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Abstract. *This paper is devoted to studying a nonlinear wave equation with boundary conditions of two-point type. First, we state two local existence theorems and under the suitable conditions, we prove that any weak solutions with negative initial energy will blow up in finite time. Next, we give a sufficient condition to guarantee the global existence and exponential decay of weak solutions. Finally, we present numerical results.*

Keyword: Nonlinear wave equation, local existence, global existence, blow up, exponential decay.

AMS subject classification: 35L05, 35L15, 35L70, 37B25.

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1 Introduction

In this paper, we consider the following nonlinear wave equation with initial conditions and boundary conditions of two-point type

$$u_{tt} - u_{xx} + u + \lambda u_t = |u|^{p-2}u,\ 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = -\ |u(0,t)|^{\alpha-2}\, u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1(t)u(1,t) + \widetilde{\lambda}_1 u_t(1,t),\ t > 0, \tag{1.2}$$

$$-u_x(1,t) = -\ |u(1,t)|^{\beta-2}\, u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0(t)u(0,t) + \widetilde{\lambda}_0 u_t(0,t),\ t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \tag{1.4}$$

where λ_0, λ_1, $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, λ, p are constants and u_0, u_1, $\widetilde{h}_0$, $\widetilde{h}_1$ are given functions satisfying conditions specified later.

The wave equation

$$u_{tt} - \Delta u = f(x,t,u,u_t), \tag{1.5}$$

with the different boundary conditions, has been extensively studied by many authors, see ([?], [?], [?] – [?]) and references therein. In these works, many interesting results about the existence, regularity and the asymptotic behavior of solutions were obtained.

In [?], J.E. Munoz-Rivera and D. Andrade dealt with the global existence and exponential decay of solutions of the nonlinear one-dimensional wave equation with a viscoelastic boundary condition.

In [?] – [?], Santos also studied the asymptotic behavior of solutions to a coupled system of wave equations having integral convolutions as memory terms. The main results show that solutions of that system decay uniformly in time, with rates depending on the rate of decay of the kernel of the convolutions.

In [?], the global existence and regularity of weak solutions for the linear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x,t),\ 0 < x < 1,\ t > 0, \tag{1.6}$$

with the initial conditions as in (1.4) and the two-point boundary conditions

$$\begin{cases} u_x(0,t) = h_0 u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1 u(1,t) + \widetilde{\lambda}_1 u_t(1,t) + g_0(t), \\ -u_x(1,t) = h_1 u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0 u(0,t) + \widetilde{\lambda}_0 u_t(0,t) + g_1(t), \end{cases} \tag{1.7}$$

were proved, where h_0, h_1, $\widetilde{h}_0$, $\widetilde{h}_1$, λ_0, λ_1, $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, K, λ are constants and u_0, u_1, g_0, g_1, f are given functions. Furthermore, the exponential decay of solutions were also given there by using Lyapunov's method.

We note more that, the following nonhomogeneous boundary conditions were considered by Hellwig ([?], p.151):

$$\begin{cases} \alpha_{01}u(0,t) + \alpha_{02}u_x(0,t) + \alpha_{03}u_t(0,t) + \beta_{01}u(1,t) + \beta_{02}u_x(1,t) + \beta_{03}u_t(1,t) = f_0(t), \\ \alpha_{11}u(0,t) + \alpha_{12}u_x(0,t) + \alpha_{13}u_t(0,t) + \beta_{11}u(1,t) + \beta_{12}u_x(1,t) + \beta_{13}u_t(1,t) = f_1(t), \end{cases} \tag{1.8}$$

where α_{ij}, β_{ij}, $i = 0,1$, $j = 1,2,3$ are constants and $f_0(t)$, $f_1(t)$ are given functions.

Let $\Delta = \alpha_{02}\beta_{12} - \alpha_{12}\beta_{02} \neq 0$, (1.8) is transformed into

$$\begin{cases} u_x(0,t) = h_0 u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1 u(1,t) + \widetilde{\lambda}_1 u_t(1,t) + g_0(t), \\ -u_x(1,t) = h_1 u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0 u(0,t) + \widetilde{\lambda}_0 u_t(0,t) + g_1(t), \end{cases} \tag{1.9}$$

in which

$$\begin{cases} h_0 = \frac{1}{\Delta}(\beta_{02}\alpha_{11} - \beta_{12}\alpha_{01}),\ h_1 = \frac{1}{\Delta}(\alpha_{02}\beta_{11} - \alpha_{12}\beta_{01}), \\ \lambda_0 = \frac{1}{\Delta}(\beta_{02}\alpha_{13} - \beta_{12}\alpha_{03}),\ \lambda_1 = \frac{1}{\Delta}(\alpha_{02}\beta_{13} - \alpha_{12}\beta_{03}), \\ \widetilde{h}_0 = \frac{1}{\Delta}(\alpha_{02}\alpha_{11} - \alpha_{12}\alpha_{01}),\ \widetilde{h}_1 = \frac{1}{\Delta}(\beta_{02}\beta_{11} - \beta_{12}\beta_{01}), \\ \widetilde{\lambda}_0 = \frac{1}{\Delta}(\alpha_{02}\alpha_{13} - \alpha_{12}\alpha_{03}),\ \widetilde{\lambda}_1 = \frac{1}{\Delta}(\beta_{02}\beta_{13} - \beta_{12}\beta_{03}), \\ g_0(t) = \frac{1}{\Delta}(\beta_{12}f_0(t) - \beta_{02}f_1(t)),\ g_1(t) = \frac{1}{\Delta}(\alpha_{12}f_0(t) - \alpha_{02}f_1(t)). \end{cases} \tag{1.10}$$

The main goal of this paper is to extend some results of [?]. Motivated by the problem of the exponential decay of solutions for (1.6) – (1.7), we establish a blow up result and a decay result for the general problem (1.1) – (1.4).

In Theorem 3.1, by applying techniques as in [?] with some necessary modifications and with some restrictions on the initial data, we prove that the solution of (1.1) – (1.4) blows up in finite time.

In Theorem 4.1, by the construction of a suitable Lyapunov functional we also prove that the solution will exponential decay if the initial energy is positive and small.

The paper consists of five sections. In Section 2, we present some preliminaries and the existence results. The proofs of Theorems 3.1 and 4.1 are done in Sections 3 and 4. Finally, in Section 5 we give numerical results.

2 Existence and uniqueness of solution

First, we put $\Omega = (0,1)$; $Q_T = \Omega \times (0,T)$, $T > 0$ and we denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, ...$ Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $||\cdot||$ stands for the norm in L^2 and we denote by $||\cdot||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1, we use the following norm $\|v\|_1 = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}$.

We have the following lemmas.

Lemma 2.1. $\|v\|_{C^0([0,1])} \leq \sqrt{2}\|v\|_1$, *for all* $v \in H^1$.

Lemma 2.2. *Let* λ_0, $\lambda_1 > 0$ *and* $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1 \in \mathbb{R}$, *such that* $\left|\widetilde{\lambda}_0 + \widetilde{\lambda}_1\right| < 2\sqrt{\lambda_0\lambda_1}$. *Then*

$$\lambda_0 x^2 + \lambda_1 y^2 + (\widetilde{\lambda}_0 + \widetilde{\lambda}_1)xy \geq \tfrac{1}{2}\mu_*\left(x^2 + y^2\right), \text{ for all } x,\ y \in \mathbb{R}, \tag{2.1}$$

where

$$\mu_* = \tfrac{1}{4}\left[-(\widetilde{\lambda}_0 + \widetilde{\lambda}_1)^2 + 4\lambda_0\lambda_1\right] \min\left\{\tfrac{1}{\lambda_0}, \tfrac{1}{\lambda_1}\right\} > 0. \tag{2.2}$$

The proofs of these lemmas are straightforward. We shall omit the details.■

Next, we state two local existence theorems. We make the following assumptions:

Suppose that p, α, β, λ, λ_0, λ_1, $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1 \in \mathbb{R}$, are constants satisfying

(A_1) $p > 2$, $\alpha > 2$, $\beta > 2$, $\lambda > 0$;

(A_2) λ_0, $\lambda_1 > 0$, $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1 \in \mathbb{R}$, with $\left|\widetilde{\lambda}_0 + \widetilde{\lambda}_1\right| < 2\sqrt{\lambda_0\lambda_1}$.

Let

(A_3) $\widetilde{h}_i \in H^1(0,T)$, $i = 1, 2$.

Then we have the following theorem about the existence of a "strong solution".

Theorem 2.3. *Suppose that* $(A_1) - (A_3)$ *hold and the initial data* $(u_0, u_1) \in H^2 \times H^1$ *satisfies the compatibility conditions*

$$\begin{cases} u_{0x}(0) = -\ |u_0(0)|^{\alpha-2}\, u_0(0) + \lambda_0 u_1(0) + \widetilde{h}_1(0)u_0(1) + \widetilde{\lambda}_1 u_1(1), \\ -u_{0x}(1) = -\ |u_0(1)|^{\beta-2}\, u_0(1) + \lambda_1 u_1(1) + \widetilde{h}_0(0)u_0(0) + \widetilde{\lambda}_0 u_1(0). \end{cases} \tag{2.3}$$

Then problem (1.1) – (1.4) *has a unique local solution*

$$\begin{cases} u \in L^{\infty}\left(0,T_{*};H^{2}\right),\ u_{t} \in L^{\infty}\left(0,T_{*};H^{1}\right),\ u_{tt} \in L^{\infty}\left(0,T_{*};L^{2}\right), \\ u(0,\cdot),\ \ u(1,\cdot) \in H^{2}\left(0,T_{*}\right), \end{cases} \tag{2.4}$$

for $T_{*} > 0$ *small enough.*■

Remark 2.1.
The regularity obtained by (2.4) shows that problem (1.1) – (1.4) has a unique strong solution

$$\begin{cases} u \in L^{\infty}\left(0,T_{*};H^{2}\right) \cap C^{0}\left(0,T_{*};H^{1}\right) \cap C^{1}\left(0,T_{*};L^{2}\right), \\ u_{t} \in L^{\infty}\left(0,T_{*};H^{1}\right) \cap C^{0}\left(0,T_{*};L^{2}\right), \\ u_{tt} \in L^{\infty}\left(0,T_{*};L^{2}\right), \\ u(i,\cdot) \in H^{2}\left(0,T_{*}\right),\ i=0,1. \end{cases} \tag{2.5}$$

With less regular initial data, we obtain the following theorem about the existence of a weak solution.

Theorem 2.4. *Suppose that* $(A_1) - (A_3)$ *hold. Let* $(u_0, u_1) \in H^1 \times L^2$.
Then problem (1.1) – (1.4) *has a unique local solution*

$$u \in C\left([0,T_{*}];H^{1}\right) \cap C^{1}\left([0,T_{*}];L^{2}\right),\ \ u(i,\cdot) \in H^{1}\left(0,T_{*}\right),\ i=0,1, \tag{2.6}$$

for $T_{*} > 0$ *small enough.*

Proof of Theorem 2.3.

The proof is established by a combination of the arguments in [**?**]. It consits of steps 1 – 4.
Step 1. *The Faedo-Galerkin approximation.* Let $\{w_j\}$ be a denumerable base of H^1. We find the approximate solution of the problem (1.1) – (1.4) in the form

$$u_{m}(t) = \textstyle\sum_{j=1}^{m} c_{mj}(t) w_{j}, \tag{2.7}$$

where the coefficient functions c_{mj}, $1 \le j \le m$, satisfy the system of ordinary differential equations

$$\begin{cases} \left\langle u_{m}''(t), w_{j}\right\rangle + \left\langle u_{mx}(t), w_{jx}\right\rangle + \left\langle u_{m}(t), w_{j}\right\rangle + \lambda\left\langle u_{m}'(t), w_{j}\right\rangle \\ \qquad\qquad + \left(\lambda_{0} u_{m}'(0,t) + \widetilde{h}_{1}(t) u_{m}(1,t) + \widetilde{\lambda}_{1} u_{m}'(1,t)\right) w_{j}(0) \\ \qquad\qquad + \left(\lambda_{1} u_{m}'(1,t) + \widetilde{h}_{0}(t) u_{m}(0,t) + \widetilde{\lambda}_{0} u_{m}'(0,t)\right) w_{j}(1) \\ \qquad = \left\langle |u_{m}|^{p-2} u_{m}, w_{j}\right\rangle + |u_{m}(0,t)|^{\alpha-2}\, u_{m}(0,t) w_{j}(0) \\ \qquad\qquad + |u_{m}(1,t)|^{\beta-2}\, u_{m}(1,t) w_{j}(1),\ 1 \le j \le m, \\ u_{m}(0) = u_{0},\ \ u_{m}'(0) = u_{1}. \end{cases} \tag{2.8}$$

From the assumptions of Theorem 2.3, system (2.8) has a solution u_m on an interval $[0, T_m] \subset [0, T]$.
Step 2. **The first estimate**. Multiplying the j^{th} equation of (2.8) by $c_{mj}'(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, after some rearrangements and using Lemma 2.2, we get

$$\begin{aligned} S_{m}(t) \le\ & S_{m}(0) + 2\textstyle\int_{0}^{t} \left\langle |u_{m}(s)|^{p-2} u_{m}(s), u_{m}'(s)\right\rangle ds \\ & + 2\textstyle\int_{0}^{t} |u_{m}(0,s)|^{\alpha-2}\, u_{m}(0,s) u_{m}'(0,s) ds + 2\int_{0}^{t} |u_{m}(1,s)|^{\beta-2}\, u_{m}(1,s) u_{m}'(1,s) ds \\ & - 2\textstyle\int_{0}^{t} \widetilde{h}_{1}(s) u_{m}(1,s) u_{m}'(0,s) ds - 2\int_{0}^{t} \widetilde{h}_{0}(s) u_{m}(0,s) u_{m}'(1,s) ds, \end{aligned} \tag{2.9}$$

where

$$S_m(t) = \left\|u'_m(t)\right\|^2 + \|u_m(t)\|_1^2 + 2\lambda \int_0^t \left\|u'_m(s)\right\|^2 ds + \mu_* \int_0^t \left(\left|u'_m(0,s)\right|^2 + \left|u'_m(1,s)\right|^2\right) ds, \quad (2.10)$$

$$S_m(0) = \|u_1\|^2 + \|u_0\|_1^2 \equiv S_0. \quad (2.11)$$

Applying the classical inequalities, we estimate the terms on the right-hand side of (2.9) and obtain

$$\begin{aligned} S_m(t) \le \overline{d}_0 + \overline{d}_1 \textstyle\int_0^t (S_m(s))^{\frac{p}{2}}\, ds + \overline{d}_2 \int_0^t (S_m(s))^{\alpha-1}\, ds \\ + \overline{d}_3 \textstyle\int_0^t (S_m(s))^{\beta-1}\, ds + \overline{d}_4(T) \int_0^t S_m(s) ds, \;\; 0 \le t \le T_m, \end{aligned} \quad (2.12)$$

where

$$\begin{cases} \overline{d}_0 = 2\overline{S}_0,\ \overline{d}_1 = 4\left(\sqrt{2}\right)^{p-1},\ \overline{d}_2 = \frac{1}{\mu_*} 2^{\alpha+3}, \\ \overline{d}_3 = \frac{1}{\mu_*} 2^{\beta+3},\ \overline{d}_4(T) = \frac{32}{\mu_*} \left(\left\|\widetilde{h}_0\right\|^2_{L^\infty(0,T)} + \left\|\widetilde{h}_1\right\|^2_{L^\infty(0,T)}\right), \\ \frac{p}{2} > 1,\ \alpha - 1 > 1,\ \beta - 1 > 1. \end{cases} \quad (2.13)$$

Then, by solving a nonlinear Volterra integral equation (based on the methods in [**?**]), we get the following lemma.

Lemma 2.5. *There exists a constant $T_* > 0$ depending on T (independent of m) such that*

$$S_m(t) \le C_T,\ \forall m \in \mathbb{N},\ \forall t \in [0, T_*], \quad (2.14)$$

where C_T is a constant depending only on T.■

Lemma 2.5 allows one to take constant $T_m = T_*$ for all m.

The second estimate.

First of all, we estimate $u''_m(0)$. By taking $t = 0$ and $w_j = u''_m(0)$ in (**??**), we assert

$$\left\|u''_m(0)\right\| \le \|u_{0xx}\| + \|u_0\| + \lambda \|u_1\| + \left\||u_0|^{p-1}\right\| = \overline{X}_0^*. \quad (2.15)$$

Now, by differentiating (2.8) with respect to t and substituting $w_j = u''_m(t)$, after integrating with respect to the time variable from 0 to t, using again Lemma 2.2, we have

$$\begin{aligned} X_m(t) \le X_m(0) - 2\textstyle\int_0^t \left(\widetilde{h}_1(s) u'_m(1,s) + \widetilde{h}'_1(s) u_m(1,s)\right) u''_m(0,s) ds \\ -2\textstyle\int_0^t \left(\widetilde{h}_0(s) u'_m(0,s) + \widetilde{h}'_0(s) u_m(0,s)\right) u''_m(1,s) ds \\ +2(\alpha-1)\textstyle\int_0^t |u_m(0,s)|^{\alpha-2}\, u'_m(0,s) u''_m(0,s) ds \\ +2(\beta-1)\textstyle\int_0^t |u_m(1,s)|^{\beta-2}\, u'_m(1,s) u''_m(1,s) ds \\ +2(p-1)\textstyle\int_0^t \left\langle |u_m(s)|^{p-2}\, u'_m(s), u''_m(s)\right\rangle ds, \end{aligned} \quad (2.16)$$

where

$$X_m(t) = \left\|u''_m(t)\right\|^2 + \left\|u'_m(t)\right\|_1^2 + 2\lambda \int_0^t \left\|u''_m(s)\right\|^2 ds + \mu_* \int_0^t \left(\left|u''_m(0,s)\right|^2 + \left|u''_m(1,s)\right|^2\right) ds, \quad (2.17)$$

$$X_m(0) = \left\|u''_m(0)\right\|^2 + \|u_1\|_1^2 \le \overline{X}_0^{*2} + \|u_1\|_1^2 \equiv X_0. \quad (2.18)$$

Estimate respectively all the terms on the right-hand side of (2.16) leads to

$$X_m(t) \le \widetilde{d}_T + 2\textstyle\int_0^t X_m(s) ds, \quad (2.19)$$

where

$$\begin{aligned}\widetilde{d}_T = 2X_0 + \tfrac{16}{\mu_*}\left[(\alpha-1)^2 2^{\alpha-2} C_T^{\alpha-1} + (\beta-1)^2 2^{\beta-2} C_T^{\beta-1}\right] \\ +(p-1)^2 2^{p-1} T C_T^{p-1} + \tfrac{32C_T}{\mu_*} d_T \left(\left\|\widetilde{h}_0\right\|^2_{H^1(0,T)} + \left\|\widetilde{h}_1\right\|^2_{H^1(0,T)}\right),\end{aligned} \tag{2.20}$$

in which d_T is a constant verifying the inequality $\frac{1}{\mu_*}\|v\|^2_{L^\infty(0,T)} + 2\|v'\|^2_{L^2(0,T)} \le d_T \|v\|^2_{H^1(0,T)}$, for all $v \in H^1(0,T)$.

By Gronwall's lemma, it follows from (2.19), that

$$X_m(t) \le \widetilde{d}_T \exp(2T) \le C_T,\ \forall t \in [0, T_*], \tag{2.21}$$

where C_T is a constant depending only on T.

Step 3. *Limiting process.* From (2.10), (2.14), (2.17) and (2.21), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,T_*;H^1) \quad \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^\infty(0,T_*;H^1) \quad \text{weakly*}, \\ u''_m \to u'' & \text{in} \quad L^\infty(0,T_*;L^2) \quad \text{weakly*}, \\ u_m(0,\cdot) \to u(0,\cdot) & \text{in} \quad H^2(0,T_*) \quad \text{weakly}, \\ u_m(1,\cdot) \to u(1,\cdot) & \text{in} \quad H^2(0,T_*) \quad \text{weakly}. \end{cases} \tag{2.22}$$

By the compactness lemma of Lions ([**?**], p. 57) and the compact imbedding $H^2(0,T_*) \hookrightarrow C^1([0,T_*])$, we can deduce from (2.22) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u & \text{strongly in} \quad L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}, \\ u'_m \to u' & \text{strongly in} \quad L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}, \\ u_m(i,\cdot) \to u(i,\cdot) & \text{strongly in} \quad C^1([0,T_*]),\ i = 0,\ 1. \end{cases} \tag{2.23}$$

Using the following inequality

$$\left|\ |x|^{p-2}x - |y|^{p-2}y\right| \le (p-1)M^{p-2}|x-y|,\ \forall x, y \in [-M, M],\ \forall M > 0,\ \forall p \ge 2, \tag{2.24}$$

with $M = \sqrt{2C_T}$, we deduce from (2.14) that

$$\left|\ |u_m|^{p-2}u_m - |u|^{p-2}u\right| \le (p-1)M^{p-2}|u_m - u|,\ \text{for all } m,\ (x,t) \in Q_{T_*}. \tag{2.25}$$

Hence, by $(2.23)_1$, we deduce from (2.25), that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly in} \quad L^2(Q_{T_*}). \tag{2.26}$$

Passing to the limit in (2.8) by (2.22), (2.23), and (2.26), we have u satisfying the problem

$$\begin{cases} \langle u''(t), v\rangle + \langle u_x(t), v_x\rangle + \langle u(t), v\rangle + \lambda\langle u'(t), v\rangle \\ \qquad + \left(\lambda_0 u'(0,t) + \widetilde{h}_1(t)u(1,t) + \widetilde{\lambda}_1 u'(1,t)\right)v(0) \\ \qquad + \left(\lambda_1 u'(1,t) + \widetilde{h}_0(t)u(0,t) + \widetilde{\lambda}_0 u'(0,t)\right)v(1) \\ \qquad = \left\langle |u|^{p-2}u, v\right\rangle + |u(0,t)|^{\alpha-2}u(0,t)v(0) + |u(1,t)|^{\beta-2}u(1,t)v(1),\ \text{for all } v \in H^1, \\ u(0) = u_0,\ \ u'(0) = u_1. \end{cases} \tag{2.27}$$

On the other hand, we have from $(2.22)_{1,2,3}$, $(2.27)_1$ that

$$u_{xx} = u'' + u + \lambda u' - |u|^{p-2}u \in L^\infty(0, T_*; L^2). \tag{2.28}$$

Thus $u \in L^\infty(0, T_*; H^2)$ and the existence of the solution is proved completely.

Step 4. *Uniqueness of the solution.* Let u_1, u_2 be two weak solutions of problem (1.1) – (1.4), such that

$$\begin{cases} u_i \in L^\infty\left(0, T_*; H^2\right), \ u_i' \in L^\infty\left(0, T_*; L^2\right), \ u_i'' \in L^\infty\left(0, T_*; L^2\right), \\ u_i(0,\cdot), \ u_i(1,\cdot) \in H^2\left(0, T_*\right), \ i = 1, 2. \end{cases} \tag{2.29}$$

Then $w = u_1 - u_2$ verifies

$$\begin{cases} \langle w''(t), v\rangle + \langle w_x(t), v_x\rangle + \langle w(t), v\rangle + \lambda \langle w'(t), v\rangle \\ \qquad + \left(\lambda_0 w'(0,t) + \widetilde{h}_1(t) w(1,t) + \widetilde{\lambda}_1 w'(1,t)\right) v(0) \\ \qquad + \left(\lambda_1 w'(1,t) + \widetilde{h}_0(t) w(0,t) + \widetilde{\lambda}_0 w'(0,t)\right) v(1) \\ \qquad = \left\langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, v\right\rangle \\ \qquad + \left[|u_1(0,t)|^{\alpha-2} u_1(0,t) - |u_2(0,t)|^{\alpha-2} u_2(0,t)\right] v(0) \\ \qquad + \left[|u_1(1,t)|^{\beta-2} u_1(1,t) - |u_2(1,t)|^{\beta-2} u_2(1,t)\right] v(1), \text{ for all } v \in H^1, \\ w(0) = w'(0) = 0. \end{cases} \tag{2.30}$$

We take $v = w = u_1 - u_2$ in (2.30) and integrating with respect to t, we obtain

$$\begin{aligned} S(t) \le & -2\textstyle\int_0^t \widetilde{h}_0(s) w(0,s) w'(1,s) ds - 2\int_0^t \widetilde{h}_1(s) w(1,s) w'(0,s) ds \\ & + 2\textstyle\int_0^t \left[|u_1(0,s)|^{\alpha-2} u_1(0,s) - |u_2(0,s)|^{\alpha-2} u_2(0,s)\right] w'(0,s) ds \\ & + 2\textstyle\int_0^t \left[|u_1(1,s)|^{\beta-2} u_1(1,s) - |u_2(1,s)|^{\beta-2} u_2(1,s)\right] w'(1,s) ds \\ & + 2\textstyle\int_0^t \left\langle |u_1(s)|^{p-2}u_1(s) - |u_2(s)|^{p-2}u_2(s), w'(s)\right\rangle ds, \end{aligned} \tag{2.31}$$

where

$$S(t) = \|w'(t)\|^2 + \|w(t)\|_1^2 + 2\lambda \textstyle\int_0^t \|w'(s)\|^2 ds + \mu_* \int_0^t \left(|w'(0,s)|^2 + |w'(1,s)|^2\right) ds. \tag{2.32}$$

It implies that

$$S(t) \le \widetilde{K}_M \textstyle\int_0^t S(s) ds, \tag{2.33}$$

where

$$\widetilde{K}_M = \tfrac{32}{\mu_*}\left(\left\|\widetilde{h}_0\right\|^2_{L^\infty(0,T)} + \left\|\widetilde{h}_1\right\|^2_{L^\infty(0,T)} + (\alpha-1)^2 M_1^{2\alpha-4} + (\beta-1)^2 M_1^{2\beta-4}\right) + 2(p-1)M_1^{p-2}, \tag{2.34}$$

with $M_1 = \sqrt{2}\left(\|u\|_{L^\infty(0,T_*;H^1)} + \|v\|_{L^\infty(0,T_*;H^1)}\right)$.

By Gronwall's lemma, it follows from (2.23), that $S \equiv 0$, i.e., $u \equiv v$. Theorem 2.3 is proved completely.■

Proof of Theorem 2.4.

In order to obtain the existence of a weak solution, we use standard arguments of density.

Let us consider $(u_0, u_1) \in H^1 \times L^2$ and let sequences $\{u_{0m}\}$ and $\{u_{1m}\}$ in H^2 and H^1, respectively, such that

$$\begin{cases} u_{0m} \to u_0 & \text{strongly in} \quad H^1, \\ u_{1m} \to u_1 & \text{strongly in} \quad L^2. \end{cases} \tag{2.35}$$

So $\{(u_{0m}, u_{1m})\}$ satisfy, for all $m \in \mathbb{N}$, the compatibility conditions

$$\begin{cases} u_{0mx}(0) = - \ |u_{0m}(0)|^{\alpha-2} u_{0m}(0) + \lambda_0 u_{1m}(0) + \widetilde{h}_1(0) u_{0m}(1) + \widetilde{\lambda}_1 u_{1m}(1), \\ -u_{0mx}(1) = - \ |u_{0m}(1)|^{\beta-2} u_{0m}(1) + \lambda_1 u_{1m}(1) + \widetilde{h}_0(0) u_{0m}(0) + \widetilde{\lambda}_0 u_{1m}(0). \end{cases} \tag{2.36}$$

Then, for each $m \in \mathbb{N}$ there exists a unique function u_m in the conditions of the Theorem 2.3. So we can verify

$$\begin{cases} \langle u_m''(t), v\rangle + \langle u_{mx}(t), v_x\rangle + \langle u_m(t), v\rangle + \lambda \langle u_m'(t), v\rangle \\ \qquad + \left(\lambda_0 u_m'(0,t) + \widetilde{h}_1(t) u_m(1,t) + \widetilde{\lambda}_1 u_m'(1,t)\right) v(0) \\ \qquad + \left(\lambda_1 u_m'(1,t) + \widetilde{h}_0(t) u_m(0,t) + \widetilde{\lambda}_0 u_m'(0,t)\right) v(1) \\ \qquad = \left\langle |u_m|^{p-2} u_m, v\right\rangle + |u_m(0,t)|^{\alpha-2} u_m(0,t) v(0) \\ \qquad + |u_m(1,t)|^{\beta-2} u_m(1,t) v(1), \text{ for all } v \in H^1, \\ u_m(0) = u_{0m}, \ \ u_m'(0) = u_{1m}, \end{cases} \tag{2.37}$$

and

$$\begin{cases} u_m \in L^\infty\left(0, T_*; H^2\right) \cap C^0\left(0, T_*; H^1\right) \cap C^1\left(0, T_*; L^2\right), \\ u_m' \in L^\infty\left(0, T_*; H^1\right) \cap C^0\left(0, T_*; L^2\right), \\ u_m'' \in L^\infty\left(0, T_*; L^2\right), \\ u_m(0,\cdot), \ u_m(1,\cdot) \in H^2\left(0, T_*\right). \end{cases} \tag{2.38}$$

By the same arguments used to obtain the above estimates, we get

$$\|u_m'(t)\|^2 + \|u_m(t)\|_1^2 + 2\lambda \int_0^t \|u_m'(s)\|^2 \, ds + \mu_* \int_0^t \left(|u_m'(0,s)|^2 + |u_m'(1,s)|^2\right) ds \le C_T, \tag{2.39}$$

$\forall t \in [0, T_*]$, where C_T is a positive constant independent of m and t.

On the other hand, we put $w_{m,l} = u_m - u_l$, from (2.37), it follows that

$$\begin{cases} \left\langle w_{m,l}''(t), v\right\rangle + \langle w_{m,lx}(t), v_x\rangle + \langle w_{m,l}(t), v\rangle + \lambda \left\langle w_{m,l}'(t), v\right\rangle \\ \qquad + \left(\lambda_0 w_{m,l}'(0,t) + \widetilde{h}_1(t) w_{m,l}(1,t) + \widetilde{\lambda}_1 w_{m,l}'(1,t)\right) v(0) \\ \qquad + \left(\lambda_1 w_{m,l}'(1,t) + \widetilde{h}_0(t) w_{m,l}(0,t) + \widetilde{\lambda}_0 w_{m,l}'(0,t)\right) v(1) \\ \quad = \left\langle |u_m|^{p-2} u_m - |u_l|^{p-2} u_l, v\right\rangle + \left[|u_m(0,t)|^{\alpha-2} u_m(0,t) - |u_l(0,t)|^{\alpha-2} u_l(0,t)\right] v(0) \\ \qquad + \left[|u_m(1,t)|^{\beta-2} u_m(1,t) - |u_l(1,t)|^{\beta-2} u_l(1,t)\right] v(1), \text{ for all } v \in H^1, \\ w_{m,l}(0) = u_{0m} - u_{0l}, \ w_{m,l}'(0) = u_{1m} - u_{1l}. \end{cases} \tag{2.40}$$

We take $v = w'_{m,l} = u'_m - u'_l$, in (2.40) and integrating with respect to t, we obtain

$$\begin{aligned} S_{m,l}(t) \leq S_{m,l}(0) &- 2\int_0^t \widetilde{h}_1(s) w_{m,l}(1,s) w'_{m,l}(0,s) ds - 2\int_0^t \widetilde{h}_0(s) w_{m,l}(0,s) w'_{m,l}(1,s) ds \\ &+2\int_0^t \left[|u_m(0,s)|^{\alpha-2} u_m(0,s) - |u_l(0,s)|^{\alpha-2} u_l(0,s) \right] w'_{m,l}(0,s) ds \\ &+2\int_0^t \left[|u_m(1,s)|^{\beta-2} u_m(1,s) - |u_l(1,s)|^{\beta-2} u_l(1,s) \right] w'_{m,l}(1,s) ds \\ &+2\int_0^t \left\langle |u_m(s)|^{p-2} u_m(s) - |u_l(s)|^{p-2} u_l(s), w'_{m,l}(s) \right\rangle ds, \end{aligned} \tag{2.41}$$

where

$$S_{m,l}(t) = \left\| w'_{m,l}(t) \right\|^2 + \left\| w_{m,l}(t) \right\|_1^2 + 2\lambda \int_0^t \left\| w'_{m,l}(s) \right\|^2 ds + \mu_* \int_0^t \left[\left| w'_{m,l}(0,s) \right|^2 + \left| w'_{m,l}(1,s) \right|^2 \right] ds, \tag{2.42}$$

$$S_{m,l}(0) = \left\| u_{1m} - u_{1l} \right\|^2 + \left\| u_{0m} - u_{0l} \right\|_1^2. \tag{2.43}$$

Hence

$$S_{m,l}(t) \leq 2\left(\left\| u_{1m} - u_{1l} \right\|^2 + \left\| u_{0m} - u_{0l} \right\|_1^2 \right) + \widetilde{K}_T \int_0^t S_{m,l}(s) ds, \tag{2.44}$$

where

$$\widetilde{K}_T = 2(p-1) M_T^{p-2} + \tfrac{32}{\mu_*} \left[\left\| \widetilde{h}_1 \right\|_{L^\infty(0,T)}^2 + \left\| \widetilde{h}_0 \right\|_{L^\infty(0,T)}^2 + (\alpha-1)^2 M_T^{2\alpha-4} + (\beta-1)^2 M_T^{2\beta-4} \right], \tag{2.45}$$

with $M_T = \sqrt{2C_T}$.

By Gronwall's lemma, it follows from (2.44), that

$$S_{m,l}(t) \leq 2\left(\left\| u_{1m} - u_{1l} \right\|^2 + \left\| u_{0m} - u_{0l} \right\|_1^2 \right) \exp(T\widetilde{K}_T), \ \ \forall t \in [0, T_*] \tag{2.46}$$

Convergences of the sequences $\{u_{0m}\}$, $\{u_{1m}\}$ imply the convergence to zero (when m, $l \to \infty$) of terms on the right hand side of (2.46). Therefore, we get

$$\begin{cases} u_m \to u & \text{strongly in} \quad C^0([0,T_*]; H^1) \cap C^1([0,T_*]; L^2), \\ u_m(i,\cdot) \to u(i,\cdot) & \text{strongly in} \quad H^1(0,T_*), \ i = 0, \ 1. \end{cases} \tag{2.47}$$

On the other hand, from (2.39), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,T_*; H^1) & \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^\infty(0,T_*; L^2) & \text{weakly*}, \\ u_m(0,\cdot) \to u(0,\cdot) & \text{in} \quad H^1(0,T_*) & \text{weakly}, \\ u_m(1,\cdot) \to u(1,\cdot) & \text{in} \quad H^1(0,T_*) & \text{weakly}, \\ u'_m(0,\cdot) \to u'(0,\cdot) & \text{in} \quad L^2(0,T_*) & \text{weakly}, \\ u'_m(1,\cdot) \to u'(1,\cdot) & \text{in} \quad L^2(0,T_*) & \text{weakly}. \end{cases} \tag{2.48}$$

By the compactness lemma of Lions ([**?**], p. 57) and the compact imbedding $H^1(0,T_*) \hookrightarrow C^0([0,T_*])$, we can deduce from $(2.48)_{1-4}$ the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u & \text{strongly in} \quad L^2(Q_{T*}) \text{ and a.e. in } Q_{T_*}, \\ u_m(i,\cdot) \to u(i,\cdot) & \text{strongly in} \quad C^0([0,T_*]), \ i = 0, \ 1. \end{cases} \tag{2.49}$$

Similarly, by (2.25), we deduce from $(2.49)_1$, that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly in} \quad L^2(Q_{T_*}). \tag{2.50}$$

Passing to the limit in (2.37) by (2.47) – (2.50), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\langle u'(t), v\rangle + \langle u_x(t), v_x\rangle + \langle u(t), v\rangle + \lambda \langle u'(t), v\rangle \\ \qquad + \left(\lambda_0 u'(0,t) + \widetilde{h}_1(t)u(1,t) + \widetilde{\lambda}_1 u'(1,t)\right) v(0) \\ \qquad + \left(\lambda_1 u'(1,t) + \widetilde{h}_0(t)u(0,t) + \widetilde{\lambda}_0 u'(0,t)\right) v(1) \\ \quad = \left\langle |u|^{p-2}u, v\right\rangle + |u(0,t)|^{\alpha-2}\, u(0,t)v(0) + |u(1,t)|^{\beta-2}\, u(1,t)v(1), \text{ for all } v \in H^1, \\ u(0) = u_0, \;\; u'(0) = u_1. \end{cases} \tag{2.51}$$

Next, the uniqueness of a weak solution is obtained by using the well-known regularization procedure due to Lions. See for example Ngoc et al. [**?**].

Theorem 2.4 is proved completely.■

Remark 2.2. In case $1 < p,\ \alpha,\ \beta \le 2$, and $\widetilde{h}_0,\ \widetilde{h}_1 \in L^\infty(0,T)$, $(u_0, u_1) \in H^1 \times L^2$, the integral inequality (**??**) leads to the following global estimation

$$S_m(t) \le C_T,\ \forall m \in \mathbb{N},\ \forall t \in [0,T],\ \forall T > 0. \tag{2.52}$$

Then, by applying a similar argument used in the proof of Theorem 2.4, we can obtain a global weak solution u of problem (1.1) – (1.4) satisfying

$$u \in L^\infty\left(0,T;H^1\right),\ u_t \in L^\infty\left(0,T;L^2\right),\ \ u(i,\cdot) \in H^1(0,T),\ i = 0,1. \tag{2.53}$$

However, in case $1 < p,\ \alpha,\ \beta < 2$, we do not imply that a weak solution obtained here belongs to $C\left([0,T];H^1\right) \cap C^1\left([0,T];L^2\right)$. Furthermore, the uniqueness of a weak solution is also not asserted.

3 Finite time blow up

In this section we show that the solution of problem (1.1) – (1.4) blows up in finite time if $\widetilde{\lambda}_0 = \widetilde{\lambda}_1 = \widetilde{\lambda}$, with $\left|\widetilde{\lambda}\right| < \sqrt{\lambda_0\lambda_1}$, and

$$-H(0) = \tfrac{1}{2}\|u_1\|^2 + \tfrac{1}{2}\|u_0\|_1^2 - \tfrac{1}{p}\|u_0\|_{L^p}^p - \tfrac{1}{\alpha}|u_0(0)|^\alpha - \tfrac{1}{\beta}|u_0(1)|^\beta + \widetilde{h}u_0(0)u_0(1) < 0. \tag{3.1}$$

First, in order to obtain the blow up result, we make the following assumptions

(A_2') $\widetilde{\lambda}_0 = \widetilde{\lambda}_1 = \widetilde{\lambda}$, with $\left|\widetilde{\lambda}\right| < \sqrt{\lambda_0\lambda_1}$.

(A_3') $\widetilde{h}_0(t) = \widetilde{h}_1(t) = \widetilde{h}$, where $\widetilde{h}$ is a constant satisfies $\left|\widetilde{h}\right| < \frac{q-2}{4(q+2)}$, $q = \min\{p, \alpha, \beta\}$;

Then we obtain the theorem.

Theorem 3.1. *Let the assumptions* (A_1), (A_2'), (A_3') *hold and* $H(0) > 0$. *Then, for any* $(u_0, u_1) \in H^1 \times L^2$, *the solution* u *of problem* (1.1) – (1.4) *blows up in finite time.*

Proof. We denote by $E(t)$ the energy associated to the solution u, defined by

$$E(t) = \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\|u(t)\|_1^2 - \frac{1}{p}\|u(t)\|_{L^p}^p - \frac{1}{\alpha}|u(0,t)|^\alpha - \frac{1}{\beta}|u(1,t)|^\beta, \tag{3.2}$$

and we put

$$H(t) = -E(t) - \widetilde{h}u(0,t)u(1,t). \tag{3.3}$$

From Lemma 2.1, it is easy to see that

$$H(t) \geq \frac{1}{p}\|u(t)\|_{L^p}^p + \frac{1}{\alpha}|u(0,t)|^\alpha + \frac{1}{\beta}|u(1,t)|^\beta - \frac{1}{2}\left\|u'(t)\right\|^2 - \left(\frac{1}{2} + 2\left|\widetilde{h}\right|\right)\|u(t)\|_1^2. \tag{3.4}$$

On the other hand, by multiplying (1.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$H'(t) = \lambda\left\|u'(t)\right\|^2 + \left\{\lambda_0\left|u'(0,t)\right|^2 + \lambda_1\left|u'(1,t)\right|^2 + 2\widetilde{\lambda}u'(0,t)u'(1,t)\right\} \geq 0,\ \forall t \in [0,T_*). \tag{3.5}$$

By Lemma 2.2, we have

$$\lambda_0\left|u'(0,t)\right|^2 + \lambda_1\left|u'(1,t)\right|^2 + 2\widetilde{\lambda}u'(0,t)u'(1,t) \geq \frac{1}{2}\mu_*\left(\left|u'(0,t)\right|^2 + \left|u'(1,t)\right|^2\right),\ \forall t \in [0,T_*), \tag{3.6}$$

where

$$\mu_* = \left(\lambda_0\lambda_1 - \widetilde{\lambda}^2\right)\min\left\{\frac{1}{\lambda_0}, \frac{1}{\lambda_1}\right\} > 0. \tag{3.7}$$

Hence, we can deduce from (3.5), (3.6) and $H(0) > 0$ that

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\|u(t)\|_{L^p}^p + \frac{1}{\alpha}|u(0,t)|^\alpha + \frac{1}{\beta}|u(1,t)|^\beta,\ \forall t \in [0,T_*). \tag{3.8}$$

Now, we define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon\Phi(t), \tag{3.9}$$

where

$$\Phi(t) = \langle u(t), u'(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_0}{2}|u(0,t)|^2 + \frac{\lambda_1}{2}|u(1,t)|^2 + \widetilde{\lambda}u(0,t)u(1,t), \tag{3.10}$$

for ε small enough and

$$0 < \eta \leq \frac{p-2}{2p} < \frac{1}{2}. \tag{3.11}$$

Lemma 3.2. *There exists a constant $d_1 > 0$ such that*

$$L'(t) \geq d_1\left(H(t) + \left\|u'(t)\right\|^2 + \|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta\right). \tag{3.12}$$

Proof of Lemma 3.2. By multiplying (1.1) by $u(x,t)$ and integrating over $[0,1]$, we get

$$\Phi'(t) = \|u'(t)\|^2 + \|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta - \|u(t)\|_1^2 - 2\widetilde{h}u(0,t)u(1,t). \tag{3.13}$$

By taking a derivative of (3.9) and using (3.13), we obtain

$$\begin{aligned} L'(t) = {} & (1-\eta)H^{-\eta}(t)H'(t) + \varepsilon\|u'(t)\|^2 + \varepsilon\|u(t)\|_{L^p}^p + \varepsilon\left(|u(0,t)|^\alpha + |u(1,t)|^\beta\right) \\ & -\varepsilon\|u(t)\|_1^2 - 2\varepsilon\widetilde{h}u(0,t)u(1,t). \end{aligned} \tag{3.14}$$

Since (3.5), (3.14) and the following inequality

$$-2\widetilde{h}u(0,t)u(1,t) \geq -4\left|\widetilde{h}\right|\|u(t)\|_1^2, \tag{3.15}$$

we deduce that

$$\begin{aligned} L'(t) \geq {} & \left[\lambda(1-\eta)H^{-\eta}(t) + \varepsilon\right]\|u'(t)\|^2 + \varepsilon\left(\|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta\right) \\ & -\varepsilon\left(1 + 4\left|\widetilde{h}\right|\right)\|u(t)\|_1^2. \end{aligned} \tag{3.16}$$

On the other hand, it follows from (3.8) and the following inequality

$$H(t) \leq \frac{1}{p}\|u(t)\|_{L^p}^p + \frac{1}{\alpha}|u(0,t)|^\alpha + \frac{1}{\beta}|u(1,t)|^\beta - \frac{1}{2}\left\|u'(t)\right\|^2 - \left(\frac{1}{2} - 2\left|\widetilde{h}\right|\right)\|u(t)\|_1^2, \tag{3.17}$$

that

$$\|u(t)\|_1^2 \leq \frac{2}{q}\frac{1}{1-4\left|\widetilde{h}\right|}\left(\|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta\right), \tag{3.18}$$

where $q = \min\{p, \alpha, \beta\}$.

Combining (3.16) and (3.18), we have

$$L'(t) \geq \varepsilon\|u'(t)\|^2 + \varepsilon\left(1 - \frac{2}{q}\frac{1+4|\widetilde{h}|}{1-4|\widetilde{h}|}\right)\left(\|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta\right). \tag{3.19}$$

Using the inequality

$$\|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta \geq qH(t),\ t \geq 0, \tag{3.20}$$

we can deduce from (3.19) that, with ε is small enough,

$$L'(t) \geq d_1\left(H(t) + \|u'(t)\|^2 + \|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta\right), \tag{3.21}$$

for d_1 is a positive constant. The lemma 3.2 is proved completely.■

Remark 3.1. From the formula of $L(t)$ and the Lemma 3.2, we can choose ε small enough such that

$$L(t) \geq L(0) > 0,\ \forall t \in [0, T_*). \tag{3.22}$$

Now we continue to prove Theorem 3.1**.**

Using the inequality

$$\left(\sum_{i=1}^6 x_i\right)^p \leq 6^{p-1}\sum_{i=1}^6 x_i^p, \text{ for all } p > 1, \text{ and } x_1, ..., x_6 \geq 0, \tag{3.23}$$

we deduce from (3.9), (3.10) that

$$\begin{aligned} L^{1/(1-\eta)}(t) \leq Const\Big(&H(t) + |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} + \|u(t)\|^{2/(1-\eta)} \\ &+|u(0,t)|^{2/(1-\eta)} + |u(1,t)|^{2/(1-\eta)} + |u(0,t)u(1,t)|^{1/(1-\eta)}\Big) \\ \leq Const\Big(&H(t) + |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} + |u(0,t)|^{2/(1-\eta)} + |u(1,t)|^{2/(1-\eta)} + \|u(t)\|_{L^p}^{2/(1-\eta)}\Big). \end{aligned} \tag{3.24}$$

On the other hand, by using the Young's inequality

$$\begin{aligned} |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} &\leq \|u(t)\|^{1/(1-\eta)}\|u'(t)\|^{1/(1-\eta)} \\ &\leq Const\ \|u(t)\|_{L^p}^{1/(1-\eta)}\|u'(t)\|^{1/(1-\eta)} \\ &\leq Const\ \left(\|u(t)\|_{L^p}^s + \|u'(t)\|^2\right), \end{aligned} \tag{3.25}$$

where $s = 2/(1-2\eta) \leq p$ by (3.11).

Now, we need the following lemma.

Lemma 3.3. *Let $2 \leq r_1 \leq p$, $2 \leq r_2 \leq \alpha$, $2 \leq r_3 \leq \beta$, we have*

$$\|v\|_{L^p}^{r_1} + |v(0)|^{r_2} + |v(1)|^{r_3} \leq 5\left(\|v\|_1^2 + \|v\|_{L^p}^p + |v(0)|^\alpha + |v(1)|^\beta\right), \tag{3.26}$$

for any $v \in H^1$.

Proof of Lemma 3.3.

(i) We consider two cases for $\|v\|_{L^p}$:
(i.1) Case 1: $\|v\|_{L^p} \leq 1$:
By $2 \leq r_1 \leq p$, we have

$$\|v\|_{L^p}^{r_1} \leq \|v\|_{L^p}^2 \leq \|v\|_1^2 \leq \|v\|_1^2 + \|v\|_{L^p}^p + |v(0)|^\alpha + |v(1)|^\beta \equiv \rho[v]. \tag{3.27}$$

(i.2) Case 2: $\|v\|_{L^p} \geq 1$: By $2 \leq r_1 \leq p$, we have

$$\|v\|_{L^p}^{r_1} \leq \|v\|_{L^p}^p \leq \rho[v]. \tag{3.28}$$

Therefore

$$\|v\|_{L^p}^{r_1} \leq \|v\|_{L^p}^p \leq \rho[v], \text{ for any } v \in H^1. \tag{3.29}$$

(ii) We consider two cases for $|v(0)|$:
(ii.1) Case 1: $|v(0)| \leq 1$:
By $2 \leq r_2 \leq \alpha$, we have

$$|v(0)|^{r_1} \leq |v(0)|^2 \leq \|v\|_{C^0([0,1])}^2 \leq 2\|v\|_1^2 \leq 2\rho[v]. \tag{3.30}$$

(ii.2) Case 2: $|v(0)| \geq 1$: By $2 \leq r_2 \leq \alpha$, we have

$$|v(0)|^{r_1} \leq |v(0)|^\alpha \leq \rho[v]. \tag{3.31}$$

Therefore

$$|v(0)|^{r_1} \leq 2\rho[v], \text{ for any } v \in H^1. \tag{3.32}$$

(iii) Similarly

$$|v(1)|^{r_2} \leq 2\rho[v], \text{ for any } v \in H^1. \tag{3.33}$$

Combining (3.29), (3.32), (3.33), we obtain

$$\|v\|_{L^p}^{r_1} + |v(0)|^{r_2} + |v(1)|^{r_3} \leq 5\rho[v] \leq 5\left(\|v\|_1^2 + \|v\|_{L^p}^p + |v(0)|^\alpha + |v(1)|^\beta\right), \forall v \in H^1. \tag{3.34}$$

Lemma 3.3 is proved completely.■

Combining (3.18), (3.24) – (3.26) and using the Lemma 3.2 we obtain

$$L^{1/(1-\eta)}(t) \leq Const\left(H(t) + \|u'(t)\|^2 + \|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta\right), \forall t \in [0, T_*). \tag{3.35}$$

This implies that

$$L'(t) \geq d_2 L^{1/(1-\eta)}(t), \ \forall t \in [0, T_*), \tag{3.36}$$

where d_2 is a positive constant. By integrating (3.36) over $(0, t)$ we deduce that

$$L^{\eta/(1-\eta)}(t) \geq \frac{1}{L^{-\eta/(1-\eta)}(0) - \frac{d_2\eta}{1-\eta}t}, \ 0 \leq t < \frac{1-\eta}{d_2\eta}L^{-\eta/(1-\eta)}(0). \tag{3.37}$$

Therefore, (3.37) shows that $L(t)$ blows up in a finite time given by

$$T_* = \frac{1-\eta}{d_2\eta}L^{-\eta/(1-\eta)}(0). \tag{3.38}$$

Theorem 3.1 is proved completely.■

4 Exponential decay

In this section we show that each solution u of (1.1) – (1.4) is global and exponential decay provided that $I(0) = \|u_0\|_1^2 - \|u_0\|_{L^p}^p - |u_0(0)|^\alpha - |u_0(1)|^\beta > 0$ and $E(0)$ is small enough.

First, we construct the following Lyapunov functional

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{4.1}$$

where $\delta > 0$ is chosen later and

$$\psi(t) = \langle u(t), u'(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 + \frac{\lambda_0}{2}|u(0,t)|^2 + \frac{\lambda_1}{2}|u(1,t)|^2. \tag{4.2}$$

Put

$$I(t) = I(u(t)) = \|u(t)\|_1^2 - \|u(t)\|_{L^p}^p - |u(0,t)|^\alpha - |u(1,t)|^\beta. \tag{4.3}$$

We make the following assumption

(A_3'') $\widetilde{h}_i \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+),\ i = 1, 2.$

Then we have the following theorem.

Theorem 4.1. *Assume that* (A_1), (A_2), (A_3'') *hold. Let* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfies*

$$\eta^* = C_p^p\left(\tfrac{2qr}{q-2}E(0)\right)^{(p-2)/2} + 2^{\alpha/2}\left(\tfrac{2qr}{q-2}E(0)\right)^{(\alpha-2)/2} + 2^{\beta/2}\left(\tfrac{2qr}{q-2}E(0)\right)^{(\beta-2)/2} < 1, \tag{4.4}$$

with $q = \min\{p, \alpha, \beta\}$, *and*

$$r = \exp\left[\frac{4q}{\mu_*(q-2)}\left(\left\|\widetilde{h}_0\right\|_{L^2(\mathbb{R}_+)}^2 + \left\|\widetilde{h}_1\right\|_{L^2(\mathbb{R}_+)}^2\right)\right], \tag{4.5}$$

and C_p *is a constant verifying the inequality* $\|v\|_{L^p} \le C_p\|v\|_1$, *for all* $v \in H^1$.

Then, there exist positive constants C, γ *such that, for* $\left\|\widetilde{h}_0\right\|_{L^2(\mathbb{R}_+)}$, $\left\|\widetilde{h}_1\right\|_{L^2(\mathbb{R}_+)}$ *sufficiently small, we have*

$$E(t) \le C\exp(-\gamma t), \text{ for all } t \ge 0. \tag{4.6}$$

Proof.

First, we need the following lemmas

Lemma 4.2. *The energy functional* $E(t)$ *satisfies*

$$E'(t) \le -\lambda\|u'(t)\|^2 + \tfrac{1}{2\mu_*}\left(\widetilde{h}_0^2(t) + \widetilde{h}_1^2(t)\right)\|u(t)\|_1^2 - \tfrac{1}{4}\mu_*\left[|u'(0,t)|^2 + |u'(1,t)|^2\right]. \tag{4.7}$$

Proof of Lemma 4.2. Multiplying (1.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned} E'(t) = &-\lambda\|u'(t)\|^2 - \left\{\lambda_0|u'(0,t)|^2 + \lambda_1|u'(1,t)|^2 + \left(\widetilde{\lambda}_0 + \widetilde{\lambda}_1\right)u'(0,t)\,u'(1,t)\right\} \\ &-\widetilde{h}_0(t)u(0,t)\,u'(1,t) - \widetilde{h}_1(t)u(1,t)\,u'(0,t). \end{aligned} \tag{4.8}$$

Again, by lemma 2.2, we have

$$\begin{aligned} &\lambda_0|u'(0,t)|^2 + \lambda_1|u'(1,t)|^2 + \left(\widetilde{\lambda}_0 + \widetilde{\lambda}_1\right)u'(0,t)\,u'(1,t) \\ &\qquad \ge \tfrac{1}{2}\mu_*\left[|u'(0,t)|^2 + |u'(1,t)|^2\right]. \end{aligned} \tag{4.9}$$

On the other hand

$$-\widetilde{h}_0(t)u\left(0,t\right)u'\left(1,t\right)\leq\tfrac{1}{4}\mu_*\left|u'\left(1,t\right)\right|^2+\tfrac{2}{\mu_*}\widetilde{h}_0^2(t)\left\|u\left(t\right)\right\|_1^2, \tag{4.10}$$

$$-\widetilde{h}_1(t)u\left(1,t\right)u'\left(0,t\right)\leq\tfrac{1}{4}\mu_*\left|u'\left(0,t\right)\right|^2+\tfrac{2}{\mu_*}\widetilde{h}_1^2(t)\left\|u\left(t\right)\right\|_1^2. \tag{4.11}$$

Combining (4.8) - (4.11), it is easy to see (4.7) holds.

Lemma 4.2 is proved completely.■

Lemma 4.3. *Suppose that* (A_1), (A_2), (A_3'') *hold. Then, if we have* $I(0)>0$ *and*

$$\eta^*=C_p^p\left(\frac{2qr}{q-2}E(0)\right)^{(p-2)/2}+2^{\alpha/2}\left(\frac{2qr}{q-2}E(0)\right)^{(\alpha-2)/2}+2^{\beta/2}\left(\frac{2qr}{q-2}E(0)\right)^{(\beta-2)/2}<1, \tag{4.12}$$

then $I(t)>0$, $\forall t\geq 0$.

Proof of Lemma 4.3. By the continuity of $I(t)$ and $I(0)>0$, there exists $T_1>0$ such that

$$I(u(t))\geq 0,\ \forall t\in[0,T_1], \tag{4.13}$$

this implies

$$J(t)\geq\tfrac{q-2}{2q}\left\|u(t)\right\|_1^2+\tfrac{1}{q}I(t)\geq\tfrac{q-2}{2q}\left\|u(t)\right\|_1^2,\ \forall t\in[0,T_1], \tag{4.14}$$

where

$$J(t)=\frac{1}{2}\left\|u(t)\right\|_1^2-\frac{1}{p}\|u(t)\|_{L^p}^p-\frac{1}{\alpha}|u(0,t)|^\alpha-\frac{1}{\beta}|u(1,t)|^\beta. \tag{4.15}$$

It follows from (4.14), (4.15) that

$$\|u(t)\|_1^2\leq\frac{2q}{q-2}J(t)\leq\frac{2q}{q-2}E(t),\ \forall t\in[0,T_1]. \tag{4.16}$$

Combining (4.7), (4.16) and using the Gronwal's inequality we have

$$\|u(t)\|_1^2\leq\frac{2q}{q-2}E(t)\leq\frac{2qr}{q-2}E(0),\ \forall t\in[0,T_1], \tag{4.17}$$

where r as in (4.5).

Hence, it follows from (4.12), (4.17) that

$$\begin{aligned}\|u(t)\|_{L^p}^p+|u(0,t)|^\alpha+|u(1,t)|^\beta&\leq C_p^p\|u(t)\|_1^p+2^{\alpha/2}\|u(t)\|_1^\alpha+2^{\beta/2}\|u(t)\|_1^\beta\\&\leq\eta^*\|u(t)\|_1^2<\|u(t)\|_1^2,\ \forall t\in[0,T_1].\end{aligned} \tag{4.18}$$

Therefore $I(t)>0$, $\forall t\in[0,T_1]$.

Now, we put $T_*=\sup\left\{T>0:I(u(t))>0,\forall t\in[0,T]\right\}$. If $T_*<+\infty$ then, by the continuity of $I(t)$, we have $I(T_*)\geq 0$. By the same arguments as in above part we can deduce that there exists $T_2>T_*$ such that $I(t)>0$, $\forall t\in[0,T_2]$. Hence, we conclude that $I(t)>0$, $\forall t\geq 0$.

Lemma 4.3 is proved completely.■

Lemma 4.4. *Let* $I(0)>0$ *and* (4.13) *hold. Then there exist the positive constants* β_1, β_2 *such that*

$$\beta_1E(t)\leq\mathcal{L}(t)\leq\beta_2E(t),\ \forall t\geq 0, \tag{4.19}$$

for δ *is small enough.*

Proof of Lemma 4.4. It is easy to see that

$$\mathcal{L}(t)\leq\frac{1+\delta}{2}\left\|u'\left(t\right)\right\|^2+\left[\frac{1}{2}+\delta\left(\frac{1+\lambda}{2}+\lambda_0+\lambda_1\right)\right]\left\|u\left(t\right)\right\|_1^2\leq\beta_2E(t), \tag{4.20}$$

where

$$\beta_2 = 1 + \delta + \frac{2q}{q-2}\left[\frac{1}{2} + \delta\left(\frac{1+\lambda}{2} + \lambda_0 + \lambda_1\right)\right]. \tag{4.21}$$

Similarly, we can prove that

$$\begin{aligned}\mathcal{L}(t) &\geq \tfrac{1-\delta}{2}\left\|u'(t)\right\|^2 + \tfrac{1}{2}(1-\delta)\left\|u(t)\right\|_1^2 - \tfrac{1}{p}\|u(t)\|_{L^p}^p - \tfrac{1}{\alpha}|u(0,t)|^\alpha - \tfrac{1}{\beta}|u(1,t)|^\beta \\ &\geq \tfrac{1-\delta}{2}\left\|u'(t)\right\|^2 + \tfrac{1}{2}\left(\tfrac{q-2}{q} - \delta\right)\left\|u(t)\right\|_1^2 + \tfrac{1}{q}I(t) \geq \beta_1 E(t),\end{aligned} \tag{4.22}$$

where

$$\beta_1 = \min\left\{1-\delta;\ \frac{q-2}{q} - \delta\right\} > 0, \quad \delta \text{ is small enough.} \tag{4.23}$$

Lemma 4.4 is proved completely.■

Lemma 4.5. *Let $I(0) > 0$ and (4.12) hold. The functional $\psi(t)$ defined by (4.2) satisfies*

$$\begin{aligned}\psi'(t) \leq \left\|u'(t)\right\|^2 &- \left[1 - \eta^* - \varepsilon_1 - 2\left|\widetilde{h}_0(t) + \widetilde{h}_1(t)\right|\right]\left\|u(t)\right\|_1^2 \\ &+ \tfrac{1}{\varepsilon_1}\left(\widetilde{\lambda}_0^2 + \widetilde{\lambda}_1^2\right)\left(|u'(0,t)|^2 + |u'(1,t)|^2\right).\end{aligned} \tag{4.24}$$

for all $\varepsilon_1 > 0$.

Proof of Lemma 4.5. By multiplying (1.1) by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) = \left\|u'(t)\right\|^2 &+ \|u(t)\|_{L^p}^p + |u(0,t)|^\alpha + |u(1,t)|^\beta - \left\|u(t)\right\|_1^2 \\ &- \left(\widetilde{h}_0(t) + \widetilde{h}_1(t)\right)u(0,t)\,u(1,t) - \widetilde{\lambda}_0 u'(0,t)\,u(1,t) - \widetilde{\lambda}_1 u(0,t)\,u'(1,t).\end{aligned} \tag{4.25}$$

Hence, the lemma 4.5 is proved by using some simple estimates.■

Now we continue to prove Theorem 4.1**.**

It follows from (4.1), (4.2), (4.7) and (4.24), that

$$\begin{aligned}\mathcal{L}'(t) \leq &-(\lambda - \delta)\left\|u'(t)\right\|^2 \\ &+ \left[\tfrac{2}{\mu_*}\left(\widetilde{h}_0^2(t) + \widetilde{h}_1^2(t)\right) + 2\delta\left|\widetilde{h}_0(t) + \widetilde{h}_1(t)\right| - \delta(1 - \eta^* - \varepsilon_1)\right]\left\|u(t)\right\|_1^2 \\ &- \left[\tfrac{1}{4}\mu_* - \tfrac{\delta}{\varepsilon_1}\left(\widetilde{\lambda}_0^2 + \widetilde{\lambda}_1^2\right)\right]\left[|u'(0,t)|^2 + |u'(1,t)|^2\right]\end{aligned} \tag{4.26}$$

for all δ, $\varepsilon_1 > 0$.

Let

$$0 < \varepsilon_1 < 1 - \eta^*. \tag{4.27}$$

Then, for δ small enough, with $0 < \delta < \lambda$ and if $\widetilde{h}_0$, $\widetilde{h}_1$ satisfy

$$\frac{2}{\mu_*}\left(\left\|\widetilde{h}_0\right\|_{L^\infty(\mathbb{R}_+)}^2 + \left\|\widetilde{h}_1\right\|_{L^\infty(\mathbb{R}_+)}^2\right) + 2\delta\left(\left\|\widetilde{h}_0\right\|_{L^\infty(\mathbb{R}_+)} + \left\|\widetilde{h}_1\right\|_{L^\infty(\mathbb{R}_+)}\right) < \delta(1 - \eta^* - \varepsilon_1), \tag{4.28}$$

we deduce from (4.19) and (4.26) that there exists a constant $\gamma > 0$ such that

$$\mathcal{L}'(t) \leq -\gamma\mathcal{L}(t), \quad \forall t \geq 0. \tag{4.29}$$

Combining (4.19) and (4.29), we get (4.6). Theorem 4.1 is proved completely.■

5 Numerical results

Consider the following problem:

$$u_{tt} - u_{xx} + u + \lambda u_t = |u|^{p-2} u + f(x,t), \tag{5.1}$$

$0 < x < 1$, $t > 0$, with boundary conditions

$$\begin{cases} u_x(0,t) + |u(0,t)|^{\alpha-2} u(0,t) = \lambda_0 u_t(0,t) + \widetilde{h}_1(t) u(1,t) + \widetilde{\lambda}_1 u_t(1,t) + g_0(t), \\ -u_x(1,t) + |u(1,t)|^{\beta-2} u(1,t) = \lambda_1 u_t(1,t) + \widetilde{h}_0(t) u(0,t) + \widetilde{\lambda}_0 u_t(0,t) + g_1(t), \end{cases} \tag{5.2}$$

and initial conditions

$$u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x), \tag{5.3}$$

where $\lambda = \lambda_0 = \lambda_1 = 1$, $\widetilde{\lambda}_0 = \widetilde{\lambda}_1 = \frac{-1}{2}$, $p = 3$, $\alpha = \beta = 4$ are constants and the functions $\widetilde{u}_0$, $\widetilde{u}_1$, $\widetilde{h}_0$, $\widetilde{h}_1$, g_0, g_1 and f are defined by

$$\begin{cases} u_0(x) = e^x, \ \widetilde{u}_1(x) = -e^x, \\ \widetilde{h}_0(t) = e^{3-2t}, \ \widetilde{h}_1(t) = -e^{-1-2t}, \\ g_0(t) = (2 - \frac{e}{2})e^{-t} + 2e^{-3t}, \ g_1(t) = -\frac{1}{2}e^{-t}, \\ f(x,t) = -e^{2x-2t}. \end{cases} \tag{5.4}$$

The exact solution of the problem (5.1) – (5.3) with $\widetilde{u}_0$, $\widetilde{u}_1$, $\widetilde{h}_0$, $\widetilde{h}_1$, g_0, g_1 and f defined in (5.4) respectively, is the function U_{ex} given by

$$U_{ex}(x,t) = e^{x-t}. \tag{5.5}$$

To solve problem (5.1) – (5.3) numerically, we consider the differential system for the unknowns $U_j(t) \equiv u(x_j,t)$, $V_j(t) = \frac{dU_j}{dt}(t)$, with $x_j = j\Delta x$, $\Delta x = \frac{1}{N}$, $j = 0, 1, ..., N$:

$$\begin{cases} \frac{dU_j}{dt}(t) = V_j(t), \ , j = 0, 1, ..., N, \\ \frac{dV_0}{dt}(t) = -\left(1 + N^2\right) U_0(t) + N^2 U_1(t) - N\widetilde{h}_1(t) U_N(t) \\ \qquad -(\lambda + N\lambda_0) V_0(t) - N\widetilde{\lambda}_1 V_N(t) + |U_0|^{p-2} U_0 + N |U_0|^{\alpha-2} U_0 - N g_0(t) + f_0(t), \\ \frac{dV_j}{dt}(t) = N^2 U_{j-1}(t) - \left(1 + 2N^2\right) U_j(t) + N^2 U_j(t) - \lambda V_j(t) \\ \qquad + \left|U_j\right|^{p-2} U_j + f_j(t), \ j = \overline{1, N-1}, \\ \frac{dV_N}{dt}(t) = -N\widetilde{h}_0(t) U_0(t) + N^2 U_{N-1}(t) - \left(1 + N^2\right) U_N(t) \\ \qquad -N\widetilde{\lambda}_0 V_0(t) - (\lambda + N\lambda_1) V_N(t) + |U_N|^{p-2} U_N + N |U_N|^{\beta-2} U_N - N g_1(t) + f_N(t), \\ U_j(0) = \widetilde{u}_0(x_j), \ V_j(0) = \widetilde{u}_1(x_j), \ j = \overline{0, N}. \end{cases} \tag{5.6}$$

To solve the nonlinear differential system (5.6), we use the following linear recursive scheme gen-

erated by the nonlinear terms

$$\begin{cases} \frac{dU_j^{(m)}}{dt}(t) = V_j^{(m)}(t),\ j = 0, 1, ..., N, \\ \frac{dV_0^{(m)}}{dt}(t) = -\left(1+N^2\right) U_0^{(m)}(t) + N^2 U_1^{(m)}(t) - N\widetilde{h}_1(t) U_N^{(m)}(t) \\ \qquad\qquad -(\lambda + N\lambda_0) V_0^{(m)}(t) - N\widetilde{\lambda}_1 V_N^{(m)}(t) \\ \qquad\qquad + \left|U_0^{(m-1)}\right|^{p-2} U_0^{(m-1)} + N\left|U_0^{(m-1)}\right|^{\alpha-2} U_0^{(m-1)} - Ng_0(t) + f_0(t), \\ \frac{dV_j^{(m)}}{dt}(t) = N^2 U_{j-1}^{(m)}(t) - \left(1+2N^2\right) U_j^{(m)}(t) + N^2 U_j^{(m)}(t) - \lambda V_j^{(m)}(t) \\ \qquad\qquad\qquad\qquad + \left|U_j^{(m-1)}\right|^{p-2} U_j^{(m-1)} + f_j(t),\ j = \overline{1, N-1}, \\ \frac{dV_N^{(m)}}{dt}(t) = -N\widetilde{h}_0(t) U_0^{(m)}(t) + N^2 U_{N-1}^{(m)}(t) - \left(1+N^2\right) U_N^{(m)}(t) \\ \qquad\qquad -N\widetilde{\lambda}_0 V_0^{(m)}(t) - (\lambda + N\lambda_1) V_N^{(m)}(t) \\ \qquad\qquad + \left|U_N^{(m-1)}\right|^{p-2} U_N^{(m-1)} + N\left|U_N^{(m-1)}\right|^{\beta-2} U_N^{(m-1)} - Ng_1(t) + f_N(t), \\ U_j^{(m)}(0) = \widetilde{u}_0(x_j),\ V_j^{(m)}(0) = \widetilde{u}_1(x_j),\ j = \overline{0, N},\ m = 1, 2, ... \end{cases} \tag{5.7}$$

Then system (5.7) is equivalent to:

$$\frac{d}{dt}\begin{bmatrix} U_0^{(m)} \\ U_1^{(m)} \\ \vdots \\ U_N^{(m)} \\ V_0^{(m)} \\ V_1^{(m)} \\ \vdots \\ \vdots \\ V_N^{(m)} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 0 & 0 & \cdots & \cdots & 0 & 1 & & & & \\ 0 & 0 & \cdots & \cdots & 0 & & 1 & & & \\ \vdots & \vdots & \cdots & \cdots & \vdots & & & \ddots & & \\ 0 & 0 & \cdots & \cdots & 0 & & & & & 1 \\ \hline \gamma+\alpha_1 & \alpha_1 & & & \widetilde{\gamma}_1(t) & \widehat{\delta}_0 & & & & \widetilde{\delta}_1 \\ \alpha_1 & \gamma & \alpha_1 & & & & -\lambda & & & \\ & \ddots & \ddots & \ddots & & & & \ddots & & \\ & & \alpha_1 & \gamma & \alpha_1 & & & & -\lambda & \\ \widetilde{\gamma}_0(t) & & & \alpha_1 & \gamma+\alpha_1 & \widetilde{\delta}_0 & & & & \widehat{\delta}_1 \end{array}\right] \begin{bmatrix} U_0^{(m)} \\ U_1^{(m)} \\ \vdots \\ U_N^{(m)} \\ V_0^{(m)} \\ V_1^{(m)} \\ \vdots \\ \vdots \\ V_N^{(m)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F_0^{(m)} \\ F_1^{(m)} \\ \vdots \\ \vdots \\ F_N^{(m)} \end{bmatrix}, \tag{5.8}$$

and

$$\left(U_0^{(m)}(0), U_1^{(m)}(0), ..., U_N^{(m)}(0)\right) = \left(\widetilde{u}_0(x_0), \widetilde{u}_0(x_1), , ..., \widetilde{u}_0(x_N)\right),$$
$$\left(V_0^{(m)}(0), V_1^{(m)}(0), ..., V_N^{(m)}(0)\right) = \left(\widetilde{u}_1(x_0), \widetilde{u}_1(x_1), , ..., \widetilde{u}_1(x_N)\right),$$

where

$$\begin{cases} \alpha_1 = N^2,\ \gamma = -1 - 2N^2 = -1 - 2\alpha_1,\ \widetilde{\gamma}_0(t) = -N\widetilde{h}_0(t),\ \widetilde{\gamma}_1(t) = -N\widetilde{h}_1(t), \\ \widehat{\delta}_0 = -\lambda - N\lambda_0,\ \widehat{\delta}_1 = -\lambda - N\lambda_1,\ \widetilde{\delta}_0 = -N\widetilde{\lambda}_0,\ \widetilde{\delta}_1 = -N\widetilde{\lambda}_1, \\ F_j^{(m)} = F_j(t, U_j^{(m-1)}) = \left|U_j^{(m-1)}\right|^{p-2} U_j^{(m-1)} + f_j(t),\ j = \overline{1, N-1}, \\ F_0^{(m)} = F_0(t, U_0^{(m-1)}) = \left|U_0^{(m-1)}\right|^{p-2} U_0^{(m-1)} + N\left|U_0^{(m-1)}\right|^{\alpha-2} U_0^{(m-1)} - Ng_0(t) + f_0(t), \\ F_N^{(m)} = F_N(t, U_N^{(m-1)}) = \left|U_N^{(m-1)}\right|^{p-2} U_N^{(m-1)} + N\left|U_N^{(m-1)}\right|^{\beta-2} U_N^{(m-1)} - Ng_1(t) + f_N(t), \\ f_j(t) = f(x_j, t),\ j = \overline{0, N}. \end{cases} \tag{5.9}$$

Rewritten (5.8)

$$\begin{cases} \frac{d}{dt}X^{(m)}(t) = A(t)X^{(m)}(t) + F^{(m)}(t, X^{(m-1)}), \\ X^{(m)}(0) = X_0, \end{cases} \tag{5.10}$$

where

$$\begin{cases} X^{(m)}(t) = \left(U_0^{(m)}(t), U_1^{(m)}(t), ..., U_N^{(m)}(t), V_0^{(m)}(t), V_1^{(m)}(t), ..., V_N^{(m)}(t)\right)^T \in \mathbb{R}^{2N+2}, \\ F^{(m)}(t) = \left(0, 0, ..., 0, F_0^{(m)}, F_1^{(m)}, ..., F_N^{(m)}\right)^T \in \mathbb{R}^{2N+2}, \\ X_0 = (\widetilde{u}_0(x_0), \widetilde{u}_0(x_1), , ..., \widetilde{u}_0(x_N), \widetilde{u}_1(x_0), \widetilde{u}_1(x_1), , ..., \widetilde{u}_1(x_N)) \in \mathbb{R}^{2N+2}, \\ A(t) = \begin{bmatrix} O & E \\ \widetilde{A}(t) & \widetilde{B} \end{bmatrix}, \end{cases} \tag{5.11}$$

$$E = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix},\ \widetilde{A}(t) = \begin{bmatrix} \gamma + \alpha_1 & \alpha_1 & & & \widetilde{\gamma}_1(t) \\ \alpha_1 & \gamma & \alpha_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_1 & \gamma & \alpha_1 \\ \widetilde{\gamma}_0(t) & & & \alpha_1 & \gamma + \alpha_1 \end{bmatrix}, \tag{5.12}$$

$$\widetilde{B} = \begin{bmatrix} \widehat{\delta}_0 & & & & \widetilde{\delta}_1 \\ & -\lambda & & & \\ & & \ddots & & \\ & & & -\lambda & \\ \widetilde{\delta}_0 & & & & \widehat{\delta}_1 \end{bmatrix}. \tag{5.13}$$

To solve the linear differential system (**??**), we use a spectral method with a time step $\Delta t = 0.08$

and a spacial step $\Delta x = 0.1$

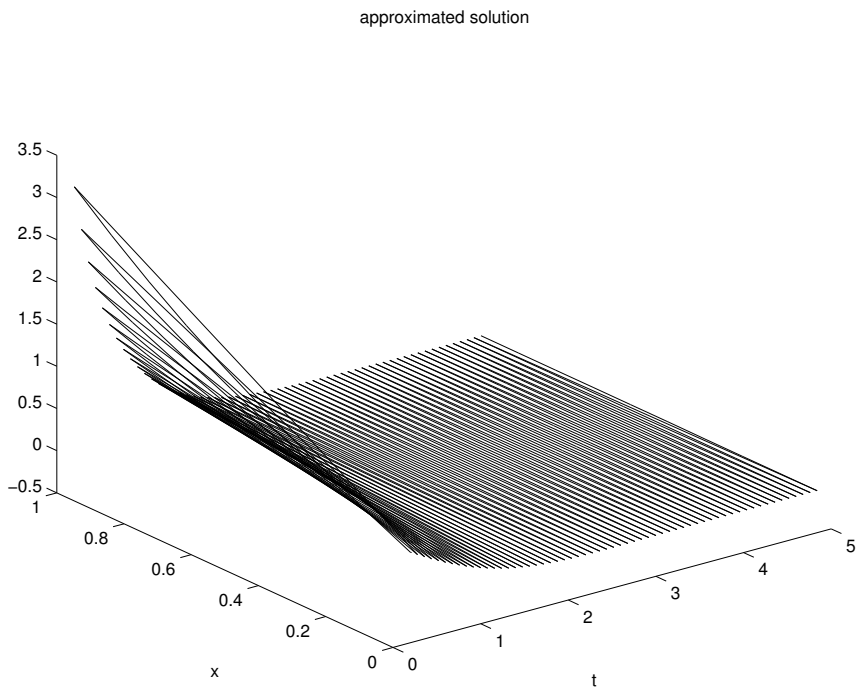

Figure 1. Approximated solution

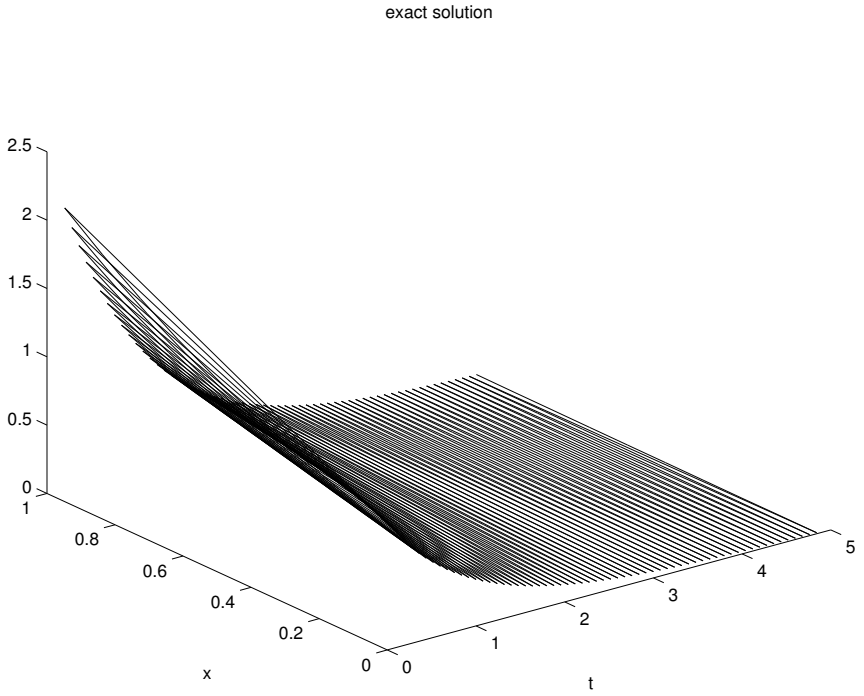

Figure 2: Exact solution

In fig. 1 we have drawn the approximated solution of the problem (5.1) – (5.3) while fig. 2 represents his corresponding exact solution (5.5). So in both cases we notice the very good decay of these surfaces from $T = 0$ to $T = 5$.

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