



**HAL**  
open science

# Divided Differences & Restriction Operator on Paley-Wiener Spaces for N-Carleson sequences

Frederic Gaunard

► **To cite this version:**

Frederic Gaunard. Divided Differences & Restriction Operator on Paley-Wiener Spaces for N-Carleson sequences. 2010. hal-00585565v1

**HAL Id: hal-00585565**

**<https://hal.science/hal-00585565v1>**

Preprint submitted on 13 Apr 2011 (v1), last revised 15 May 2012 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DIVIDED DIFFERENCES & RESTRICTION OPERATOR ON PALEY-WIENER SPACES $PW_\tau^p$ FOR $N$ -CARLESON SEQUENCES

FRÉDÉRIC GAUNARD

ABSTRACT. We study the restriction operator  $R_\Lambda, f \mapsto (f(\lambda))_{\lambda \in \Lambda}$  defined on Paley-Wiener spaces  $PW_\tau^p$  ( $1 < p < \infty$ ), where  $\Lambda$  is a sequence of complex numbers. Lyubarskii and Seip gave necessary and sufficient conditions for  $R_\Lambda$  to be an isomorphism between  $PW_\tau^p$  and some weighted  $l^p$  space, involving Carleson's and Muckenhoupt's  $(A_p)$  conditions. Here, we deal with  $N$ -Carleson sequences (finite unions of disjoint Carleson sequences) and use the methods of Lyubarskii and Seip to give necessary and sufficient conditions for  $R_\Lambda$  to be an isomorphism between  $PW_\tau^p$  and some space of sequences, constructed with the help of divided differences. For  $p = 2$ , this characterization coincides with a result of Avdonin and Ivanov on Riesz bases of divided differences of exponentials in  $L^2(0, \tau)$ .

## 1. INTRODUCTION

Let  $X$  be a Banach space of analytic functions on a domain  $\Omega$ , and  $\Lambda \subset \Omega$  a sequence of complex numbers. The restriction operator  $R_\Lambda$  associated to  $\Lambda$  is defined on  $X$  by  $R_\Lambda f := (f(\lambda))_{\lambda \in \Lambda}$ . We want to describe the range of  $R_\Lambda$ , that we will denote by  $X|\Lambda$ , and to study its injectivity. This problem is of course related with interpolation in  $X$ . The central result was shown by Carleson [Ca58] for  $X = H^\infty$  the Hardy space of bounded holomorphic functions on the unit disk, and  $\Lambda \subset \mathbb{D}$ . By definition,  $X|\Lambda \subset l^\infty$ , but Carleson showed that

$$H^\infty|\Lambda = l^\infty \iff \Lambda \in (C)$$

where

$$(C) \quad \inf_{\lambda \in \Lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} |b_\mu(\lambda)| > 0.$$

---

1991 *Mathematics Subject Classification.* 30E05, 42A15, 44A15.

*Key words and phrases.* Divided differences, Carleson sequences, interpolation, Paley-Wiener spaces, Discrete Muckenhoupt condition.

Here,  $b_\mu(z) = \frac{|\mu|}{\mu} \frac{\mu-z}{1-\bar{\mu}z}$  denotes the Blaschke factor (of the unit disc) associated to  $\mu$ . Three years later, Shapiro and Shields [SS61] generalized this result to  $X = H^p$ ,  $1 \leq p \leq \infty$ ,

$$H^p|_\Lambda = l^p(1 - |\lambda|^2) \iff \Lambda \in (C).$$

Of course, these results are still true in the Hardy spaces of the upper half-plane where the weight  $(1 - |\lambda|^2)$  becomes  $\text{Im}(\lambda)$ . In the Hardy space, it is obvious that  $R_\Lambda$  cannot be injective : if  $\Lambda$  is a Blaschke sequence (this is the case in particular when  $\Lambda$  is a Carleson sequence), then for every  $h \in H^p$ , the function  $Bh$ , where

$$B = \prod_{\lambda \in \Lambda} b_\lambda$$

is the Blaschke product associated to  $\Lambda$ , is still in  $H^p$  and also vanishes on  $\Lambda$ .

Let us turn to the situation in half-planes. We will say that a sequence  $\Lambda$  lying in a half-plane

$$\mathbb{C}_a^+ := \{z \in \mathbb{C} : \text{Im}(z) > a\} \text{ or } \mathbb{C}_a^- := \{z \in \mathbb{C} : \text{Im}(z) < a\} \quad (a \in \mathbb{R})$$

satisfies the Carleson condition (C) if

$$\inf_{\lambda \in \Lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left| \frac{\lambda - \mu}{\lambda - \bar{\mu} - 2ia} \right| > 0.$$

The Hardy spaces associated with the half-planes  $\mathbb{C}^\pm := \mathbb{C}_0^\pm$  will be denoted by  $H_\pm^p$ .

We consider now the Paley-Wiener spaces  $PW_\tau^p$  which consist of all entire functions of exponential type at most  $\tau$  satisfying

$$\|f\|_p^p = \int_{-\infty}^{+\infty} |f(t)|^p dt < \infty.$$

Writing  $\mathcal{F}$  the Fourier transform, it is well known in the case  $p = 2$ ,  $PW_\tau^2 = \mathcal{F}L^2(-\tau, \tau) \simeq L^2(-\tau, \tau)$ . These spaces are connected with so-called model spaces. More precisely, we recall that  $H_+^\infty$  denotes the Hardy space of bounded analytic functions in the upper half-plane and that an inner function  $I$  is a function of  $H_+^\infty$  such that

$$|I(x)| = 1 \text{ a.e } x \in \mathbb{R}.$$

The model space  $K_I^p$  is defined by  $K_I^p := H_+^p \cap \bar{I}H_-^p$ . It can be shown that  $PW_\tau^p \simeq K_{I_\tau}^p$  with  $I_\tau(z) := \exp(2i\tau z)$ . For more details, see e.g. [Ni02b] or [Se04]. So  $PW_\tau^p$  can be seen as a subspace of the Hardy space of the upper half plane  $H_+^p$  (modulo analytic continuation in the lower half-plane). This point of view permits to make a link with the results

of Hrushev, Nikolski and Pavlov [HNP81] about bases of exponentials  $(e^{\lambda t})_{\lambda \in \Lambda}$  in  $L^2(-\tau, \tau)$ , assuming that  $\Lambda \subset \mathbb{C}_\eta^+ := \{z \in \mathbb{C} : \text{Im}(z) > \eta\}$ , where  $\eta$  is some real number. Minikin managed to get rid of this restriction ([Mi92]).

The technics in [HNP81] are based on the invertibility of a certain Toeplitz operator  $T_{I, \overline{B}}$ , and a criterion of invertibility for Toeplitz operators is known. This is the theorem of Widom-Devinatz for  $p = 2$  (see [Ni02a, B4.3.1]) and Rochberg ([Ro77]) for  $1 < p < \infty$  and involves the Muckenhoupt condition  $(A_p)$  (or equivalently the Helson-Szeg condition for  $p = 2$ ).

With a different method, Luybarskii and Seip [LS97] gave necessary and sufficient conditions for  $R_\Lambda$  to be an isomorphism (with  $\Lambda$  lying in the whole complex plane) between  $PW_\tau^p$  and the weighted space  $l^p(e^{-p\tau|\text{Im}(\lambda)}(1 + |\text{Im}(\lambda)|))$ . Their proof is based on the boundedness of the Hilbert transform in certain weighted Hardy space.

The Hilbert transform  $\mathcal{H}$  is defined by

$$\mathcal{H}f(z) = \int_{-\infty}^{+\infty} \frac{f(t)}{t - z} dt.$$

It is known (see e.g [HMW73] and [Gar81]) that, if  $w > 0$ ,  $\mathcal{H}$  is bounded from the weighted space

$$L^p(w) := \left\{ f \text{ meas. on } \mathbb{R} : \int_{\mathbb{R}} |f|^p w dm < \infty \right\}$$

into itself, if and only if  $w$  satisfies the Muckenhoupt  $(A_p)$  condition

$$(A_p) \quad \sup_I \left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{\frac{p}{p-1}} < \infty,$$

where the supremum is taken over all intervals of finite length. Moreover, if  $f \in L^p(\mathbb{R})$ ,  $1 < p < \infty$ , we deduce from properties of Hardy spaces of the upper half-plane (see e.g. [Ko80, p. 116]) that  $\mathcal{H}f = P_+ f$ , where  $P_+$  denotes the Riesz projection from  $L^2$  onto  $H_+^2$ . In [LS97], the authors also introduce the *discrete Hilbert transform* as follows. For fixed  $\epsilon > 0$  and two sequences  $\Gamma := \{\gamma_n\}_n$  and  $\Sigma := \{\sigma_n\}_n$  satisfying  $|\gamma_n - \sigma_n| = \epsilon$ , and  $a = (a_n)_n$ ,

$$(\mathcal{H}_{\Gamma, \Sigma}(a))_n := \sum_j \frac{a_j}{\gamma_j - \sigma_n}.$$

It is proved in the same paper that  $\mathcal{H}_{\Gamma, \Sigma}$  is bounded from  $l^p(w_n)$  into itself if and only if  $(w_n)_n$  satisfies the *discrete Muckenhoupt condition*

$$(\mathfrak{A}_p) \quad \sup_{\substack{k \in \mathbb{Z} \\ n > 0}} \left( \frac{1}{n} \sum_{j=k+1}^{k+n} w_j \right) \left( \frac{1}{n} \sum_{j=k+1}^{k+n} w_j^{-1/(p-1)} \right)^{p-1} < \infty.$$

We are now in a position to state Lyubarskii and Seip's result. We say that a sequence  $\Lambda \subset \mathbb{C}$  satisfies the *(LS) conditions* ( $\Lambda \in (LS)$ ) if the four following conditions are fulfilled

- (i)  $\forall a \in \mathbb{R}$ ,  $\Lambda \cap \mathbb{C}_a^\pm$  satisfies the Carleson condition in the corresponding half-plane;
- (ii) The sequence is weakly dense:  $\exists r > 0$ ,  $\forall x \in \mathbb{R}$ ,

$$d(x, \Lambda) := \inf_{\lambda \in \Lambda} |x - \lambda| < r;$$

- (iii) The limit

$$S(z) = \lim_{R \rightarrow \infty} \prod_{|\lambda| < R} \left( 1 - \frac{z}{\lambda} \right)$$

exists and defines an entire function of exponential type  $\tau$ ;

- (iv) The function  $x \mapsto \left( \frac{|S(x)|}{d(x, \Lambda)} \right)^p$  satisfies  $(A_p)$ .

The condition (iv) can be replaced by the condition (iv)'

- (iv)' There is a subsequence  $\Gamma = (\gamma_n)_n \subset \Lambda$ , satisfying (ii), such that the sequence  $|S'(\gamma_n)|^p$  satisfies the discrete Muckenhoupt condition  $(\mathfrak{A}_p)$ .

Note that if  $0 \in \Lambda$ , then the corresponding factor in (iii) is just  $z$ . In order not to complicate the notation we shall assume in all what follows that  $0 \notin \Lambda$  which we can do without loss of generality (for instance, by shifting the sequence).

We can now state Lyubarski and Seip's theorem.

**Theorem 1.** ([LS97, Theorem 1])

Let  $\Lambda \subset \mathbb{C}$ . The following assertions are equivalent.

- (1)  $R_\Lambda$  is an isomorphism between  $PW_\tau^p$  and the weighted  $l^p$  space  $l^p(e^{-p\tau|\operatorname{Im}(\lambda)|} (1 + |\operatorname{Im}(\lambda)|))$ ;
- (2)  $\Lambda \in (LS)$ .

The aim of this paper is to replace the Carleson condition in (i) by the  $N$ -Carleson condition and to describe the range of  $R_\Lambda$ , using results of Hartmann ([Ha96b]) about  $N$ -Carleson sequences and interpolation in Hardy spaces, involving divided differences.

This paper is organized as follows. The next section will be devoted to divided differences, the following section will deal with  $N$ -Carleson

sequences. We will state and prove our main result (Theorem 14) in the fourth section. Finally, in the last section we will discuss the necessity of the  $N$ -Carleson condition with an appropriate definition of the trace  $PW_\tau^p|_\Lambda$ .

A final word on our notation. If  $\sigma$  and  $\tau$  are two disjoint subsets, the union will be denoted by  $\sigma \uplus \tau$ . If  $\delta$  is a metric, we will denote by  $\text{diam}_\delta(E)$  the  $\delta$ -diameter of  $E$  and shortly  $\text{diam}(E)$  when  $\delta$  is the Euclidian distance.

## 2. DIVIDED DIFFERENCES

Divided differences appear in many results about interpolation or bases of exponentials (see e.g. [Va84], [Ha96b], [BNO96] or [AI01]). We give here the definition and some properties that we will need in the following. We recall that the (non-normalized) Blaschke factors of a half-plane  $\mathbb{C}_a^\pm$  are given by

$$b_\mu^{\pm,a}(z) = \frac{z - \mu}{z - \bar{\mu} - 2ia}.$$

(They are actually the same for the upper and the lower half-plane). The associated pseudohyperbolic distance will be denoted by

$$\rho_{\pm,a}(z, \mu) := |b_\mu^{\pm,a}(z)|.$$

For  $\mathbb{C}^+$ , we will write  $b_\mu = b_\mu^{+,0}$  and use  $\rho$  for  $\rho_{+,0}$  and  $\rho_{-,0}$ .

Notice that, in order to simplify to the notation, we keep the same notation  $b_\mu$  as already introduced for the disk and we hope that it will be clear from the context which one we have in mind. The same will be true for the definition of divided differences.

Let  $\Gamma := \{\mu_i : 1 \leq i \leq |\Gamma| < \infty\} \subset \mathbb{C}^+$ . For a finite set  $a = \{a_i\}_{1 \leq i \leq |\Gamma|}$ , we can construct the sequence of (*pseudohyperbolic*) *divided differences* of  $a$  relatively to  $\Gamma$  as follows

$$\Delta_\Gamma^0(a_i) := a_i, \quad \Delta_\Gamma^1(a_i, a_j) := \frac{a_j - a_i}{b_{\mu_i}(\mu_j)},$$

and

$$\Delta_\Gamma^k(a_{i_1}, \dots, a_{i_{k+1}}) := \frac{\Delta_\Gamma^{k-1}(a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}) - \Delta_\Gamma^{k-1}(a_{i_1}, \dots, a_{i_k})}{b_{\mu_{i_k}}(\mu_{i_{k+1}})}.$$

The following properties will be useful in the next sections. The first lemma gives us the expression of the divided differences of a product, in terms of sums of products of divided differences.

**Lemma 2.** *We have*

$$\Delta_{\Gamma}^{j-1}(a_{i_1}b_{i_1}, \dots, a_{i_j}b_{i_j}) = \sum_{l=0}^{j-1} \Delta_{\Gamma}^l(a_{i_1}, \dots, a_{i_{l+1}}) \Delta_{\Gamma}^{j-l-1}(b_{i_{l+1}}, \dots, b_{i_j}).$$

For the proof, we refer to [BNO96] where the computation is done in  $\mathbb{D}$ , but the proof remains valid in  $\mathbb{C}^+$ .

We will also need estimates of the divided differences when  $\Gamma$  lies in a compact set  $K$  and  $a = \{f(\mu) : \mu \in \Gamma\}$  for  $f$  an analytic function bounded in  $K$ . Here  $K$  is supposed to be the closure of a non void open connected set. Then, we say that  $f \in H^{\infty}(K)$  if  $f$  is holomorphic in  $K$  and

$$\|f\|_{\infty, K} := \sup_{z \in K} |f(z)| < \infty.$$

We have the following lemma.

**Lemma 3.** *Suppose that  $\Gamma$  lies in a compact set  $K$  with the properties mentioned above, and assume that there exists  $\eta > 0$  such that  $\rho(\Gamma, \partial K) \geq \eta$ . Then, for each function  $f \in H^{\infty}(K)$ , we have*

$$|\Delta_{\Gamma}^j(f(\mu^{(j+1)}))| \leq \left(\frac{2}{\eta}\right)^j \prod_{k=0}^j \left(\frac{1}{1 - \frac{k}{2M}}\right) \|f\|_{\infty, K}$$

where

$$\mu^{(j+1)} = (\mu_1, \dots, \mu_{j+1}) \text{ and } f(\mu^{(j+1)}) = (f(\mu_1), \dots, f(\mu_{j+1})).$$

*Proof.* Let us introduce

$$A_j := \left\{ z \in K : \rho(z, \partial K) \geq \frac{j}{2N}\eta \right\}, \quad 0 \leq j \leq N-1.$$

We show by induction over  $j$  that for every  $z \in A_j$ ,

$$|\Delta_{\Gamma}^j(f(\mu^{(j)}, z))| \leq c_j \|f\|_{\infty, K}$$

with the right coefficient  $c_j$ . Since  $\Gamma \subset A_{N-1} \subset \dots \subset A_1 \subset A_0$ , the result will follow. The claim is obviously true for  $j = 0$  since  $A_0 = K$  and  $f \in H^{\infty}(K)$ . Now, the function

$$z \mapsto \Delta_{\Gamma}^{j+1}(f(\mu^{(j)}, z))$$

is holomorphic on  $A_{j+1}$  and by the maximum principle and the definition of divided differences, we have for  $z \in A_{j+1}$ ,

$$(2.1) \quad |\Delta_{\Gamma}^{j+1}(f(\mu^{(j+1)}, z))| \leq \sup_{\xi \in \partial A_{j+1}} \left| \frac{\Delta_{\Gamma}^j(f(\mu^{(j)}, \xi)) - \Delta_{\Gamma}^j(f(\mu^{(j+1)}))}{\rho(\xi, \mu_{j+1})} \right|.$$

Let  $\xi \in \partial A_{j+1}$ . It is possible to find a point  $\zeta \in \partial K$  such that

$$\rho(\zeta, \xi) = \left( \frac{j+1}{2N} \right) \eta$$

and so, since  $\mu_{j+1} \in \Gamma$  and  $\rho(\Gamma, \partial K) \geq \eta$ , we have, by the triangle inequality,

$$(2.2) \quad \rho(\xi, \mu_{j+1}) \geq \rho(\zeta, \mu_{j+1}) - \rho(\xi, \zeta) \geq \eta \left( 1 - \frac{j+1}{2N} \right).$$

From (2.1), (2.2) and the induction hypothesis, we finally obtain

$$|\Delta_{\Gamma}^{j+1}(f(\mu^{(j+1)}, \xi))| \leq \frac{2}{\eta} \left( \frac{1}{1 - \frac{j+1}{2N}} \right) c_j \|f\|_{\infty, K}$$

which gives the required estimate.  $\square$

The next lemma will be very important in the sequel; we can define a rational Newton type interpolating function which interpolates the values  $\{a(\mu) : \mu \in \Gamma\}$  on  $\Gamma$ .

**Lemma 4.** *The holomorphic function*

$$P_{\Gamma, a}(z) := \sum_{k=1}^{|\Gamma|} \Delta_{\Gamma}^{k-1}(a(\mu^{(k)})) \prod_{l=1}^{k-1} b_{\mu_l}(z)$$

satisfies

$$P_{\Gamma, a}(\mu) = a(\mu), \quad \mu \in \Gamma.$$

The proof is quite straightforward (see also [Ha96a, p.80]).

*Remark 5.* Divided differences with respect to the pseudohyperbolic metric can be found in [BNO96, Ha96b, Va84] but we can also find other divided differences, based on the euclidian metric. This is what is done in [AI01] where *euclidian divided differences* are defined by

$$\square_{\Gamma}^0 := a_i, \quad \square_{\Gamma}^1(a_i, a_j) := \frac{a_j - a_i}{\mu_j - \mu_i},$$

and

$$\square_{\Gamma}^k(a_{i_1}, \dots, a_{i_{k+1}}) := \frac{\square_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}) - \square_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_k})}{\mu_{k+1} - \mu_k}.$$

It is possible to see that, if  $\Gamma$  lies in a strip parrallel to the real axis, then the pseudohyperbolic and the euclidian divided differences are equivalent.



3.  $N$ -CARLESON SEQUENCES

We now deal with  $N$ -Carleson sequences, i.e with disjoint unions of  $N$  Carleson sequences,  $N$  being a natural number. Let us make a link between the  $N$ -Carleson condition and the Generalized Carleson condition ( $CG$ ), also called Carleson-Vasyunin condition (see e.g. [Ni86] and references therein). The following result has originally been stated in  $\mathbb{D}$ .

**Proposition 6.** ([Ha96b, Proposition 3.1])

Let  $\Lambda$  be a sequence of complex numbers, lying in  $\mathbb{C}^+$ . The following assertions are equivalent

- (i)  $\Lambda = \bigsqcup_{i=1}^N \Lambda^i$ , with  $\Lambda^i \in (C)$  ;
- (ii) There exists a  $\delta > 0$  and a sequence of disjoint subsets  $\{\sigma_n\}_{n \geq 1}$ ,  $\Lambda = \bigsqcup_{n \geq 1} \sigma_n$ ,  $|\sigma_n| \leq N$ , such that, if  $B$  (resp.  $B_n$ ) denotes the Blaschke product associated to  $\Lambda$  (resp.  $\sigma_n$ ), then the sequence  $(B_n)_{n \geq 1}$  satisfies the Generalized Carleson condition ( $CG$ )

$$(3.1) \quad |B(z)| > \delta \inf_{n \geq 1} |B_n(z)|, \quad z \in \mathbb{C}^+.$$

When  $(B_n)_n$  satisfies (3.1), we write  $(B_n)_n \in (CG)$ .

*Remark 7.* The subsets  $\sigma_n$  can for instance be obtained as intersections  $\tau_n^\epsilon \cap \Lambda$  for  $\epsilon$  small enough, where  $\tau_n^\epsilon$  are the connected components of  $L(B, \epsilon) := \{z : |B(z)| < \epsilon\}$ . Choosing  $\epsilon$  in a suitable way, it is possible to assume that the pseudohyperbolic diameter of  $\sigma_n$  is arbitrarily small.

*Remark 8.* If  $\Lambda$  is  $N$ -Carleson in  $\mathbb{C}^+$ , and so we have the previous decomposition  $\Lambda = \bigsqcup_n \sigma_n$ , it is possible to find a sequence  $(R_n)_n$  of rectangles of  $\mathbb{C}^+$  such that

- (i)  $\sigma_n \subset R_n$ ;
- (ii) if  $l_n$  and  $L_n$  denote the length and width of  $R_n$ , we have  $L_n \asymp l_n$ ;
- (iii)  $d(\partial R_n, \mathbb{R}) \asymp l_n$ ;
- (iv) the following estimates hold

$$0 < \inf_{n \geq 1} \rho(\sigma_n, \partial R_n) \leq \sup_{\substack{n \geq 1 \\ \lambda \in \sigma_n}} \rho(\lambda, \partial R_n) < \infty.$$

Moreover, since the diameter of  $\sigma_n$  can be chosen arbitrarily small by the preceding remark, we can suppose the  $R_n$  disjoint and even

$$\inf_{n \neq k} d(R_n, R_k) > 0.$$

Results of Vasyunin ([Va84]), for  $p = \infty$ , and Hartmann ([Ha96b]), for  $1 < p < \infty$ , describe  $H^p|\Lambda$  for  $N$ -Carleson sequences, using divided differences constructed on  $\sigma_n$ .

**Theorem 9.** ([Ha96b, Va84])

Let  $\Lambda = \bigcup_{i=1}^N \Lambda^i, \Lambda^i \subset \mathbb{D}$ ,  $\Lambda^i \in (C)$  and  $1 \leq p \leq \infty$ . There exists a decomposition  $\Lambda = \bigcup_{n \geq 1} \sigma_n$  such that  $\sup_{n \geq 1} |\sigma_n| \leq N$ , and if we choose definitely  $\lambda_{n,0} \in \sigma_n$  ( $n \geq 1$ ), then

$$H^p|\Lambda = X^p(\Lambda) := \{a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X^p(\Lambda)} < \infty\}$$

where, for  $1 < p < \infty$ ,

$$\|a\|_{X^p(\Lambda)} := \left\{ \sum_{n \geq 1} (1 - |\lambda_{n,0}|^2)^{\frac{|\sigma_n|}{k=1}} \left| \Delta_{\sigma_n}^{k-1} (a(\lambda_{n,1}, \dots, \lambda_{n,k})) \right|^p \right\}^{\frac{1}{p}},$$

and

$$\|a\|_{X^\infty(\Lambda)} := \sup_{n \geq 1} \sum_{k=1}^{|\sigma_n|} \left| \Delta_{\sigma_n}^{k-1} (a(\lambda_{n,1}, \dots, \lambda_{n,k})) \right|.$$

Let us translate this result to the upper half-plane. If  $\Lambda \subset \mathbb{C}^+$  is  $N$ -Carleson and can be written as the union of  $\sigma_n$ . For each  $n$ , let  $\lambda_{n,0}$  be any point from  $\sigma_n$  and write  $\lambda_n^{(j)} = (\lambda_{n,1}, \dots, \lambda_{n,j})$ . We introduce the space

$$X_+^p(\Lambda) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \sum_{n \geq 1} \text{Im}(\lambda_{n,0}) \sum_{k=1}^{|\sigma_n|} \left| \Delta_{\sigma_n}^{k-1} (a(\lambda_n^{(k)})) \right|^p < \infty \right\}.$$

**Corollary 10.** Under the above hypotheses, we have

$$H_+^p|\Lambda = X_+^p(\Lambda).$$

*Proof.* We introduce the conformal mapping

$$\begin{aligned} \gamma : \mathbb{D} &\rightarrow \mathbb{C}^+ \\ z &\mapsto i \frac{1+z}{1-z} \end{aligned}$$

and set  $\tilde{\Lambda} := \gamma^{-1}(\Lambda)$ . Observe that  $\gamma$  conserves the pseudohyperbolic metric and hence the (CG) condition and so, we deduce from Theorem 9 that

$$H^p|\tilde{\Lambda} = X^p(\tilde{\Lambda}) = \tilde{X}_+^p(\Lambda),$$

where

$$\tilde{X}_+^p(\Lambda) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \sum_{n \geq 1} \frac{\text{Im}(\lambda_{n,0})}{|\lambda_{n,0} + i|^2} \sum_{k=1}^{|\sigma_n|} \left| \Delta_{\sigma_n}^{k-1} (a(\lambda_n^{(k)})) \right|^p < \infty \right\}.$$

Since (see e.g. [Ko80, p. 118])

$$f \in H^p \iff \left( z \mapsto \frac{f \circ \gamma^{-1}(z)}{(z+i)^{\frac{2}{p}}} \in H_+^p \right),$$

it is enough to show

$$\left( (a(\lambda)(\lambda+i)^{\frac{2}{p}})_{\lambda \in \Lambda} \right) \in \tilde{X}_+^p(\Lambda) \iff (a(\lambda))_{\lambda \in \Lambda} \in X_+^p(\Lambda),$$

which follows directly from Lemmas 2 and 3.  $\square$

#### 4. MAIN RESULT

Let  $\Lambda$  be a sequence in the complex plane. In this section we will always assume that there is some integer  $N \geq 1$  such that for every  $a \in \mathbb{R}$ , the sequence  $\Lambda_a^\pm := \Lambda \cap \mathbb{C}_a^\pm$  is  $N$ -Carleson in the corresponding half-plane. In particular, it will be possible to write

$$\Lambda_a^\pm = \bigsqcup_{n \geq 1} \sigma_{n,a}^\pm,$$

where  $\left( B_{\sigma_{n,a}^\pm}^{\pm,a} \right)_n$  satisfies the generalized Carleson condition in the corresponding half-plane  $\mathbb{C}_a^\pm$  ( $B_{\sigma_{n,a}^\pm}^{\pm,a}$  being the Blaschke product in  $\mathbb{C}_a^\pm$  vanishing on  $\sigma_{n,a}^\pm$ ). To simplify the notation, we will omit  $a$  if  $a = 0$  and write

$$\sigma_n := \begin{cases} \sigma_n^+, & n \geq 0 \\ \sigma_n^-, & n < 0 \end{cases}.$$

The reader might notice that  $\sigma_n^+$  and  $\sigma_m^-$  can come very close for certain values of  $n$  and  $m$ . This issue will be fixed below. Let us distinguish the sets of points close to the real axis and the ones far away from it. More precisely, for  $\epsilon > 0$ , we set

$$M_\epsilon := \{n \in \mathbb{Z} : \sigma_n \cap \{|\operatorname{Im}(z)| < \epsilon\} \neq \emptyset\} \text{ and } M_{\epsilon,\infty} := \mathbb{Z} \setminus M_\epsilon,$$

$$\Lambda_{\epsilon,\infty} := \bigsqcup_{n \in M_{\epsilon,\infty}} \sigma_n$$

and

$$\Lambda_\epsilon := \Lambda \setminus \Lambda_{\epsilon,\infty}.$$

Notice that  $\Lambda_\epsilon$  is in the strip  $\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \epsilon\}$  which is in general not true for  $\bigcup_{n \in M_\epsilon} \sigma_n$ . Now, since  $\Lambda_\epsilon$  is still  $N$ -Carleson in  $\mathbb{C}_{-2\epsilon}^+$ ,  $\Lambda_\epsilon$  breaks up into a disjoint union

$$\Lambda_\epsilon = \bigsqcup_{n \geq 1} \sigma_n'$$

such that the sequence of Blaschke products  $(B_{\sigma'_n}^{+,-2\epsilon})_{n \geq 1} \in (GC)$  in  $\mathbb{C}_{-2\epsilon}^+$ . So,

$$\Lambda = \left( \bigsqcup_{n \in M_{\epsilon, \infty}} \sigma_n \right) \uplus \left( \bigsqcup_{n \geq 1} \sigma'_n \right) =: \bigsqcup_{n \in \mathbb{Z}} \tau_n.$$

We will need both definitions of divided differences.

$$\tilde{\Delta}_{\tau_n} := \begin{cases} \Delta_{\tau_n} & \text{if } \exists k \text{ s.t. } \tau_n = \sigma_k \\ \square_{\tau_n} & \text{if } \exists k \text{ s.t. } \tau_n = \sigma'_k \end{cases}.$$

It is now possible to introduce a space of sequences that will be, assuming some hypotheses on  $\Lambda$ , the range of  $R_\Lambda$ . Choosing arbitrarily  $\lambda_{n,0} \in \tau_n$ ,  $n \in \mathbb{Z}$ , we define, for  $1 < p < \infty$ ,

$$X_\tau^p(\Lambda) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X_\tau^p(\Lambda)} < \infty \right\},$$

with

$$\|a\|_{X_\tau^p(\Lambda)}^p := \sum_{n \in \mathbb{Z}} (1 + |\operatorname{Im}(\lambda_{n,0})|) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (ae^{\pm i\tau \cdot} (\lambda_n^{(k)})) \right|^p,$$

and

$$e^{\pm i\tau\lambda} = \begin{cases} e^{i\tau\lambda} & \text{if } \lambda \in \tau_n, \quad n \in N_+, \\ e^{-i\tau\lambda} & \text{if } \lambda \in \tau_n, \quad n \in N_-, \end{cases}$$

where

$$N_+ := \{n \in \mathbb{Z} : \tau_n \cap (\mathbb{C}^+ \cup \mathbb{R}) \neq \emptyset\}$$

and

$$N_- := \mathbb{Z} \setminus N_+.$$

(The factor  $e^{\pm i\tau\lambda}$  does not really matter close to  $\mathbb{R}$ .) We will need some well known facts about Paley-Wiener spaces that we recall here. First, we have the Plancherel-Polya inequality (see e.g. [Le96] or [Se04, p. 95]).

**Proposition 11.** *Let  $f \in PW_\tau^p$  and  $a \in \mathbb{R}$ . Then,*

$$\int_{-\infty}^{+\infty} |f(x + ia)|^p dx \leq e^{\tau p |a|} \|f\|_p^p.$$

It follows that for every  $f \in PW_\tau^p$ , the function  $z \mapsto e^{i\tau z} f(z)$  belongs to  $H_+^p$ . It also follows that translation is an isomorphism from  $PW_\tau^p$  onto itself. The second fact is a pointwise estimate; there exists a constant  $C = C(p)$  such that for every  $f \in PW_\tau^p$ , we have

$$(4.1) \quad |f(z)| \leq C \|f\|_p (1 + |\operatorname{Im}(z)|)^{-\frac{1}{p}} e^{\tau |\operatorname{Im}(z)|}, \quad z \in \mathbb{C}.$$

For a sequence  $\Lambda$  whose restrictions to every half-plane is  $N$ -Carleson, and  $n \in \mathbb{Z}$ , with corresponding decomposition  $\Lambda = \uplus_{n \in \mathbb{Z}} \tau_n$ , we introduce the products  $p_n(x) := \prod_{\lambda \in \tau_n} |x - \lambda|$  and we define the function

$$d_N(x) := \inf_{n \in \mathbb{Z}} p_n(x), \quad x \in \mathbb{R}.$$

We will say that  $\Lambda$  satisfies  $(H_N)$  if the following three conditions are fulfilled

- (i) The sequence is weakly dense: there is some  $\epsilon_0 > 0$  such that  $d(x, \Lambda) < \epsilon_0$ , for every  $x \in \mathbb{R}$ .

In all what follows we will do the previous decomposition with  $\epsilon = \epsilon_0$ :

$$\Lambda = \Lambda_{\epsilon_0} \uplus \Lambda_{\epsilon_0, \infty}$$

and suppose (which is possible in view of the Remark 7) that

$$\max(\rho_0, \rho'_0) < \frac{\epsilon_0}{2}$$

where

$$\rho_0 := \sup_{n \in \mathbb{Z}} \text{diam}_\rho(\sigma_n)$$

and

$$\rho'_0 := \sup_{n \geq 1} \text{diam}(\sigma'_n),$$

which are both finite in view of the  $(GC)$  condition and the fact that the pseudohyperbolic diameter of  $\sigma'_n$  in  $\mathbb{C}_{-2\epsilon}^+$  is equivalent to its Euclidian diameter close to  $\mathbb{R}$ .

- (ii) The limit

$$S(z) := \lim_{R \rightarrow \infty} \prod_{|\lambda| < R} \left(1 - \frac{z}{\lambda}\right)$$

exists and defines an entire function of exponential type  $\tau$ .

- (iii) The function  $x \mapsto \left(\frac{|S(x)|}{d_N(x)}\right)^p$  satisfies the (continuous) Muckenhoupt condition  $(A_p)$ .

We will see in the following that (iii) can be replaced by (iii)', which is

- (iii)' The sequence

$$\left( \frac{|S'(\lambda_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} |\lambda_{n,0} - \lambda|} \right)_{n \geq 1}$$

satisfies the discrete Muckenhoupt condition  $(\mathfrak{A}_p)$ .

It is clear that for  $N = 1$ ,  $d_1(x) = d(x, \Lambda)$  and  $(H_1)$  with the Carleson condition corresponds exactly to the  $(LS)$  conditions.

*Remark 12.* In the construction of the sets  $\tau_n$ , it is actually possible to assume that for some  $\delta > 0$ , the sets

$$\Omega_n := \left\{ z \in \mathbb{C} : \prod_{\lambda \in \sigma'_n} |z - \lambda| \leq \delta \right\}, \quad n \geq 1$$

satisfy

$$\inf_{n \neq m} d(\Omega_n, \Omega_m) > 0.$$

(Cf. also Remarks 7 and 8, in particular 8(iv).)

*Remark 13.* The weak density condition  $(H_N) - (i)$  implies that

$$\sup_{x \in \mathbb{R}} d_N(x) < \infty.$$

Clearly, in the definition of  $d_N$ , the infimum is actually a minimum. Also, by the  $(GC)$  condition, we have

$$\delta'_0 := \inf_{n \neq m} d(\sigma'_n, \sigma'_m) > 0,$$

and so, for each  $x \in \mathbb{R}$ , there is  $n_x \geq 1$  such that  $d_N(x) = p_{n_x}(x)$  and we can also notice that

$$\inf_{x \in \mathbb{R}} \inf_{m \neq n_x} p_m(x) \geq \left( \frac{\delta'_0}{2} \right)^N > 0.$$

Indeed, for  $x \in \mathbb{R}$  and  $m \neq n_x$ , if  $p_m(x) < \left( \frac{\delta'_0}{2} \right)^N$  then, there is some  $\lambda_1 \in \sigma_m$  such that

$$|x - \lambda_1| < \frac{\delta'_0}{2},$$

and so, for  $\lambda \in n_x$ , we have

$$|x - \lambda| \geq |\lambda - \lambda_1| - |x - \lambda_1| \geq \delta'_0 - \frac{\delta'_0}{2} = \frac{\delta'_0}{2}$$

and it follows that

$$p_m(x) \geq p_{n_x}(x) \geq \left( \frac{\delta'_0}{2} \right)^N$$

from the infimum property of  $n_x$ . This contradiction gives the required estimate.

It is now possible to state our main result.

**Theorem 14.** *Let  $N \geq 1$  and  $\Lambda$  be a sequence of complex numbers such that, for every  $a \in \mathbb{R}$ ,  $\Lambda_a^\pm$  is  $N$ -Carleson in the corresponding half-plane. Then,  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda)$  if and only if  $\Lambda$  satisfies  $(H_N)$ .*

We will discuss below the necessity of the  $N$ -Carleson condition in Theorem 17. It requires a different approach to the definition of the trace which relies here on the decomposition  $\Lambda = \bigsqcup_n \tau_n$  coming from the  $N$ -Carleson condition.

**4.1. Necessary conditions.** The necessity of (i) and (ii) can be shown exactly as in [LS97] and so we omit it here. We show that the condition (iii)' is necessary for  $R_\Lambda$  to be an isomorphism between  $PW_\tau^p$  and  $X_\tau^p(\Lambda)$ . Then, with a technical lemma, adapted from [LS97], we prove that (iii)' implies (iii).

Since  $R_\Lambda$  is onto, for each  $\lambda \in \Lambda$ , there is a unique function  $f_\lambda \in PW_\tau^p$  such that

$$f_\lambda(\mu) = \begin{cases} 1, & \text{if } \mu = \lambda \\ 0, & \text{if } \mu \neq \lambda \end{cases}.$$

As in [LS97], it can be shown that  $f_\lambda$  only vanishes on  $\Lambda \setminus \{\lambda\}$ . Moreover,  $z \mapsto (z - \lambda) f_\lambda(z)$  is a function of the Cartwright Class  $\mathcal{C}$  vanishing exactly on  $\Lambda$  (see e.g. [Le96] for definition and general results on  $\mathcal{C}$ ). Hence,  $S(z) = c_\lambda (z - \lambda) f_\lambda(z)$ ,  $z \in \mathbb{C}$ , or

$$f_\lambda(z) = \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

For each  $n \geq 1$ , the holomorphic function

$$g_n : z \mapsto \frac{S(z)}{\prod_{\lambda \in \sigma'_n} (z - \lambda)}$$

does not vanish in  $\Omega_n$  (see Remark 12). Moreover, choosing  $\lambda'_{n,0} \in \sigma'_n$ ,

$$g_n(\lambda_{n,0}) = \frac{S'(\lambda'_{n,0})}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} (\lambda'_{n,0} - \lambda)}.$$

Hence, it follows from the maximum and the minimum principle that

$$\inf_{\xi \in \partial\Omega_n} \left| \frac{S(\xi)}{\prod_{\lambda \in \sigma'_n} (\xi - \lambda)} \right| \leq \left| \frac{S'(\lambda'_{n,0})}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} (\lambda'_{n,0} - \lambda)} \right| \leq \sup_{\xi \in \partial\Omega_n} \left| \frac{S(\xi)}{\prod_{\lambda \in \sigma'_n} (\xi - \lambda)} \right|.$$

From the intermediate values theorem, we deduce the existence of a point  $\theta_n \in \partial\Omega_n$  such that

$$(4.2) \quad |S(\theta_n)| = \delta \frac{|S'(\lambda'_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda'_{n,0} - \lambda|} =: \delta \omega_n.$$

Setting  $\Gamma := (\lambda'_{n,0})_{n \geq 1}$  and  $\Theta := (\theta_n)_{n \geq 1}$ , we show that the discrete Hilbert transform  $\mathcal{H}_{\Gamma, \Theta}$  is bounded from  $l^p(\omega)$  into itself. Indeed, let  $(a_n)_{n \geq 1}$  be a finite sequence of  $l^p(\omega)$ . Then, the sequence

$$a(\lambda) := \begin{cases} a_n S'(\lambda'_{n,0}) & , \text{ if } \lambda = \lambda'_{n,0} \\ 0 & , \text{ if } \lambda \in \Lambda \setminus \bigcup_{n \geq 1} \{\lambda'_{n,0}\} \end{cases}$$

belongs to  $X_\tau^p(\Lambda)$  because, if  $\lambda'_{n,0}$  is chosen as the “last” point of  $\sigma'_n$ ,

$$\tilde{\Delta}_{\sigma_n}^{k-1} (ae^{i\tau \cdot} (\lambda_n^{(k)})) = 0, \quad k < |\sigma_n|$$

and

$$\left| \tilde{\Delta}_{\sigma_n}^{|\sigma_n|-1} (ae^{i\tau \cdot} (\lambda_n^{(|\sigma_n|)})) \right| = \frac{|a_n S'(\lambda'_{n,0})| e^{-\tau |\text{Im}(\lambda'_{n,0})|}}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda - \lambda'_{n,0}|}.$$

Thus, from (4.2), we obtain, observing that  $1 + |\text{Im}(\lambda'_{n,0})|$  and  $|e^{i\tau \lambda}|$ ,  $\lambda \in \sigma'_n$ , are comparable to a constant since  $\sigma'_n$  is close to  $\mathbb{R}$ ,

$$(4.3) \quad \begin{aligned} \|a\|_{X_\tau^p(\Lambda)}^p &= \sum_n \left( 1 + |\text{Im}(\lambda'_{n,0})| \right) \left| \tilde{\Delta}_{\sigma'_n}^{|\sigma'_n|-1} (ae^{i\tau \cdot} (\lambda_n^{(|\sigma'_n|)})) \right|^p \\ &\asymp \sum_n \left( \frac{|a_n S'(\lambda'_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda'_{n,0} - \lambda|} \right)^p = \sum_n \omega_n^p |a_n|^p. \end{aligned}$$

So, let  $f \in PW_\tau^p$  be the (unique) solution of the interpolation problem  $f|_\Lambda = a$ . Notice that, since  $R_\Lambda$  is an isomorphism onto  $X_\tau^p(\Lambda)$ , then

$$(4.4) \quad \|f\|_p^p \lesssim \|a\|_{X_\tau^p(\Lambda)}^p.$$

This function is of the form  $f(z) = \sum_j a_j \frac{S(z)}{z - \lambda_{j,0}}$  and so, with (4.2) we have

$$(4.5) \quad \sum_n |f(\theta_n)|^p = \delta^p \sum_n \omega_n^p \left| \left( \mathcal{H}_{\Gamma, \Theta}((a_j)_{j \geq 1}) \right)_n \right|^p.$$



On the other hand, the Polya inequality (see [Le96, Lecture 20]), and the inequalities (4.4) and (4.3) give

$$(4.6) \quad \sum_n |f(\theta_n)|^p \lesssim \|f\|_p^p \lesssim \|a\|_{X_\tau^p(\Lambda)}^p \lesssim \sum_n \omega_n^p |a_n|^p.$$

From (4.5) and (4.6), we deduce that  $\mathcal{H}_{\Gamma, \Theta}$  is bounded from  $l^p(\omega^p)$  into itself. The Lemma 1 of [LS97] allows us to conclude that the weight  $(\omega_n^p)_{n \geq 1}$  satisfies the discrete Muckenhoupt condition  $(\mathfrak{A}_p)$ . Notice that we do not have exactly the required condition  $|\theta_n - \lambda'_{n,0}| = \delta$ . However, since  $\sup_{n \geq 1} \text{diam}(\sigma'_n) < \infty$  and  $\prod |\lambda - \theta_n| = \delta$ , it is easy to see that  $|\theta_n - \lambda'_{n,0}| \asymp 1$  and this is enough for the lemma to remain valid. Now, in order to prove (iii), we use the following lemma, adapted from [LS97, Lemma 2]. For technical reasons, we set  $\gamma := (\gamma_n)_{n \in \mathbb{Z}}$  in a way such that  $\{\gamma_n : n \in \mathbb{Z}\} = \Gamma$  and  $\text{Re}(\gamma_n) \leq \text{Re}(\gamma_{n+1})$ . We then define  $\sigma_{\gamma_n}$  to be the  $\sigma'_n$  such that  $\gamma_n \in \sigma_{\gamma_n}$  and  $w_n$  will denote

$$\frac{|S'(\gamma_n)|}{\prod_{\substack{\lambda \in \sigma_{\gamma_n} \\ \lambda \neq \gamma_n}} |\lambda - \gamma_n|}.$$

(The sequence  $(w_n)_n$  is a permutation of the sequence  $(\omega_n)_n$ ).

*Remark 15.* It follows from the weak density condition  $((H_N) - (i))$ , the  $(GC)$  condition on  $(B_{\sigma_{\gamma_n}})_n$  and the growth of the sequence  $(\text{Re}(\gamma_n))_n$  that we have  $\text{Re}(\gamma_{n+1}) - \text{Re}(\gamma_n) \leq 4\epsilon_0$ . This implies that

$$\delta'_0 \leq |\gamma_n - \gamma_{n+1}| \leq 5\epsilon_0.$$

**Lemma 16.** *Suppose  $x \in \mathbb{R}$  and  $\text{Re}(\gamma_n) \leq x \leq \text{Re}(\gamma_{n+1})$ . Then, there exists an  $\alpha = \alpha(x) \in [0, 1]$  such that*

$$w_n^\alpha w_{n+1}^{1-\alpha} \asymp \frac{|S(x)|}{d_N(x)},$$

*uniformly with respect to  $x \in \mathbb{R}$ .*

Assuming this lemma to hold, (iii) follows directly from (iii)' and the inequality  $t^\alpha s^{1-\alpha} \leq t + s$ ,  $t, s > 0$  and  $\alpha \in [0, 1]$ .

*Proof.* For  $x \in [\text{Re}(\gamma_n), \text{Re}(\gamma_{n+1})]$ , we set  $N(x) := \{n : d(\sigma'_n, x) < \epsilon_0\}$  and

$$\Lambda(x) := \left( \bigsqcup_{n \in N(x)} \sigma'_n \right) \cup \sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}}.$$

Notice that  $\sigma_{\gamma_n}$  and  $\sigma_{\gamma_{n+1}}$  may be subsets of  $\biguplus_{n \in N(x)} \sigma'_n$ . Observe also that since  $\Lambda$  is a finite union of Carleson sequences, we have

$$\sup_{x \in \mathbb{R}} |N(x)| < \infty.$$

For  $\alpha \in [0, 1]$ , we want to show that  $\vartheta \asymp 1$ , where

$$\vartheta := \frac{w_n^\alpha w_{n+1}^{1-\alpha} d_N(x)}{|S(x)|},$$

and  $x \notin \Lambda$  (this is not restrictive since the expression extends continuously to  $\Lambda$ ). From the definition of  $S$ , we have that

$$S'(\lambda) = -\frac{1}{\lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left(1 - \frac{\lambda}{\mu}\right), \quad \lambda \in \Lambda.$$

In order to not overcharge notation, all infinite products occurring below will be understood as symmetric limits of finite products.

Thus,

$$\vartheta = \left( \frac{\left| \frac{1}{\gamma_n} \prod_{\lambda \in \Lambda \setminus \{\gamma_n\}} \left(1 - \frac{\gamma_n}{\lambda}\right) \right|^\alpha \left| \frac{1}{\gamma_{n+1}} \prod_{\lambda \in \Lambda \setminus \{\gamma_{n+1}\}} \left(1 - \frac{\gamma_{n+1}}{\lambda}\right) \right|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda} \left(1 - \frac{x}{\lambda}\right) \prod_{\lambda \in \sigma_{\gamma_n} \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_{n+1}} \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha}} \right).$$

For  $\lambda \in \Lambda \setminus \{\gamma_n, \gamma_{n+1}\}$ ,

$$\frac{\left|1 - \frac{\gamma_n}{\lambda}\right|^\alpha \left|1 - \frac{\gamma_{n+1}}{\lambda}\right|^{1-\alpha}}{\left|1 - \frac{x}{\lambda}\right|} = \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|x - \lambda|}.$$

Note also that for the remaining two points  $\gamma_n, \gamma_{n+1}$  we have:

$$\frac{\left| \frac{1}{\gamma_n} \left(1 - \frac{\gamma_n}{\gamma_{n+1}}\right) \right|^\alpha \left| \frac{1}{\gamma_{n+1}} \left(1 - \frac{\gamma_{n+1}}{\gamma_n}\right) \right|^{1-\alpha}}{\left| \left(1 - \frac{x}{\gamma_n}\right) \left(1 - \frac{x}{\gamma_{n+1}}\right) \right|} = \frac{|\gamma_{n+1} - \gamma_n|^\alpha |\gamma_n - \gamma_{n+1}|^{1-\alpha}}{|\gamma_n - x| |\gamma_{n+1} - x|}.$$

Now, we split  $\vartheta$  in two products  $\vartheta = \Pi_1(x) \cdot \Pi_2(x)$  corresponding essentially to zeros in  $\Lambda(x)$  and zeros in  $\Lambda \setminus \Lambda(x)$  ( $d_N(x)$  appearing in

$\Pi_1$ ):

$$\begin{aligned} \Pi_1(x) &:= \frac{\prod_{\lambda \in \Lambda(x) \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \Lambda(x) \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda(x)} |\lambda - x| \prod_{\lambda \in \sigma_{\gamma_n} \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_{n+1}} \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha}} \\ &= \frac{\prod_{\lambda \in \Lambda(x) \setminus \sigma_{\gamma_n}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \Lambda(x) \setminus \sigma_{\gamma_{n+1}}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda(x)} |\lambda - x|} \end{aligned}$$

and

$$\Pi_2(x) := \prod_{\lambda \in \Lambda \setminus \Lambda(x)} \left( \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|\lambda - x|} \right).$$

We can write

$$\begin{aligned} \Pi_1(x) &= \left( \frac{\prod_{\lambda \in \sigma_{\gamma_{n+1}}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_n}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}}} |x - \lambda|} \right) \\ &\quad \times \left( \prod_{\Lambda(x) \setminus (\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}})} \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|x - \lambda|} \right) \end{aligned}$$

and notice that if  $\lambda \in \Lambda(x) \setminus (\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}})$ , then  $\lambda \in \sigma_{\gamma_l}$  for a suitable  $l \in N(x)$ , so that

$$1 \lesssim d(\sigma_{\gamma_n}, \sigma_{\gamma_l}) \leq |\lambda - \gamma_n| \leq 2\rho'_0 + 2\epsilon_0 \lesssim 1$$

and, in view of Remark 15, for  $\lambda \in \sigma_{\gamma_n}$  and  $\mu \in \sigma_{\gamma_{n+1}}$ , we have

$$|\lambda - \gamma_{n+1}| \asymp 1 \text{ and } |\mu - \gamma_n| \asymp 1.$$

These three relations imply that

$$\Pi_1(x) \asymp \frac{d_N(x)}{\prod_{\lambda \in \Lambda(x)} |x - \lambda|}.$$

Now, let  $n_x$  be such that  $d_N(x) = p_{n_x}(x)$  (we refer to Remark 13). Clearly  $n_x \in N(x)$ . Note also that for  $\lambda \in \sigma'_m$ ,  $m \in N(x)$ , we have  $|\lambda - x| \leq d(\sigma'_m, x) + \text{diam}(\sigma'_m) \leq \epsilon_0 + \rho'_0$ . Hence

$$\frac{1}{(\epsilon_0 + \rho'_0)^{|N(x)|-1}} \leq \frac{d_N(x)}{\prod_{\lambda \in \Lambda(x)} |x - \lambda|} = \frac{1}{\prod_{\lambda \in \Lambda(x) \setminus \sigma_{n_x}} |\lambda - x|} \leq \left( \frac{2}{\delta'_0} \right)^{N \cdot (|N(x)|-1)}$$

and, from the end of Remark 13, we obtain that

$$\Pi_1(x) \asymp 1.$$

The relation

$$\Pi_2(x) \asymp 1$$

is shown exactly in the same way as in [LS97], using the  $N$ -Carleson condition. The lemma is proved.  $\square$

**4.2. Sufficient conditions.** We show the converse of the theorem in two parts; first, the injectivity of  $R_\Lambda$  and then its surjectivity.

**4.2.1. Injectivity of  $R_\Lambda$ .** Let  $f \in PW_\tau^p$  such that  $f(\lambda) = 0$ ,  $\lambda \in \Lambda$ . We want to show that  $f \equiv 0$ . Let us introduce  $\phi := f/S$ . It can be shown that  $\phi$  is an entire function of exponential type 0. The idea of the proof, given by Lyubarskii and Seip in [LS97], is to bound  $\phi$  by a constant on the imaginary axis and to use a Phragmen-Lindelf theorem to obtain that  $\phi$  is a constant. Then, for integrability reasons, the only possible value for the constant will be zero.

We will proceed as follows: since  $\phi$  is analytic, it is bounded on the compact  $[-2i\epsilon_0, 2i\epsilon_0]$ . In order to bound  $\phi$  on  $i\mathbb{R} \setminus [-2i\epsilon_0, 2i\epsilon_0]$ , we will use a lower estimate for  $S$  in a certain area of  $\mathbb{C}$ . Let us introduce

$$A_n := \{z \in \mathbb{C} : |\operatorname{Im}(z)| \geq 2\epsilon_0, \rho(\lambda_{n,0}, z) < 2\rho_0 < \epsilon_0\}, \quad n \in \mathbb{Z}.$$

We begin to show that for  $z \in (\mathbb{C}_{2\epsilon_0}^+ \cup \mathbb{C}_{-2\epsilon_0}^-) \setminus (\bigcup_n A_n)$ ,

$$(4.7) \quad |S(z)| \gtrsim e^{\tau|\operatorname{Im}(z)|} (|\operatorname{Im}(z)|)^{\frac{1}{q}} (1 + |z|)^{-1}.$$

Indeed, let us introduce

$$B_{\epsilon_0}(z) := \prod_{\lambda \in \Lambda_{\epsilon_0}} \left( c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 3i\epsilon_0} \right) \quad \text{and} \quad S_1(z) := (S/B_{\epsilon_0})(z), \quad z \in \mathbb{C}^+,$$

where  $B_{\epsilon_0}$  is the Blaschke product in  $\mathbb{C}_{-\frac{3}{2}\epsilon_0}^+$ , and  $c_\lambda$  is the unimodular normalizing constant which ensures the convergence of the Blaschke product (we do not need the explicit value here). Let  $x \in \mathbb{R}$ . Observe that for  $n \geq 1$  and  $\lambda \in \sigma'_{n_x}$ , we have

$$|x - \bar{\lambda}| = |x - \lambda| \leq \epsilon_0 + \operatorname{diam}(\sigma'_{n_x}) \leq \epsilon_0 + \rho'_0 \leq 3\frac{\epsilon_0}{2}.$$

Hence,

$$\frac{3}{2}\epsilon_0 \leq |x - \bar{\lambda} + 3i\epsilon_0| \leq 5\epsilon_0.$$

It follows from these inequalities that

$$\left( \prod_{\lambda \in \sigma_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon_0} \right| \right) \asymp d_N(x).$$

Writing

$$|B_{\epsilon_0}(x)| = \left( \prod_{\lambda \in \sigma_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon_0} \right| \right) \left( \prod_{\lambda \in \Lambda_{\epsilon_0} \setminus \sigma_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon_0} \right| \right)$$

and using the fact that  $\Lambda_{\epsilon_0}$  is  $N$ -Carleson in  $\mathbb{C}_{-\frac{3}{2}\epsilon}^+$ , we have then that

$$(4.8) \quad |B_{\epsilon_0}(x)| \asymp d_N(x),$$

and so  $x \mapsto |S_1(x)|^p \in (A_p)$ .

In particular, the function  $z \mapsto e^{i\tau z} \frac{S_1(z)}{z+i} = e^{i\tau z} \frac{S}{B_{\epsilon_0}(z+i)}$  belongs to  $H_+^p$  and the function  $z \mapsto e^{i\tau z} S_1(z)$  is a function of  $\mathcal{N}^+$ , the Smirnov Class in the upper half-plane (for definition and general results, see e.g. [Ni02a, A.4]). Hence, we can write

$$S_1(z) = e^{-i\tau z} B_1(z) G_1(z), \quad z \in \mathbb{C}^+,$$

where  $B_1$  is the Blaschke product associated to  $\Lambda^+ \setminus \Lambda_{\epsilon_0}$  and  $G_1$  is an outer function in  $\mathbb{C}^+$ . Thus,  $x \mapsto |G_1(x)|^p \in (A_p)$  or equivalently,  $x \mapsto |G_1(x)|^{-q} \in (A_q)$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ . So, it follows from properties of functions satisfying Muckenhoupt's  $(A_p)$  condition, that

$$z \mapsto \frac{1}{G_1(z)(z+i)} \in H_+^q$$

and we can write

$$\frac{1}{G_1(z)(z+i)} = \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{1}{G_1(t)(t+i)} \frac{dt}{t-z}$$

from where we get for  $z \in \mathbb{C}^+$ ,

$$\left| \frac{1}{G_1(z)} \right| \lesssim (1 + |z|) (\operatorname{Im}(z))^{-\frac{1}{q}}.$$

Moreover, because of the  $N$ -Carleson condition of  $\Lambda^+ \setminus \Lambda_{\epsilon_0}$ , we have that

$$|B_1(z)| \gtrsim 1, \quad z \in \mathbb{C}^+ \setminus \left( \bigcup_{n \geq 0} A_n \right)$$

and so we do have the lower bound for  $S_1$  stated in (4.7). We notice that  $|S(z)| \asymp |S_1(z)|$ ,  $\operatorname{Im}(z) > 2\epsilon_0$  and so we have the same bound for  $S$  in  $\mathbb{C}_{2\epsilon_0}^+$ . A similar reasoning gives us the estimate in  $\mathbb{C}_{-2\epsilon_0}^-$ .

Using now (4.1) and (4.7), we have for  $z \in (\mathbb{C}_{2\epsilon_0}^+ \cup \mathbb{C}_{-2\epsilon_0}^-) \setminus (\bigcup_n A_n)$ ,

$$\begin{aligned} |\phi(z)| &= \left| \frac{f(z)}{S(z)} \right| \lesssim \frac{(1+|z|)}{e^{\tau|\operatorname{Im}(z)|} |\operatorname{Im}(z)|^{\frac{1}{q}} (1+|\operatorname{Im}(z)|)^{\frac{1}{p}}} \frac{e^{\tau|\operatorname{Im}(z)|}}{e^{\tau|\operatorname{Im}(z)|} |\operatorname{Im}(z)|^{\frac{1}{q}} (1+|\operatorname{Im}(z)|)^{\frac{1}{p}}} \\ &\asymp \frac{(1+|z|)}{|\operatorname{Im}(z)|^{\frac{1}{q}} (1+|\operatorname{Im}(z)|)^{\frac{1}{p}}} =: \psi(z). \end{aligned}$$

We notice then that if  $A_n \cap i\mathbb{R} \neq \emptyset$ , then

$$A_n \subset S_{\pm} := \left\{ z \in \mathbb{C}^{\pm} : \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right| < \eta \right\},$$

where  $\eta$  is a suitable constant. Note that  $S_{\pm}$  are Stolz angles in  $\mathbb{C}^{\pm}$  at  $x = 0$ . Since  $A_n$  is far from  $\mathbb{R}$  and has uniformly bounded pseudohyperbolic diameter, every  $A_n$  hitting the imaginary axis will be in the Stolz angle  $S_+$  or  $S_-$ . Obviously, there is some  $M > 0$  such that for every  $z \in \mathbb{C}_{\pm 2\epsilon_0}^{\pm} \cap S_{\pm}$ , we have

$$|\psi(z)| \leq M.$$

In particular,  $|\psi(z)| \leq M$  for  $z \in \partial A_n$  and by the maximum principle,

$$|\psi(iy)| \leq M \text{ for } iy \in A_n \cap i\mathbb{R}.$$

Hence,  $\phi$  is uniformly bounded on  $i\mathbb{R}$  and it follows, by a Phragmen-Lindelf principle that  $\phi \equiv K$ . Let us now show that  $K = 0$ . Because  $x \mapsto |S_1(x)|^p \in (A_p)$  we have

$$\int |S_1(x)|^p = \infty$$

and, applying the Plancherel-Polya inequality, we also have

$$\int |S_1(x + 2i\epsilon_0)|^p = \infty$$

but  $|S_1(x + 2i\epsilon_0)| \asymp |S(x + 2i\epsilon_0)|$ , so

$$\int |S(x + 2i\epsilon_0)|^p = \infty.$$

We apply again the Plancherel-Polya inequality to obtain

$$\int |S(x)|^p = \infty.$$

From the fact that  $f \in PW_{\tau}^p$ , we have by definition that  $f \in L^p$  and since  $f = \phi S = KS$ , the only possibility is  $K = 0$  and so  $f \equiv 0$ , which ends the proof of the injectivity of  $R_{\Lambda}$ . Now, we can show the last part of the proof.

4.2.2. *Surjectivity of  $R_\Lambda$ .* Let  $a$  be a finitely supported sequence. It suffices to bound the norm of the solution of the interpolation problem  $f(\lambda) = a(\lambda)$ ,  $\lambda \in \Lambda$ , by a constant times the norm of  $a$  in  $X_\tau^p(\Lambda)$ . The general case follows from density. Let

$$f(z) = \sum_{\lambda \in \Lambda} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

We want to split this sum in two pieces. We recall that we have the decomposition  $\Lambda = \bigsqcup_{n \in \mathbb{Z}} \tau_n$  and we have already introduced

$$N_+ = \{n \in \mathbb{Z} : \tau_n \cap (\mathbb{C}^+ \cup \mathbb{R}) \neq \emptyset\} \text{ and } N_- = \mathbb{Z} \setminus N_+.$$

We set

$$\Lambda_+ := \bigsqcup_{n \in N_+} \tau_n \text{ and } \Lambda_- := \bigsqcup_{n \in N_-} \tau_n = \Lambda \setminus \Lambda_+.$$

(Observe that since  $\text{diam}(\tau_n) < \frac{\epsilon_0}{2}$ , we have  $\Lambda_+ \subset \mathbb{C}_{-\frac{\epsilon_0}{2}}^+$ ). Now, we can write  $f = f^+ + f^-$ , with

$$f^\pm(z) := \sum_{\lambda \in \Lambda_\pm} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)} = \sum_{n \in N_\pm} \sum_{\lambda \in \tau_n} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)}$$

and we will estimate the norm of each sum separately. Here we will only estimate the norm of  $f^+$ , the method is the same for  $f^-$ . In the following,  $\beta$  will be the Blaschke product associated to  $\Lambda_{-\epsilon_0}^+$

$$\beta(z) = \prod_{\lambda \in \Lambda_{-\epsilon_0}^+} \left( c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon_0} \right), \quad z \in \mathbb{C}_{-\epsilon_0}^+,$$

where again  $c_\lambda$  is a suitable normalizing factor. For  $z \in \mathbb{C}_{-\epsilon_0}^+$ , we write  $S(z) = e^{-i\tau z} \beta(z) G(z)$ , which implies, using  $\beta(0) = \prod_{\lambda \in \Lambda} c_\lambda \frac{\lambda}{\bar{\lambda} - 2i\epsilon_0}$  (recall that we have assumed  $0 \notin \Lambda$ ) that

$$\begin{aligned} G(z) &= e^{i\tau z} S(z) \beta(z)^{-1} \\ &= e^{i\tau z} \prod_{\lambda \in \Lambda} \left( \frac{\lambda - z}{\lambda} \right) \prod_{\lambda \in \Lambda_{-\epsilon_0}^+} \left( c_\lambda \frac{z - \bar{\lambda} + 2i\epsilon_0}{z - \lambda} \right) \\ &= \beta(0)^{-1} e^{i\tau z} \prod_{\tilde{\lambda} \in \tilde{\Lambda}} \left( 1 - \frac{z}{\tilde{\lambda}} \right), \end{aligned}$$

with  $\tilde{\Lambda} := (\Lambda \setminus \Lambda_{-\epsilon_0}^+) \cup (\overline{\Lambda_{-\epsilon_0}^+} - 2i\epsilon_0)$ . The function  $G$  is outer function in  $\mathbb{C}_{-\epsilon_0}^+$ . As in (4.8), we obtain  $|\beta(x)| \asymp d_N(x)$ . In particular, we have

$|G(x)|^p \in (A_p)$ . And, since  $|e^{i\tau z}\beta(z)| \lesssim 1$  on every fixed line parallel to the real axis, it is enough to estimate the  $L^p$  norm of

$$f_0^+(z) := \sum_{n \in N_+} \sum_{\lambda \in \tau_n} a(\lambda) \frac{G(\lambda)}{S'(\lambda)(z - \lambda)}$$

instead of the norm of  $f^+$ . Let  $\eta$  be such that  $\frac{\epsilon_0}{2} < \eta < \epsilon_0$ . Since  $\tilde{\Lambda}$  is a (not necessarily disjoint) union of two  $N$ -Carleson sequences in  $\mathbb{C}_{\frac{\epsilon_0}{2}}^-$ , so that in particular  $\text{Im}(\tilde{\lambda} + i\eta) \leq \eta - \epsilon_0 < 0$  and thus every  $x \in \mathbb{R}$  is far from  $\tilde{\Lambda}$ , we obtain

$$|G(x - i\eta)| = e^{\tau\eta} |G(x)| \left( \prod_{\tilde{\lambda} \in \tilde{\Lambda}} \left| \frac{x - \tilde{\lambda} - i\eta}{x - \tilde{\lambda}} \right| \right) \asymp |G(x)|.$$

So  $x \mapsto |G(x - i\eta)|^p$  also satisfies the Muckenhoupt condition  $(A_p)$ . According to the Plancherel-Polya inequality, it is possible to estimate the norm of  $f_0^+$  on the axis  $\{\text{Im}(z) = -\eta\}$ .

By duality, we need to estimate

$$\sup_{\substack{h \in H^q(\mathbb{C}_{-\eta}^+) \\ \|h\|_q = 1}} N(h),$$

with

$$\begin{aligned} N(h) &:= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} \int \frac{G(x - i\eta)h(x - i\eta)}{x - i\eta - \lambda} dx \right| \\ &= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} \mathcal{H}(\tilde{G}\tilde{h})(\lambda + i\eta) \right| \\ &= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} P_+(\tilde{G}\tilde{h})(\lambda + i\eta) \right|, \end{aligned}$$

where  $z \mapsto \tilde{G}(z) = G(z - i\eta)$  is an outer function in  $\mathbb{C}^+$  and the function  $z \mapsto \tilde{h}(z) = h(z - i\eta)$  belongs to  $H_+^q$ . In order to compute  $S'(\lambda)$ , let us recall that

$$S(z) = e^{-i\tau z}\beta(z)G(z), \quad z \in \mathbb{C}_{-\eta}^+.$$

For  $\lambda \in \tau_n$ ,  $n \in N_+$ , we have

$$S'(\lambda) = c_\lambda \frac{e^{-i\tau\lambda}}{\lambda - \bar{\lambda} + 2i\epsilon_0} G(\lambda) \frac{\beta}{b_\lambda^{\epsilon_0}}(\lambda),$$



where  $b_\lambda^{\epsilon_0}(z) = c_\lambda \frac{z-\lambda}{z-\bar{\lambda}+2i\epsilon_0}$ . Using that  $G(\lambda) = \tilde{G}(\lambda + i\eta)$ , and setting

$$\psi := \frac{P_+(\tilde{G}\tilde{h})}{\tilde{G}} \text{ and } \alpha(\lambda) := a(\lambda)e^{i\tau\lambda}, \quad \lambda \in \Lambda^+,$$

the expression becomes

$$N(h) = \left| \sum_{n \in N_+} \sum_{\lambda \in \tau_n} \frac{\alpha(\lambda)\psi(\lambda + i\eta)}{\prod_{\mu \neq \lambda} b_\mu^{\epsilon_0}(\lambda)} (\lambda - \bar{\lambda} + 2i\epsilon_0) \right|.$$

Writing

$$N_+ = N_{\epsilon_0} \uplus N_{+\infty}, \text{ with } N_{\epsilon_0} := \{n \in N_+ : \tau_n \cap \{|\operatorname{Im}(z)| < \epsilon_0\} \neq \emptyset\},$$

we set, with the help of the functions of Lemma 4,

$$P_{\tau_n, \alpha}(z) := \sum_{k=1}^{|\tau_n|} \Delta_{\tau_n}^{k-1}(\alpha(\lambda_{n,1}, \dots, \lambda_{n,k})) \prod_{l=1}^{k-1} b_{\lambda_{n,l}}(z), \quad n \in N_{+\infty},$$

$$P_{\tau_n, \alpha}(z) := \sum_{k=1}^{|\tau_n|} \square_{\tau_n}^{k-1}(\alpha(\lambda_{n,1}, \dots, \lambda_{n,k})) \prod_{l=1}^{k-1} (z - \lambda_{n,l}), \quad n \in N_{\epsilon_0}$$

and setting  $\tilde{\tau}_n := \tau_n + i\eta$

$$Q_{\tilde{\tau}_n, \psi}(z) := \sum_{k=1}^{|\tilde{\tau}_n|} \Delta_{\tilde{\tau}_n}^{k-1}(\psi(\lambda_{n,1} + i\eta, \dots, \lambda_{n,k} + i\eta)) \prod_{l=1}^{k-1} b_{\lambda_{n,l} + i\eta}(z).$$

We notice that

$$N(h) = \left| \sum_{n \in N_+} \sum_{\lambda \in \tau_n} \frac{P_{\tau_n, \alpha}(\lambda) Q_{\tilde{\tau}_n, \psi}(\lambda + i\eta)}{\prod_{\mu \neq \lambda} b_\mu^{\epsilon_0}(\lambda)} (\lambda - \bar{\lambda} + 2i\epsilon_0) \right|.$$

Recall now that  $\tau_n \subset R_n$ , where  $(R_n)_n$  are the disjoint rectangles (constructed here in the half-plane  $\mathbb{C}_{-\eta}^+$  so that in particular satisfying  $d(\partial R_n, \mathbb{R} - i\eta) \asymp l_n \asymp L_n$ ) introduced in Remark 8. (Note also that here we have that  $\Lambda_+ \subset \mathbb{C}_{-\frac{\epsilon_0}{2}}^+$  and in particular,  $\Lambda_+$  is far from  $\{\operatorname{Im}(z) = -\eta\}$ ). Then, if  $\Gamma_n := \partial R_n$ , the function

$$z \mapsto h_n(z) := \frac{P_{\tau_n, \alpha}(z) Q_{\tilde{\tau}_n, \psi}(z + i\eta)}{\beta(z)}$$

is a meromorphic function in  $\overset{\circ}{R}_n$  with simple poles at  $\lambda \in \tau_n$ . Thus, the residue theorem implies that

$$\int_{\Gamma_n} h_n(z) dz = 2i\pi \sum_{\lambda \in \tau_n} \operatorname{Res}(h_n, \lambda)$$

and

$$\operatorname{Res}(h_n, \lambda) = P_{\tau_n, \alpha}(\lambda) Q_{\tilde{\tau}_n, \psi}(\lambda + i\eta) \left( \frac{\beta}{b_{\lambda}^{\epsilon_0}}(\lambda) \right)^{-1} \cdot (\lambda - \bar{\lambda} + 2i\epsilon_0).$$

It follows that

$$N(h) = \left| \frac{1}{2i\pi} \sum_{n \in N_+} \int_{\Gamma_n} \frac{P_{\tau_n, \alpha} Q_{\tilde{\tau}_n, \psi}(z) dz}{\beta} \right|.$$

Obviously  $|b_{\lambda_{n,l}}(z)| \leq 1$ . Observe also that by condition (iv) of Remark 8 for  $z \in \Gamma_n$ ,  $n \in N_{\epsilon_0, \infty}$ , we have that  $|z - \lambda_{n,l}|$  is bounded by a fixed constant. Hence for every  $n \in N_+$ ,

$$|P_{\tau_n, \alpha}| \lesssim \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1}(\alpha(\lambda_{n,1}, \dots, \lambda_{n,k})) \right|.$$

Also

$$|Q_{\tilde{\tau}_n, \psi}| \lesssim \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1}(\psi(\lambda_{n,1} + i\eta, \dots, \lambda_{n,k} + i\eta)) \right|,$$

and we obtain that

$$N(h) \lesssim \sum_{n \in N_+} \left[ \left( \int_{\Gamma_n} \left| \frac{dz}{\beta(z)} \right| \right) \left( \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1}(\alpha) \right| \right) \left( \sum_{l=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{l-1}(\psi) \right| \right) \right].$$

For  $z \in \Gamma_n$ , we see that

$$\begin{aligned} |\beta(z)| &= \left( \prod_{\lambda \in \Lambda_+ \setminus \tau_n} \left| \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon_0} \right| \right) \cdot \left( \prod_{\lambda \in \tau_n} \left| \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon_0} \right| \right) \\ &=: \Pi_1(z) \cdot \Pi_2(z). \end{aligned}$$

Since  $\Lambda_+$  is  $N$ -Carleson in  $\mathbb{C}_{-\epsilon_0}^+$ , it follows from the fact that  $R_n$  is far from  $\tau_k$ ,  $k \neq n$  that

$$\Pi_1(z) \asymp 1$$

and from the fact that  $R_n$  is far from  $\tau_n$  that

$$\Pi_2(z) \asymp 1.$$

Hence, choosing arbitrarily  $\lambda_{n,0} \in \tau_n$ , the construction of  $R_n$  gives

$$\int_{\Gamma_n} \left| \frac{dz}{\beta(z)} \right| \lesssim \int_{\Gamma_n} |dz| \lesssim \operatorname{Im}(\lambda_{n,0}) + \eta \lesssim 1 + |\operatorname{Im}(\lambda_{n,0})|.$$

Applying Hlder's inequality, we obtain

$$N(h) \lesssim \left( \sum_{n \in N_+} (1 + \operatorname{Im}(\lambda_{n,0})) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (e^{i\tau \cdot} a) \right|^p \right)^{\frac{1}{p}} \\ \times \left( \sum_{n \in N_+} \operatorname{Im}(\lambda_{n,0} + i\eta) \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1} (\psi) \right|^q \right)^{\frac{1}{q}}.$$

Now, notice that by the Muckenhoupt condition on  $|\tilde{G}|^{-q}$  and thus the boundedness of  $\mathcal{H}$  on

$$H_+^q \left( \left| \frac{1}{\tilde{G}} \right|^q \right) := \left\{ f \in \mathcal{N}^+ : f|_{\mathbb{R}} \in L^q \left( \left| \frac{1}{\tilde{G}} \right|^q \right) \right\},$$

we get that  $\psi \in H_+^q$  and  $\|\psi\|_q \lesssim \|\tilde{h}\|_{H_+^q} = 1$ . But, since

$$\bigcup_{n \in N_+} \tilde{\tau}_n = \Lambda^+ + i\eta$$

is in fact  $N$ -Carleson in  $\mathbb{C}_{-\frac{\epsilon_0}{2} + \eta}^+ \subset \mathbb{C}^+$  and  $\psi \in H_+^q$ , Theorem 9 implies that

$$\left( \sum_{n \in N_+} \operatorname{Im}(\lambda_{n,0} + i\eta) \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1} (\psi) \right|^q \right)^{\frac{1}{q}} \lesssim \|\psi\|_{H_+^q} \lesssim \|\tilde{h}\|_{H_+^q} = 1.$$

Finally, we obtain

$$N(h) \lesssim \left( \sum_{n \in N_+} (1 + \operatorname{Im}(\lambda_{n,0})) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (e^{i\tau \cdot} a) \right|^p \right)^{\frac{1}{p}} = \|a\|_{X_\tau^p(\Lambda)},$$

which ends the proof.

## 5. ABOUT THE $N$ -CARLESON CONDITION

It is clear that the definition of  $X_\tau^p(\Lambda)$  depends on the  $N$ -Carleson hypothesis, and more precisely for the construction of the groups  $\tau_n$ . In this last section, we show that in a certain way, the  $N$ -Carleson condition is necessary.

It will be convenient to introduce the distance function

$$\delta(z, \xi) := \frac{|z - \xi|}{1 + |z - \bar{\xi}|}$$

which expresses that locally we deal with Euclidian geometry close to the real axis and pseudohyperbolic geometry far away from the real axis (see *e.g.* [Se98, page 715]). Let  $\Lambda = \{\lambda_n\}_{n \geq 1}$  be a sequence of complex numbers. Let  $N \geq 1$  be an integer and  $\eta \in (0, \frac{1}{2})$ . For  $\lambda \in \Lambda$ , we define

$$D_{\lambda, \eta} := \{z \in \mathbb{C} : \delta(\lambda, z) < \eta\},$$

$$N_\lambda := \{\mu_{\lambda, i} : 1 \leq i \leq N\} \subset \Lambda$$

as the set of  $N$  closest neighbors of  $\lambda$  (including in particular  $\lambda$ ) with respect to the distance  $\delta$ . Then we set

$$\sigma_\lambda := D_{\lambda, \eta} \cap N_\lambda, \quad n_\lambda := |\sigma_\lambda| \leq N.$$

Note that the set  $N_\lambda$ , and consequently  $\sigma_\lambda$ , is not unique. It is now natural to introduce the space (for  $1 < p < \infty$ )

$$X_\tau^p(\Lambda, N) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X_\tau^p(\Lambda, N)} < \infty \right\},$$

where

$$\|a\|_{X_\tau^p(\Lambda, N)}^p := \sum_{\lambda \in \Lambda} (1 + |\operatorname{Im}(\lambda)|) \sum_{k=1}^{n_\lambda} \left| \tilde{\Delta}_{\sigma_\lambda}^{k-1} (ae^{\pm i\tau \cdot} (\mu^{(k)})) \right|^p$$

with

$$\tilde{\Delta}_{\sigma_\lambda} = \begin{cases} \Delta_{\sigma_\lambda}, & \text{if } \sigma_\lambda \cap \{z \in \mathbb{C} : |\operatorname{Im}(z)| < 1\} = \emptyset \\ \square_{\sigma_\lambda}, & \text{if not} \end{cases}$$

and

$$e^{\pm i\tau \mu} = \begin{cases} e^{i\tau \mu} & , \text{ if } \mu \in \sigma_\lambda \text{ and } \sigma_\lambda \cap \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\} \neq \emptyset \\ e^{-i\tau \mu} & , \text{ otherwise} \end{cases}.$$

It can be shown that if  $\Lambda \cap \mathbb{C}_a^\pm$  is  $N$ -Carleson in the corresponding half-plane, for each  $a \in \mathbb{R}$ , then this norm is equivalent to the previously defined norm  $\|\cdot\|_{X_\tau^p(\Lambda)}$  of the previous section. The result is the following one.

**Theorem 17.** *If  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda, N)$ , then for every  $a \in \mathbb{R}$ ,  $\Lambda \cap \mathbb{C}_a^\pm$  is  $N'$ -Carleson in the corresponding half-plane, with  $N' \leq N$ .*

The proof is in two parts. We begin to show that if  $R_\Lambda$  is such an isomorphism, then  $\Lambda_a^\pm$  is  $N'$ -Carleson for some  $N' \in \mathbb{N}$ . This only requires the boundedness of  $R_\Lambda$ . We first notice that by the Plancherel-Polya theorem (Proposition 11) the map

$$\tau_a : \begin{array}{ccc} PW_\tau^p & \rightarrow & PW_\tau^p \\ f & \mapsto & f(\cdot + i(1 + |a|)) \end{array}$$

is an isomorphism and so  $\tilde{R}_\Lambda := R_\Lambda \circ \tau_a$  is still an isomorphism. Obviously,  $\tilde{R}_\Lambda = R_{\tilde{\Lambda}}$ , where  $\tilde{\Lambda} := \Lambda + i(1 + |a|) =: \left\{ \tilde{\lambda} \right\}_{\tilde{\lambda} \in \tilde{\Lambda}}$ . Note that for  $\lambda \in \Lambda_a^+$

$$|a_\lambda|^p e^{-p\text{Im}(\lambda)} \leq \sum_{k=1}^{n_\lambda} \left| \tilde{\Delta}_{\sigma_\lambda}^{k-1} (ae^{\pm i\tau \cdot} (\mu^{(k)})) \right|^p$$

and so  $X_\tau^p(\tilde{\Lambda}, N)$  injects into  $l^p\left(\left(1 + \left|\text{Im}(\tilde{\lambda})\right|\right) e^{-p|\text{Im}(\tilde{\lambda})}\right)$  so that

$$R_{\tilde{\Lambda}} : PW_\tau^p \rightarrow l^p\left(\left(1 + \left|\text{Im}(\tilde{\lambda})\right|\right) e^{-p|\text{Im}(\tilde{\lambda})}\right)$$

is bounded. We set  $\tilde{\Lambda}_a^+ := \Lambda_a^+ + i(1 + |a|)$  and reintroduce the inner function  $I_\tau(z) = e^{2i\tau z}$ . We have mentioned in the beginning of the paper that  $PW_\tau^p$  is isomorphic to  $K_{I_\tau}^p$ , so

$$R_{\tilde{\Lambda}_a^+}^{I_\tau} := R_{\tilde{\Lambda}_a^+} \Big| K_{I_\tau}^p : K_{I_\tau}^p \rightarrow L^p(\mu_{\tilde{\Lambda}_a^+})$$

is bounded, where

$$\mu_{\tilde{\Lambda}_a^+} := \sum_{\tilde{\lambda} \in \tilde{\Lambda}_a^+} \text{Im}(\tilde{\lambda}) \delta_{\tilde{\lambda}}.$$

In order to show the claim, it is sufficient to show that  $\mu_{\tilde{\Lambda}_a^+}$  is a Carleson measure for  $H_+^p$ . Since in particular  $\tilde{\Lambda}_a^+ \subset \mathbb{C}_1^+$ , it is possible to find  $\epsilon \in (0, 1)$  such that

$$\tilde{\Lambda}_a^+ \subset L(I_\tau, \epsilon) := \{z \in \mathbb{C}^+ : |I_\tau(z)| < \epsilon\}.$$

Now, from a result of Treil and Volberg (see [TV95] or [Al97]), the boundedness of  $R_{\tilde{\Lambda}_a^+}^{I_\tau}$  implies that

$$(5.1) \quad \sup_I \frac{\mu_{\tilde{\Lambda}_a^+}(\omega_I)}{m(I)} < \infty,$$

where the supremum is taken over all the intervals of finite length such that the Carleson window  $\omega_I$  constructed on  $I$  satisfies

$$\omega_I \cap L(I_\tau, \epsilon) \neq \emptyset.$$

Observe that  $L(I_\tau, \epsilon)$  is in the upper half plane  $\mathbb{C}_b^+$ ,  $b = \log(1/\epsilon)$ , so that if the length of the Carleson window is less than  $b$ , then we have  $\omega_I \cap L(I_\tau, \epsilon) = \emptyset$ . Hence,  $\omega_I \cap \tilde{\Lambda}_a^+ = \emptyset$  and so  $\mu_{\tilde{\Lambda}_a^+}(\omega_I) = 0$ . It follows that (5.1) is true for all finite length intervals  $I$ , which is equivalent to the fact that  $\mu_{\tilde{\Lambda}_a^+}$  is a Carleson measure or also that  $\tilde{\Lambda}_a^+$  is  $N'$ -Carleson and hence  $\Lambda_a^+$  in the corresponding half-plane. Considering the map

$$s : \begin{array}{ccc} PW_\tau^p & \rightarrow & PW_\tau^p \\ f & \mapsto & f(\cdot) \end{array}$$

which is also an isomorphism, we will also have the result for  $\Lambda_a^-$ .

Now, we want to prove that  $N' \leq N$ . In the following, if  $\Lambda_a^+$  is  $(N+k)$ -Carleson, we write

$$\Lambda_a^+ = \bigcup_{n \geq 1} \tau_n^k,$$

where the groups  $\tau_n^k$  come from the Generalized Carleson condition, and so it is possible to assume that

$$\text{diam}_\delta(\tau_n^k) < \frac{\eta}{4}$$

(which in particular implies that  $\tau_n^k \subset D_{\lambda, \eta}$ ) and

$$\gamma := \inf_{n \neq m} \delta(\tau_n^k, \tau_m^k) > 0.$$

We need the following lemma and its corollary. For technical reasons, let us assume (without loss of generality) that  $\Lambda_a^+ \subset \mathbb{C}_1^+$  so that we can deal with the pseudohyperbolic metric and the corresponding divided differences.

**Lemma 18.** *If  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda, N)$  and  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson,  $k \geq 0$ , then it is possible to find  $\eta > 0$  such that for every subsequence  $(\tilde{\tau}_j)_j$  of  $(\tau_n^{k+1})$  with  $|\tilde{\tau}_j| = N+k+1$ , we have  $\text{diam}_\rho(\tilde{\tau}_j) > \eta$ .*

*Proof.* Let us suppose to the contrary that we can find a subsequence  $(\tilde{\tau}_j)$  of  $(\tau_n^{k+1})$  such that  $|\tilde{\tau}_j| = N+k+1$  and  $\text{diam}_\rho(\tilde{\tau}_j) \rightarrow 0$ ,  $j \rightarrow \infty$ . We set  $\tilde{\tau}_j = \{\lambda_i^j : i = 0, \dots, N+k\}$ . Let us now introduce the sequence  $a^j = (a^j(\lambda))_{\lambda \in \Lambda}$  defined by

$$a^j(\lambda) := 0, \quad \lambda \neq \lambda_{N+k}^j, \quad a^j(\lambda_{N+k}^j) := \frac{\prod_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)| e^{\tau \text{Im}(\lambda_{N+k}^j)}}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)| \text{Im}(\lambda_{N+k}^j)^{\frac{1}{p}}}.$$

Let

$$M_j := \{\lambda \in \Lambda_a^+ : \lambda_{N+k}^j \in \sigma_\lambda\}.$$

Since  $\text{diam}(\tilde{\tau}_j) < \frac{\eta}{4}$  and  $\lambda_{N+k}^j \in \tilde{\tau}_j$  we have for every  $\lambda \in \tilde{\tau}_j$  that  $\lambda_{N+k}^j \in \sigma_\lambda$ . So  $\tilde{\tau}_j \subset M_j$ . Let  $B_j := M_j \setminus \tilde{\tau}_j$ . Also, since  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson, and thus  $D_{\lambda_{N+k}^j, \eta}$  can only contain a uniformly bounded number of points of  $\Lambda_a^+$ , it follows that

$$\sup_j |M_j| < \infty.$$

By construction,

$$\|a^j\|_{X_\tau^p(\Lambda, N)}^p = \sum_{\lambda \in M_j} (1 + \operatorname{Im}(\lambda)) \sum_{l=1}^{n_\lambda} |\Delta_{\sigma_\lambda}^{l-1} (a^j e^{i\tau \cdot} (\mu^{(l)}))|^p.$$

We have to evaluate this expression. Take  $\lambda \in M_j$ . We recall that  $n_\lambda = |\sigma_\lambda|$ . Note also that for every  $1 \leq l \leq n_\lambda$ , the divided difference

$$|\Delta_{\sigma_\lambda}^{l-1} (a^j e^{\pm i\tau \cdot} (\lambda^{(l)}))|$$

will be equal either to 0 or to

$$\left| a^j (\lambda_{N+k}^j) e^{\pm i\tau \lambda_{N+k}^j} \prod_{m \in \omega_l} b_{\lambda_m^j} (\lambda_{N+k}^j) \right|,$$

where  $\omega_l \subset \sigma_\lambda$  contains  $l - 1$  points. Now,  $\omega_l = \omega_{l,1} \uplus \omega_{l,2}$  where  $\omega_{l,1} = \sigma_\lambda \cap \tilde{\tau}_j$  and  $\omega_{l,2}$  are the other points. Note that  $\omega_l$  cannot contain  $\lambda_{N+k}^j$ . By assumption, for  $\mu \in \omega_{l,2}$ ,  $|b_\mu(\lambda_{N+k}^j)| \geq \gamma$ . Hence,

$$\begin{aligned} & \sum_{l=1}^{n_\lambda} |\Delta_{\sigma_\lambda}^{l-1} (a^j e^{\pm i\tau \cdot} (\lambda^{(l)}))|^p \\ & \leq \sum_{l=1}^{n_\lambda} \frac{\prod_{i \neq N+k} |b_{\lambda_i^j} (\lambda_{N+k}^j)|^p}{\max_{i \neq N+k} |b_{\lambda_i^j} (\lambda_{N+k}^j)|^p} \cdot \frac{1}{\operatorname{Im} (\lambda_{N+k}^j) \prod_{\mu \in \omega_l} |b_\mu (\lambda_{N+k}^j)|^p} \\ & \leq \sum_{l=1}^{n_\lambda} \frac{1}{\gamma^{p|\omega_{l,2}|}} \frac{\prod_{\xi \in \Omega_l} |b_\xi (\lambda_{N+k}^j)|^p}{\max_{i \neq N+k} |b_{\lambda_i^j} (\lambda_{N+k}^j)|^p} \cdot \frac{1}{\operatorname{Im} (\lambda_{N+k}^j)} \\ & \lesssim \frac{N}{\operatorname{Im} (\lambda_{N+k}^j)}, \end{aligned}$$

where  $\Omega_l = \{\lambda_i^j : i = 0, \dots, N+k-1\} \setminus \omega_{l,1}$  are subsets of  $\tilde{\tau}_j$ . The last of the above inequalities comes from the observation that  $\Omega_l$  contains at least:

$$N+k - |\omega_{l,1}| \geq N+k - (n_\lambda - 1) \geq N+k - (N-1) = k+1 \geq 1$$

points. We deduce that  $a^j \in X_\tau^p(\Lambda, N)$  and that its norm is uniformly bounded. Now, since  $R_\Lambda$  is onto, there is  $f^j \in PW_\tau^p$  such that  $f^j|_\Lambda = a^j$  and

$$\|f^j\|_{PW_\tau^p} \lesssim \|a^j\|_{X_\tau^p(\Lambda, N)} \lesssim 1.$$

Setting  $\tilde{f}^j := e^{i\tau \cdot} f^j$ , it follows from the Plancherel-Polya inequality that  $\tilde{f}^j \in H_+^p$  and since  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson in  $\mathbb{C}^+$ , Theorem (ref:Hartmann) implies in particular that

$$\operatorname{Im}(\lambda_{N+k}^j) \left| \Delta_{\tilde{\tau}_j}^{N+k} \left( \tilde{f}^j(\lambda_0^j, \dots, \lambda_{N+k}^j) \right) \right|^p \lesssim \|f^j\| \lesssim 1.$$

But by construction, we have

$$\operatorname{Im}(\lambda_{N+k}^j) \left| \Delta_{\tilde{\tau}_j}^{N+k} \left( \tilde{f}^j(\lambda_0^j, \dots, \lambda_{N+k}^j) \right) \right|^p = \frac{1}{\max_{i \neq N+k} \rho(\lambda_{N+k}^j, \lambda_i^j)}$$

which tends to  $\infty$ ,  $j \rightarrow \infty$  because  $\operatorname{diam}_{\rho} \tilde{\tau}_j$  tends to 0,  $j \rightarrow \infty$ , which gives the required contradiction.  $\square$

The following corollary to the previous lemma allows us to end the proof of our theorem.

**Corollary 19.** *If  $R_\Lambda$  is an isomorphism from  $PW_\tau^p$  onto  $X_\tau^p(\Lambda, N)$  and  $\Lambda_a^+$  is  $(N+k+1)$ -Carleson,  $k \geq 0$ , then  $\Lambda_a^+$  is  $(N+k)$ -Carleson.*

*Proof.* We write  $\Lambda_a^+ = \bigcup_{n \geq 1} \tau_n^{k+1}$  with  $|\tau_n^{k+1}| \leq N+k+1$ . Let us suppose that there is an infinity of  $n$  for which we have  $|\tau_n^{k+1}| = N+k+1$  and let  $Z$  be the set of such  $n$ . Because of the previous lemma, we can find  $\eta > 0$  such that  $\operatorname{diam}_{\rho}(\tau_n^{k+1}) > \eta$  for  $n \in Z$ . Then, for every  $n \in Z$ , it is possible to write  $\tau_n^{k+1} = \{\lambda_i^n : i = 1, \dots, N+k+1\}$  such that

$$\rho(\lambda_i, \lambda_{N+k+1}^n) \geq \frac{\eta}{2(N+k)}, \quad i = 1, \dots, N+k.$$

It follows that

$$\Lambda_a^+ = \bigsqcup_{n \notin Z} \tau_n^{k+1} \sqcup \left( \bigsqcup_{n \in Z} \tau_n^{k+1} \setminus \{\lambda_{N+k+1}^n\} \right) \sqcup \left( \bigsqcup_{n \in Z} \{\lambda_{N+k+1}^n\} \right)$$

is a disjoint union of sets  $\sigma_n$  with  $|\sigma_n| \leq N+k$  and it can be shown that the sequence of Blascke products  $(B_{\sigma_n})_n$  satisfies the Generalized Carleson condition and hence that  $\Lambda_a^+$  is  $(N+k)$ -Carleson.  $\square$

I would like to thank Andreas Hartmann for his very helpful and permanent support during this research and, more generally, from the beginning of my thesis.



## REFERENCES

- [Al97] A.B. ALEKSANDROV, *A simple proof of a theorem of Volberg and Treil on the embedding of coinvariant subspaces of the shift operator*, J. Math. Sci. **85-2** (1997), 1773-1778.
- [AI01] S.A. AVDONIN AND S.A. IVANOV, *Exponential Riesz bases of subspaces and divided differences*, St. Petersburg. Math. J. **93-3** (2001), 339-351.
- [BNO96] J. BRUNA, A. NICOLAU AND K. OYMA, *A note on interpolation in the Hardy spaces of the unit disc*, Proc. Amer. Math. Soc. **124-4** (1996), 1197-1204.
- [Ca58] L. CARLESON, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921-930.
- [Gar81] J.B. GARNETT, *Bounded analytic functions (Revised first edition)*, Graduate Texts in Math. **236** (2007), Springer-Verlag. First edition in Pure and applied Mathematics **86** (1981), Academic Press.
- [Ha96a] A. HARTMANN, *Interpolation libre et caractrisation des traces des fonctions holomorphes sur les unions finies de suites de Carleson*, Thse de l'Universit Bordeaux 1 (1996).
- [Ha96b] A. HARTMANN, *Une approche de l'interpolation libre gnralise par la thorie des oprateurs et caractrisations des traces  $H^p|\Lambda$* , J. Operator Theory **35-2** (1996), 281-316.
- [HMW73] R. HUNT, B. MUCKENHOUPT AND R. WHEEDEN, *Weighted norm inequalities for the conjugate Hilbert transform*, Proc. Amer. Math. Soc. **176** (1973), 227-251.
- [HNP81] S.V. HRUSCEV, N.K. NIKOLSKII AND B.S PAVLOV, *Unconditional bases of exponentials and of reproducing kernels in Complex analysis and spectral theory*, Lectures Notes in Math. **864** (1981), 214-335.
- [Ko80] P. KOOSIS, *Introduction to  $H^p$  spaces, second edition*, Cambridge tracts in Math. **115** (1998), First edition published in 1980.
- [Le96] B.Y. LEVIN, *Lectures on entire functions*, Math. Monographs **150** (1996), Amer. Math. Soc.
- [LS97] Y.L. LYUBARSKII AND K. SEIP, *Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt's  $(A_p)$  condition*, Rev. Mat. Iber. **13-2** (1997), 361-376.
- [Mi92] A.M. MINKIN, *The reflection of indices and unconditionnal bases of exponentials*, St. Petersburg Math. J. **3-5** (1992), 1043-1064.
- [Ni86] N.K. NIKOLSKII, *A treatise on the shift operator*, Grundlehren der mathematischen Wissenschaften **273** (1986), Springer-Verlag.
- [Ni02a] N.K. NIKOLSKII, *Operators, functions and systems: An easy reading, volume 1*, Mathematical Surveys and Monographs **92** (2002), Amer. Math. Soc.
- [Ni02b] N.K. NIKOLSKII, *Operators, functions and systems: An easy reading, volume 2*, Mathematical Surveys and Monographs **93** (2002), Amer. Math. Soc.
- [Ro77] R. ROCHBERG, *Toeplitz operators on weighted  $H^p$  spaces*, Indiana Univ. Math. J. **26-2** (1977), 291-298.
- [Se98] K. SEIP, *Developments from nonharmonic Fourier series*. Proceedings of the International Congress of Mathematicians, Doc. Math. Extra Vol. **II** (1998) , 713-722.

- [Se04] K. SEIP, *Interpolation and sampling in spaces of analytic functions*, Univ. Lect. Series **33** (2004), Amer. Math. Soc.
- [SS61] H.S. SHAPIRO AND A.L. SHIELDS, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513-532.
- [TV95] S.R. TREIL AND A.L. VOLBERG, *Weighted embeddings and weighted norm inequalities for the Hilbert transform and the maximal operator*, Algebra i Analiz **7-6** (1995), 205-226; translation in St. Petersburg Math. J. **7-6** (1996), 1017-1032.
- [Va84] V.I. VASYUNIN, *Traces of bounded analytic functions on finite unions of Carleson sets*, J. Soviet Math. **27-1** (1984), 2448-2450.