma_n$ ,  $1 \asymp |\operatorname{Im}(\lambda) - a| / |\operatorname{Im}(\lambda_{n,0}) - a|$  and so the definition of  $X_{\pm a}^p(\Lambda)$  does not depend on the choice of  $\lambda_{n,0}$ . The following result was originally stated in  $\mathbb{D}$  (see [8]) but it is not hard to check that it holds in  $\mathbb{C}_a^\pm$ . The reader will find details in [6, p. 92].

**Theorem 13.** (Hartmann). *Let  $\Lambda$  be  $N$ -Carleson in  $\mathbb{C}_a^\pm$  and  $1 < p < \infty$ . Then,  $R_\Lambda$  is continuous and surjective from  $H^p(\mathbb{C}_a^\pm)$  onto  $X_{\pm a}^p(\Lambda)$ .*

#### 4. MAIN RESULT

Let  $\Lambda$  be a sequence in the complex plane. In this section we assume that there is an integer  $N \geq 1$  such that for every  $a \in \mathbb{R}$ , the sequence

$$\Lambda_a^\pm := \Lambda \cap \mathbb{C}_a^\pm$$

is  $N$ -Carleson in the corresponding half-plane. Note that the partitions discussed in the previous section were adapted to sequences in a half-plane. Here, we will start discussing a ‘‘right’’ partition of  $\Lambda$  taking into account the fact that  $\Lambda$  lies in the whole complex plane

**4.1. An adapted partition.** From our above discussions it is possible to write

$$\Lambda_a^\pm = \bigcup_{n \geq 1} \sigma_{n,a}^\pm,$$

where  $(B_{\sigma_{n,a}^\pm}^{\pm,a})_n$  satisfies the generalized Carleson condition in the corresponding half-plane  $\mathbb{C}_a^\pm$  ( $B_{\sigma_{n,a}^\pm}^{\pm,a}$  being the Blaschke product in  $\mathbb{C}_a^\pm$  vanishing on  $\sigma_{n,a}^\pm$ ). To simplify the notation, we will omit  $a$  if  $a = 0$  and write

$$\sigma_n := \begin{cases} \sigma_{n+1}^+, & n \geq 0 \\ \sigma_n^-, & n < 0 \end{cases}.$$

The reader might notice that  $\sigma_n^+$  and  $\sigma_m^-$  can come very close for certain values of  $n$  and  $m$ . This issue will be fixed below. Let us distinguish

the sets of points close to the real axis and the ones far away from it. Let us fix  $\epsilon > 0$  for all what follows. We can assume that

$$\rho_0 := \sup_{n \in \mathbb{Z}} \text{diam}_\rho(\sigma_n) < \frac{\epsilon}{2}.$$

(Observe that  $\rho_0$  is well defined by the Generalized Carleson condition). Next introduce

$$M_{\epsilon, \infty} := \{n \in \mathbb{Z} : \sigma_n \cap \{|\text{Im}(z)| < \epsilon\} = \emptyset\},$$

$$\Lambda_{\epsilon, \infty} := \bigcup_{n \in M_{\epsilon, \infty}} \sigma_n$$

(corresponding to the points for which the corresponding set  $\sigma_n$  does not intersect the previous strip) and

$$\Lambda_\epsilon := \Lambda \setminus \Lambda_{\epsilon, \infty}.$$

Notice that  $\Lambda_\epsilon$  contains the points of  $\Lambda$  lying in the real axis and moreover

$$\Lambda_\epsilon \subset \{z \in \mathbb{C} : |\text{Im}(z)| < 3\epsilon\}.$$

Indeed, if  $\lambda \in \Lambda_\epsilon$  and  $\lambda \notin \mathbb{R}$ , then there is  $n_\lambda \in \mathbb{Z} \setminus M_{\epsilon, \infty}$  such that  $\lambda \in \sigma_{n_\lambda}$ . Hence, it is possible to find  $\mu \in \sigma_{n_\lambda}$  such that  $|\text{Im}(\mu)| < \epsilon$ . It follows that

$$\begin{aligned} |\lambda - \mu| &= \frac{|\lambda - \mu|}{|\lambda - \bar{\mu}|} |\lambda - \bar{\mu}| \\ &\leq \rho_0 (2|\text{Im}(\mu)|) + |\lambda - \mu| \\ &\leq \frac{3}{2}\epsilon^2 < \frac{3}{2}\epsilon, \end{aligned}$$

which implies that  $|\text{Im}(\lambda)| < 5\epsilon/2$ . Now, since  $\Lambda_\epsilon$  is contained in a strip, parallel to the real axis, of finite width and is  $N$ -Carleson in  $\mathbb{C}_{-3\epsilon}^+$ ,  $\Lambda_\epsilon$  breaks up into a disjoint union

$$\Lambda_\epsilon = \bigcup_{n \geq 1} \sigma'_n$$

with

$$\rho'_0 := \sup_{n \geq 1} \text{diam}(\sigma'_n) < \frac{\epsilon}{2}$$

and moreover, for some  $\delta > 0$ , the subsets

$$\Omega_n := \left\{ z \in \mathbb{C} : \prod_{\lambda \in \sigma'_n} |z - \lambda| \leq \delta \right\}, \quad n \geq 1,$$

satisfy

$$(4.1) \quad \inf_{n \neq m} d(\Omega_n, \Omega_m) > 0.$$

This is possible in view of Remarks 10 and 12. It follows that we can write  $\Lambda$  as the following disjoint union

$$\Lambda = \left( \bigcup_{n \in M_{\epsilon, \infty}} \sigma_n \right) \cup \left( \bigcup_{n \geq 1} \sigma'_n \right) =: \bigcup_{n \in \mathbb{Z}} \tau_n.$$

Now that the partition is done, it is possible to construct divided differences. Since we will need both definitions of divided differences, we set

$$\tilde{\Delta}_{\tau_n} := \begin{cases} \Delta_{\tau_n} & \text{if } \exists k \text{ s.t. } \tau_n = \sigma_k \\ \square_{\tau_n} & \text{if } \exists k \text{ s.t. } \tau_n = \sigma'_k \end{cases}.$$

It is now possible to introduce a space of sequences that will be, assuming some hypotheses on  $\Lambda$ , the range of  $R_\Lambda$ . Naturally, we write

$$\tau_n = \{\lambda_{n,k} : 1 \leq k \leq |\sigma_n|\} \text{ and } \lambda_n^{(k)} := (\lambda_{n,1}, \dots, \lambda_{n,k}).$$

As previously, we choose, in an arbitrarily way,  $\lambda_{n,0} \in \tau_n$ , for every  $n \in \mathbb{Z}$ . We define, for  $1 < p < \infty$ ,

$$X_{\tau, \epsilon}^p(\Lambda) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X_{\tau, \epsilon}^p(\Lambda)} < \infty \right\},$$

with

$$\|a\|_{X_{\tau, \epsilon}^p(\Lambda)}^p := \sum_{n \in \mathbb{Z}} (1 + |\text{Im}(\lambda_{n,0})|) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (a e^{\pm i\tau \cdot} (\lambda_n^{(k)})) \right|^p,$$

and

$$e^{\pm i\tau \lambda} = \begin{cases} e^{i\tau \lambda} & \text{if } \lambda \in \tau_n, \quad n \in N_+, \\ e^{-i\tau \lambda} & \text{if } \lambda \in \tau_n, \quad n \in N_-, \end{cases}$$

where

$$N_+ := \{n \in \mathbb{Z} : \tau_n \cap (\mathbb{C}^+ \cup \mathbb{R}) \neq \emptyset\}$$

and

$$N_- := \mathbb{Z} \setminus N_+.$$

(The factor  $e^{\pm i\tau \lambda}$  does not really matter close to  $\mathbb{R}$ .) Next proposition will be proved in Section 5.

**Proposition 14.** *If there exists  $\epsilon > 0$  such that  $R_\Lambda$  is an isomorphism between  $PW_\tau^p$  and  $X_{\tau, \epsilon}^p(\Lambda)$  then  $\Lambda$  is relatively dense, i.e. there exists  $r > 0$  such that for every  $x \in \mathbb{R}$ ,  $d(x, \Lambda) < r$ .*

It follows from the conclusion of the previous proposition that the relative density is necessary. Thus, we will assume in all what follows that  $\Lambda$  is relatively dense:

$$\exists r > 0, \forall x \in \mathbb{R}, d(x, \Lambda) < r.$$

Still relative to the previous partition of  $\Lambda$ , we introduce, for  $n \geq 1$ , the products

$$p_n(x) := \prod_{\lambda \in \tau_n} |x - \lambda|$$

which permit us to define the function

$$d_N(x) := \inf_{n \in \mathbb{Z}} p_n(x), \quad x \in \mathbb{R}.$$

*Remark 15.* From the definition of the function  $d_N$ , we can do the following observations.

- (1) The relative density condition implies that

$$\sup_{x \in \mathbb{R}} d_N(x) \leq (r + \delta'_0)^N < \infty,$$

where

$$\delta'_0 := \inf_{n \neq m} d(\sigma'_n, \sigma'_m) > 0.$$

- (2) It is clear that, in the definition of  $d_N$ , the infimum is actually a minimum. So, for each  $x \in \mathbb{R}$ , there is  $n_x \in \mathbb{Z}$  such that  $d_N(x) = p_{n_x}(x)$ . It is not difficult to see that

$$\inf_{x \in \mathbb{R}} \inf_{m \neq n_x} p_m(x) \geq \left(\frac{\delta'_0}{2}\right)^N > 0.$$

- (3) Using the relative density, a similar reasoning as the one that can be used to show (2) yields that, with an other partition (and in particular with an other choice of  $\epsilon$ ), the function obtained is equivalent to  $d_N$ .

#### 4.2. The theorem.

**Definition 16.** Let  $\Lambda$  be  $N$ -Carleson in every half-plane and relatively dense. We say that  $\Lambda$  satisfies the conditions  $(H_N)_{\tau, p}$  (for  $\tau > 0$  and  $1 < p < \infty$ ) if

- (i) The limit

$$S(z) := \lim_{R \rightarrow \infty} \prod_{|\lambda| < R} \left(1 - \frac{z}{\lambda}\right)$$

exists and defines an entire function of exponential type  $\tau$ .

- (ii) The function  $x \mapsto \left(\frac{|S(x)|}{d_N(x)}\right)^p$  satisfies the (continuous) Muckenhoupt condition  $(A_p)$ .

The reader would notice that, in view of Remark 15–(3), the definition of the conditions  $(H_N)_{\tau,p}$  do not depend on the partition of  $\Lambda$ .

**Theorem 17.** *Let  $N \geq 1$ ,  $\tau > 0$ ,  $1 < p < \infty$  and  $\Lambda$  be  $N$ –Carleson in every half-plane and relatively dense (for some  $r > 0$ ). Then, the restriction operator  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_{\tau,r}^p(\Lambda)$  if and only if  $\Lambda$  satisfies  $(H_N)_{\tau,p}$ .*

*Remark 18.* We will see in the following that  $(H_N)_{\tau,p}$  – (ii) can be replaced by (ii)', which is

- (ii)' There exists a subsequence  $\Gamma = \{\gamma_n : n \geq 1\} \subset \Lambda$ , still relatively dense, such that, if  $\sigma_{\gamma_n}$  is the set containing  $\gamma_n$ , the sequence

$$\left( \frac{|S'(\gamma_n)|^p}{\prod_{\substack{\lambda \in \sigma_{\gamma_n} \\ \lambda \neq \gamma_n}} |\gamma_n - \lambda|^p} \right)_{n \geq 1}$$

satisfies the discrete Muckenhoupt condition  $(\mathfrak{A}_p)$ .

It is clear that for  $N = 1$ ,  $d_1(x) = d(x, \Lambda)$  and  $(H_1)_{\tau,p}$  with the Carleson condition and the relative density corresponds exactly to the  $(LS)_{\tau,p}$  conditions. The proof of Theorem 17 will be done in Section 5.

*Remark 19.* The choice of  $\epsilon = r$  in our construction ensures that, for every  $x \in \mathbb{R}$ ,  $\tau_{n_x} = \sigma'_{n_x}$  and permits us to avoid tedious considerations but the conclusion of Theorem 17 is still true with any choice of  $\epsilon > 0$ .

We will discuss below the necessity of the  $N$ –Carleson condition in Theorem 24. In Theorem 17, the definition of the range of  $R_\Lambda$  definitely depends on the partition of  $\Lambda$  which is possible because of the  $N$ –Carleson condition. In Section 6, we will construct a space without the *a priori* assumption that  $\Lambda$  is  $N$ –Carleson in every half-plane.

## 5. PROOFS

## 5.1. Proof of Proposition 14.

*Proof.* Let us suppose to the contrary that there exists a real sequence  $\{x_j\}_{j \geq 1}$  and a sequence of positive numbers  $\{r_j\}_{j \geq 1}$  such that  $r_j \rightarrow \infty$ ,  $j \rightarrow \infty$  and

$$B(x_j, r_j) \cap \Lambda = \emptyset.$$

We consider the functions of  $PW_\tau^p$  defined by

$$f_j(z) := \frac{\sin \tau(z - x_j)}{\tau(z - x_j)}, \quad z \in \mathbb{C}, \quad j \geq 1.$$

Since  $R_\Lambda$  is an isomorphism, we obtain that

$$1 \asymp \|f_j\|_p^p \asymp \|R_\Lambda f_j\|_{X_{\tau, \epsilon}^p(\Lambda)}^p.$$

We will show that  $\|R_\Lambda f_j\|_{X_{\tau, \epsilon}^p(\Lambda)}^p \rightarrow 0$ ,  $j \rightarrow \infty$ , which implies the required contradiction. From the definition, we have

$$\|R_\Lambda f_j\|_{X_{\tau, \epsilon}^p(\Lambda)}^p = \sum_{n \geq 1} (1 + |\operatorname{Im}(\lambda_{n,0})|) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (f_j e^{\pm i \tau \cdot} (\lambda_n^{(k)})) \right|^p.$$

Using Lemme 5 (see [6, p. 95] for details), we can see that, for every  $n \geq 1$  and every  $1 \leq k \leq |\tau_n|$ ,

$$\left| \tilde{\Delta}_{\tau_n}^{k-1} (f_j e^{\pm i \tau \cdot} (\lambda_n^{(k)})) \right|^p \lesssim \frac{1}{|\lambda_{n,0} - x_j|^p},$$

which implies

$$\|R_\Lambda f_j\|_{X_{\tau, \epsilon}^p(\Lambda)}^p \lesssim \sum_{n \geq 1} \frac{1 + |\operatorname{Im}(\lambda_{n,0})|}{|\lambda_{n,0} - x_j|^p}.$$

On the other hand,  $p > 1$  and so we can find  $\alpha > 0$  such that  $p - \alpha > 1$ . Recall that  $|\lambda_{n,0} - x_j| \geq r_j$  and let us write

$$\|R_\Lambda f_j\|_{X_{\tau, \epsilon}^p(\Lambda)}^p \lesssim \frac{1}{r_j^\alpha} \sum_{n \geq 1} \frac{1 + |\operatorname{Im}(\lambda_{n,0})|}{|\lambda_{n,0} - x_j|^{p-\alpha}}.$$

We split this sum in two parts, writing  $\{\lambda_{n,0} : n \geq 1\} = A^+ \cup A^-$ , where

$$A^+ \subset (\mathbb{C}^+ \cup \mathbb{R}) \subset \mathbb{C}_{-\frac{1}{2}}^+$$

and

$$A^- \subset \mathbb{C}^- \subset \mathbb{C}_{\frac{1}{2}}^-.$$



Since  $r_j \rightarrow \infty$ ,  $j \rightarrow \infty$ , we obtain  $|\lambda_{n,0} - x_j| \asymp |\lambda_{n,0} - x_j \pm i|$ . It follows that the functions

$$g^\pm : z \mapsto \frac{1}{z - x_j \pm i} \in H^{p-\alpha} \left( \mathbb{C}_{\mp \frac{1}{2}}^\pm \right).$$

Now,  $A^\pm$  is Carleson in  $\mathbb{C}_{\mp \frac{1}{2}}^\pm$ , thus

$$\sum_{\lambda \in A^\pm} \frac{1 + |\operatorname{Im}(\lambda)|}{|\lambda - x_j \pm i|^{p-\alpha}} = \sum_{\lambda \in A^\pm} \frac{1 + |\operatorname{Im}(\lambda)|}{|g^\pm(\lambda)|^{p-\alpha}} \lesssim \|g\|_{H^{p-\alpha}(\mathbb{C}_{\mp \frac{1}{2}}^\pm)}^{p-\alpha} \lesssim 1.$$

We finally obtain that

$$\|R_\Lambda f_j\|_{X_{r,\epsilon}^p(\Lambda)}^p \lesssim \frac{1}{r_j^\alpha} \rightarrow 0, j \rightarrow \infty,$$

which is the required contradiction and ends the proof. □

**5.2. Proof of Theorem 17.** The proof of Theorem 17 follows the main ideas of Lyubarskii and Seip's paper but needs an important technical work to characterize this more general case.

**5.2.1. Paley-Wiener Spaces.** We will need some well known facts about Paley-Wiener spaces that we recall here. First, we have the Plancherel-Polyà inequality (see e.g. [11] or [18, p. 95]).

**Proposition 20.** (*Plancherel-Polyà*). *Let  $f \in PW_\tau^p$  and  $a \in \mathbb{R}$ . Then,*

$$\int_{-\infty}^{+\infty} |f(x + ia)|^p dx \leq e^{\tau p|a|} \|f\|_p^p.$$

It follows that for every  $f \in PW_\tau^p$ , the function  $z \mapsto e^{i\tau z} f(z)$  belongs to  $H_+^p$ . It also follows that translation is an isomorphism from  $PW_\tau^p$  onto itself. The second fact is a pointwise estimate; there exists a constant  $C = C(p)$  such that for every  $f \in PW_\tau^p$ , we have

$$(5.1) \quad |f(z)| \leq C \|f\|_p (1 + |\operatorname{Im}(z)|)^{-\frac{1}{p}} e^{\tau |\operatorname{Im}(z)|}, \quad z \in \mathbb{C}.$$

**5.2.2. Necessary conditions.** Let us do the construction of subsection 4.1 with  $\epsilon = r$  and suppose that  $R_\Lambda$  is an isomorphism between  $PW_\tau^p$  and  $X_{r,\epsilon}^p(\Lambda)$ . The necessity of  $(H_N) - (i)$  can be shown exactly as in [12] and so we do not prove it here. We first show that the condition  $(ii)'$  is necessary. Then, with a technical lemma, adapted from [12], we prove that  $(ii)'$  implies  $(ii)$ .

Since  $R_\Lambda$  is bijective, for each  $\lambda \in \Lambda$ , there is a unique function  $f_\lambda \in PW_\tau^p$  such that

$$f_\lambda(\mu) = \begin{cases} 1, & \text{if } \mu = \lambda \\ 0, & \text{if } \mu \neq \lambda \end{cases}.$$

As in [12], it can be shown that  $f_\lambda$  only vanishes on  $\Lambda \setminus \{\lambda\}$  and that  $f_\lambda$  is of exponential type  $\tau$  (if its type was  $\tau' < \tau$  then considering the function  $e^{i(\tau-\tau')(-\lambda)} f_\lambda$ , we would obtain a contradiction with the injectivity of  $R_\Lambda$ ). Moreover,  $z \mapsto (z - \lambda) f_\lambda(z)$  is a function of the Cartwright Class  $\mathcal{C}$  vanishing exactly on  $\Lambda$  (see e.g. [11] for definition and general results on  $\mathcal{C}$ ). Hence, since  $S$  is also of exponential type  $\tau$ ,  $S(z) = c_\lambda (z - \lambda) f_\lambda(z)$ ,  $z \in \mathbb{C}$ , or

$$f_\lambda(z) = \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

For each  $n \geq 1$ , the holomorphic function

$$g_n : z \mapsto \frac{S(z)}{\prod_{\lambda \in \sigma'_n} (z - \lambda)}$$

does not vanish in  $\Omega_n$  (see Formula 4.1). Moreover, choosing  $\lambda'_{n,0} \in \sigma'_n$ ,

$$g_n(\lambda_{n,0}) = \frac{S'(\lambda'_{n,0})}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} (\lambda'_{n,0} - \lambda)}.$$

Hence, it follows from the maximum and the minimum principle that

$$\inf_{\xi \in \partial\Omega_n} \left| \frac{S(\xi)}{\prod_{\lambda \in \sigma'_n} (\xi - \lambda)} \right| \leq \left| \frac{S'(\lambda'_{n,0})}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} (\lambda'_{n,0} - \lambda)} \right| \leq \sup_{\xi \in \partial\Omega_n} \left| \frac{S(\xi)}{\prod_{\lambda \in \sigma'_n} (\xi - \lambda)} \right|.$$

From the intermediate values theorem, we deduce the existence of a point  $\theta_n \in \partial\Omega_n$  such that

$$(5.2) \quad |S(\theta_n)| = \delta \frac{|S'(\lambda'_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} |\lambda'_{n,0} - \lambda|} =: \delta \omega_n.$$

Let us consider now a subsequence  $\Gamma := (\gamma_n)_{n \geq 1}$  of  $\{\lambda'_{n,0} : n \geq 1\}$  which is still relatively dense and such that

$$\inf_{n \geq 1} (\operatorname{Re}(\gamma_{n+1}) - \operatorname{Re}(\gamma_n)) > 0.$$

We define  $\sigma_{\gamma_n}$  as the set containing  $\gamma_n$ . The sequence  $\Theta := (\theta_n)_{n \geq 1}$  denotes the previous  $\theta_n$ , corresponding to  $\gamma_n$ , and for  $n \geq 1$ , we set

$$\omega_n := \left| \frac{S'(\gamma_n)}{\prod_{\substack{\lambda \in \sigma_{\gamma_n} \\ \lambda \neq \gamma_n}} (\gamma_n - \lambda)} \right|$$

so that

$$|S(\theta_n)| = \delta \omega_n.$$

We show that the discrete Hilbert transform  $\mathcal{H}_{\Gamma, \Theta}$  is bounded from  $l^p(\omega)$  into itself. Indeed, let  $(a_n)_{n \geq 1}$  be a finite sequence of  $l^p(\omega)$ . Then, the sequence

$$a(\lambda) := \begin{cases} a_n S'(\gamma_n) & , \text{ if } \lambda = \gamma_n \\ 0 & , \text{ if } \lambda \in \Lambda \setminus \Gamma \end{cases}$$

belongs to  $X_\tau^p(\Lambda)$  because, if  $\gamma_k = \lambda'_{n,0} = \lambda_{n,|\sigma_{\gamma_n}|}$  is chosen as the “last” point of  $\sigma'_n$ ,

$$\tilde{\Delta}_{\sigma_n}^{k-1} (a e^{i\tau \cdot} (\lambda_n^{(k)})) = 0, \quad k < |\sigma_n|$$

and

$$\left| \tilde{\Delta}_{\sigma_n}^{|\sigma_n|-1} (a e^{i\tau \cdot} (\lambda_n^{(|\sigma_n|)})) \right| = \frac{|a_n S'(\lambda'_{n,0})| e^{-\tau |\operatorname{Im}(\lambda'_{n,0})|}}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda - \lambda'_{n,0}|}.$$

Thus, from (5.2), we obtain, observing that  $1 + |\operatorname{Im}(\lambda'_{n,0})|$  and  $|e^{i\tau \lambda}|$ ,  $\lambda \in \sigma'_n$ , are comparable to a constant since  $\sigma'_n$  is close to  $\mathbb{R}$ ,

$$\begin{aligned} \|a\|_{X_\tau^p(\Lambda)}^p &= \sum_n \left( 1 + |\operatorname{Im}(\lambda'_{n,0})| \right) \left| \tilde{\Delta}_{\sigma'_n}^{|\sigma'_n|-1} (a e^{i\tau \cdot} (\lambda_n^{(|\sigma'_n|)})) \right|^p \\ (5.3) \quad &\asymp \sum_n \left( \frac{|a_n S'(\lambda'_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda'_{n,0} - \lambda|} \right)^p = \sum_n \omega_n^p |a_n|^p. \end{aligned}$$

So, let  $f \in PW_\tau^p$  be the (unique) solution of the interpolation problem  $f|_\Lambda = a$ . Notice that, since  $R_\Lambda$  is an isomorphism onto  $X_\tau^p(\Lambda)$ , then

$$(5.4) \quad \|f\|_p^p \lesssim \|a\|_{X_\tau^p(\Lambda)}^p.$$

This function is of the form  $f(z) = \sum_j a_j \frac{S(z)}{z - \gamma_j}$  and so, with (5.2) we have

$$\sum_n |f(\theta_n)|^p = \sum_n \left| \sum_j a_j \frac{S(\theta_n)}{\theta_n - \gamma_j} \right|^p = \sum_n |S(\theta_n)|^p \left| \sum_j \frac{a_j}{\theta_n - \gamma_j} \right|^p$$

and, from the construction of  $\Theta$ , we obtain

$$(5.5) \quad \sum_n |f(\theta_n)|^p = \delta^p \sum_n \omega_n^p \left| \left( \mathcal{H}_{\Gamma, \Theta}((a_j)_{j \geq 1}) \right)_n \right|^p.$$

On the other hand, the Polyà inequality (see [11, Lecture 20]), and the inequalities (5.4) and (5.3) give

$$(5.6) \quad \sum_n |f(\theta_n)|^p \lesssim \|f\|_p^p \lesssim \|a\|_{X_r^p(\Lambda)}^p \lesssim \sum_n \omega_n^p |a_n|^p.$$

From (5.5) and (5.6), we deduce that  $\mathcal{H}_{\Gamma, \Theta}$  is bounded from  $l^p(\omega^p)$  into itself. Using a slight modified version of [12, Lemma 1], we can conclude that the weight  $(\omega_n^p)_{n \geq 1}$  satisfies the discrete Muckenhoupt condition  $(\mathfrak{A}_p)$ .

*Remark 21.* It follows from the weak density condition  $((H_N) - (i))$ , the Generalized Carleson condition (3.1) on  $(B_{\sigma_{\gamma_n}})_n$  and the growth of the sequence  $(\operatorname{Re}(\gamma_n))_n$  that we have  $\operatorname{Re}(\gamma_{n+1}) - \operatorname{Re}(\gamma_n) \leq 3\epsilon$ . This implies that

$$\delta'_0 \leq |\gamma_n - \gamma_{n+1}| \leq 4\epsilon.$$

Now, in order to prove (iii), we use the following lemma, adapted from [12, Lemma 2].

**Lemma 22.** *Suppose  $x \in \mathbb{R}$  and  $\operatorname{Re}(\gamma_n) \leq x \leq \operatorname{Re}(\gamma_{n+1})$ . Then, there exists an  $\alpha = \alpha(x) \in [0, 1]$  such that*

$$\omega_n^\alpha \omega_{n+1}^{1-\alpha} \asymp \frac{|S(x)|}{d_N(x)},$$

*uniformly with respect to  $x \in \mathbb{R}$ .*

Assuming this lemma to hold, (iii) follows from (iii)' and the inequality  $t^\alpha s^{1-\alpha} \leq t + s$ ,  $t, s > 0$  and  $\alpha \in [0, 1]$  (we still refer to [6] for details).

*Proof.* For  $x \in [\operatorname{Re}(\gamma_n), \operatorname{Re}(\gamma_{n+1})]$ , we set  $N(x) := \{n : d(\sigma'_n, x) < \epsilon\}$  and

$$\Lambda(x) := \left( \bigcup_{n \in N(x)} \sigma'_n \right) \cup \sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}}.$$

Notice that  $\sigma_{\gamma_n}$  and  $\sigma_{\gamma_{n+1}}$  may be subsets of  $\bigcup_{n \in N(x)} \sigma'_n$ . Observe also that since  $\Lambda$  is a finite union of Carleson sequences, we have

$$\sup_{x \in \mathbb{R}} |N(x)| < \infty.$$

For  $\alpha \in [0, 1]$ , we want to show that  $\vartheta \asymp 1$ , where

$$\vartheta := \frac{\omega_n^\alpha \omega_{n+1}^{1-\alpha} d_N(x)}{|S(x)|},$$

and  $x \notin \Lambda$  (this is not restrictive since the expression extends continuously to  $\Lambda$ ). From the definition of  $S$ , we have that

$$S'(\lambda) = -\frac{1}{\lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left(1 - \frac{\lambda}{\mu}\right), \quad \lambda \in \Lambda.$$

In order to not overcharge notation, all infinite products occurring below will be understood as symmetric limits of finite products:

$$\prod_{\lambda \in \Lambda} a(\lambda) = \lim_{R \rightarrow \infty} \prod_{|\lambda| \leq R} a(\lambda).$$

Thus,

$$\vartheta = \left( \frac{\left| \frac{1}{\gamma_n} \prod_{\lambda \in \Lambda \setminus \{\gamma_n\}} \left(1 - \frac{\gamma_n}{\lambda}\right) \right|^\alpha \left| \frac{1}{\gamma_{n+1}} \prod_{\lambda \in \Lambda \setminus \{\gamma_{n+1}\}} \left(1 - \frac{\gamma_{n+1}}{\lambda}\right) \right|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda} \left(1 - \frac{x}{\lambda}\right) \prod_{\lambda \in \sigma_{\gamma_n} \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_{n+1}} \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha}} \right).$$

For  $\lambda \in \Lambda \setminus \{\gamma_n, \gamma_{n+1}\}$ ,

$$\frac{\left|1 - \frac{\gamma_n}{\lambda}\right|^\alpha \left|1 - \frac{\gamma_{n+1}}{\lambda}\right|^{1-\alpha}}{\left|1 - \frac{x}{\lambda}\right|} = \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|x - \lambda|}.$$

Note also that for the remaining two points  $\gamma_n, \gamma_{n+1}$  we have:

$$\frac{\left| \frac{1}{\gamma_n} \left(1 - \frac{\gamma_n}{\gamma_{n+1}}\right) \right|^\alpha \left| \frac{1}{\gamma_{n+1}} \left(1 - \frac{\gamma_{n+1}}{\gamma_n}\right) \right|^{1-\alpha}}{\left| \left(1 - \frac{x}{\gamma_n}\right) \left(1 - \frac{x}{\gamma_{n+1}}\right) \right|} = \frac{|\gamma_{n+1} - \gamma_n|^\alpha |\gamma_n - \gamma_{n+1}|^{1-\alpha}}{|\gamma_n - x| |\gamma_{n+1} - x|}.$$

Now, we split  $\vartheta$  in two products  $\vartheta = \Pi_1(x) \cdot \Pi_2(x)$  corresponding essentially to zeros in  $\Lambda(x)$  and zeros in  $\Lambda \setminus \Lambda(x)$  ( $d_N(x)$  appearing

in  $\Pi_1$ ):

$$\begin{aligned} \Pi_1(x) &:= \frac{\prod_{\lambda \in \Lambda(x) \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \Lambda(x) \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda(x)} |\lambda - x| \prod_{\lambda \in \sigma_{\gamma_n} \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_{n+1}} \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha}} \\ &= \frac{\prod_{\lambda \in \Lambda(x) \setminus \sigma_{\gamma_n}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \Lambda(x) \setminus \sigma_{\gamma_{n+1}}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda(x)} |\lambda - x|} \end{aligned}$$

and

$$\Pi_2(x) := \prod_{\lambda \in \Lambda \setminus \Lambda(x)} \left( \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|\lambda - x|} \right).$$

We can write

$$\begin{aligned} \Pi_1(x) &= \left( \frac{\prod_{\lambda \in \sigma_{\gamma_{n+1}}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_n}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}}} |x - \lambda|} \right) \\ &\quad \times \left( \prod_{\Lambda(x) \setminus (\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}})} \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|x - \lambda|} \right) \end{aligned}$$

and notice that if  $\lambda \in \Lambda(x) \setminus (\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}})$ , then  $\lambda \in \sigma'_l$  for a suitable  $l \in N(x)$ , so that

$$1 \lesssim d(\sigma_{\gamma_n}, \sigma'_l) \leq |\lambda - \gamma_n| \leq 2\rho'_0 + 2\epsilon \lesssim 1$$

and, in view of Remark 21, for  $\lambda \in \sigma_{\gamma_n}$  and  $\mu \in \sigma_{\gamma_{n+1}}$ , we have

$$|\lambda - \gamma_{n+1}| \asymp 1 \text{ and } |\mu - \gamma_n| \asymp 1.$$

These three relations imply that

$$\Pi_1(x) \asymp \frac{d_N(x)}{\prod_{\lambda \in \Lambda(x)} |x - \lambda|}.$$

Now, let  $n_x$  be such that  $d_N(x) = p_{n_x}(x)$  (we refer to Remark 15). Clearly  $n_x \in N(x)$ . Note also that for  $\lambda \in \sigma'_m$ ,  $m \in N(x)$ , we have  $|\lambda - x| \leq d(\sigma'_m, x) + \text{diam}(\sigma'_m) \leq \epsilon + \rho'_0$ . Hence

$$\frac{1}{(\epsilon + \rho'_0)^{|N(x)|-1}} \leq \frac{d_N(x)}{\prod_{\lambda \in \Lambda(x)} |x - \lambda|} = \frac{1}{\prod_{\lambda \in \Lambda(x) \setminus \sigma_{n_x}} |\lambda - x|} \leq \left( \frac{2}{\delta'_0} \right)^{N \cdot (|N(x)|-1)}$$

and, from the end of Remark 15, we obtain that

$$\Pi_1(x) \asymp 1.$$

The relation

$$\Pi_2(x) \asymp 1$$

is shown exactly in the same way as in [12], using the  $N$ -Carleson condition. The lemma is proved.  $\square$

5.2.3. *Sufficient conditions.* We show the converse of the theorem in two parts; first, the injectivity of  $R_\Lambda$  and then its surjectivity.

Let  $f \in PW_\tau^p$  such that  $f(\lambda) = 0$ ,  $\lambda \in \Lambda$ . We want to show that  $f \equiv 0$ . Let us introduce  $\phi := f/S$ . It can be shown that  $\phi$  is an entire function of exponential type 0 (see [6, pp. 96-98] for details). The idea of the proof, given by Lyubarskii and Seip in [12], is to bound  $\phi$  by a constant on the imaginary axis and to use a Phragmen-Lindelöf theorem to obtain that  $\phi$  is a constant. Then, for integrability reasons, the only possible value for the constant will be zero.

We will proceed as follows: since  $\phi$  is analytic, it is bounded on the compact  $[-2i\epsilon, 2i\epsilon]$ . In order to bound  $\phi$  on  $i\mathbb{R} \setminus [-2i\epsilon, 2i\epsilon]$ , we will use a lower estimate for  $S$  in a certain area of  $\mathbb{C}$ . Let us introduce

$$A_n := \{z \in \mathbb{C} : |\operatorname{Im}(z)| \geq 2\epsilon, \rho(\lambda_{n,0}, z) < 2\rho_0 < \epsilon\}, \quad n \in \mathbb{Z}.$$

We begin to show that for  $z \in (\mathbb{C}_{2\epsilon}^+ \cup \mathbb{C}_{-2\epsilon}^-) \setminus (\bigcup_n A_n)$ ,

$$(5.7) \quad |S(z)| \gtrsim e^{\tau|\operatorname{Im}(z)|} (|\operatorname{Im}(z)|)^{\frac{1}{q}} (1 + |z|)^{-1}.$$

Indeed, let us introduce

$$S_1(z) := (S/B_\epsilon)(z),$$

where

$$B_\epsilon(z) := \prod_{\lambda \in \Lambda_\epsilon} \left( c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 3i\epsilon} \right),$$

is the Blaschke product in  $\mathbb{C}_{-\frac{3}{2}\epsilon}^+$  associated to  $\Lambda_\epsilon$ , and  $c_\lambda$  is the unimodular normalizing constant which ensures the convergence of the Blaschke product (we do not need the explicit value here). Let  $x \in \mathbb{R}$ . Observe that for  $n \geq 1$  and  $\lambda \in \sigma'_{n_x}$ , we have

$$|x - \bar{\lambda}| = |x - \lambda| \leq \epsilon + \operatorname{diam}(\sigma'_{n_x}) \leq \epsilon + \rho'_0 \lesssim 1.$$

Hence,

$$|x - \bar{\lambda} + 3i\epsilon| \asymp 1.$$

It follows from these inequalities that

$$\left( \prod_{\lambda \in \sigma'_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon} \right| \right) \asymp d_N(x).$$

Writing

$$|B_\epsilon(x)| = \left( \prod_{\lambda \in \sigma'_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon} \right| \right) \left( \prod_{\lambda \in \Lambda_\epsilon \setminus \sigma'_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon} \right| \right)$$

and using the fact that  $\Lambda_\epsilon$  is  $N$ -Carleson in  $\mathbb{C}^+_{-\frac{3}{2}\epsilon}$ , we have then that

$$(5.8) \quad |B_\epsilon(x)| \asymp d_N(x),$$

and so  $x \mapsto |S_1(x)|^p$  satisfies  $(A_p)$ .

In particular, the function  $z \mapsto e^{i\tau z} \frac{S_1(z)}{z+i} = e^{i\tau z} \frac{S(z)}{B_\epsilon(z+i)}$  belongs to  $H^p_+$  and the function  $z \mapsto e^{i\tau z} S_1(z)$  is a function of  $\mathcal{N}^+$ , the Smirnov Class in the upper half-plane (for definition and general results, see e.g. [15, A.4]). Hence, we can write

$$S_1(z) = e^{-i\tau z} B_1(z) G_1(z), \quad z \in \mathbb{C}^+,$$

where  $B_1$  is the Blaschke product associated to  $\Lambda^+ \setminus \Lambda_\epsilon$  and  $G_1$  is an outer function in  $\mathbb{C}^+$  (observe that  $e^{i\tau \cdot} S_1$  cannot contain any inner singular factor). Thus,  $x \mapsto |G_1(x)|^p$  satisfies  $(A_p)$  or equivalently,  $x \mapsto |G_1(x)|^{-q}$  satisfies  $(A_q)$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ . So, it follows from properties of functions satisfying Muckenhoupt's  $(A_p)$  condition, that

$$\phi_{G_1} : z \mapsto \frac{1}{G_1(z)(z+i)} \in H^q_+$$

and, from well known estimates in  $H^q_+$ , we get

$$|\phi_{G_1}(z)| \lesssim \frac{1}{(\operatorname{Im}(z))^{\frac{1}{q}}},$$

and so, for  $z \in \mathbb{C}^+$ ,

$$\left| \frac{1}{G_1(z)} \right| \lesssim (1 + |z|) (\operatorname{Im}(z))^{-\frac{1}{q}}.$$

Moreover, because of the  $N$ -Carleson condition of  $\Lambda^+ \setminus \Lambda_\epsilon$ , we have that

$$|B_1(z)| \gtrsim 1, \quad z \in \mathbb{C}^+ \setminus \left( \bigcup_{n \geq 0} A_n \right)$$



and so we do have the lower bound for  $S_1$  stated in (5.7). We notice that  $|S(z)| \asymp |S_1(z)|$ ,  $\text{Im}(z) > 2\epsilon$  and so we have the same bound for  $S$  in  $\mathbb{C}_{2\epsilon}^+$ . A similar reasoning gives us the estimate in  $\mathbb{C}_{-2\epsilon}^-$ .

Using now (5.1) and (5.7), we have for  $z \in (\mathbb{C}_{2\epsilon}^+ \cup \mathbb{C}_{-2\epsilon}^-) \setminus (\bigcup_n A_n)$ ,

$$\begin{aligned} |\phi(z)| &= \left| \frac{f(z)}{S(z)} \right| \lesssim \frac{(1+|z|)}{e^{\tau|\text{Im}(z)|} |\text{Im}(z)|^{\frac{1}{q}} (1+|\text{Im}(z)|)^{\frac{1}{p}}} \frac{e^{\tau|\text{Im}(z)|}}{e^{\tau|\text{Im}(z)|} |\text{Im}(z)|^{\frac{1}{q}} (1+|\text{Im}(z)|)^{\frac{1}{p}}} \\ &\asymp \frac{(1+|z|)}{|\text{Im}(z)|^{\frac{1}{q}} (1+|\text{Im}(z)|)^{\frac{1}{p}}} =: \psi(z). \end{aligned}$$

We notice then that if  $A_n \cap i\mathbb{R} \neq \emptyset$ , then

$$A_n \subset S_{\pm} := \left\{ z \in \mathbb{C}^{\pm} : \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right| < \eta \right\},$$

where  $\eta$  is a suitable constant. Note that  $S_{\pm}$  are Stolz angles in  $\mathbb{C}^{\pm}$  at  $x = 0$ . Since  $A_n$  is far from  $\mathbb{R}$  and has uniformly bounded pseudohyperbolic diameter, every  $A_n$  hitting the imaginary axis will be in the Stolz angle  $S_+$  or  $S_-$ . Obviously, there is some  $M > 0$  such that for every  $z \in \mathbb{C}_{\pm 2\epsilon}^{\pm} \cap S_{\pm}$ , we have

$$|\psi(z)| \leq M.$$

In particular,  $|\phi(z)| \leq M$  for  $z \in \partial A_n$  and by the maximum principle,

$$|\phi(iy)| \leq M \text{ for } iy \in A_n \cap i\mathbb{R}.$$

Hence,  $\phi$  is uniformly bounded on  $i\mathbb{R}$  and it follows, by a Phragmen-Lindelöf principle that  $\phi \equiv K$ , and  $f = KS$ . Let us now show that  $K = 0$ . Because  $x \mapsto |S_1(x)|^p$  satisfies  $(A_p)$  we have

$$\int |S_1(x)|^p = \infty$$

and, applying the Plancherel-Polyà inequality, we also have

$$\int |S_1(x + 2i\epsilon)|^p = \infty$$

but  $|S(x + 2i\epsilon)| \asymp |S_1(x + 2i\epsilon)|$ , so

$$\int |S_1(x + 2i\epsilon)|^p = \infty.$$

We apply again the Plancherel-Polyà inequality to obtain

$$\int |S(x)|^p = \infty.$$

From the fact that  $f \in PW_r^p$ , we have by definition that  $f \in L^p$  and since  $f = \phi S = KS$ , the only possibility is  $K = 0$  and so  $f \equiv 0$ , which

ends the proof of the injectivity of  $R_\Lambda$ . Now, we can show the last part of the proof.

Let us consider a finite sequence  $a = (a(\lambda))_{\lambda \in \Lambda}$  and the solution of the interpolation problem  $f(\lambda) = a(\lambda)$ ,  $\lambda \in \Lambda$ , given by

$$f(z) = \sum_{\lambda \in \Lambda} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

Since the sum is finite,  $f$  is an entire function of type at most  $\tau$ . We want to split this sum according to the localization of the points of  $\Lambda$ . More precisely, we recall that we have the decomposition  $\Lambda = \bigcup_{n \in \mathbb{Z}} \tau_n$  and we have already introduced

$$N_+ = \{n \in \mathbb{Z} : \tau_n \cap (\mathbb{C}^+ \cup \mathbb{R}) \neq \emptyset\} \text{ and } N_- = \mathbb{Z} \setminus N_+.$$

We set

$$\Lambda_+ := \bigcup_{n \in N_+} \tau_n \text{ and } \Lambda_- := \bigcup_{n \in N_-} \tau_n = \Lambda \setminus \Lambda_+.$$

(Observe that since  $\text{diam}(\tau_n) < \frac{\epsilon}{2}$ , we have  $\Lambda_+ \subset \mathbb{C}_{-\frac{\epsilon}{2}}^+$ ). Now, we can write  $f = f^+ + f^-$ , with

$$f^\pm(z) := \sum_{\lambda \in \Lambda_\pm} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)} = \sum_{n \in N_\pm} \sum_{\lambda \in \tau_n} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

We want to estimate, separately,

$$(5.9) \quad \inf \left\{ \|f^\pm - g\|_p : g \in PW_\tau^p, g|_\Lambda = 0 \right\}.$$

Here we will only consider  $f^+$ , the method is the same for  $f^-$ . In the following,  $\beta$  will be the Blaschke product associated to  $\Lambda_{-\epsilon}^+ := \Lambda \cap \mathbb{C}_{-\epsilon}^+$

$$\beta(z) = \prod_{\lambda \in \Lambda_{-\epsilon}^+} \left( c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon} \right), \quad z \in \mathbb{C}_{-\epsilon}^+,$$

where again  $c_\lambda$  is a suitable normalizing factor. For  $z \in \mathbb{C}_{-\epsilon}^+$ , we write  $S(z) = e^{-i\tau z} \beta(z) G(z)$ . Observe that  $\beta(0) = \prod_{\lambda \in \Lambda} c_\lambda \frac{\lambda}{\bar{\lambda} - 2i\epsilon}$  (recall that we have assumed  $0 \notin \Lambda$ ). Thus, we can write

$$\begin{aligned} G(z) &= e^{i\tau z} S(z) \beta(z)^{-1} \\ &= e^{i\tau z} \prod_{\lambda \in \Lambda} \left( \frac{\lambda - z}{\lambda} \right) \prod_{\lambda \in \Lambda_{-\epsilon}^+} \left( c_\lambda \frac{z - \bar{\lambda} + 2i\epsilon}{z - \lambda} \right) \\ &= \beta(0)^{-1} e^{i\tau z} \prod_{\bar{\lambda} \in \bar{\Lambda}} \left( 1 - \frac{z}{\bar{\lambda}} \right), \end{aligned}$$

with  $\tilde{\Lambda} := (\Lambda \setminus \Lambda_{-\epsilon}^+) \cup (\overline{\Lambda_{-\epsilon}^+} - 2i\epsilon) \subset \mathbb{C}_{-\epsilon}^-$ . The function  $G$  is outer in  $\mathbb{C}_{-\epsilon}^+$ . As in (5.8), we obtain  $|\beta(x)| \asymp d_N(x)$ . In particular, we have  $|G(x)|^p \in (A_p)$ . Let then be  $\eta$  such that  $\frac{\epsilon}{2} < \eta < \epsilon$ . Since  $\tilde{\Lambda}$  is the union (not necessarily disjoint) of two  $N$ -Carleson sequences in  $\mathbb{C}_{-\epsilon}^-$ , and in particular

$$\operatorname{Im}(\tilde{\lambda} + i\eta) \leq \eta - \epsilon < 0, \quad \tilde{\lambda} \in \tilde{\Lambda},$$

(which implies in particular that every real  $x$  is far from  $\tilde{\Lambda}$ ), we obtain that

$$|G(x - i\eta)| = e^{\tau\eta} |G(x)| \left( \prod_{\tilde{\lambda} \in \tilde{\Lambda}} \left| \frac{x - \tilde{\lambda} - i\eta}{x - \tilde{\lambda}} \right| \right) \asymp |G(x)|.$$

So  $x \mapsto |G(x - i\eta)|^p$  also satisfies the Muckenhoupt condition  $(A_p)$ . According to the Plancherel-Polyà inequality, it is possible to estimate (5.9) on the axis  $\{\operatorname{Im}(z) = -\eta\}$ .

By duality arguments (see [19, p. 576] or [6, p. 94]), we need to estimate

$$\sup_{\substack{h \in H^q(\mathbb{C}_{-\eta}^+) \\ \|h\|_q = 1}} N(h),$$

with

$$\begin{aligned} N(h) &:= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} \int \frac{G(x - i\eta)h(x - i\eta)}{x - i\eta - \lambda} dx \right| \\ &= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} \mathcal{H}(\tilde{G}\tilde{h})(\lambda + i\eta) \right| \end{aligned}$$

where  $z \mapsto \tilde{G}(z) = G(z - i\eta)$  is an outer function in  $\mathbb{C}^+$  and the function  $z \mapsto \tilde{h}(z) = h(z - i\eta)$  belongs to  $H_+^q$ . In order to compute  $S'(\lambda)$ , let us recall that

$$S(z) = e^{-i\tau z} \beta(z) G(z), \quad z \in \mathbb{C}_{-\eta}^+.$$

For  $\lambda \in \tau_n$ ,  $n \in N_+$ , we have

$$S'(\lambda) = c_\lambda \frac{e^{-i\tau\lambda}}{\lambda - \bar{\lambda} + 2i\epsilon} G(\lambda) \frac{\beta}{b_\lambda^\epsilon}(\lambda),$$

where  $b_\lambda^\epsilon(z) = c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon}$ . Using that  $G(\lambda) = \tilde{G}(\lambda + i\eta)$ , and setting

$$\psi := \frac{\mathcal{H}(\tilde{G}\tilde{h})}{\tilde{G}} \quad \text{and} \quad \alpha(\lambda) := a(\lambda)e^{i\tau\lambda}, \quad \lambda \in \Lambda^+,$$

where we recall that  $\mathcal{H}$  denotes the Hilbert transform (see 1.2 for definition) the expression becomes

$$N(h) = \left| \sum_{n \in N_+} \sum_{\lambda \in \tau_n} \frac{\alpha(\lambda) \psi(\lambda + i\eta)}{\prod_{\mu \neq \lambda} b_\mu^\epsilon(\lambda)} (\lambda - \bar{\lambda} + 2i\epsilon) \right|.$$

Writing

$$N_+ = N_\epsilon \cup N_\infty, \text{ with } N_\epsilon := \{n \in N_+ : \tau_n \cap \{|\operatorname{Im}(z)| < \epsilon\} \neq \emptyset\},$$

we set, with the help of the functions of Lemma 6,

$$P_{\tau_n, \alpha}(z) := \sum_{k=1}^{|\tau_n|} \Delta_{\tau_n}^{k-1} (\alpha(\lambda_n^{(k)})) \prod_{l=1}^{k-1} b_{\lambda_{n,l}}(z), \quad n \in N_\infty,$$

$$P_{\tau_n, \alpha}(z) := \sum_{k=1}^{|\tau_n|} \square_{\tau_n}^{k-1} (\alpha(\lambda_n^{(k)})) \prod_{l=1}^{k-1} (z - \lambda_{n,l}), \quad n \in N_\epsilon$$

and setting  $\tilde{\tau}_n := \tau_n + i\eta$

$$Q_{\tilde{\tau}_n, \psi}(z) := \sum_{k=1}^{|\tilde{\tau}_n|} \Delta_{\tilde{\tau}_n}^{k-1} (\psi(\lambda_{n,1} + i\eta, \dots, \lambda_{n,k} + i\eta)) \prod_{l=1}^{k-1} b_{\lambda_{n,l} + i\eta}(z).$$

We notice that

$$N(h) = \left| \sum_{n \in N_+} \sum_{\lambda \in \tau_n} \frac{P_{\tau_n, \alpha}(\lambda) Q_{\tilde{\tau}_n, \psi}(\lambda + i\eta)}{\prod_{\mu \neq \lambda} b_\mu^\epsilon(\lambda)} (\lambda - \bar{\lambda} + 2i\epsilon) \right|.$$

Recall now that  $\tau_n \subset R_n$ , where  $(R_n)_n$  are the disjoint rectangles (constructed here in the half-plane  $\mathbb{C}_{-\eta}^+$  so that in particular satisfying  $d(\partial R_n, \mathbb{R} - i\eta) \asymp l_n \asymp L_n$ ) introduced in Remark 12. (Note also that here we have that  $\Lambda_+ \subset \mathbb{C}_{-\frac{\epsilon}{2}}^+$  and in particular,  $\Lambda_+$  is far from  $\mathbb{R} - i\eta$ ). Then, if  $\Gamma_n := \partial R_n$ , the function

$$z \mapsto h_n(z) := \frac{P_{\tau_n, \alpha}(z) Q_{\tilde{\tau}_n, \psi}(z + i\eta)}{\beta(z)}$$

is a meromorphic function in  $\overset{\circ}{R}_n$  with simple poles at  $\lambda \in \tau_n$ . Thus, the residue theorem implies that

$$\int_{\Gamma_n} h_n(z) dz = 2i\pi \sum_{\lambda \in \tau_n} \operatorname{Res}(h_n, \lambda)$$

and

$$\operatorname{Res}(h_n, \lambda) = P_{\tau_n, \alpha}(\lambda) Q_{\tilde{\tau}_n, \psi}(\lambda + i\eta) \left( \frac{\beta}{b_\lambda^\epsilon}(\lambda) \right)^{-1} \cdot (\lambda - \bar{\lambda} + 2i\epsilon).$$

It follows that

$$N(h) = \left| \frac{1}{2i\pi} \sum_{n \in N_+} \int_{\Gamma_n} \frac{P_{\tau_n, \alpha}(z) Q_{\tilde{\tau}_n, \psi}(z + i\eta)}{\beta} dz \right|.$$

Obviously  $|b_{\lambda_{n,l}}(z)| \leq 1$ . Observe also that by condition (3.5) of Remark 12 for  $z \in \Gamma_n$ ,  $n \in N_\epsilon$ , we have that  $|z - \lambda_{n,l}|$  is bounded by a fixed constant. Hence for every  $n \in N_+$ ,

$$|P_{\tau_n, \alpha}| \lesssim \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1}(\alpha(\lambda_n^{(k)})) \right|.$$

Also

$$|Q_{\tilde{\tau}_n, \psi}| \lesssim \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1}(\psi(\lambda_{n,1} + i\eta, \dots, \lambda_{n,k} + i\eta)) \right|,$$

and we obtain that

$$N(h) \lesssim \sum_{n \in N_+} \left[ \left( \int_{\Gamma_n} \left| \frac{dz}{\beta(z)} \right| \right) \left( \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1}(\alpha) \right| \right) \left( \sum_{l=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{l-1}(\psi) \right| \right) \right].$$

For  $z \in \Gamma_n$ , we see that

$$\begin{aligned} |\beta(z)| &= \left( \prod_{\lambda \in \Lambda_+ \setminus \tau_n} \left| \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon} \right| \right) \cdot \left( \prod_{\lambda \in \tau_n} \left| \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon} \right| \right) \\ &=: \Pi_1(z) \cdot \Pi_2(z). \end{aligned}$$

Since  $\Lambda_+$  is  $N$ -Carleson in  $\mathbb{C}_{-\epsilon}^+$ , it follows from the fact that  $R_n$  is “far” from  $\tau_k$ ,  $k \neq n$  that

$$\Pi_1(z) \asymp 1$$

and from the fact that  $R_n$  is “far” from  $\tau_n$  that

$$\Pi_2(z) \asymp 1.$$

Hence, choosing arbitrarily  $\lambda_{n,0} \in \tau_n$ , the construction of  $R_n$  gives

$$\int_{\Gamma_n} \left| \frac{dz}{\beta(z)} \right| \lesssim \int_{\Gamma_n} |dz| \lesssim \operatorname{Im}(\lambda_{n,0}) + \eta \lesssim 1 + |\operatorname{Im}(\lambda_{n,0})|.$$

Applying Hölder's inequality, we obtain

$$N(h) \lesssim \left( \sum_{n \in N_+} (1 + \operatorname{Im}(\lambda_{n,0})) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (e^{i\tau \cdot} a) \right|^p \right)^{\frac{1}{p}} \\ \times \left( \sum_{n \in N_+} \operatorname{Im}(\lambda_{n,0} + i\eta) \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1} (\psi) \right|^q \right)^{\frac{1}{q}}.$$

Now, notice that by the Muckenhoupt condition on  $|\tilde{G}|^{-q}$  and thus the boundedness of  $\mathcal{H}$  on

$$H_+^q \left( \left| \frac{1}{\tilde{G}} \right|^q \right) := \left\{ f \in \mathcal{N}^+ : f|_{\mathbb{R}} \in L^q \left( \left| \frac{1}{\tilde{G}} \right|^q \right) \right\},$$

( $\mathcal{N}^+$  denotes the Smirnov class) we get that  $\psi \in H_+^q$  and  $\|\psi\|_q \lesssim \|\tilde{h}\|_{H_+^q} = 1$ . But, since

$$\bigcup_{n \in N_+} \tilde{\tau}_n = \Lambda^+ + i\eta$$

is in fact  $N$ -Carleson in  $\mathbb{C}_{\eta - \frac{\epsilon}{2}}^+ \subset \mathbb{C}^+$  and  $\psi \in H_+^q$ , Theorem 13 implies that

$$\left( \sum_{n \in N_+} \operatorname{Im}(\lambda_{n,0} + i\eta) \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1} (\psi) \right|^q \right)^{\frac{1}{q}} \lesssim \|\psi\|_{H_+^q} \lesssim \|\tilde{h}\|_{H_+^q} = 1.$$

Finally, we obtain

$$N(h) \lesssim \left( \sum_{n \in N_+} (1 + \operatorname{Im}(\lambda_{n,0})) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (e^{i\tau \cdot} a) \right|^p \right)^{\frac{1}{p}} = \|a\|_{X_{\tau, \epsilon}^p(\Lambda)},$$

which ends the proof.

## 6. ABOUT THE $N$ -CARLESON CONDITION

It is clear that the definition of  $X_{\tau, \epsilon}^p(\Lambda)$  depends on the  $N$ -Carleson hypothesis, and more precisely for the construction of the groups  $\tau_n$ . In this last section, we show that in a certain way, the  $N$ -Carleson condition is necessary.

It will be convenient to introduce the distance function

$$\delta(z, \xi) := \frac{|z - \xi|}{1 + |z - \bar{\xi}|}, \quad z, \xi \in \mathbb{C},$$

which expresses that locally we deal with Euclidian geometry close to the real axis and pseudohyperbolic geometry far away from the real axis (see *e.g.* [17, page 715]). Let  $\Lambda = \{\lambda_n\}_{n \geq 1}$  be a sequence of complex numbers. Let  $N \geq 1$  be an integer and  $\eta \in (0, \frac{1}{2})$ . For  $\lambda \in \Lambda$ , we define

$$D_{\lambda, \eta} := \{z \in \mathbb{C} : \delta(\lambda, z) < \eta\},$$

$$N_\lambda := \{\mu_{\lambda, i} : 1 \leq i \leq N\} \subset \Lambda$$

as the set of  $N$  closest neighbors of  $\lambda$  (including in particular  $\lambda$ ) with respect to the distance  $\delta$ . Then we set

$$\sigma_\lambda := D_{\lambda, \eta} \cap N_\lambda, \quad n_\lambda := |\sigma_\lambda| \leq N.$$

Note that the set  $N_\lambda$ , and consequently  $\sigma_\lambda$ , is not unique. It is now natural to introduce the space (for  $1 < p < \infty$ )

$$X_\tau^p(\Lambda, N) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X_\tau^p(\Lambda, N)} < \infty \right\},$$

where

$$\|a\|_{X_\tau^p(\Lambda, N)}^p := \sum_{\lambda \in \Lambda} (1 + |\operatorname{Im}(\lambda)|) \sum_{k=1}^{n_\lambda} \left| \tilde{\Delta}_{\sigma_\lambda}^{k-1} (ae^{\pm i\tau \cdot} (\mu^{(k)})) \right|^p$$

with

$$\tilde{\Delta}_{\sigma_\lambda} = \begin{cases} \Delta_{\sigma_\lambda}, & \text{if } \sigma_\lambda \cap \{z \in \mathbb{C} : |\operatorname{Im}(z)| < 1\} = \emptyset \\ \square_{\sigma_\lambda}, & \text{if not} \end{cases}$$

and

$$e^{\pm i\tau \mu} = \begin{cases} e^{i\tau \mu} & , \text{ if } \mu \in \sigma_\lambda \text{ and } \sigma_\lambda \cap \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\} \neq \emptyset \\ e^{-i\tau \mu} & , \text{ otherwise} \end{cases}.$$

*Remark 23.* It can be shown that if  $\Lambda \cap \mathbb{C}_a^\pm$  is  $N$ -Carleson in the corresponding half-plane, for each  $a \in \mathbb{R}$ , then this norm is equivalent to the previously norm  $\|\cdot\|_{X_{\tau, \epsilon}^p(\Lambda)}$  (for every  $\epsilon > 0$ ) defined in the above section. For the proof, we refer to [7, pp. 36-38].

The result is the following one.

**Theorem 24.** *If  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda, N)$ , then for every  $a \in \mathbb{R}$ ,  $\Lambda \cap \mathbb{C}_a^\pm$  is  $N'$ -Carleson in the corresponding half-plane, with  $N' \leq N$ .*

The proof is in two parts. We begin by showing that if  $R_\Lambda$  is such an isomorphism, then  $\Lambda_a^\pm$  is  $N'$ -Carleson for some  $N' \in \mathbb{N}$ . This only requires the boundedness of  $R_\Lambda$ . We first notice that by the Plancherel-Polyà theorem (Proposition 20) the map

$$\begin{aligned} \tau_a : PW_\tau^p &\rightarrow PW_\tau^p \\ f &\mapsto f(\cdot + i(1 + |a|)) \end{aligned}$$

is an isomorphism and so  $\tilde{R}_\Lambda := R_\Lambda \circ \tau_a$  is still an isomorphism. Obviously,  $\tilde{R}_\Lambda = R_{\tilde{\Lambda}}$ , where

$$\tilde{\Lambda} := \Lambda + i(1 + |a|).$$

Note that for  $\lambda \in \Lambda_a^+$ , with the notations of Lemma 6,

$$|a_\lambda|^p e^{-p\text{Im}(\lambda)} = |P_{\sigma_\lambda, e^{\pm i\tau \cdot a}}| \leq \sum_{k=1}^{n_\lambda} \left| \tilde{\Delta}_{\sigma_\lambda}^{k-1} (ae^{\pm i\tau \cdot} (\mu^{(k)})) \right|^p$$

and so  $X_\tau^p(\tilde{\Lambda}, N)$  injects into  $l^p\left(\left(1 + |\text{Im}(\tilde{\lambda})|\right) e^{-p|\text{Im}(\tilde{\lambda})}\right)$  so that

$$R_{\tilde{\Lambda}} : PW_\tau^p \rightarrow l^p\left(\left(1 + |\text{Im}(\tilde{\lambda})|\right) e^{-p|\text{Im}(\tilde{\lambda})}\right)$$

is bounded. We set  $\tilde{\Lambda}_a^+ := \Lambda_a^+ + i(1 + |a|)$  and reintroduce the inner function  $I_\tau(z) = \exp(2i\tau z)$ . We have mentioned in the beginning of the paper that  $PW_\tau^p$  is isomorphic to  $K_{I_\tau}^p$ , so

$$R_{\tilde{\Lambda}_a^+}^{I_\tau} := R_{\tilde{\Lambda}_a^+} \Big| K_{I_\tau}^p : K_{I_\tau}^p \rightarrow L^p(\mu_{\tilde{\Lambda}_a^+})$$

is bounded, where

$$\mu_{\tilde{\Lambda}_a^+} := \sum_{\tilde{\lambda} \in \tilde{\Lambda}_a^+} \text{Im}(\tilde{\lambda}) \delta_{\tilde{\lambda}}.$$

In order to show that  $\Lambda_a^\pm$  is  $N'$ -Carleson, it is sufficient to show that  $\mu_{\tilde{\Lambda}_a^+}$  is a Carleson measure for  $H_+^p$ . Since in particular  $\tilde{\Lambda}_a^+ \subset \mathbb{C}_1^+$ , it is possible to find  $\epsilon \in (0, 1)$  such that

$$\tilde{\Lambda}_a^+ \subset L(I^\tau, \epsilon) := \{z \in \mathbb{C}^+ : |I^\tau(z)| < \epsilon\}.$$

Now, from a result of Treil and Volberg (see [20] or [1]), the boundedness of  $R_{\tilde{\Lambda}_a^+}^{I_\tau}$  implies that

$$(6.1) \quad \sup_I \frac{\mu_{\tilde{\Lambda}_a^+}(\omega_I)}{m(I)} < \infty,$$

where the supremum is taken over all the intervals of finite length such that the Carleson window  $\omega_I$  constructed on  $I$  satisfies

$$\omega_I \cap L(I^\tau, \epsilon) \neq \emptyset.$$

Observe that  $L(I^\tau, \epsilon)$  is in the upper half plane  $\mathbb{C}_b^+$ ,  $b = \log(1/\epsilon)$ , so that if the length of the Carleson window is less than  $b$ , then we have  $\omega_I \cap L(I^\tau, \epsilon) = \emptyset$ . Hence,  $\omega_I \cap \tilde{\Lambda}_a^+ = \emptyset$  and so  $\mu_{\tilde{\Lambda}_a^+}(\omega_I) = 0$ . It follows that (6.1) is true for all finite length intervals  $I$ , which is equivalent to



the fact that  $\mu_{\tilde{\Lambda}_a^+}$  is a Carleson measure or also that  $\tilde{\Lambda}_a^+$  is  $N'$ -Carleson and hence  $\Lambda_a^+$  in the corresponding half-plane. Considering the map

$$s : \begin{array}{ccc} PW_\tau^p & \rightarrow & PW_\tau^p \\ f & \mapsto & f(-\cdot) \end{array}$$

which is also an isomorphism, we will also have the result for  $\Lambda_a^-$ .

Now, we want to prove that  $N' \leq N$ . In the following, if  $\Lambda_a^+$  is  $(N+k)$ -Carleson, we write

$$\Lambda_a^+ = \bigcup_{n \geq 1} \tau_n^k,$$

where the groups  $\tau_n^k$  come from the Generalized Carleson condition, and so it is possible to assume that

$$\text{diam}_\delta(\tau_n^k) < \frac{\eta}{4}$$

(which in particular implies that  $\tau_n^k \subset D_{\lambda, \eta}$ ) and

$$\gamma := \inf_{n \neq m} \delta(\tau_n^k, \tau_m^k) > 0.$$

We need the following lemma and its corollary. For technical reasons, let us assume (without loss of generality) that  $\Lambda_a^+ \subset \mathbb{C}_1^+$  so that we can deal with the pseudohyperbolic metric and the corresponding divided differences.

**Lemma 25.** *If  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda, N)$  and  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson,  $k \geq 0$ , then it is possible to find  $\vartheta > 0$  such that every  $\tau_n^{k+1}$  with  $|\tau_n^{k+1}| = N+k+1$  satisfies  $\text{diam}_\rho(\tau_n^{k+1}) > \vartheta$ .*

*Proof.* Let us suppose to the contrary that we can find a subsequence  $(\tilde{\tau}_j)$  of  $(\tau_n^{k+1})$  such that  $|\tilde{\tau}_j| = N+k+1$  and  $\text{diam}_\rho(\tilde{\tau}_j) \rightarrow 0$ ,  $j \rightarrow \infty$ . We set  $\tilde{\tau}_j = \{\lambda_i^j : i = 0, \dots, N+k\}$ . Let us now introduce the sequence  $a^j = (a^j(\lambda))_{\lambda \in \Lambda}$  defined by

$$a^j(\lambda) := 0, \quad \lambda \neq \lambda_{N+k}^j,$$

and

$$a^j(\lambda_{N+k}^j) := e^{\tau \text{Im}(\lambda_{N+k}^j)} \text{Im}(\lambda_{N+k}^j)^{-\frac{1}{p}} \frac{\prod_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|}.$$

Let

$$M_j := \{\lambda \in \Lambda_a^+ : \lambda_{N+k}^j \in \sigma_\lambda\}$$

the set of the points of  $\Lambda_a^+$  close to  $\lambda_{N+k}^j$ . Since  $\text{diam}_\rho(\tilde{\tau}_j) < \frac{\eta}{4}$  and  $\lambda_{N+k}^j \in \tilde{\tau}_j$  we have for every  $\lambda \in \tilde{\tau}_j$  that  $\lambda_{N+k}^j \in \sigma_\lambda$ , *i.e.*  $\tilde{\tau}_j \subset M_j$ . So, let  $B_j := M_j \setminus \tilde{\tau}_j$ . Also, since  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson,

$$\sup_j \left| \Lambda_a^+ \cap D_{\lambda_{N+k}^j} \right| < \infty,$$

which implies that

$$\sup_j |M_j| < \infty.$$

By construction,

$$\|a^j\|_{X_\tau^p(\Lambda, N)}^p = \sum_{\lambda \in M_j} (1 + \text{Im}(\lambda)) \sum_{l=1}^{n_\lambda} |\Delta_{\sigma_\lambda}^{l-1}(a^j e^{i\tau \cdot}(\mu^{(l)}))|^p.$$

(Observe that we only consider in the sum the points containing  $\lambda_{N+k}^j$  in their neighborhood.)

We have to evaluate this expression. Take  $\lambda \in M_j$ . We recall that  $n_\lambda = |\sigma_\lambda|$ . Note also that for every  $1 \leq l \leq n_\lambda$ , the divided difference

$$|\Delta_{\sigma_\lambda}^{l-1}(a^j e^{\pm i\tau \cdot}(\lambda^{(l)}))|$$

will be equal either to 0 or to

$$\left| a^j(\lambda_{N+k}^j) e^{\pm i\tau \lambda_{N+k}^j} \prod_{m \in \omega_l} b_{\lambda_m^j}(\lambda_{N+k}^j) \right|,$$

where  $\omega_l \subset \sigma_\lambda$  contains  $l-1$  points. Now,  $\omega_l = \omega_{l,1} \cup \omega_{l,2}$  where  $\omega_{l,1} = \sigma_\lambda \cap \tilde{\tau}_j$  and  $\omega_{l,2}$  are the other points. Note that  $\omega_l$  cannot contain  $\lambda_{N+k}^j$ . By assumption, for  $\mu \in \omega_{l,2}$ ,  $|b_\mu(\lambda_{N+k}^j)| \geq \gamma$ . Hence,

$$\begin{aligned} & \sum_{l=1}^{n_\lambda} |\Delta_{\sigma_\lambda}^{l-1}(a^j e^{\pm i\tau \cdot}(\lambda^{(l)}))|^p \\ & \leq \sum_{l=1}^{n_\lambda} \frac{\prod_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|^p}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|^p} \cdot \frac{1}{\text{Im}(\lambda_{N+k}^j) \prod_{\mu \in \omega_l} |b_\mu(\lambda_{N+k}^j)|^p} \\ & \leq \sum_{l=1}^{n_\lambda} \frac{1}{\gamma^{p|\omega_{l,2}|}} \frac{\prod_{\xi \in \Omega_l} |b_\xi(\lambda_{N+k}^j)|^p}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|^p} \cdot \frac{1}{\text{Im}(\lambda_{N+k}^j)} \\ & \lesssim \frac{N}{\text{Im}(\lambda_{N+k}^j)}, \end{aligned}$$

where  $\Omega_l = \{\lambda_i^j : i = 0, \dots, N+k-1\} \setminus \omega_{l,1}$  are subsets of  $\tilde{\tau}_j$ . The last of the above inequalities comes from the observation that  $\Omega_l$  contains at least:

$$N+k - |\omega_{l,1}| \geq N+k - (n_\lambda - 1) \geq N+k - (N-1) = k+1 \geq 1$$

points. We deduce that  $a^j \in X_\tau^p(\Lambda, N)$  and that its norm is uniformly bounded. Now, since  $R_\Lambda$  is onto, there is  $f^j \in PW_\tau^p$  such that  $f^j|_\Lambda = a^j$  and

$$\|f^j\|_{PW_\tau^p} \lesssim \|a^j\|_{X_\tau^p(\Lambda, N)} \lesssim 1.$$

Setting  $\tilde{f}^j := e^{i\tau \cdot} f^j$ , it follows from the Plancherel-Polyà inequality that  $\tilde{f}^j \in H_+^p$  and since  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson in  $\mathbb{C}^+$ , Theorem (13) implies in particular that

$$\operatorname{Im}(\lambda_{N+K}^j) \left| \Delta_{\tilde{\tau}_j}^{N+k} \left( \tilde{f}^j \left( (\lambda^j)^{(N+k+1)} \right) \right) \right|^p \lesssim \|f^j\| \lesssim 1.$$

But by construction, we have

$$\operatorname{Im}(\lambda_{N+K}^j) \left| \Delta_{\tilde{\tau}_j}^{N+k} \left( \tilde{f}^j \left( (\lambda^j)^{(N+k+1)} \right) \right) \right|^p = \frac{1}{\max_{i \neq N+k} \rho(\lambda_{N+k}^j, \lambda_i^j)}$$

which tends to  $\infty$ ,  $j \rightarrow \infty$  because  $\operatorname{diam}_\rho \tilde{\tau}_j$  tends to 0,  $j \rightarrow \infty$ , which gives the required contradiction.  $\square$

The following corollary to the previous lemma allows us to end the proof of our theorem.

**Corollary 26.** *If  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda, N)$  and  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson,  $k \geq 0$ , then  $\Lambda_a^+$  is  $(N+k)$ -Carleson.*

*Proof.* We write  $\Lambda_a^+ = \bigcup_{n \geq 1} \tau_n^{k+1}$  with  $|\tau_n^{k+1}| \leq N+k+1$ . Let us suppose that there are infinitely many  $n$  for which we have  $|\tau_n^{k+1}| = N+k+1$  and let  $Z$  be the set of such  $n$ . Because of the previous lemma, we can find  $\vartheta > 0$  such that  $\operatorname{diam}_\rho(\tau_n^{k+1}) > \vartheta$  for  $n \in Z$ . Then, for every  $n \in Z$ , it is possible to write  $\tau_n^{k+1} = \{\lambda_i^n : i = 1, \dots, N+k+1\}$  such that

$$\rho(\lambda_i, \lambda_{N+k+1}^n) \geq \frac{\vartheta}{2(N+k)}, \quad i = 1, \dots, N+k.$$

It follows that

$$\Lambda_a^+ = \bigcup_{n \notin Z} \tau_n^{k+1} \cup \left( \bigcup_{n \in Z} \tau_n^{k+1} \setminus \{\lambda_{N+k+1}^n\} \right) \cup \left( \bigcup_{n \in Z} \{\lambda_{N+k+1}^n\} \right)$$

is a disjoint union of sets  $\sigma_n$  with  $|\sigma_n| \leq N+k$  and it can be shown that the sequence of Blaschke products  $(B_{\sigma_n})_n$  satisfies the Generalized Carleson condition and hence that  $\Lambda_a^+$  is  $(N+k)$ -Carleson.  $\square$

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