Central limit theorems for additive functionals of ergodic Markov diffusions processes

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CENTRAL LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF ERGODIC MARKOV DIFFUSIONS PROCESSES

PATRICK CATTIAUX, DJALIL CHAFAÏ, AND ARNAUD GUILLIN

Dedicated to the Memory of Naoufel Ben Abdallah

Abstract. We revisit functional central limit theorems for additive functionals of ergodic Markov diffusion processes. Translated in the language of partial differential equations of evolution, they appear as diffusion limits in the asymptotic analysis of Fokker-Planck type equations. We focus on the square integrable framework, and we provide tractable conditions on the infinitesimal generator, including degenerate or anomalously slow diffusions. We take advantage on recent developments in the study of the trend to the equilibrium of ergodic diffusions. We discuss examples and formulate open problems.

Contents

Date: Preprint April, 2011. Compiled April 12, 2011.

2010 *Mathematics Subject Classification.* :60F05, 60G44, 60J25, 60J60.

Key words and phrases. Functional central limit theorem; invariance principle; diffusion process; Markov semigroup; Markov process; Lyapunov criterion; long time behavior; Fokker-Planck equation.

1. Introduction

Let $(X_t)_{t\geq 0}$ be a continuous time strong Markov process with state space $\mathbb{R}^d$, non explosive, irreducible, positive recurrent, with unique invariant probability measure μ. Following [MT59, Theorem 5.1 page 170], for every $f \in \mathbb{L}^1(\mu)$, if almost surely (a.s.) the function $s \in \mathbb{R}_+ \mapsto f(X_s)$ is locally Lebesgue integrable, then

$$\frac{S_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \int f\,d\mu \quad \text{where} \quad S_t := \int_0^t f(X_s)\,ds. \tag{1.1}$$

If $X_0 \sim \mu$ then by the Fubini theorem (1.1) holds for all $f \in \mathbb{L}^1(\mu)$ and the convergence holds additionally in $\mathbb{L}^1$ thanks to the dominated convergence theorem. The statement (1.1) which relates an average in time with an average in space is an instance of the ergodic phenomenon. It can be seen as a strong law of large numbers for the additive functional $(S_t)_{t\geq 0}$ of the Markov process $(X_t)_{t\geq 0}$. The asymptotic fluctuations are described by a central limit theorem which is the subject of this work. Let us assume that $X_0 \sim \mu$ and $f \in \mathbb{L}^2(\mu)$ with $\int f\,d\mu = 0$ and $f \neq 0$. Then for all $t \geq 0$ we have $S_t \in \mathbb{L}^2(\mu) \subset \mathbb{L}^1(\mu)$ and $\mathbb{E}(S_t) = 0$. We say that $(S_t)_{t\geq 0}$ satisfies to a *central limit theorem* (CLT) when

$$\frac{S_t}{s_t} \xrightarrow[t\to\infty]{\text{law}} \mathcal{N}(0,1) \tag{CLT}$$

for a deterministic positive function $t \mapsto s_t$ which may depend on f. Here $\mathcal{N}(0,1)$ stands for the standard Gaussian law on $\mathbb{R}$ with mean 0 and variance 1. By analogy with the CLT for i.i.d. sequences one may expect that $s_t^2 = \mathrm{Var}(S_t)$ and that this variance is of order t as $t \to \infty$. A standard strategy for proving (CLT) consists in representing $(S_t)_{t\geq 0}$ as a sum of an $\mathbb{L}^2$-martingale plus a remainder term which vanishes in the limit, reducing the proof to a central limit theorem for martingales. This strategy is particularly simple under mild assumptions [JS03, VII.3 p. 486]. Namely, if L is the infinitesimal generator of $(X_t)_{t\geq 0}$ with domain $\mathbb{D}(L) \subset \mathbb{L}^2(\mu)$ and if $g \in \mathbb{D}(L)$ then $(M_t)_{t\geq 0}$ defined by

$$M_t := g(X_t) - g(X_0) - \int_0^t (Lg)(X_s)\,ds$$

is a local $\mathbb{L}^2$ martingale. Now if $g^2 \in \mathbb{D}(L)$ and $\Gamma(g) := L(g^2) - 2gLg \in \mathbb{L}^1(\mu)$, then

$$\langle M \rangle_t = \int_0^t \Gamma(g)(X_s)\,ds.$$

The law of large numbers (1.1) yields $\lim_{t\to\infty} t^{-1}\langle M \rangle_t = \int \Gamma(g)\,d\mu$. As a consequence, for a prescribed f, if the Poisson equation $Lg = f$ admits a mild enough solution g then

$$\frac{M_t}{s_t} = \frac{g(X_t) - g(X_0)}{s_t} - \frac{S_t}{s_t}.$$

This suggests to deduce (CLT) from a CLT for martingales. We will revisit this strategy. Beyond (CLT), we say that $(S_t)_{t\geq 0}$ satisfies to a *Functional Central Limit Theorem* (FCLT) or *Invariance Principle* when for every finite sequence $0 < t_1 \leq \cdots \leq t_n < \infty$,

$$\left(\frac{S_{t_1/\varepsilon}}{s_{t_1/\varepsilon}}, \ldots, \frac{S_{t_n/\varepsilon}}{s_{t_n/\varepsilon}}\right) \xrightarrow[\varepsilon\to 0]{\text{law}} \mathcal{L}((B_{t_1}, \ldots, B_{t_n})) \tag{FCLT}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian Motion on $\mathbb{R}$. Taking $n = 1$ gives (CLT). To capture multitime correlations, one may upgrade the convergence in law in (FCLT) to an $\mathbb{L}^2$ convergence. The statement (FCLT) means that as $\varepsilon \to 0$, the rescaled process $(S_{t/\varepsilon}/s_{t/\varepsilon})_{t\geq 0}$ converges in law to a Brownian Motion, for the topology of finite dimensional marginal laws. At the level of Chapman-Kolmogorov-Fokker-Planck equations, (FCLT) is a *diffusion limit* for a weak topology.

In this work, we focus on the case where $(X_t)_{t\geq 0}$ is a Markov *diffusion* process on $E = \mathbb{R}^d$, and we seek for conditions on f and on the infinitesimal generator in order to get (CLT) or even (FCLT). We shall revisit the renowned result of Kipnis and Varadhan [KV86], and provide an alternative approach which is not based on the resolvent. Our results cover fully degenerate situations such as the kinetic model studied in [GJS$^+$09, DM08, CCM10]. More generally, we believe that a whole category of diffusion limits which appear in the asymptotic analysis of evolution partial differential equations of Fokker-Planck type enters indeed the framework of the central limit theorems we shall discuss. We also explain how the behavior *out of equilibrium* (i.e. $X_0 \not\sim \mu$) may be recovered from the behavior *at equilibrium* (i.e. $X_0 \sim \mu$) by using propagation of chaos (decorrelation), for instance via Lyapunov criteria ensuring a quick convergence in law of X_t to μ as $t \to \infty$. Note that since we focus on an $\mathbb{L}^2$ framework, the natural normalization is the square root of the variance and we can only expect Gaussian fluctuations. We believe however that stable limits that are not Gaussian, also known as "anomalous diffusion limits", can be studied using similar tools (one may take a look at the works [JKO09, MMM08] in this direction).

The literature on central limit theorems for discrete or continuous Markov processes is immense and possesses many connected components. Some instructive entry points for ergodic Markov processes are given by [DL01a, DL01b, DL03, CL09, HP04, KM03, Kut04, KM05, GM96, PV01, PV03, PV05, Lan03]. We refer to [KLO] and [HL03] for null recurrent Markov processes. Central limit theorems for additive functionals of Markov chains can be traced back to the works of Kolmogorov and Doeblin [Doe38]. The discrete time allows to decompose the sample paths into excursions. The link with stationary sequences goes back to Gordin [Gor69], see also Ibragimov and Linnik [IL65] and Nagaev [Nag57] (only stable laws can appear at the limit). The link with martingales goes back to Gordin and Lifsic [GL78]. For diffusions, the martingale method was developed by Kipnis and Varadhan [KV86], see also [Hel82] (the Poisson equation is solved via the resolvent).

Outline. Section 2 provides some notations and preliminaries including a discussion on the variance of S_t. Section 3 is devoted to FCLT at equilibrium and contains a lot of known results. We recall how to use the Poisson equation and compare with the known results on stationary sequences, which seems more powerful. In particular, we give in section 3.1 a direct new proof of the renowned FCLT of Kipnis and Varadhan [KV86, Corollary 1.9] in the reversible case. In section 4.3 we provide a non-reversible version of the Kipnis-Varadhan theorem. Actually some of the results of section 4 are written in the CLT situation, but under mild assumptions, they can be extended to a general FCLT (see Proposition 8.1). All these general results are illustrated by the examples discussed in Section 5. In sections 6 and 7 we exhibit a particularly interesting behavior, i.e. a possible anomalous rate of convergence to a Gaussian limit. This behavior is a consequence of a not too slow decay to equilibrium in the ergodic theorem. Finally we give in the next section some results concerning fluctuations out of equilibrium.

Acknowledgments. This work benefited from discussions with N. Ben Abdallah, M. Puel and S. Motsch, in the Institut de Mathématiques de Toulouse.

2. The framework

Unless otherwise stated $(X_t)_{t\geq 0}$ is a continuous time strong Markov process with state space $\mathbb{R}^d$, non explosive, irreducible, positive recurrent, with unique invariant probability measure μ. We realize the process on a canonical space and we denote by $\mathbb{P}_\nu$ the law of the process with initial law $\nu = \mathcal{L}(X_0)$. In particular $\mathbb{P}_x := \mathbb{P}_{\delta_x} = \mathcal{L}((X_t)_{t\geq 0}|X_0 = x)$ for all $x \in E$. We denote by $\mathbb{E}_\nu$ and Var_ν the expectation and variance under $\mathbb{P}_\nu$. For all $t \geq 0$, all $x \in E$, and every $f : E \to \mathbb{R}$ integrable for $\mathcal{L}(X_t|X_0 = x)$, we define the function $P_t(f) : x \mapsto \mathbb{E}(f(X_t)|X_0 = x)$. One can check that $P_t(f)$ is well defined for all $f : E \to \mathbb{R}$ which is measurable and positive, or in $\mathbb{L}^p(\mu)$ for $1 \leq p \leq \infty$. On each $\mathbb{L}^p(\mu)$

with $1 \le p \le \infty$, the family $(P_t)_{t\ge 0}$ forms a Markov semigroup of linear operators of unit norm, leaving stable each constant function and preserving globally the set of non negative functions. We denote by L the infinitesimal generator of this semigroup in $\mathbb{L}^2(\mu)$, defined by $Lf := \lim_{t\to 0} t^{-1}(P_t(f) - f)$. We assume that $(X_t)_{t\ge 0}$ is a diffusion process (this implies that for all $x \in E$ the law $\mathbb{P}_x$ is supported in the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ taking the value x at time 0) and that there exists an algebra $\mathbb{D}(L)$ of uniformly continuous and bounded functions, containing constant functions, which is a core for the extended domain $\mathbb{D}_e(L)$ of the generator, see e.g. [CL96, DM87]. Following [CL96], one can then show that there exists a countable orthogonal family (C^n) of local martingales and a countable family (∇^n) of operators such that for all $f \in \mathbb{D}_e(L)$, the stochastic process $(M_t)_{t\ge 0}$ defined from f by

$$M_t := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds = \sum_n \int_0^t \nabla^n f(X_s)\,dC^n_s, \tag{2.1}$$

is a square integrable local martingale for all probability measure on E. Its bracket is

$$\langle M \rangle_t = \int_0^t \Gamma(f)(X_s)\,ds.$$

where $\Gamma(f)$ is the carré-du-champ functional quadratic form defined for any $f \in \mathbb{D}(L)$ by

$$\Gamma(f) := \sum_n \nabla^n f\, \nabla^n f. \tag{2.2}$$

We write for convenience $M_t = \int_0^t \nabla f(X_s)\,dC_s$. With these definitions, for $f \in \mathbb{D}(L)$,

$$\mathcal{E}(f) := \int \Gamma(f)\,d\mu = -2\int f\,Lf\,d\mu \; = -\partial_{t=0}\|P_t f\|^2_{\mathbb{L}^2(\mu)}. \tag{2.3}$$

The diffusion property states that for every smooth $\Phi : \mathbb{R}^n \to \mathbb{R}$ and $f_1, \dots, f_n \in \mathbb{D}(L)$,

$$L\Phi(f_1,\dots,f_n) = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(f_1,\dots,f_n)\,Lf_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 \Phi}{\partial x_i\,\partial x_j}(f_1,\dots,f_n)\,\Gamma(f_i,f_j)$$

where $\Gamma(f,g) = L(fg) - f\,Lg - g\,Lf$ is the bilinear form associated to the carré-du-champ. We shall also use the adjoint L^* of L in $\mathbb{L}^2(\mu)$ given for all $f, g \in \mathbb{D}(L)$ by

$$\int f Lg\,d\mu = \int gL^* f\,d\mu$$

and the corresponding semigroup $(P^*_t)_{t\ge 0}$. We shall mainly be interested by diffusion processes with generator of the form

$$L = \frac{1}{2}\sum_{i,j=1}^d A_{ij}(x)\,\partial^2_{i,j} \; + \sum_{i=1}^d B_i(x)\,\partial_i \tag{2.4}$$

where $x \mapsto A(x) := (A_{i,j}(x))_{1\le i,j\le d}$ is a smooth field of symmetric positive semidefinite matrices, and $x \mapsto b(x) := (b_i(x))_{1\le i\le d}$ is a smooth vector field. If we denote by $(X^x_t)_{t\ge 0}$ a process of law $\mathbb{P}_x$ then it is the solution of the stochastic differential equation

$$dX^x_t = b(X^x_t)\,dt + \sqrt{A}(X^x_t)dB_t, \quad \text{with} \quad X^x_0 = x \tag{2.5}$$

where $(B_t)_{t\ge 0}$ is a d-dimensional standard Brownian Motion, and we have also

$$\Gamma(f) = \langle A\nabla f, \nabla f \rangle.$$

Note that since the process admits a unique invariant probability measure μ, the process is positive recurrent. We say that the invariant probability measure μ is *reversible* when $L = L^*$ (and thus $P_t = P^*_t$ for all $t \ge 0$).

In practice, the initial data consists in the operator L. We give below a criterion on L ensuring the existence of a unique probability measure and thus positive recurrence.

Definition 2.6 (Lyapunov function)**.** *Let $\varphi : [1,+\infty[\to]0,\infty[$. We say that $V \in D_e(L)$ (the extended domain of the generator, see* [CL96, DM87]*) is a φ-Lyapunov function if $V \geq 1$ and if there exist a constant κ and a closed petite set C such that for all x*

$$LV(x) \leq -\varphi(V(x)) + \kappa \mathbf{1}_C(x)\,.$$

Recall that C is a petite set if there exists some probability measure $p(dt)$ on $\mathbb{R}_+$ such that for all $x \in C$, $\int_0^\infty P_t(x,\cdot)\,p(dt) \geq \nu$ for a non trivial positive measure ν.

In the $\mathbb{R}^d$ situation with L given by (2.4) with smooth coefficients, compact subsets are petite sets and we have the following [Kha80]:

Proposition 2.7. *If L is given by* (2.4) *a sufficient condition for positive recurrence is the existence of a φ-Lyapunov function with $\varphi(u) = 1$ and for C some compact subset. In addition, for all $x \in \mathbb{R}^d$ the law of* (2.5) *denoted by $P_t(x,.)$ converges to the unique invariant probability measure μ in total variation distance, as $t \to +\infty$.*

We say that an invariant probability measure μ is ergodic if the only invariant functions (i.e. such that $P_t f = f$ for all t) are the constants. In this case the ergodic theorem says that the Cesàro means $\frac{1}{t}\int_0^t f(X_s)\,ds$ converge, as $t \to \infty$, $\mathbb{P}_\mu$ almost surely and in $\mathbb{L}^1$, to $\int f\,d\mu$ for any $f \in \mathbb{L}^1(\mu)$. We say that the process is strongly ergodic if $P_t f \to \int f\,d\mu$ in $\mathbb{L}^2(\mu)$ for any $f \in \mathbb{L}^2(\mu)$ (this immediately extends to $\mathbb{L}^p(\mu)$, $1 \leq p < +\infty$) and recall that $t \mapsto \|P_t f\|_{\mathbb{L}^2(\mu)}$ is always non increasing. If μ is ergodic and reversible then the process is strongly ergodic. We say that the Dirichlet form is non degenerate if $\mathcal{E}(f,f) = 0$ if and only if f is constant. Again the reversible ergodic case is non degenerate, but kinetic models will be degenerate. We refer to section 5 in [Cat04] for a detailed discussion of these notions.

Lemma 2.8 (Variance in the reversible case)**.** *Assume that μ is reversible and $0 \neq f \in \mathbb{L}^2(\mu)$ with $\int f\,d\mu = 0$. Then we have the following properties:*

1. $\liminf_{t\to\infty} \frac{1}{t}\operatorname{Var}_\mu(S_t) > 0$
2. $\limsup_{t\to\infty} \frac{1}{t}\operatorname{Var}_\mu(S_t) < \infty$ *iff the Kipnis-Varadhan condition is satisfied:*

$$V := \int_0^\infty \left(\int (P_s f)^2\,d\mu\right) ds < \infty, \tag{2.9}$$

and in this case $\lim_{t\to\infty} \frac{1}{t}\operatorname{Var}_\mu(S_t) = 4V$

The quantity $4V$ is the *asymptotic variance of the scaled additive functional* $\frac{1}{t}S_t$.

Proof. By using the Markov property, and the invariance of μ, we can write

$$\begin{aligned}
\operatorname{Var}_\mu(S_t) &= \mathbb{E}(S_t^2) \\
&= 2\int_{0\leq u\leq s\leq t} \mathbb{E}[f(X_s)f(X_u)]\,duds \\
&= 2\int_{0\leq u\leq s\leq t} \left(\int f P_{s-u} f\,d\mu\right) duds \\
&= 2\int_{0\leq u\leq s\leq t} \left(\int f P_u f\,d\mu\right) duds \\
&= 2\int_{0\leq u\leq s\leq t} \left(\int P^*_{u/2} f P_{u/2} f\,d\mu\right) duds \\
&= 4\int_0^{t/2} (t-2s)\left(\int P^*_s f P_s f\,d\mu\right) ds.
\end{aligned}$$

Using now the reversibility of μ and the decay of the $\mathbb{L}^2$ norm, we obtain

$$2t \int_0^{t/4} \left(\int (P_s f)^2 \, d\mu \right) ds \leq \mathrm{Var}_\mu(S_t) \leq 4t \int_0^{t/2} \left(\int (P_s f)^2 \, d\mu \right) ds.$$

This implies the first property. The second property follows from the Cesàro rule and

$$\frac{\mathrm{Var}_\mu(S_t)}{t} = \frac{2}{t} \int_{0 \leq u \leq s \leq t} \left(\int P_{u/2}^2 f \, d\mu \right) du \, ds.$$

□

Remark 2.10 (Non reversible case). *If μ is not reversible, we do not even know whether $\int P_s^* f P_s f \, d\mu$ is non-negative or not. Nevertheless we may define V_- and V_+ by*

$$V_- := \liminf_{t \to \infty} \int_0^t \left(\int P_s f P_s^* f \, d\mu \right) ds \quad \text{and} \quad V_+ := \limsup_{t \to \infty} \int_0^t \left(\int P_s f P_s^* f \, d\mu \right) ds$$

abridged into V if $V_+ = V_-$. As in the reversible case, if $V_+ < +\infty$ then $V_+ = V_-$ and $\lim_{t \to \infty} t^{-1} \mathrm{Var}_\mu(S_t) = 4V$. We ignore if $V_-(f) > 0$ as in the reversible case. We have thus a priori to face two type of situations: either $V_+ < +\infty$ and the asymptotic variance exists and $\mathrm{Var}_\mu(S_t)$ is of order t as $t \to \infty$, or $V_+ = +\infty$ and $\mathrm{Var}_\mu(S_t)$ is much larger.

Remark 2.11 (Possible limits). *For every sequence $(\nu_n)_{n \geq 1}$ of probability measure on $\mathbb{R}$ with unit second moment and zero mean, it can be shown by using for instance the Skorokhod representation theorem that all adherence values of $(\nu_n)_{n \geq 1}$ for the weak topology (with respect to continuous bounded functions) have second moment ≤ 1 and mean 0. In particular, if an adherence value is a stable law then it is necessarily a centered Gaussian with variance ≤ 1. As a consequence, if $(S_t/\sqrt{\mathrm{Var}_\mu(S_t)})_{t \geq 0}$ converges in law to a probability measure as $t \to \infty$, then this probability measure has second moment ≤ 1 and mean 0, and if it is a stable law, then it is a centered Gaussian with variance ≤ 1.*

3. Poisson equation and martingale approximation

We present in this section a strategy to prove (FCLT) which consists in a reduction to a more standard result for a family of martingales. We start by solving the Poisson equation: we fix $0 \neq f \in \mathbb{L}^2(\mu)$, $\int f \, d\mu = 0$, and we seek for g solving

$$Lg = f. \tag{3.1}$$

The Poisson equation (3.1) corresponds to a so called *coboundary* in ergodic theory. If (3.1) admits a regular enough solution g, then by Itô's formula, for every $t \geq 0$ and $\varepsilon > 0$,

$$S_{\varepsilon^{-1}t} = \int_0^{\varepsilon^{-1}t} f(X_s) \, ds = g(X_{\varepsilon^{-1}t}) - g(X_0) - M_t^\varepsilon \tag{3.2}$$

where $(M_t^\varepsilon)_{t \geq 0}$ is a local martingale with brackets

$$\langle M^\varepsilon \rangle_t = \int_0^{\varepsilon^{-1}t} \Gamma(g)(X_s) \, ds. \tag{3.3}$$

Now the Rebolledo FCLT for $\mathbb{L}^2$ local martingales (see [Reb80] or [Whi07]) says that if

$$v^2(\varepsilon) \langle M^\varepsilon \rangle_t \xrightarrow[\varepsilon \to 0]{\mathbb{P}} h^2(t) \tag{3.4}$$

for all $t \geq 0$, where v and h are deterministic functions which may depend on f via g, then

$$(v(\varepsilon) M_t^\varepsilon)_{t \geq 0} \xrightarrow[\varepsilon \to 0]{\text{Law}} \left(\int_0^t h(s) \, dW_s \right)_{t \geq 0} \tag{3.5}$$

where $(W_t)_{t \geq 0}$ is a standard Brownian Motion, the convergence in law being in the sense of finite dimensional process marginal laws. To obtain (FCLT), it suffices to show the convergence in probability to 0 of $v(\varepsilon)g(X_{\varepsilon^{-1}t})$ as $\varepsilon \to 0$, for any fixed $t \geq 0$. Moreover, if this convergence holds in $\mathbb{L}^2$ then the normalization factor v can be chosen such that

$$\lim_{\varepsilon\to 0} v^2(\varepsilon)\mathbb{E}\big[S^2_{\varepsilon^{-1}t}\big] = \lim_{\varepsilon\to 0} v^2(\varepsilon)\mathbb{E}[\langle M^\varepsilon\rangle_t] = \lim_{\varepsilon\to 0} v^2(\varepsilon)\frac{t}{\varepsilon}\mathcal{E}(g) = h^2(t) \tag{3.6}$$

i.e. we recover $v(\varepsilon) = \sqrt{\varepsilon}$ and $V = \lim_{t\to\infty} t^{-1}\mathrm{Var}_\mu(S_t) = \frac{1}{4}\mathcal{E}(g)$. To summarize, this martingale approach reduces the proof of (FCLT) to the following three steps:

- solve the Poisson equation $Lg = f$ in the g variable
- control the regularity of g in order to use Itô's formula (3.2)
- check the convergence to 0 of $g(X_{\varepsilon^{-1}t})$ as $\varepsilon \to 0$ in an appropriate way.

Let us start with a simple proposition which follows from the discussion above.

Theorem 3.7 (FCLT via Poisson equation in $\mathbb{L}^2$). *If $0 \neq f \in \mathbb{L}^2(\mu)$ with $\int f\,d\mu = 0$, and if $f \in \mathbb{D}(L^{-1})$ i.e. there exists $g \in \mathbb{D}(L)$ such that $Lg = f$ where L is seen as an unbounded operator, then* $\mathrm{Var}_\mu(S_t) \sim_{t\to\infty} t\mathcal{E}(g,g)$ *and* (FCLT) *holds under $\mathbb{P}_\mu$ with $s_t^2(f) = t\mathcal{E}(g,g)$.*

Let us examine a natural candidate to solve the Poisson equation. Assume that $Lg = f$ in $\mathbb{L}^2(\mu)$ and that $\int g\,d\mu = 0$ (note that since $L1 = 0$ we may always center g). Then

$$P_t g - g = \int_0^t \partial_s P_s g\,ds = \int_0^t LP_s g\,ds = \int_0^t P_s Lg\,ds = \int_0^t P_s f\,ds$$

so that, if the process is strongly ergodic, $\lim_{t\to\infty} P_t g = \int g\,d\mu = 0$, and thus

$$g = -\int_0^\infty P_s f\,ds. \tag{3.8}$$

For the latter to be well defined in $\mathbb{L}^2(\mu)$, it is enough to have some quantitative controls for the convergence of $P_s f$ to 0 as $s \to \infty$. Conversely, for a deterministic $T > 0$ we set

$$g_T := -\int_0^T P_s f\,ds \tag{3.9}$$

which is well defined in $\mathbb{L}^2(\mu)$ and satisfies to

$$Lg_T = \lim_{u\to 0}\frac{P_u g_T - g_T}{u} = -\partial_{u=0}\int_u^{u+T} P_s f\,ds = f - P_T f.$$

If g_T converges in $\mathbb{L}^2$ to g then $Lg = f$. In particular, we obtain the following.

Corollary 3.10 (Solving the Poisson equation in $\mathbb{L}^2$). *Let $0 \neq f \in \mathbb{L}^2(\mu)$ with $\int f\,d\mu = 0$.*

(1) *If we have*

$$\int_0^\infty s\|P_s f\|_{\mathbb{L}^2(\mu)}\,ds < \infty, \tag{3.11}$$

then $f \in \mathbb{D}(L^{-1})$ and g in (3.8) *is in $\mathbb{L}^2(\mu)$ and solves the Poisson equation* (3.1)

(2) *If μ is reversible then $f \in D(L^{-1})$ if and only if*

$$\int_0^\infty s\|P_s f\|^2_{\mathbb{L}^2(\mu)}\,ds < \infty, \tag{3.12}$$

and in this case the Poisson equation (3.1) *has a unique solution g given by* (3.8). *Moreover, condition* (3.11) *implies condition* (3.12).

Proof. The existence of $g \in \mathbb{L}^2(\mu)$ in the case (3.11) is immediate. For (3.12) consider g_T defined in (3.9). For $a > 0$ we then have, using reversibility

$$\begin{aligned}\int |g_{T+a} - g_T|^2 \, d\mu &= 2 \int \left(\int_T^{T+a} P_s f \int_T^s P_u f \, du \, ds \right) d\mu \\ &= 2 \int \left(\int_T^{T+a} \int_T^s \left(P_{\frac{s+u}{2}} f \right)^2 du \, ds \right) d\mu \\ &= 4 \int \left(\int_T^{T+a} (u - T) \, (P_u f)^2 \, du \right) d\mu,\end{aligned}$$

so that $(g_T)_T$ is Cauchy, hence convergent, if and only if (3.12) is satisfied. In addition, taking $T = 0$ above gives

$$\int g_T^2 \, d\mu = 4 \int_0^T u \left(\int (P_u f)^2 \, d\mu \right) du.$$

Hence the family $(g_T)_T$ is bounded in $\mathbb{L}^2$ only if (3.12) is satisfied, i.e. here convergence and boundedness of $(g_T)_T$ are equivalent.

To deduce (3.12) from (3.11), we note that $t \mapsto \|P_t f\|_{\mathbb{L}^2(\mu)}$ is non-increasing, and hence,

$$t \, \|P_t f\|_{\mathbb{L}^2(\mu)} \leq \int_0^t \|P_s f\|_{\mathbb{L}^2(\mu)} ds \leq \int_0^\infty \|P_s f\|_{\mathbb{L}^2(\mu)} ds$$

so that $\|P_t f\|_{\mathbb{L}^2(\mu)} = O(1/t)$ by (3.11), which gives (3.12). We remark by the way that conversely, (3.12) implies $\|P_t f\|_{\mathbb{L}^2(\mu)} = O(1/t)$ since by the same reasoning,

$$\frac{1}{2} t^2 \, \|P_t f\|^2_{\mathbb{L}^2(\mu)} \leq \int_0^{+\infty} s \, \|P_s f\|^2_{\mathbb{L}^2(\mu)} \, ds.$$

□

Recent results on the asymptotic behavior of such semigroups can be used to give tractable conditions and general examples. We shall recall them later. In particular for $\mathbb{R}^d$ valued diffusion processes we will compare them with [GM96, PV01, PV03, PV05].

Actually one can (partly) improve on this result. For instance if μ is a reversible measure, the same FCLT holds under the weaker assumption $f \in \mathbb{D}(L^{-1/2})$ as shown in [KV86] and revisited in the next subsection too. For non-reversible Markov chains, a systematic study of fractional Poisson equation is done in [DL01b]. The connection with the rate of convergence of $P_t f$ is also discussed therein, and the result "at equilibrium" is extended to an initial δ_x Dirac mass in [DL01a, DL03] extending [MW00] for the central limit theorem (i.e. for each marginal of the process). The previous $f \in \mathbb{D}(L^{-1/2})$ is however no more sufficient (see the final discussion in [DL03]). It is thus more natural to look at the rate of convergence (as in [DL03, MW00]) rather than at fractional operators.

3.1. **Reversible case and Kipnis-Varadhan theorem.** In this section we assume that μ is reversible. Corollary 3.10 states that (2.9) (equivalent to the existence of the asymptotic variance) is not sufficient to solve the Poisson equation, even in a weak sense. Nevertheless it is enough to get (FCLT), the result below is Corollary 1.9 of [KV86].

Theorem 3.13 (FCLT from the existence of asymptotic variance)**.** *Assume that μ is reversible, that $0 \neq f \in \mathbb{L}^2(\mu)$ with $\int f \, d\mu = 0$, and that f satisfies the Kipnis-Varadhan condition* (2.9)*. Then* (FCLT) *holds under $\mathbb{P}_\mu$ with $s_t^2 = 4tV$, and* $\mathrm{Var}_\mu(S_t) \sim_{t \to \infty} s_t^2$.

Proof. For $T>0$ introduce g_T by (3.9), and the corresponding family $((\nabla^n g_T))_{T>0}$ (recall (2.1)). We thus have $Lg_T = f - P_T f$ and, for all $S \le T$,

$$\begin{aligned}\int \Gamma(g_T - g_S)\, d\mu &= 2\int (-L(g_T-g_S))\,(g_T-g_S)\, d\mu \\ &= 2\int_S^T \int (P_S f - P_T f)\, P_s f\, d\mu\, ds \\ &= 2\int_S^T \int (P^2_{(s+S)/2} f - P^2_{(s+T)/2} f)\, d\mu\, ds \\ &\le 4\int_S^\infty \int P_s^2 f\, d\mu\, ds,\end{aligned}$$

so that according to (2.9), the family $((\nabla^n g_T))_{T>0}$ is Cauchy in $\mathbb{L}^2(\mu)$. It follows that it strongly converges to h in $\mathbb{L}^2(\mu)$. On the other hand, using Itô's formula,

$$\begin{aligned} S^T_{t/\varepsilon} &= g_T(X_{t/\varepsilon}) - g_T(X_0) - M_t^T + \int_0^{t/\varepsilon} P_T f(X_s)\, ds \\ &= g_T(X_{t/\varepsilon}) - g_T(X_0) - M_t^T + S^T_{t/\varepsilon}\end{aligned} \tag{3.14}$$

where $(M_t^T)_{t\ge 0}$ is a martingale with brackets $\langle M^T\rangle_t = \int_0^{t/\varepsilon} \Gamma(g_T)(X_s)\, ds$ (recall (2.2)).

According to what precedes and the framework (recall (2.1)) we may replace $(M_t^T)_{t\ge0}$ by another martingale $(N_t^h)_{t\ge 0}$ with brackets $\langle N^h\rangle_t = \int_0^{t/\varepsilon} |h|^2(X_s)\, ds$ such that

$$\varepsilon \mathbb{E}_\mu\left(\sup_{0\le s\le t} |M_s^T - N_s^h|^2\right) \le t\|\nabla g_T - h\|^2_{\mathbb{L}^2(\mu)} \to 0 \text{ as } T\to\infty \text{ uniformly in } \varepsilon.$$

In addition the ergodic theorem tells us that

$$\lim_{\varepsilon\to 0} \varepsilon \left\langle N^h \right\rangle_t = t\int h^2\, d\mu.$$

Thus we may again apply Rebolledo's FCLT, taking first the limit in T and then in ε. It remains to control the others terms. But

$$\begin{aligned}\mathrm{Var}_\mu(S^T_{t/\varepsilon}) &= 2\int_0^{t/\varepsilon}\int_0^s \left(P^2_{T+(u/2)} f\, d\mu\right) du\, ds \\ &= 4\int_0^{t/\varepsilon}\int_T^{T+(s/2)} \left(\int P_u^2 f\, d\mu\right) du\, ds \\ &\le 4\int_0^{t/\varepsilon}\int_T^\infty \left(\int P_u^2 f\, d\mu\right) du\, ds \\ &\le 4(t/\varepsilon)\int_T^\infty \left(\int P_u^2 f\, d\mu\right) du.\end{aligned}$$

Since $\lim_{T\to\infty} \int_T^\infty \left(\int P_u^2 f\, d\mu\right) du = 0$ according to (2.9), we have, uniformly in ε,

$$\lim_{T\to\infty} \varepsilon \mathrm{Var}_\mu(S^T_{t/\varepsilon}) = 0.$$

Next,

$$\int g_T^2\, d\mu = 4\int_0^T u\left(\int P_u^2 f\, d\mu\right) du \le 4T\int_0^\infty \left(\int P_u^2 f\, d\mu\right) du.$$

Hence $\lim_{\varepsilon\to 0} \varepsilon \|g_T\|^2_{\mathbb{L}^2(\mu)} = 0$. The desired result follows by taking T large enough. □

Remark 3.15. *Our proof is different from the original one by Kipnis and Varadhan and is perhaps simpler. Indeed we have chosen to use the natural approximation of what should be the solution of the Poisson equation (i.e g_t), rather than the approximating R_ε*

resolvent as in [KV86]*. Let us mention at this point the work by Holzmann* [Hol05] *giving a necessary and sufficient condition for the so called "martingale approximation" property (we get some in our proof), thanks to an approximation procedure using the resolvent.*

Remark 3.16 (By D. Bakry)**.** *The condition* (2.9) *is satisfied if Assumption (1.14) in* [KV86] *is satisfied i.e. there exists a constant c_f such that for all F in the domain of $\mathcal{E}$,*

$$\left(\int f\,F\,d\mu\right)^2 \leq -c_f^2 \int FLF\,d\mu. \tag{3.17}$$

Indeed, if we define $\varphi(t) := -\int f\,g_t\,d\mu$ where as usual $g_t = -\int_0^t P_s f\,ds$, and if we take $F = g_t$, then $-LF = -Lg_t = P_t f - f$, and using (3.17) *we get $\varphi^2(t) \leq c_f^2(2\varphi(t) - \varphi(2t))$. Using that $\varphi(2t) \geq 0$ we obtain $2c_f^2\varphi(t) - \varphi^2(t) \geq 0$ which implies that φ is bounded hence $\varphi(+\infty) < +\infty$. Taking the limit as $t \to \infty$ and using $2V(f) = \varphi(+\infty)$, we obtain*

$$V(f) \leq \frac{1}{2}c_f^2.$$

All this can be interpreted in terms of the domain of $(-L)^{-1/2}$ (which is formally the gradient ∇) i.e. condition (2.9) *can be seen to be equivalent to the existence in $\mathbb{L}^2(\mu)$ of*

$$(-L)^{-1/2} f = c\int_0^\infty s^{-\frac{1}{2}}\,P_s f\,ds$$

for an ad-hoc constant c. Indeed, for some constant $C > 0$,

$$\left\|\int_0^\infty s^{-\frac{1}{2}}\,P_s f\,ds\right\|^2_{\mathbb{L}^2(\mu)} \leq C\int\int_0^\infty P_s^2 f\left(\int_s^{2s}(2u-s)^{-1/2}\,u^{-1/2}\,du\right)ds\,d\mu$$

and $\int_s^{2s}(2u-s)^{-1/2}\,u^{-1/2}\,du$ is bounded. Note that (2.9) *implies that $\|P_t f\|_{\mathbb{L}^2(\mu)} \leq C(f)/\sqrt{t}$.*

We shall come back later to the method we used in the previous proof, for more general situations including anomalous rate of convergence.

3.2. Poisson equation in $\mathbb{L}^q$ with $q \leq 2$ for diffusions.

What has been done before is written in a $\mathbb{L}^2$ framework. But the method can be extended to a more general setting. Indeed, what is really needed is

1. a solution $g \in \mathbb{L}^q(\mu)$ of the Poisson equation, for some $q \geq 1$,
2. sufficient smoothness of g in order to apply Itô's formula,
3. control the brackets i.e. give a sense to the following quantities

$$\int \Gamma(g)\,d\mu = -2\int f\,g\,d\mu.$$

Definition 3.18 (Ergodic rate of convergence)**.** *For any $r \geq p \geq 1$ and $t \geq 0$ we define*

$$t \mapsto \alpha_{p,r}(t) := \sup_{\substack{\|g\|_{\mathbb{L}^r(\mu)}=1 \\ \int g\,d\mu=0}} \|P_t g\|_{\mathbb{L}^p(\mu)}.$$

The uniform decay rate is $\alpha := \alpha_{2,\infty}$. We denote by α^* the uniform decay rate of L^*. We say that the process is uniformly ergodic if $\lim_{t\to\infty}\alpha(t) = 0$.

We shall discuss later how to get some estimates on these decay rates.

Proposition 3.19 (Solving the Poisson equation in $\mathbb{L}^q$)**.** *Let $p \geq 2$ and $q := p/(p-1)$. If*

$$f \in \mathbb{L}^p(\mu) \quad and \quad \int f\,d\mu = 0 \quad and \quad \int_0^\infty \alpha_{2,p}^*(t)\,\|P_t f\|_{\mathbb{L}^2(\mu)}\,dt < \infty$$

then $g := -\int_0^\infty P_s f\,ds$ belongs to $\mathbb{L}^q(\mu)$ and solves the Poisson equation $Lg = f$.

The assumption of Proposition 3.19 is satisfied for any μ-centered $f \in \mathbb{L}^p(\mu)$ if

$$\int_0^\infty \alpha^*_{2,p}(t)\alpha_{2,p}(t)\,dt < \infty.$$

In the reversible case, we recover a version of the Kipnis-Varadhan statement implying a stronger result (the existence of a solution of the Poisson equation). The results of this section are mainly interesting in the non-reversible situation.

Proof. Let $h \in \mathbb{L}^p(\mu)$, $\bar{h} := h - \int h\,d\mu$, $T > 0$ and $g_T := -\int_0^T P_t f\,dt$. Then

$$\begin{aligned}\left|\int h\,(g_{T+a} - g_T)\,d\mu\right| &= \left|\int \bar{h}\,(g_{T+a} - g_T)\,d\mu\right| \\ &= \left|\int_T^{T+a}\left(\int P^*_{t/2}\bar{h}\,P_{t/2}f\,d\mu\right)dt\right| \\ &\leq \left(\int_T^{T+a}\alpha^*_{2,p}(t/2)\,\|P_{t/2}f\|_{\mathbb{L}^2(\mu)}\,dt\right)\|h\|_{\mathbb{L}^p(\mu)}.\end{aligned}$$

As in the proof of Corollary 3.10, g_T is Cauchy, hence convergent in $\mathbb{L}^q(\mu)$ and solves the Poisson equation. □

The previous proof "by duality" can be improved, just calculating the $\mathbb{L}^q(\mu)$ norm of g_T, for some $1 \leq q \leq 2$ which is not necessarily the conjugate of p.

Proposition 3.20 (Solving the Poisson equation in $\mathbb{L}^q$)**.** *Let $p \geq 2$ and $1 \leq q \leq 2$. If*

$$f \in \mathbb{L}^p(\mu) \quad \text{and} \quad \int f\,d\mu = 0 \quad \text{and} \quad \int_0^\infty t^{q-1}\,\alpha^*_{2,p/(q-1)}(t)\,\|P_t f\|_{\mathbb{L}^2(\mu)}\,dt < \infty$$

then $g = -\int_0^\infty P_s f\,ds$ belongs to $\mathbb{L}^q(\mu)$ and solves the Poisson equation $Lg = f$.

Proof. We have

$$\begin{aligned}\int |g_T|^q\,d\mu &= q\int\left(\int_0^T P_s f\,(\mathbf{1}_{g_s<0} - \mathbf{1}_{g_s>0})\left|\int_0^s P_u f\,du\right|^{q-1} ds\right)d\mu \\ &\leq q\int_0^T \|P_{s/2}f\|_{\mathbb{L}^2(\mu)}\left\|P^*_{s/2}\bar{h}_s\right\|_{\mathbb{L}^2(\mu)}\,ds \\ &\leq q\int_0^T \|P_{s/2}f\|_{\mathbb{L}^2(\mu)}\alpha^*_{2,m}(s/2)\|\bar{h}_s\|_{\mathbb{L}^m(\mu)}\,ds\end{aligned}$$

for an arbitrary $m \geq 2$, where

$$h_s := (\mathbf{1}_{g_s<0} - \mathbf{1}_{g_s>0})\left|\int_0^s P_u f\,du\right|^{q-1} \quad \text{and} \quad \bar{h}_s := h_s - \int h_s\,d\mu.$$

It remains to choose the best m. But of course $\|\bar{h}_s\|_{\mathbb{L}^m(\mu)} \leq 2\|h_s\|_{\mathbb{L}^m(\mu)}$ and

$$\begin{aligned}\left(\int |h_s|^m\,d\mu\right)^{\frac{1}{m}} &= s^{(q-1)}\left(\int\left(\int_0^s |P_u f|\,\frac{du}{s}\right)^{(q-1)m} d\mu\right)^{\frac{1}{m}} \\ &\leq s^{(q-1)}\left(\int |f|^{(q-1)m}\,d\mu\right)^{\frac{1}{m}}.\end{aligned}$$

The best choice is $m = p/(q-1)$. We then proceed as in the proof of proposition 3.19. □

In view of FCLT, the main difficulty is to apply Itô's formula in the non $\mathbb{L}^2$ context. Though things can be done in some abstract setting, we shall restrict ourselves here to the diffusion setting (2.5). For simplicity again we shall consider rather regular settings.

Proposition 3.21 (FCLT via the Poisson equation)**.** *Assume that*

- $0 \neq f \in \mathbb{L}^2(\mu)$ *with* $\int f\,d\mu = 0$
- L *is given by* (2.4) *with smooth coefficients and is hypoelliptic*
- μ *has positive Lesbegue density* $\frac{d\mu}{dx} = e^{-U}$ *for some locally bounded* U
- f *is smooth and belongs to* $\mathbb{L}^p(\mu)$ *for some* $2 \leq p$ *and, with,* $q = p/(p-1)$,

$$\int_0^\infty \alpha^*_{2,p}(t)\,\|P_t f\|_{\mathbb{L}^2(\mu)}\,dt < \infty \quad or \quad \int_0^\infty t^{q-1}\,\alpha^*_{2,p/(q-1)}(t)\,\|P_t f\|_{\mathbb{L}^2(\mu)}\,dt < \infty$$

then $g := -\int_0^\infty P_s f\,ds$ *is well defined in* $\mathbb{L}^q(\mu)$, *is smooth, and solves the Poisson equation* $Lg = f$, *and hence* (FCLT) *holds under* $\mathbb{P}_\mu$ *with* $s_t^2 = -t\int f\,g\,d\mu$.

Proof. The only thing to do is to show that g (obtained in proposition 3.19) satisfies $Lg = f$ in the Schwartz space of distributions $\mathcal{D}'$. To see the latter just write for $h \in \mathcal{D}$,

$$\int L^* h\, g_T\,d\mu = \int h\,Lg_T\,d\mu = \int h\,(f - P_T f)\,d\mu$$

and use that $P_T f$ goes to 0 in $\mathbb{L}^1(\mu)$. It follows that $e^{-U} g_T$ converges in $\mathcal{D}'$ to some Schwartz distribution we may write $e^{-U} g$, since e^{-U} is everywhere positive and smooth. Furthermore since the adjoint operator of $e^{-U}L^*$ (defined on $\mathcal{D}$) is $e^{-U}L$ (defined on $\mathcal{D}'$), we get that g solves the Poisson equation $Lg = f$ in $\mathcal{D}'$. Using hypoellipticity, we deduce that g is smooth and satisfies $Lg = f$ in the usual sense. Finally (FCLT) follows from the usual strategy, provided $\int \Gamma(g)\,d\mu$ is finite. That is why we have to restrict ourselves (in the second case) to q the conjugate of p, ensuring that $\int |fg|\,d\mu < \infty$. □

Remark 3.22. *If* $f \in \mathbb{L}^p(\mu)$ *for some* $p \geq 1$ *(*f *being still smooth), one can immediately adapt the proof of the previous proposition to show that the Poisson equation* $Lg = f$ *has a solution* $g \in \mathbb{L}^1(\mu)$ *as soon as* $\int_0^{+\infty} \alpha^*_{q,\infty}(t)\,dt < +\infty$. ♢

In the hypoelliptic context one can go a step further. First of all, as before we may and will assume that f is of C^∞ class, so that g_t is also smooth. Next, if $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int Lg_t\,\varphi\,p\,dx = \int Lg_t\,\varphi\,d\mu \to_{t\to+\infty} \int f\,\varphi\,d\mu = \int f\,\varphi\,p\,dx$$

so that $p\,Lg_t \to_{t\to+\infty} p\,f$ in $\mathcal{D}'(\mathbb{R}^d)$, hence $Lg_t \to_{t\to+\infty} f$ in $\mathcal{D}'(\mathbb{R}^d)$, since p is smooth and positive.

Assume in addition that there exists a solution $\psi \in \mathbb{L}^2(\mu)$ of the Poisson equation $L^*\psi = \varphi$. Thanks to the assumptions, ψ belongs to C^∞ and solves the Poisson equation in the usual sense. Hence

$$\int g_t\,\varphi\,d\mu = \int g_t\,L^*\psi\,d\mu = \int Lg_t\,\psi\,d\mu \to_{t\to+\infty} \int f\,\psi\,d\mu\,.$$

It follows that for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle p\,g_t\,,\,\varphi\rangle \to_{t\to+\infty} a(\varphi) = \int f\,\psi\,d\mu$$

where the bracket denotes the duality bracket between $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$. Thanks to the uniform boundedness principle it follows that there exists an element $\nu \in \mathcal{D}'(\mathbb{R}^d)$ such that $p\,g_t \to \nu$ in $\mathcal{D}'(\mathbb{R}^d)$, and using again smoothness and positivity of p, we have that $g_t \to g = \nu/p$. We immediately deduce that $Lg = f$ in $\mathcal{D}'(\mathbb{R}^d)$, hence thanks to (H3) that $g \in C^\infty$. Let us summarize all this

Lemma 3.23. *Consider the assumptions of proposition 3.21 and assume that for all* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *there exists a solution* $\psi \in \mathbb{L}^2(\mu)$ *of the Poisson equation* $L^*\psi = \varphi$. *Then for all smooth* f *there exists some smooth function* g *such that* $Lg = f$.

Of course in the cases we are interested in, g does not belong to $\mathbb{L}^q(\mu)$ if $f \in \mathbb{L}^p(\mu)$, so that we cannot use previous results. We shall give sufficient conditions ensuring that the dual Poisson equation has a solution for all smooth functions with compact support (see Theorem 5.12 in section 5).

Remark 3.24 (The Kipnis Varadhan situation). *If $\varphi \in \mathcal{D}(\mathbb{R})$, we thus have*

$$\int f\varphi \, d\mu = \int Lg\varphi \, d\mu = \int \nabla g \nabla \varphi \, d\mu \leq \left(\int |\nabla g|^2 \, d\mu \right)^{\frac{1}{2}} \left| \int |\nabla \varphi|^2 \, d\mu \right|^{\frac{1}{2}}$$

so that (3.17) *is satisfied as soon as $\nabla g \in \mathbb{L}^2(\mu)$, since $\mathcal{D}(\mathbb{R})$ is everywhere dense in $\mathbb{L}^2(\mu)$.*

Remark 3.25 (Time reversal, duality, forward-backward martingale decomposition). *We have just seen that it could be useful to work with L^* too. Actually if the process is strongly ergodic, we do not know whether $\lim_{t\to+\infty} P_t^* f = 0$ for centered f's or not (the limit taking place in the $\mathbb{L}^2$ strong sense). However if the process is uniformly ergodic (i.e. $\lim_{t\to+\infty} \alpha(t) = 0$ recall definition 3.18) then $\lim_{t\to+\infty} \alpha^*(t) = 0$, as will be shown in Proposition 4.5 in section 4. Now remark that:*

$$\int_0^t f(X_s) \, ds = \int_0^t f(X_{t-s}) \, ds \, .$$

Since the infinitesimal generator of the process $s \mapsto X_{t-s}$ (for $s \leq t$) is given by L^* we can use the previous strategy replacing L by L^* and the process $X_.$ by its time reversal up to time t. It is then known that, similarly to the standard forward decomposition (2.1)*, one can associate a backward one*

$$g(X_0) - g(X_t) - (M^*)_t = \int_0^t L^* g(X_s) \, ds \, , \tag{3.26}$$

where $((M^*)_t - (M^*)_{t-s})_{0\leq s\leq t}$ is a backward martingale with the same brackets as M (in the reversible case this is just the time reversal of M). The solution to the dual Poisson equation $L^* g = f$ thus furnishes a triangular array of local martingales to which Rebolledo's FCLT applies. Thus, **all the results we have shown with the solution of the Poisson equation are still true with the dual Poisson equation, at least in the uniformly ergodic case**. *The previous remark yields another possible improvement, which is a standard tool in the reversible case, namely the so called Lyons-Zheng decomposition. If g is smooth enough, summing up the standard forward decomposition* (2.1) *and the backward decomposition* (3.26)*, we obtain the forward-backward decomposition*

$$\int_0^t (L + L^*) g(X_s) \, ds = - \left(M_t + (M^*)_t \right)$$

so that if one can solve the Poisson equation for the symmetrized operator $L^S := L + L^*$ the previous decomposition can be used to study the behavior of our additive functional. This is done in e.g. [Wu99]*, but of course what can be obtained is only a tightness result since the addition is not compatible with convergence in distribution. However, the forward-backward decomposition will be useful in the sequel.*

4. Comparison with general results on stationary sequences

The CLT and FCLT theory for stationary sequences can be used in our context. Indeed, let us assume as usual that $X_0 \sim \mu$, $0 \neq f \in \mathbb{L}^2(\mu)$, $\int f \, d\mu = 0$. We may introduce the stationary sequence of random variables $(Y_n)_{n\geq 0}$:

$$Y_n := \int_n^{n+1} f(X_s) \, ds. \tag{4.1}$$

and the partial sum $S_n := \sum_{k=0}^{n-1} Y_k$. If $f \in \mathbb{L}^1(\mu)$ and $\beta(t) \to 0$ as $t \to +\infty$, denoting by $[t]$ the integer part of t, we have that $\beta(t) \int_{[t]}^t f(X_s)\, ds \to 0$ in $\mathbb{P}_\mu$ probability as $t \to +\infty$, so that the control of the law of our additive functional reduces to the one of S_n as $n \to +\infty$. We may thus use the known results for convergence of sums of stationary sequences.

At the process level we may similarly consider the random variables $S_{[nt]}$ where $[\cdot]$ denotes the integer part again, and for $n \leq (1/\varepsilon) < (n+1)$. The remainder $S_{t/\varepsilon} - S_{[nt]}$ multiplied by a quantity going to 0 will converge to 0 in probability, so that for any k-uple of times $t_1, \dots, t_k$ we will obtain the convergence (in distribution) of the corresponding k-uple, provided the usual FCLT holds for $S_{[nt]}$.

Hence we may apply the main results in [MPU06] for instance. In particular a renowned result of Maxwell and Woodroofe ([MW00] and (18) in [MPU06]) adapted to the present situation tells us that (CLT) holds under $\mathbb{P}_\mu$ as soon as $0 \neq f \in \mathbb{L}^2(\mu)$ with $\int f\, d\mu = 0$ and

$$\int_1^\infty t^{-\frac{3}{2}} \left(\int \left(\int_0^t P_s f\, ds \right)^2 d\mu \right)^{\frac{1}{2}} dt < \infty. \tag{4.2}$$

This has been improved for chains [CL09]. For (FCLT) we recall [MPU06, Cor. 12]:

Theorem 4.3 (FCLT). *Assume that $0 \neq f \in \mathbb{L}^2(\mu)$ with $\int f\, d\mu = 0$ and that*

$$\int_1^\infty t^{-\frac{1}{2}} \, \|P_t f\|_{\mathbb{L}^2(\mu)} \, dt < \infty. \tag{4.4}$$

Then (FCLT) *holds true under $\mathbb{P}_\mu$ with $s_t^2 := \mathrm{Var}_\mu(S_t)$ and $s^2 := \lim_{t\to\infty} \frac{1}{t} s_t^2$ exists.*

Condition (4.4) is much better than both (3.11) and (3.12) when $P_t f$ goes slowly to 0. In the reversible case however, (4.4) is stronger that the Kipnis-Varadhan condition (2.9) (if one prefers Theorem 4.3 is implied by Theorem 3.13), according to what we said in Remark 3.16. Also note that in full generality it is worse than the one in Proposition 3.19 as soon as $\alpha^*_{2,p}(t) \leq c/\sqrt{t}$ and $f \in \mathbb{L}^p$. Additionally, an advantage of the previous section is the simplicity of proofs, compared with the intricate block decomposition used in the proof of the CLT for general stationary sequences.

4.1. **Mixing.** Following [CG08] (Section 3, Proposition 3.4), let $\mathcal{F}_s$ (resp. $\mathcal{G}_s$) be the σ-field generated by $(X_u)_{u\leq s}$ (resp. $(X_u)_{u\geq s}$). The strong mixing coefficient $\alpha_{mix}(r)$ is

$$\alpha_{mix}(r) = \sup_{s,F,G} \{|\mathrm{Cov}(F,G)|\}$$

where the sup runs over s and F (resp. G) $\mathcal{F}_s$ (resp. $\mathcal{G}_{s+r}$) measurable, non-negative and bounded by 1. If $\lim_{r\to\infty} \alpha_{mix}(r) = 0$ then we say that the process is strongly mixing.

Proposition 4.5. *Let α be as in definition 3.18. The following correspondence holds :*

$$\alpha^2(t) \vee (\alpha^*)^2(t) \leq \alpha_{mix}(t) \leq \alpha(t/2)\alpha^*(t/2).$$

Hence the process is strongly mixing if and only if it is uniformly ergodic (or equivalently if and only if its dual is uniformly ergodic).

Proof. For the first inequality, it suffices to take $F = P_r f(X_0)$ and $G = f(X_r)$ (respectively $F = f(X_0)$ and $G = P_r^* f(X_r)$) for f μ-centered and bounded by 1. For the second inequality, let F and G be centered and bounded by 1, respectively $\mathcal{F}_s$ and $\mathcal{G}_{s+r}$ measurable. We may apply the Markov property to get

$$\mathbb{E}_\mu[FG] = \mathbb{E}_\mu[F\, \mathbb{E}_\mu[G|X_{s+r}]] = \mathbb{E}_\mu[F\, P_r g(X_s)]$$

where g is μ-centered and bounded by 1. Indeed since the state space E is Polish, we may find a measurable g such that $\mathbb{E}_\mu[G|X_{s+r}] = g(X_{s+r})$ (disintegration of measure). But

$$\mathbb{E}_\mu[F\, P_r g(X_s)] = \mathbb{E}^*_\mu[F(X_{s-.})\, P_r g(X_0)] = \mathbb{E}^*_\mu[f(X_0)\, P_r g(X_0)] = \int P^*_{r/2} f\, P_{r/2} g\, d\mu$$

where f is similarly obtained by desintegration of the measure. Here we have used the notation $\mathbb{E}^*_\mu$ for the expectation with respect to the law of the dual process at equilibrium, which is equal to the law of the reversed process on each interval $[0,s]$ (and conversely). We conclude using Cauchy-Schwarz inequality since f and g are still bounded by 1. □

Remark 4.6. *The preceding proposition implies the following comparison:*

$$\frac{(\alpha^*)^2(2t)}{\alpha^*(t)} \leq \alpha(t)\,.$$

In particular if we know that α^* is "slowly" decreasing (i.e. there exists $c>0$ such that $\alpha^*(t) \leq c\,\alpha^*(2t)$), then $\alpha(t) \geq (1/c)\,\alpha^*(2t) \geq (1/c^2)\,\alpha^*(t)$. If both α and α^* are slowly decreasing, then they are of the same order. More generally, for $t \geq 2$ (for instance)

$$\alpha^2(t) \;\leq\; \alpha(t/2)\,\alpha^*(t/2) \;\leq\; c\,\alpha(1)\,\alpha^*(t)$$

so that $\alpha(t) \leq c_1\,(\alpha^*(t))^{1/2}$. Plugging this new bound in the previous inequality we obtain

$$\alpha^2(t) \;\leq\; \alpha(t/2)\,\alpha^*(t/2) \;\leq\; c_1\,(\alpha^*(t/2))^{3/2} \;\leq\; c_1\,c^{3/2}\,(\alpha^*(t))^{3/2}$$

i.e. $\alpha(t) \leq c_2\,(\alpha^*(t))^{3/4}$. By induction, for all $\varepsilon > 0$ there exists a constant c_ε such that

$$\alpha(t) \;\leq\; c_\varepsilon\,(\alpha^*(t))^{1-\varepsilon}.$$

Again we shall mainly use the recent survey [MPU06] in order to compare and extend the results of the previous section. Notice that $f \in \mathbb{L}^p(\mu)$ implies that $Y \in \mathbb{L}^p$.

The first main result is due to Dedecker and Rio [DR00, MPU06]: if $\int_0^t f\,P_s f\,ds$ converges in $\mathbb{L}^1(\mu)$ then (FCLT) holds true under $\mathbb{P}_\mu$ with $s_t^2 = \mathrm{Var}_\mu(S_t)$ and

$$s^2 := \lim_{t\to\infty} \frac{1}{t} s_t^2 = 2 \int \left(\int_0^{+\infty} f\,P_t f\,dt \right) d\mu.$$

In the reversible case this assumption is similar to $f \in \mathbb{D}(L^{-1/2})$ (see Remark 3.16). Using some covariance estimates due to Rio, one gets ([MPU06] page 16 (37)) the following.

Proposition 4.7 (FCLT via mixing)**.** *If $0 \neq f \in \mathbb{L}^p(\mu)$ for some $p > 2$ with $\int f\,d\mu = 0$ and $\int_1^{+\infty} t^{2/(p-2)}\,\alpha(t)\,\alpha^*(t)\,dt < \infty$, then* (FCLT) *holds true under $\mathbb{P}_\mu$ with*

$$\frac{1}{t} s_t^2 = 2 \int \left(\int_0^{\infty} f\,P_t f\,dt \right) d\mu.$$

We shall compare all these results with the one obtained in the previous section later, in particular by giving some explicit comparison results between α and $\alpha_{p,q}$ introduced in definition 3.18. But we shall below give some others nice consequences of mixing.

4.2. **Self normalization with the variance and uniform integrability.** The following characterization of the CLT goes back at least to [Den86]. The FCLT seems to be less understood [MPU06, MP06].

Theorem 4.8 (CLT)**.** *Assume that $\alpha(t)$ (or $\alpha^*(t)$) goes to 0 as $t \to +\infty$ (i.e. the process is "strongly" mixing). Then for all $0 \neq f \in \mathbb{L}^2(\mu)$ such that $\int f d\mu = 0$ and $\lim_{t\to\infty} \mathrm{Var}_\mu(S_t(f)) = \infty$, the following two conditions are equivalent:*

(1) $\left(\frac{S_t^2}{\mathrm{Var}(S_t(t))} \right)_{t\geq 1}$ *is uniformly integrable*

(2) $\left(\frac{S_t}{\sqrt{\mathrm{Var}(S_t(t))}} \right)_{t\geq 1}$ *converges in distribution to a standard Gaussian law as $t \to \infty$.*

Note that if the process is not reversible, the asymptotic behavior of $\int_0^s \left(\int f\, P_u f\, d\mu\right) du$ in unknown in general, and thus $\mathrm{Var}_\mu(S_t)$ is possibly bounded.

We turn to the main goal of this section. Our aim is to show how to use the general martingale approximation strategy (as in section 3.1) in order to get sufficient conditions for $S_t^2/\mathrm{Var}_\mu(S_t)$ to be uniformly integrable. To this end let us introduce some notation.

$$\beta(s) = \int P_s f\, P_s^* f\, d\mu \quad \text{and} \quad \eta(t) = \int_0^t \beta(s)\, ds \tag{4.9}$$

$$\mathrm{Var}_\mu(S_t) = 4\int_0^{t/2} (t-2s)\,\beta(s)\, ds = th(t). \tag{4.10}$$

If the (possibly infinite) limit exists we denote $\lim_{t\to+\infty} h(t) = 2V \le +\infty$.

Assumption 4.11. *We shall say that (Hpos) is satisfied if $\beta(s) \ge 0$ for all s large enough.*

Assumption (Hpos) is satisfied is the reversible case, in the non reversible case we only know that $\int_0^t \eta(s)\, ds > 0$. Notice that if (Hpos) is satisfied

$$2t\int_0^{t/4} \beta(s)\, ds \le \mathrm{Var}_\mu(S_t) \le 4t\int_0^{t/2} \beta(s)\, ds + O_{t\to\infty}(1), \tag{4.12}$$

for t large enough similarly to the reversible case, so that

$$2\,\eta(t/4) \le h(t) \le 4\,\eta(t/2) + O_{t\to\infty}(1).$$

Denker's theorem 4.8 allows us to obtain new results, at least CLTs, using the natural symmetrization of the generator and the forward-backward martingale decomposition.

To this end consider the symmetrized generator $L^S = \frac{1}{2}(L + L^*)$. We shall assume that the closure of L^S (again denoted by L^S) is the infinitesimal generator of a μ-stationary Markov semigroup $P_\cdot^S$, which in addition is ergodic. This will be the case in many concrete situations (see e.g [Wu99]). It is then known that the Dirichlet form associated to L^S is again $\mathcal{E}(f,g) = \int \Gamma(f,g)\, d\mu$. We use systematically the superscript ^{S} for all concerned with this symmetrization.

According to Corollary 3.10 (2), we know that for a centered $f \in \mathbb{L}^2(\mu)$ there exists a $\mathbb{L}^2(\mu)$ solution of the Poisson equation $L^S g = f$ if and only if

$$\int_0^{+\infty} t\,\left\|P_t^S f\right\|^2_{\mathbb{L}^2(\mu)}\, dt \;<\; +\infty\,. \tag{4.13}$$

According to remark 3.25 we thus have

$$\int_0^t f(X_s)\, ds = -\left(M_t + (M^*)_t\right)\,,$$

for a forward (resp. backward) martingale M_t (resp. $(M^*)_t$). In order to use Denker's theorem, it is enough to get sufficient conditions for both $(M_t)^2/\mathrm{Var}_\mu(S_t)$ and $((M^*)_t)^2/\mathrm{Var}_\mu(S_t)$ to be uniformly integrable.

To this end recall first that uniform integrability of a family F_t is equivalent (La Vallée-Poussin theorem) to the existence of a non-decreasing convex function γ such that $\lim_{u\to+\infty} \gamma(u)/u = +\infty$ and

$$\sup_t \mathbb{E}_\mu\left(\gamma(F_t)\right) < +\infty\,.$$

Recall now the following strong version of Burkholder-Davis-Gundy inequalities (see [DM80], chap. VII, Theorem 92 p.304)

Proposition 4.14. *Let γ be a C^1 convex function such that $p := \sup_{u>0} \frac{u\gamma'(u)}{\gamma(u)}$ is finite (i.e. γ is moderate). For any continuous $\mathbb{L}^2$ martingale $N_.$ define $N_t^* = \sup_{s\leq t}|N_s|$. Then the following inequalities hold*

$$\frac{1}{4p}\,\|N_t^*\|_\gamma \leq \left\| \langle N\rangle_t^{\frac{1}{2}} \right\|_\gamma \leq 6p\,\|N_t^*\|_\gamma\,,$$

where $\|A\|_\gamma = \inf\{\lambda>0\,,\,\mathbb{E}[\gamma(|A|/\lambda)]\leq 1\}$ denotes the Orlicz gauge norm.

In addition Doob's inequality tells us that the Orlicz norms of N_t^* and N_t are equivalent (with constants independent of t).

Since the brackets of the forward and the backward martingales are the same, we are reduced to show that $\int_0^t \Gamma(g)(X_s)\,ds/\mathrm{Var}_\mu(S_t)$ is a $\mathbb{P}_\mu$ uniformly integrable family. But according to the ergodic theorem

$$\frac{1}{t}\int_0^t \Gamma(g)(X_s)\,ds \text{ converges as } t\to+\infty \text{ to } \int \Gamma(g)d\mu \text{ in } \mathbb{L}^1(\mathbb{P}_\mu). \tag{4.15}$$

It follows first that $\mathrm{Var}_\mu(S_t) = O(t)$. Otherwise $(M_t)^2/\mathrm{Var}_\mu(S_t)$ would converge to 0 in $\mathbb{L}^1(\mathbb{P}_\mu)$ (the same for the backward martingale), implying the same convergence for $S_t^2/\mathrm{Var}_\mu(S_t)$ whose $\mathbb{L}^1$ norm is equal to 1, hence a contradiction. If (Hpos) is satisfied, according to (4.12) we thus have that $\eta(t) = O(1)$ (and accordingly $h(t) = O(1)$), hence $(M_t)^2/\mathrm{Var}_\mu(S_t)$ and $((M^*)_t)^2/\mathrm{Var}_\mu(S_t)$ are uniformly integrable. But we do not really need (Hpos) here, only a lower bound $\liminf \mathrm{Var}_\mu(S_t)/t \geq c > 0$. Summarizing all this we have shown

Proposition 4.16. *Assume that the process is strongly mixing and that* (4.13) *is satisfied. Assume in addition that* $\liminf \mathrm{Var}_\mu(S_t)/t > 0$*. Then $S_t/\sqrt{\mathrm{Var}_\mu(S_t)}$ converges in distribution to a standard normal law, as $t\to+\infty$.*

Notice that in this situation one can find some positive constants c and d such that $0 < c \leq \mathrm{Var}_\mu(S_t)/t \leq d$ for large t's, and that the latter is ensured if (Hpos) holds.

4.3. **A non-reversible version of Kipnis-Varadhan result.** Finally what happens if one cannot solve the symmetrized Poisson equation, but if $f \in \mathbb{D}((-L^S)^{-1/2})$, i.e. if one can apply Kipnis-Vardahan theorem to the symmetrized process $X_.^S$?

Coming back to the proof of Theorem 3.13 we may introduce g_T^S so that ∇g_T^S converges to some h in $\mathbb{L}^2$ as T goes to $+\infty$.

We thus have an approximate forward-backward decomposition

$$S_t = -\frac{1}{2}\left(M_t^T + (M^*)_t^T\right) + \int_0^t P_T^S f(X_s)\,ds\,. \tag{4.17}$$

We first look at the corresponding forward martingale M_t^T whose bracket is given by

$$\langle M^T\rangle_t = \int_0^t |\nabla g_T^S|^2(X_s)\,ds\,.$$

We then have for a convex function γ,

$$\begin{aligned}
\mathbb{E}_\mu\left[\gamma(\langle M^T\rangle_t/t)\right] &= \mathbb{E}_\mu\left[\gamma\left(\frac{1}{t}\int_0^t |\nabla g_T^S|^2(X_s)\,ds\right)\right] \\
&\leq \frac{1}{t}\,\mathbb{E}_\mu\left[\int_0^t \gamma(|\nabla g_T^S|^2)(X_s)\,ds\right] \\
&\leq \int \gamma(|\nabla g_T^S|^2)\,d\mu\,.
\end{aligned}$$

Since $|\nabla g_T^S|$ is strongly convergent in $\mathbb{L}^2$, it is uniformly integrable. So we can find a function γ as in Proposition 4.14 such that the right hand side of the previous inequality is bounded by some $K < +\infty$ for all T. Hence applying Proposition 4.14 we see that $((M_t^T)^2/t))_{(T,t)}$ is uniformly integrable. The same holds for the backward martingale.

It remains to control

$$A(T,t) = \mathbb{E}_\mu \left[\gamma \left(\frac{1}{t} \left(\int_0^t P_T^S f(X_s)\, ds \right)^2 \right) \right] .$$

But we know that $P_T^S f$ goes to 0 in $L^2(\mu)$. So there exists some γ such that $\gamma((P_T^S f)^2)$ is uniformly integrable. Up to a subsequence (we already work with subsequences) we may assume that the convergence holds true μ almost surely, applying Vitali's convergence theorem we thus have (we may choose $\gamma(0) = 0$) that

$$\int \gamma\left((P_T^S f)^2\right) \, d\mu \to 0 \ \text{ as } T \to +\infty .$$

We thus may apply Cesàro's theorem, which furnishes some non-decreasing function $T(t)$ such that $\sup_t A(T(t), t) < +\infty$.

We may now conclude as for the proof of Proposition 4.16, obtaining the following reinforcement which is some non-reversible version of Kipnis-Varadhan theorem (at the CLT level), since we already proved that

$$\int_0^{+\infty} \left\| P_t^S f \right\|^2_{\mathbb{L}^2(\mu)} dt < \infty$$

is ensured by the condition (3.17):

Theorem 4.18. *Assume that the process is strongly mixing and that* (3.17) *is satisfied. Assume in addition that* $\liminf \mathrm{Var}_\mu(S_t)/t \geq c > 0$ *(or equivalently that* $V_- > 0$*). Then* $S_t/\sqrt{\mathrm{Var}_\mu(S_t)}$ *converges in distribution to a standard normal law, as* $t \to +\infty$*.*

Notice that in this situation one can find some positive constants c *and* d *such that* $0 < c \leq \mathrm{Var}_\mu(S_t)/t \leq d$ *for large* t*'s, again this is satisfied if (Hpos) holds.*

According to the discussion after Proposition 4.16, the upper bound for the rate of convergence for $\mathbb{L}^p$ functions is the worse in the reversible situation. In a sense the previous Theorem is not so surprising. But here the condition is written for the sole function f, for which we cannot prove any comparison result.

5. Complements and examples

In this section we shall first discuss in a quite "general" framework how to compare all the results described in the preceding two sections. This will be done by studying the asymptotic behavior of P_t. Next we shall describe explicit examples

5.1. **Trends to equilibrium.** In order to apply corollary 3.10 we thus have to find tractable conditions on the generator in order to control the decay of the $\mathbb{L}^2$ norm of $P_t f$. Such controls are usually obtained for all functions in a given class. The general smallest possible class is $\mathbb{L}^\infty$ so that it is natural to introduce Definition 3.18.

The uniform decay rate furnishes a first p, r-decay rate as follows

Lemma 5.1. *If* $1 \leq p \leq 2$

$$\alpha_{p,r}(t) \leq 2^{1+(p/r)} \, \alpha^{\frac{r-p}{r}}(t) ,$$

while if $2 \leq p$*,*

$$\alpha_{p,r}(t) \leq 2^{1+(p/r)} \, \alpha^{\frac{2}{p}\,\frac{r-p}{r}}(t) .$$

Proof. The proof is adapted from [CG09]. Pick some $K > 1$ and define $g_K = g \wedge K \vee -K$. Since $\int g d\mu = 0$, defining $m_K = \int g_K \, d\mu$ it holds

$$|m_K| = \left| \int g_K \, d\mu \right| = \left| \int (g_K - g) \, d\mu \right| \leq \int (|g| - K) \, \mathbf{1}_{|g| \geq K} \, d\mu \leq \|g\|_r^r / K^{(r-1)} .$$

Similarly,

$$\|g-g_K\|_p^p \leq \int |g|^p \,\mathbf{1}_{|g|\geq K}\, d\mu \leq \|g\|_r^r/K^{r-p}.$$

Using the contraction property of P_t in $\mathbb{L}^p(\mu)$ we have

$$\begin{aligned}
\|P_tg\|_p &\leq \|P_tg - P_tg_K\|_p + \|P_t(g_K-m_K)\|_p + |m_K| \\
&\leq \|P_t(g_K-m_K)\|_p + \|g-g_K\|_p + |m_K| \\
&\leq \mathrm{Var}_\mu^{1/2}(P_tg_K) + \|g\|_r^{r/p}/K^{(r-p)/p} + \|g\|_r^r/K^{(r-1)} \\
&\leq \mathrm{Var}_\mu^{1/2}(P_tg_K) + \left(2/K^{(r-p)/p}\right),
\end{aligned}$$

the latter being a consequence of $\|g\|_r = 1$ and $K > 1$. It follows

$$\|P_tg\|_p \leq \alpha(t)\, K + 2\, K^{-(r-p)/p}.$$

It remains to optimize in K. Actually up to a factor 2 we know that the optimum is attained for $\alpha(t)\, K = 2\, K^{-(r-p)/p}$ i.e. for $K = (2/\alpha(t))^{p/r}$ (which is larger than one), hence the first result.

The second one is immediate since for $p \geq 2$, $\alpha_{p,\infty}(t) \leq \alpha^{\frac{2}{p}}(t)$, and we may follow the same proof without introducing the variance. □

Note that up to a factor 2 due to the proof, the result is coherent for $r = +\infty$.

We can complete the result by the following well known consequence of the semigroup property

Lemma 5.2. *For $r = p \geq 1$, either $\alpha_{p,p}(t) = 1$ for all $t \geq 0$, or there exist positive constants c_p and C_p such that $\alpha_{p,p}(t) \leq C(p)\, e^{-c_pt}$.*

When the second statement is in force we shall (abusively in the non-reversible case) say that L has a spectral gap. We shall discuss in the next section conditions for the existence of a spectral gap or for the obtention of the optimal uniform decay rate.

Of course for $f \in \mathbb{L}^p$ for some $p \geq 2$ a sufficient condition for (3.11) to hold is

$$\int_0^{+\infty} \alpha_{2,p}(t)\, dt \;<\; +\infty\,. \tag{5.3}$$

Remark 5.4. *Specialists in interpolation theory certainly will use Riesz-Thorin theorem in order to evaluate $\alpha_{p,r}$. Let us see what happens.*

Consider the linear operator $T_tf = P_tf - \int f\, d\mu$. As an operator defined in $\mathbb{L}^2(\mu)$ with values in $\mathbb{L}^2(\mu)$, T_t is bounded with an operator norm equal to 1. As on operator defined in $\mathbb{L}^\infty(\mu)$ with values in $\mathbb{L}^2(\mu)$, T_t is bounded with an operator norm equal to $2\,\alpha(t)$. Hence T_t is bounded from $\mathbb{L}^r(\mu)$ to $\mathbb{L}^2(\mu)$ (for $r \geq 2$) with an operator norm smaller than or equal to $2^{2(1-\frac{1}{r})}\, \alpha^{\frac{r-2}{r}}(t)$, which is (up to a slightly worse constant) the same result as the one obtained in lemma 5.1. The same holds for the pair $(1,r)$, and then for all (p,r). The main advantage of the previous lemma is that the proof is elementary. See also [CGR10] *for further developments on this subject.* ♢

In section 3.2 we used $\alpha_{2,p}$ for $p > 2$. It seems that in full generality the relation $\alpha_{2,p}(t) = c_p\, \alpha^{\frac{p-2}{p}}(t)$ is the best possible. However it is interesting to notice the following duality result

Lemma 5.5. *For all pair $1 \leq p < r \leq +\infty$ there exists $c(p,r)$ such that*

$$\alpha_{p,r}(t) \leq c(p,r)\, \alpha^*_{\frac{r}{r-1},\frac{p}{p-1}}(t)\,.$$

Proof. If $f \in \mathbb{L}^r$ is such that $\int f d\mu = 0$, for all $g \in \mathbb{L}^{\frac{p}{p-1}}$, we have

$$\int P_t f\, g\, d\mu = \int P_t f \left(g - \int g d\mu\right) d\mu = \int f\, P_t^* \left(g - \int g d\mu\right) d\mu$$

hence the result. □

As a consequence we obtain that

Lemma 5.6. *For $1 < p \leq 2$, $\alpha_{1,p}(t) \leq c(p)\ (\alpha^*(t))^{\frac{2(p-1)}{p}}$.*

This result is of course much better (up to a square) than the one obtained in lemma 5.1 in this situation, since we know that for slowly decreasing α and α^* these functions are equivalent (up to some constants). It can also be compared with similar results obtained in [CG09].

Remark 5.7. *These results allow us to compare conditions obtained in Proposition 3.19, Proposition 3.21 on one hand, and Theorem 4.3 or Proposition 4.7 on the other hand.*

For example, if we use the bound obtained in lemma 5.1, proposition 3.21 tells that convergence to a brownian motion holds provided

$$\int_0^{+\infty} (\alpha(t)\,\alpha^*(t))^{\frac{p-2}{p}}\, dt\ < +\infty\,.$$

(Remark that it is exactly the condition in [Jon04] *Theorem 5). Notice that as soon as $\alpha(t)\,\alpha^*(t)\ <\ 1/t$ this bound is worse than the one in proposition 4.7, so that the mixing approach seems to be at least as interesting as the usual one.*

However, in the diffusion case we shall obtain in proposition 5.10 below a better bound for $\alpha^*_{2,p}$. Combined with remark 4.6, it yields (under the appropriate hypotheses) the condition

$$\int_0^{+\infty} (\alpha^*(t))^{\varepsilon + \frac{2(p-2)}{p-1}}\, dt\ < +\infty\,,$$

for some $\varepsilon \geq 0$ (0 is allowed in the slowly decreasing case), which is better than the mixing condition in proposition 4.5 as long as $\alpha^*(t) > (1/t)^{(\frac{p-1}{p-2})-\eta}$ for some $\eta \geq 0$. ♢

The question is: how to find α ?

5.2. **Rate of convergence for diffusions.** In "non degenerate" situations, α is given by weak Poincaré inequalities:

Definition 5.8. *μ satisfies a weak Poincaré inequality (WPI) for Γ with rate β if for all $s > 0$ and all f in the domain of Γ (or some core) the following holds,*

$$\mathrm{Var}_\mu(f)\ \leq\ \beta(s)\,\mathcal{E}(f,f) + s\mathrm{Osc}^2(f)$$

where $\mathrm{Osc}(f) = esssupf - essinff$ is the oscillation of f.

Proposition 5.9. *(*[RW01] ***Theorem 2.1 and Theorem 2.3)*** *If μ satisfies (WPI) with rate β then both $\alpha(t)$ and $\alpha^*(t)$ are less than $2\,\xi^{\frac{1}{2}}(t)$ where $\xi(t) = \inf\{s > 0, \beta(s)\, \log(1/s) \leq t\}$.*

If L is μ-reversible (or more generally normal) some converse holds, i.e. decay with uniform decay rate α implies some corresponding (WPI).

It is actually quite hard to check, in the reversible case, whether starting with some (WPI) one obtains a ξ which in return furnishes the *same* (WPI) (see the quite intricate expression of β in [RW01] Theorem 2.3). It seems that in general one can loose some slowly varying term (like a log for instance).

Notice that (WPI) implies the following: $\mathcal{E}(f,f) = 0 \Rightarrow f$ constant i.e. the Dirichlet form is non degenerate. In the degenerate case of course, the uniform decay rate cannot be controlled via a functional inequality. The most studied situation being the diffusion case we now focus on it.

First we recall the following explicit control proved in [BCG08] Theorem 2.1 (using the main result of [DFG09])

Proposition 5.10. *Let L be given by (2.4). Assume that there exists a φ-Lyapunov function V (belonging to the domain $\mathbb{D}(L)$) for some smooth increasing concave function φ and for C some compact subset. Define $H_\varphi(t) = \int_1^t (1/\varphi(s))ds$ and assume that $\int V d\mu < +\infty$.*

Then, if $\lim_{u\to+\infty} \varphi'(u) = 0$,

$$(\alpha^*)^2(t) \leq C\left(\int V\,d\mu\right)\frac{1}{\varphi\circ H_\varphi^{-1}(t)}.$$

If for $p > 2$ and q its conjugate, $V \in \mathbb{L}^q(\mu)$ then

$$\alpha^*_{2,p}(t) \leq C(p, \|V\|_q)(\alpha^*)^{\frac{p-2}{p-1}}(t).$$

If φ is linear, $\alpha^*(t)$ and $\alpha(t)$ are decaying like $e^{-\lambda t}$ for some $\lambda > 0$ (see [DMT95, BCG08, BBCG08]*).*

Note that the latter bound is better than the general one obtained in lemma 5.1. Of course we may use either remark 3.25 (telling that we may use α^* instead of α) or Remark 4.6 (comparing both rates) to apply this result.

In the same spirit we shall also recall a beautiful result due to Glynn and Meyn [GM96] or more precisely the version obtained in Gao-Guillin-Wu [GGW10]:

We introduce the Lyapunov control condition, as in [GM96, GGW10]

Assumption 5.11. *there exist a positive function F, a compact set C, a constant b and a (smooth) function θ, going to infinity at infinity such that*

$$L^*\theta \;\leq\; -F \,+\, b\,\mathbf{1}_C\,.$$

Then we have the following (Theorem 3.2 in [GM96] and its refined version Lemma 6.2 in [GGW10])

Theorem 5.12. *If Assumption 5.11 is satisfied and $\theta^2 \in \mathbb{L}^1(\mu)$, the Poisson equation $Lg = f$ admits a solution in $\mathbb{L}^2$, provided $|f| \leq F$. Hence the usual FCLT holds*

The authors get the FCLT in Theorem 4.3 of [GM96], but we know how to do in this situation.

Assumption 5.11 is thus enough in order to ensure the existence of a $\mathbb{L}^2$ solution of the Poisson equation for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, so that if this assumption is satisfied we may use Lemma 3.23 (i.e. the existence of a smooth solution (but non necessarily $\mathbb{L}^2(\mu)$) to the Poisson equation for any smooth f).

We shall continue this section by providing several families of examples, starting with the one-dimensional case. These examples are then extended to n-dimensional reversible Langevin stochastic differential equations using Lyapunov conditions and results of [BCG08, BBCG08, CGGR10] to recover Poincaré inequalities or weak Poincaré inequalities through the use of Lyapunov conditions, and so the rate α^* or α.
We will then consider elliptic (non necessarily reversible) examples for which result of [DFG09], recalled in Proposition 5.10, furnishes the rate α^* and then existence of the solution of Poisson equation and CLT where the usual Kipnis-Varadhan condition cannot be used. Comparisons with the recent results of Pardoux-Veretennikov [PV01] will be made.
We will end with some hypoelliptic cases such as the kinetic Fokker-Planck equation or oscillator chains for which results of [DFG09, BCG08] still apply, and results of [PV05] are harder to consider. It is of particular interest in PDE theory.

One of the main strategy to get explicit convergence controls are Lyapunov conditions as explained before.

5.3. **Reversible case in dimension one.**

5.3.1. *General criterion for weak Poincaré inequalities.* We recall here results of [BCR05] giving necessary and sufficient conditions for a one dimensional measure $d\mu(x) = e^{-V(x)}dx$, associated to the one dimensional diffusion

$$dX_t = \sqrt{2}dB_t - V'(X_t)dt$$

to satisfy a weak Poincaré inequality.

Proposition 5.13. [BCR05, Theorem 3] *Let m be a median of μ, and $\beta : (0, 1/2) \to \mathbb{R}_+$ be non increasing. Let C be the optimal constant such that for all f and $0 < s < 1/4$*

$$\mathrm{Var}_\mu(f) \le C\,\beta(s) \int f'^2 d\mu + s\,\mathrm{Osc}(f)^2$$

then $1/4 \max(b_-, b_+) \le C \le 12 \max(B_+, B_-)$ where, with m a median for μ

$$\begin{aligned} b_+ &= \sup_{x>m} \frac{\mu([x,\infty[)}{\beta(\mu([x,\infty[)/4)} \int_m^x e^V\,dx \\ B+ &= \sup_{x>m} \frac{\mu([x,\infty[)}{\beta(\mu([x,\infty[))} \int_m^x e^V\,dx \end{aligned}$$

and the corresponding ones for b_-, B_- with the left hand side of the median.

5.3.2. *A first particular family : general Cauchy laws.* Consider the diffusion process on the line

$$dX_t = \sqrt{2}\,dB_t - \left(\frac{\alpha\,x}{1+x^2} + \frac{2\beta\,x}{(e+x^2)\,\log(e+x^2)} \right) dt \tag{5.14}$$

for some parameters $\alpha > 1$ and $\beta \ge 0$. The model is slightly more general than the usual Cauchy laws considering $\beta = 0$, but the difference allows interesting behaviors. The corresponding generator is

$$L = \partial^2_{x^2} - \left(\frac{\alpha\,x}{1+x^2} + \frac{2\beta\,x}{(e+x^2)\,\log(e+x^2)} \right) \partial_x$$

so that L is μ-reversible for

$$\mu(dx) = \frac{c(\alpha,\beta)}{(1+x^2)^{\alpha/2}\,\log^\beta(e+x^2)}\,dx\,.$$

It is immediate that $V(x) = x^2$ satisfies

$$LV(x) = 2\,\frac{1-(\alpha-1)x^2}{1+x^2} - \frac{4\beta x^2}{(e+x^2)\,\log(e+x^2)} \tag{5.15}$$

hence verifies the assumption in proposition 2.7. So the process defined by (5.14) does not explode (is conservative if one prefers), and is ergodic with unique invariant measure μ, which satisfies a local Poincaré inequality on any interval.

The rate $\alpha_{2,\infty}$ is known in this situation. Indeed, according to Proposition 5.13, μ satisfies a weak Poincaré inequality (recall definition 5.8) with optimal rate

$$\beta(s) = d(\alpha,\beta)\,s^{-2/(\alpha-1)}\,\log^{-2\beta/(\alpha-1)}(1/s)\,.$$

According to Proposition 5.9 (and its converse in the reversible case), for large t,

$$\alpha_{2,\infty}(t) \simeq \xi^{\frac{1}{2}}(t) \quad \text{with} \quad \xi(t) = \frac{1}{t^{\frac{(\alpha-1)}{2}}}\,\log^{\frac{(\alpha-1)}{2}-\beta}(t)\,.$$

In the sequel we shall only consider bounded functions f.

If $\alpha > 3$ or $\alpha = 3$ and $\beta > 2$, $\alpha_{2,\infty}^2$ is integrable, and so we may apply Kipnis-Varadhan theorem to all bounded functions f.

Interesting cases are $\alpha = 3$ and $\beta \leq 2$.

If $\beta > 1$, $\theta(x) = |x|$ for large $|x|$'s satisfies the assumptions in Theorem 5.12, and accordingly the usual FCLT holds provided $|f(x)| \leq c/|x|$ at infinity. If $\beta \leq 1$ a similar result holds but this time for $|f(x)| \leq c/|x|^{1+\varepsilon}$ at infinity, for any $\varepsilon > 0$.

But it should be interesting to know what happens for bounded f's that do not go to 0 at infinity.

5.3.3. *A second general family: subexponential laws.* Let us consider the process on the line

$$dX_t = \sqrt{2}dB_t - \alpha x\,|x|^{\alpha-2}dt$$

for $\alpha < 1$ with the generator

$$L = \partial^2_{x^2} - \alpha x\,|x|^{\alpha-2}\,\partial_x$$

which is ν_α reversible where

$$\nu_\alpha(dx) = C(\alpha)\,e^{-|x|^\alpha}dx.$$

It is well known the process does not explode and ergodic with unique invariant measure μ. By Proposition 5.13, one easily gets that ν_α satisfies a weak Poincaré inequality with $\beta(s) = k_\alpha \log(2/s)^{\frac{2}{\alpha}-2}$. According to Proposition 5.9 (and its converse in the reversible case), for large t,

$$\alpha_{2,\infty}(t) \simeq \xi^{\frac{1}{2}}(t) \quad \text{with} \quad \xi(t) = e^{-ct^\alpha}.$$

It is then of course immediate by Kipnis-Varadhan theorem, and Proposition 3.19 for tractable conditions, to get that as soon as $f \in \mathbb{L}^p$ for $p > 2$ then it satisfies the FLCT. Of course, the interesting examples are in unbounded test functions like $f(x) = e^{\frac{1}{2}|x|^\alpha}g(x) - c$ for g in $\mathbb{L}^2(dx)$ but not in any $\mathbb{L}^p(dx)$ for any $p > 2$. We believe that in this context, one may exhibit anomalous speed in the FCLT, as in the Cauchy case explored in the following sections. It does not seem that interesting new examples may be sorted out using Glynn-Meyn's result.

5.4. **Reversible case in general.** We quickly give here multidimensional Langevin-Kolmogorov reversible diffusions example (say in $\mathbb{R}^n$), that may be treated as in the one-dimensional case using the appropriate Lyapunov conditions and weak Poincaré inequalities.

5.4.1. *Cauchy type measures.* Let us consider with $\alpha > n$

$$\mu_\alpha(dx) := Z\,(1+|x|^2)^{\alpha/2}\,dx$$

associated to the generator

$$L = \Delta - \frac{\alpha x}{1+|x|^2}.\nabla$$

reversible with respect to μ. In fact one may use as in the one dimensional case Lyapunov functions $W(x) = |x|^k$ for large $|x|$ so that for large $|x|$

$$LW = (nk + k(k-2))\,|x|^{k-2} - k\alpha\frac{|x|^k}{1+|x|^2}$$

so that to get a Lyapunov condition we have to impose the compatibility condition $\alpha > n + k - 2$.

Use now Theorems 2.8 and 5.1 in [CGGR10] to get a weak Poincaré inequality with $\beta(s) = c(n,\alpha)s^{-\frac{2}{\alpha-n}}$ leading to

$$\alpha_{2,\infty}(t) = c'(\alpha,n)\frac{\log^{\frac{\alpha-n}{2}}(t)}{t^{\frac{\alpha-n}{2}}}.$$

We then get that if $\alpha > n+2$ then $\alpha^2_{2,\infty}$ is integrable and thus Kipnis-Varadhan theorem may be used for all bounded functions. Note that in this case, one does not recover the optimal speed of decay via the results of [DFG09].
We may also use Theorem 5.12 to consider unbounded function: for $k \geq 2$, if $\alpha > n+2k$ and $\alpha > n+k-2$ then the usual FCLT holds for all centered function f such that $|f| \leq c(1+|x|^{k-2})$.
One may also, in the setting where $K \geq 2$, f is centered with $|f| \leq c(1+|x|^{k-2})$ and $\alpha > n+2(k-2)$ (so that $f \in \mathbb{L}^\beta$ for $\beta < \frac{\alpha-n}{k-2}$), use Prop. 3.19: if $\alpha > n+2k-3$ then the FCLT holds. Note that it gives better results than Theorem 5.12.
One may of course generalize the model ($\beta \neq 0$) as in the one-dimensional case, which would lead to the same discussion as in the one-dimensional case.

5.4.2. *Subexponential measures.* Let us consider for $0 < \alpha < 1$,

$$\nu_\alpha(dx) = C(\alpha)\, e^{-|x|^\alpha} dx$$

associated to the ν_α-reversible generator

$$L = \Delta - \alpha x\, |x|^{\alpha-2} .\nabla.$$

With $W(x) = e^{a|x|^\alpha}$ for large $|x|$, one easily gets that for large $|x|$

$$LW(x) \leq -c\alpha^2 a(a-1)\, |x|^{2\alpha-2} e^{a|x|^\alpha}$$

so that by Theorems 2.8 and 5.1 in [CGGR10], we get that ν_α verifies a weak Poincaré inequality with $\beta(s) = k_{n,\alpha} \log(2/s)^{\frac{2}{\alpha}-2}$. We may then mimic the results given in the one dimensional case.

5.5. **Beyond reversible diffusions.** We will focus here on general diffusion models on $\mathbb{R}^n$, with the notations of [PV01, PV05] for easier comparisons,

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

with generator

$$L = \sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_i,x_j} + \sum_{i=1}^n b_i(x)\partial_{x_i},$$

and $a = \sigma\sigma^*/2$. We will suppose that σ is bounded and b, σ locally (bounded) Lipschitz functions. We assume moreover a condition on the diffusion matrix

$$(H_\sigma): \qquad \left\langle a(x)\frac{x}{|x|}, \frac{x}{|x|}\right\rangle \leq \lambda_+\,, \qquad Tr(\sigma\sigma^*)/n \leq \Lambda.$$

Note that Pardoux and Veretennikov also impose an ellipticity condition in [PV01], or a local Doeblin condition in [PV05] preventing however too degenerate models like kinetic Fokker-Planck ones. We also introduce the following family of recurrence conditions

$$(H_b(r,\alpha)): \qquad \forall |x| \geq M, \qquad \left\langle b(x), \frac{x}{|x|}\right\rangle \leq -r|x|^\alpha.$$

We suppose $M > 0$, $\alpha \geq -1$, and when $\alpha = -1$, that the process does not explode (it will be a consequence of the Lyapunov conditions given later). We also define when $\alpha = -1$, $r_0 = (r - \Lambda n)/2)/\lambda_+$. We may then use the results of [DMT95, DFG09] and [PV01] to get that

$$\alpha_*(t)^2 \leq \begin{cases} C\, e^{-ct} & \text{if } \alpha \geq 0, \\ C\, e^{-ct^{\frac{1+\alpha}{1-\alpha}}} & \text{if } -1 \leq \alpha < 0, \\ C\,(1+t)^{-k} & \text{if } \alpha = -1 \text{ and } 0 < k < r_0, \end{cases}$$

for some (usually non explicit) constants $C, c > 0$. Note that these results are obtained using Lyapunov functions $W_1(x) = e^{a|x|}$, $W_2(x) = e^{a|x|^{1+\alpha}}$ and $W_3(x) = 1 + |x|^{2k+2}$ respectively, for some $a < \frac{2r}{\lambda_+(1+\alpha)}$ whenever $\alpha > -1$). Namely outside a large ball, for some positive λ

$$\begin{aligned} \alpha \geq 0, \qquad & LW_1 \leq -\lambda W_1, \\ -1 < \alpha < 0, \qquad & LW_2 \leq -\lambda\, W_2\, [\ln W_2]^{2\frac{\alpha}{1+\alpha}}, \\ \alpha = -1, \qquad & LW_3 \leq -\lambda W_3^{\frac{m-2}{m}}. \end{aligned}$$

All this shows that the process is positive recurrent. We denote by μ its invariant probability measure. Remark that the convergence rate in the last case is slightly better than the one in Pardoux-Veretennikov. Note that a direct consequence of these Lyapunov conditions is that W_1 is $\mathbb{L}^1(\mu)$, $W_2\,[\ln W_2]^{2\frac{\alpha}{1+\alpha}} \in \mathbb{L}^1(\mu)$ and $W_3^{\frac{k}{k+1}} \in \mathbb{L}^1(\mu)$. These last two integrability results are presumably not optimal, indeed results of [PV01, Proposition 1] give us in the case $\alpha = -1$ that for every $m < 2r_0 - 1$, $W_4(x) = 1 + |x|^m$ is in $\mathbb{L}^1$.

We may then use results of Proposition 3.19, or more precisely Proposition 3.21 to get results on the solution of the Poisson equation and the FCLT that we may compare with [PV01, Theorem 1]. Comparison is not so easy as Pardoux-Veretennikov's results consider function f with polynomial growth and obtain polynomial control of the solution of the Poisson equation, when our results deal with $\mathbb{L}^p$ control. Glynn-Meyn's result will help us in this direction. We will only consider here examples for $\alpha = -1$ and $-1 < \alpha < 0$, i.e. sub-exponential cases.

Case $\alpha = -1$. Pardoux-Veretennikov's result, assuming some ellipticity condition (namely the existence of a $\lambda_- > 0$ for the corresponding lower bound in (H_σ)) establishes that if $|f(x)| \leq c(1 + |x|^\beta)$ for $\beta < 2r_0 - 3$ then the solution of the Poisson equation g exists with a polynomial control in $|x|^{\beta+2+\varepsilon}$ ($\varepsilon > 0$ arbitrary) just ensuring that $g \in \mathbb{L}^1$. They also obtain a polynomial upper control of $|\nabla g|$. We have not pushed too much further in this last direction but elements of the next sections may give integrability results for $|\nabla g|$.
To use Proposition 3.21 in our context, one has to verify, for smooth f in $\mathbb{L}^p$ for simplicity, that $\alpha(t)\alpha^*(t)$ is sufficiently decreasing. Using Remark 4.6, one gets here that for all $k < r_0$

$$\alpha(t)\alpha^*(t) \leq c_k t^{-k}$$

and we have thus to impose the condition that $k(p-2) > p$. Our results are then weaker than Pardoux-Veretennikov as it enables us only to consider f to be in $\mathbb{L}^p$ for $p > 2$ whereas they consider f in $\mathbb{L}^m$ for $m < (2r_0 - 1)/(2r_0 - 3)$.

Note however that we have no ellipticity assumption, and we refer to examples in the next paragraph, which cannot be obtained using the results of Pardoux-Veretennikov.
Remark finally that our results do not only apply to the existence of the solution of the Poisson equation but also to the FCLT, with a finite variance, which is not at all ensured by Pardoux-Veretennikov's results. In this perspective, if we want to use Pardoux-Veretennikov result to get a finite variance, we will have to impose that there exists $p \geq 1$ such that $\max(p\beta, \frac{p}{p-1}(\beta+2)) < 2r_0 - 1$, which will imply that for $p \geq 2$ one has to impose $(r_0 - 1/2)(p-2) > p$ which is slightly stronger than our conditions.

Case $-1 < \alpha < 0$. In fact, by the results of Pardoux-Veretennikov, one has that for f bounded by a polynomial, then g is also bounded by a polynomial and thus at least in $\mathbb{L}^1$. We get much more general results here as we allow, for example, smooth f such that there exists $C > 0$ with

$$|f(x)| \leq C\, e^{\left(\frac{r}{\lambda_+(1+\alpha)} - \varepsilon\right)|x|^{1+\alpha}}$$

for $\varepsilon > 0$.

Note also that no additional ellipticity condition is supposed, and even in the subsequent work [PV05], the local Doeblin condition and condition (A_T) (see [PV05, Page 1113] seems to be verified in only slightly degenerate case. We will then give here particular examples that may be reached through our work.

5.6. **Kinetic models.** Consider a kinetic system, where v is the velocity (in $\mathbb{R}^d$) and x is the position. The motion of v is perturbed by a Brownian noise, i.e. we consider the diffusion process $(X_t, V_t)_{t\geq 0}$ with state space $\mathbb{R}^d \times \mathbb{R}^d$ solution of the kinetic stochastic differential equation

$$\begin{cases} dx_t &= v_t\, dt, \\ dv_t &= H(v_t, x_t)dt + \sqrt{2}dB_t. \end{cases}$$

If the initial law of (x_0, v_0) is ν we denote by $P(t, \nu, dx, dv)$ the law at time t of the process. A standard scaling (see e.g. [DM08]) is to consider

$$P^\varepsilon(t, \nu, dx, dv) = \varepsilon^{-d}\, P\left(\frac{t}{\varepsilon^2}, \nu^\varepsilon, \frac{dx}{\varepsilon}, dv\right)$$

i.e. the law of the scaled process $(\varepsilon\, x_{t/\varepsilon^2}\,,\, v_{t/\varepsilon^2})$ (also rescale the initial law), solution of

$$\varepsilon\partial_t P + v\cdot\nabla_x P - \frac{1}{\varepsilon}\left(\Delta_v P + \mathrm{div}_v(H\,P)\right) = 0\,. \tag{5.16}$$

The FCLT with $v(\varepsilon) = \sqrt{\varepsilon}$, if it holds, combined with a standard argument of propagation of chaos (see [CCM10] for more details) implies that as ε goes to 0, $P^\varepsilon(t, dx, dv)$ converges to the product $N(t, dx)\, M(dv)$ where $M(dv)$ is the projection of the invariant measure of the diffusion on the velocities space and $N(t, dx)$ is the solution of the appropriate (depending on the asymptotic variance) heat equation on the positions space.

Let us present more concrete examples where we can use the results of the paper just using $f(v) = v$ or $f(x, v) = v$, as well as the possible necessity of using another scaling in space (anomalous rate of convergence), via explicit speed of convergence obtained as previously via Lyapunov conditions.

Kinetic Fokker-Planck equation.
Let us consider the following stochastic differential system

$$\begin{aligned} dx_t &= v_t\, dt, \\ dv_t &= \sqrt{2}\, dB_t - v_t\, dt - \nabla F(X_t)\, dt, \end{aligned}$$

where (B_t) is a $\mathbb{R}^d$-Brownian motion. The invariant (but non-reversible) probability measure is then $\mu(dx, dv) = Z^{-1}\, e^{-\left(\frac{1}{2}|v|^2 + F(x)\right)}\, dv\, dx$.

If $F(x)$ behaves like $|x|^p$ for large $|x|$ with $0 < p < 1$ then one can build a Lyapunov function $W(x, v)$ behaving at infinity as $e^{a(|v|^2 + |x|^p)}$ (for s sufficiently small) and such that outside a large ball (see [DFG09, BCG08])

$$LW \leq -\lambda\, W\, [\ln W]^{2\frac{p-1}{p}}\,.$$

We may thus apply the results explained in the previous case $-1 < \alpha < 0$.

Oscillator chains.
We present here the model studied by Hairer-Mattingly [HM09]: 3-oscillator chains

$$\begin{aligned}
dq_0 &= p_0\,dt\\
dp_0 &= -\gamma_0 p_0\,dt - q_0|q_0|^{2k-2}\,dt - (q_0-q_1)\,dt + \sqrt{2\gamma_0 T_0}dB_t^0\\
dq_1 &= p_1\,dt\\
dp_1 &= -q_1|q_1|^{2k-2} - (2q_1-q_0-q_2)dt\\
dq_2 &= p_2\,dt\\
dp_2 &= -\gamma_2 p_2\,dt - q_2|q_2|^{2k-2}\,dt - (q_2-q_1)\,dt + \sqrt{2\gamma_2 T_2}dB_t^2
\end{aligned}$$

where B^0 and B^2 are two independent brownian motions. Then by Theorem 5.6 in [HM09], if $k > 3/2$, one can give a Lyapunov function W for which $LW \leq -\lambda W^r + C$ for some $r < 1$ so that we may use the results presented before in the polynomial rate case.

6. An example of anomalous rate of convergence

In all the examples developed before, the asymptotic variance was existing. We shall try now to investigate the possible anomalous rates of convergence, i.e. cases where the variance of S_t is super-linear. Instead of studying the full generality, we shall first focus on a simple example, namely the one discussed in section 5.3.2.

We consider the generator L defined in (5.15) in the critical situation $\alpha = 3$ and $\beta \leq 2$ or the supercritical one i.e $\alpha < 3$ (but $\alpha > 1$). For simplicity we shall here directly introduce the function g and choose $g(x) = x^2$, so that $f = Lg$ is bounded but does not go to 0 at infinity (hence we cannot use Theorem 5.12).

Since $\nabla g(x) = 2x$, $\nabla g \in \mathbb{L}^2(\mu)$ if and only if $\alpha = 3$ and $\beta > 1$.

According to Remark 3.24 we may thus apply Kipnis-Varadhan result, so that from now on these cases are excluded. Remark that for this particular case, Kipnis-Varadhan result applies for $\beta > 1$, while for the general bounded case (i.e. f bounded) we have to assume that $\beta > 2$. This is presumably due to the non exact correspondence between (WPI) and the decay rate ξ as noticed just after Proposition 5.9.

Our goal in this section will be to evaluate $\mathrm{Var}_\mu(S_t)$ and to see that one can apply Denker's Theorem 4.8, i.e. obtain a CLT with an anomalous explicit rate.

In the sequel, c will denote a universal constant that may change from place to place.

For $K > 0$ we introduce a truncation function ψ_K such that, $\mathbf{1}_{[-K,K]} \leq \psi_K' \leq \mathbf{1}_{[-K-1,K+1]}$ and all ψ_K'' are bounded by c (ψ_K is thus an approximation of $x \wedge K \vee -K$).

We then define $g_K = \psi_K(g)$, $f_K = Lg_K$ which is still bounded by c and such that

$$|f_K - f| \;\leq\; c\,\mathbf{1}_{|x|\geq K}\,.$$

In what follows, we shall use repeatedly the fact that, for large K

$$\begin{aligned}
\int_e^K x^a \log^\beta(x)\,dx &\simeq c(a,\beta)\left(1 + K^{a+1}\log^\beta(K)\right) \quad \text{if } a \neq -1\\
\int_e^K x^{-1} \log^\beta(x)\,dx &\simeq c(\beta)\left(1 + \log^{\beta+1}(K)\right) \quad \text{if } \beta \neq -1\\
\int_e^K x^{-1} \log^{-1}(x)\,dx &\simeq c\,(1 + \log\log(K))\,.
\end{aligned}$$

These estimates follow easily by integrating by parts (integrate x^a and differentiate the log).

Now we can write (we are using the notation in section 4.2, in particular (4.10) and (4.9)):

$$\begin{aligned}(S_t)^2 &\leq 2\,(S_t - S_t^{f_K})^2 + 2\,(S_t^{f_K})^2 \\ &\leq 2\,(S_t - S_t^{f_K})^2 + (M_t^{g_K})^2 + ((M^*)_t^{g_K})^2\,, \end{aligned} \tag{6.1}$$

or

$$(S_t)^2 \leq 2\,(S_t - S_t^{f_K})^2 + 8\,(g_K^2(X_t) + g_K^2(X_0)) + 4\,(M_t^{g_K})^2\,, \tag{6.2}$$

and

$$(S_t)^2 \geq 4\,(M_t^{g_K})^2 - 2\,(S_t - S_t^{f_K})^2 - 8\,(g_K^2(X_t) + g_K^2(X_0))\,. \tag{6.3}$$

Recall that

$$2t\,\eta(t/4) \leq \mathrm{Var}_\mu(S_t) \leq 4t\,\eta(t/2)$$

with η given in (4.9) which is non-decreasing since L is reversible. Hence we know that $\mathrm{Var}_\mu(S_t)/t$ is bounded below. This will allow us to improve on the results in section 5.3.2.

Indeed for $K > K_0$ where K_0 is large enough,

$$\begin{aligned}\mathbb{E}_\mu\left[(S_t - S_t^{f_K})^2\right] &\leq c\,\mathbb{E}_\mu\left[\int_0^t \int_0^s \mathbf{1}_{|X_s|\geq K}\,\mathbf{1}_{|X_u|\geq K}\,du\,ds\right] \\ &\leq c\,\mathbb{E}_\mu\left[\int_0^t s\,\mathbf{1}_{|X_s|\geq K}\,ds\right] \\ &\leq c\,t^2\,\mu(|x|\geq K) \leq c''(\alpha,\beta)\,t^2\,K^{1-\alpha}\,\log^{-\beta}(K)\,. \end{aligned} \tag{6.4}$$

$$\begin{aligned}\mathbb{E}_\mu\left[(M_t^{g_K})^2\right] &\leq c\,\mathbb{E}_\mu\left[\int_0^t X_s^2\,\mathbf{1}_{|X_s|\leq K+1}\,ds\right] \\ &\leq c\,t\int_{-K-1}^{K+1} x^2\,\mu(dx) \\ &\leq c(\alpha,\beta)\,t\,(1+\varphi(K))\,, \end{aligned} \tag{6.5}$$

with $\varphi(K) = K^{3-\alpha}\log^{-\beta}(K)$ if $\alpha \neq 3$, $\varphi(K) = \log^{1-\beta}(K)$ if $\alpha = 3$ and $\beta \neq 1$, and finally $\varphi(K) = \log\log(K)$ if $\alpha = 3$ and $\beta = 1$. Note that similarly

$$\begin{aligned}\mathbb{E}_\mu\left[(M_t^{g_K})^2\right] &\geq \mathbb{E}_\mu\left[\int_0^t X_s^2\,\mathbf{1}_{|X_s|\leq K}\,ds\right] \\ &\geq c\,t\int_{-K}^{K} x^2\,\mu(dx) \\ &\geq c'(\alpha,\beta)\,t\,(1+\varphi(K))\,. \end{aligned} \tag{6.6}$$

In addition

$$\begin{aligned}\int g_K^2\,d\mu &\leq c\int_{-K-1}^{K+1} \frac{x^4}{(1+|x|^\alpha)\,\log^\beta(e+|x|^2)}\,dx + 2\,K^4\,\mu(|x|>K) \\ &\leq c\,(1+K^{5-\alpha}\,\log^{-\beta}(K))\,. \end{aligned} \tag{6.7}$$

According to lemma 2.8 we already know that $\mathrm{Var}_\mu(S_t)/t$ is bounded if and only if we are in the Kipnis-Varadhan situation (in particular as we already saw if $\alpha = 3$ and $\beta > 1$). In order to get the good order for $\mathrm{Var}_\mu(S_t)/t$ by using (6.2) and (6.3) we have to choose $K(t)$ in such a way that

$$\mathbb{E}_\mu\left[(M_t^{g_K})^2\right] \gg \int g_K^2\,d\mu$$

and

$$\mathbb{E}_\mu\left[(M_t^{g_K})^2\right] \gg \mathbb{E}_\mu\left[(S_t - S_t^{f_K})^2\right]\,.$$

Hence, according to (6.5) and (6.6) as well as (6.4) and (6.7) we need for $(\alpha,\beta)\neq(3,1)$

$$t\,\left(K^{3-\alpha}\,\mathbf{1}_{\alpha>3}+\log(K)\,\mathbf{1}_{\alpha=3}\right)\,\log^{-\beta}(K)\,\gg\,\max(K^{5-\alpha}\,\log^{-\beta}(K)\,;\,t^2\,K^{1-\alpha}\,\log^{-\beta}(K))\,, \tag{6.8}$$

We immediately see that the unique favorable situation is obtained for

$$\alpha=3\text{ and }\beta\neq 1\quad\text{ and }\quad K^2\,\log(K)\gg t\gg K^2/\log(K)\,. \tag{6.9}$$

In this situation the leading term $\mathbb{E}_\mu\left[(M_t^{g_K})^2\right]$ is of order $t\,\log^{1-\beta}(K)$ i.e. of order $t\,\log^{1-\beta}(t)$.

If $\alpha=3$ and $\beta=1$ we get

$$K^2\,\log(K)\,\log\log(K)\gg t\gg K^2/\log(K)\,\log\log(K) \tag{6.10}$$

yielding this time $\mathbb{E}_\mu\left[(M_t^{g_K})^2\right]\simeq t\,\log\log(t)$.

So we now consider the cases $\alpha=3$ and $\beta\leq 1$.

Notice that it corresponds to the rate of convergence described in the next section 7.

We thus have

$$\mathrm{Var}_\mu(S_t)/t\simeq\log^{1-\beta}(t)\quad(\text{or }\log\log t\text{ if }\beta=1)\,. \tag{6.11}$$

Any choice of $K(t)$ satisfying (6.9) (or (6.10)) yields that $(S_t-S_t^{f_K})^2/t\,\log^{1-\beta}(t)$ (or $t\,\log\log t$) goes to 0 in $\mathbb{L}^1(\mu)$. Hence, thanks to (6.1), it remains to show that $(M_t^{g_K})^2/t\,\log^{1-\beta}(t)$ (or $t\,\log\log t$) is uniformly integrable i.e. that the bracket

$$\int_0^t|\nabla g_K|^2(X_s)\,ds/t\,\log^{1-\beta}(t)\qquad\text{or }t\,\log\log(t)$$

is uniformly integrable, according to Proposition 4.14. Due to the form of g_K it is thus enough to show that

$$H(t,X,K(t))\,:=\,\int_0^t\,X_s^2\,\mathbf{1}_{|X_s|\leq 1+K(t)}\,ds/t\,\log^{1-\beta}(t)\qquad(\text{ or }t\,\log\log(t)\text{ if }\beta=1) \tag{6.12}$$

is uniformly integrable.

Remark 6.13. *One can remark that in the situation described above, $\beta(t)\ll\alpha^2(t)$, that is the decay of the $\mathbb{L}^2$ norm of P_tf is faster than the worse possible one. Indeed, as we know, $\eta(t)\sim\mathrm{Var}_\mu(S_t)/t\sim\log^{1-\beta}(t)$ (or $\log\log t$ for $\beta=1$) while $\alpha^2(t)\sim\log^{1-\beta}(t)\,t^{-1}$ so that its primitive behaves like $\log^{2-\beta}(t)$.* ♢

To this end, denote by $u(x,M)=|x|^2\,\mathbf{1}_{|x|\leq 1+M}$ for $M\geq 1$, and $\bar{u}(x,M)=u(x,M)-\int u(.,M)\,d\mu$, and $U(t,X,M)=\int_0^t u(X_s,M)\,ds$.

We know that if $\beta\leq 1$, and $t>1$ for instance,

$$\mathrm{Var}_\mu(U(t,X,M))=4\int_0^{t/2}(t-2s)\,\left(\int P_s^2(\bar{u}(.,M))\,d\mu\right)\,ds\,.$$

Recall that $\alpha^2(s)=\alpha^2_{2,\infty}(s)$ is the mixing coefficient whose expression is recalled in section 5.3.2, i.e. $\alpha^2(s)\simeq\log^{1-\beta}(s)\,s^{-1}$.

A direct calculation thus yields (for $t\geq 1$)

$$\begin{aligned}\mathrm{Var}_\mu(U(t,X,M))\;&\leq\;4\int_0^{t/2}(t-2s)\,\alpha^2(s)\,(1+M)^4\,ds\\&\leq\;4c\,(1+M)^4\int_0^{t/2}(t-2s)\,\frac{\log^{1-\beta}(1+s)}{1+s}\,ds\\&\leq\;4c\,(1+M)^4\,t\,\log^{2-\beta}(1+t)\,.\end{aligned}$$

Hence if we choose $M(t)=t^a$ with $a<1/4$,

$$\mathrm{Var}_\mu(U(t,X,t^a))/t^2\,\log^{2(1-\beta)t}\quad\left(\text{ or }(\log\log t)^2\text{ if }\beta=1\right)\quad\to 0\text{ as }t\to+\infty\,.$$

We can also calculate the mean

$$\mathbb{E}_\mu(U(t,X,t^a)) \simeq c(\beta)\, t\, \log^{1-\beta}(t) \quad (\text{ or } \log\log t \text{ if } \beta = 1)$$

i.e. is asymptotically equivalent to the mean of $U(t,X,K(t))$, so that

$$\mathbb{E}_\mu(U(t,X,t^a))/t\, \log^{1-\beta}(t) \quad (\text{ or } \log\log t \text{ if } \beta = 1)$$

is bounded.

It follows that $U(t,X,t^a)/t\, \log^{1-\beta}(t)$ or $U(t,X,t^a)/t\, \log\log(t)$ when $\beta = 1$, is uniformly integrable.

We claim that

$$\left(U(t,X,K(t)) - U(t,X,t^a)\right)/t\, \log^{1-\beta}(t) \quad (\text{ or } \log\log t \text{ if } \beta = 1) \quad \to 0 \text{ in } \mathbb{L}^1(\mathbb{P}_\mu)\,,$$

so that it is uniformly integrable. According to what precedes, it immediately follows that $H(t,X,K(t)) = U(t,X,K(t))/t\, \log^{1-\beta}(t)$ (with the ad hoc normalization if $\beta = 1$) is also uniformly integrable.

It remains to prove our claim. For simplicity we choose $K(t) = t^{1/2}$ (any allowed $K(t)$ furnishes the result but calculations are easier). Since $U(t,X,K(t)) - U(t,X,t^a) \geq 0$ it is enough to calculate for large t

$$\mathbb{E}_\mu\left(U(t,X,K(t)) - U(t,X,t^a)\right) = t \int_{t^a}^{K(t)} x^2\, \mu(dx)\,.$$

If $\beta \neq 1$, the right hand side is equal to

$$\frac{1}{1-\beta}\left(\log^{1-\beta}(K(t)) - \log^{1-\beta}(t^a)\right) \simeq (\log(1/2) - \log(a))\, \log^{-\beta}(t)\,.$$

If $\beta = 1$ it is equal to

$$\log\log(K(t)) - \log\log t^a \simeq \log(1/2) - \log(a)\,.$$

Our claim immediately follows in both cases.

Let us collect the results we have obtained:

Theorem 6.14. *Let*

$$\mu_\beta(dx) = p_\beta(x)\, dx = c(\beta)\, (1+x^2)^{-3/2} \log^{-\beta}(e+x^2)\, dx$$

be a probability measure on the line and $L_\beta = \partial^2_{x^2} + \nabla(\log p_\beta)\, \partial_x$ the associated diffusion generator for which μ_β is reversible and ergodic. $X^\beta_.$ denotes the associated diffusion process.

For $g(x) = x^2$, $f_\beta = L_\beta g$ is a bounded function with μ-mean equal to 0. We consider the associated additive functional $S^{f_\beta}_t = \int_0^t f_\beta(X^\beta_s)\, ds$.

If $\beta > 1$ we may apply Kipnis-Varadhan result (Theorem 3.13).

If $\beta = 1$, $\lim_{t\to+\infty} \mathrm{Var}_{\mu_\beta}(S^{f_\beta}_t)/t\, \log\log t = c$ for some constant $c > 0$ and we may apply Denker's theorem 4.8.

If $\beta < 1$, $\lim_{t\to+\infty} \mathrm{Var}_{\mu_\beta}(S^{f_\beta}_t)/t\, \log^{1-\beta}(t) = c$ for some constant $c > 0$ and we may again apply Denker's theorem 4.8.

The previous theorem is really satisfactory and in a sense generic. We shall try in the next sections to exhibit general properties yielding to an anomalous rate of convergence.

7. Anomalous rate of convergence. Some hints

The standard strategy we used for the CLT is to reduce the problem to the use of the ergodic theorem for the brackets of a well chosen martingale. This requires to approximate the solution of the Poisson equation, i.e. to obtain a decomposition of S_t into some martingale terms, whose brackets may be controlled, and remaining but negligible "boundary" terms. In this section we shall address the problem of using this strategy for super-linear variance. Hence we have to choose a correct approximation of the solution of the Poisson equation, and to replace the ergodic theorem for the martingale brackets, by some uniform integrability property. Again we are using the notation (4.9) and (4.10).

As before, for $T > 0$ depending on t to be chosen later, introduce again $g_T = -\int_0^T P_s f\, ds$. We thus have $Lg_T = f - P_T f$ and using Itô's formula

$$\begin{aligned} S_t = \int_0^t f(X_s)\, ds &= g_T(X_t) - g_T(X_0) - M_t^T + \int_0^t P_T f(X_s)\, ds \qquad (7.1)\\ &= g_T(X_t) - g_T(X_0) - M_t^T + S_t^T \\ &= -\frac{1}{2}\left(M_t^T + (M^*)_t^T\right) + S_t^T\,, \end{aligned}$$

where $\langle M^T\rangle_t = \int_0^t \Gamma(g_T)(X_s)\, ds$. In order to prove that $S_t^2(f)/\mathrm{Var}(S_t(f))$ is uniformly integrable when $X_0 \sim \mu$, we shall find conditions for the following three propositions:

$$\lim_{t\to\infty} \frac{1}{\mathrm{Var}(S_t)} \int (g_T)^2\, d\mu = 0 \qquad (7.2)$$

$$\lim_{t\to\infty} \frac{1}{\mathrm{Var}(S_t)} \mathrm{Var}_\mu(S_t^T) = 0 \qquad (7.3)$$

$$\lim_{t\to\infty} \frac{1}{\mathrm{Var}(S_t)} (M_t^T)^2 \quad \text{is uniformly integrable.} \qquad (7.4)$$

We can replace (7.2) by

$$\frac{1}{\mathrm{Var}(S_t)} ((M^*)_t^T)^2 \quad \text{is uniformly integrable.} \qquad (7.5)$$

7.1. **Study of** $\int (g_T)^2\, d\mu/\mathrm{Var}(S_t)$.

We already saw that in the reversible case

$$\mathrm{Var}_\mu(g_T) = 4 \int_0^T s\, \beta(s)\, ds \;\le\; 4T\, \eta(T),$$

We immediately see using (4.12) that if $\frac{T}{t} \to 0$, then $\int (g_T)^2\, d\mu/\mathrm{Var}(S_t) \to 0$ as $t \to +\infty$.

If $t \ll T$ then β has to decay quickly enough for $\int (g_T)^2\, d\mu/\mathrm{Var}(S_t)$ to be bounded. The limiting case $T = ct$ will be the more interesting in view of the second "boundary" term. Note that actually we only need to study the uniform integrability of $(g_T)^2/\mathrm{Var}(S_t)$, but the material we have developed do not furnish any better result in this direction.

7.2. **Study of** $\mathrm{Var}_\mu(S_t^T)/\mathrm{Var}(S_t)$.

If μ is reversible, we have

$$\begin{aligned} \mathrm{Var}_\mu(S_t^T) &= 2 \int_0^t \int_0^s \left(\int P_T f\, P_{u+T} f\, d\mu \right) du\, ds \\ &= 4 \int_0^{\frac{t}{2}} (t-s)\, \beta(s+T)\, ds \\ &\le 4t\left(\eta(T + (t/2)) - \eta(T)\right), \end{aligned}$$

so that, for $\mathrm{Var}_\mu(S_t^T)/\mathrm{Var}(S_t)$ to go to 0, it is enough to have

$$\frac{\eta(T+\frac{t}{2}) - \eta(T)}{\eta(\frac{t}{4})} \to 0.$$

A similar estimate holds in the non-reversible case provided (Hpos) holds. This time we see that the good situation is the one where $t \ll T$.

7.3. **The martingale brackets.** It remains to calculate the expectation of the martingale brackets $\langle M^T \rangle_t$.

$$\begin{aligned} \mathbb{E}_\mu\left(\langle M^T \rangle_t\right) &= t \int \Gamma(g_T)\, d\mu \\ &= 2t \int \left(\int_0^t P_s f\,(f - P_T f)\, ds\right) d\mu \\ &= 4t\, \left(2\,\eta(T/2) - \eta(T)\right) . \end{aligned}$$

Hence we certainly need $\left(2\,\eta(T/2) - \eta(T)\right)/\eta(t/4)$ to be bounded. As for the first term this requires at least that t is of the same order as T.

7.4. **The good rates.** According to what precedes, we have to consider the case when T and t are comparable. For simplicity we shall choose $T = t/2$, so that the final condition in section 7.3 will be automatically satisfied. The final condition in section 7.2 becomes

$$\lim_{t\to+\infty} \frac{\eta(t) - \eta(t/2)}{\eta(t/4)} = 0\,, \tag{7.6}$$

while the discussion in section 7.1 yields to

$$\lim_{t\to+\infty} \frac{\int_0^t s\,\beta(s)\, ds}{t\, \int_0^{t/2} \beta(s)\, ds} = 0\,, \tag{7.7}$$

It is thus interesting to get a family of $\beta's$ satisfying (7.7) and (7.6). Actually since β is non increasing,

$$\int_{t/2}^t \beta(s) ds \le \int_0^{t/2} \beta(s) ds$$

so that

$$\int_0^{t/2} \beta(s) ds \le \int_0^t \beta(s) ds \le 2 \int_0^{t/2} \beta(s) ds\,.$$

Hence, (7.7) is equivalent to

$$\lim_{t\to+\infty} \frac{\int_0^t s\,\beta(s)\, ds}{t\, \int_0^t \beta(s)\, ds} = 0\,. \tag{7.8}$$

Functions satisfying this property are known, according to Karamata's theory (see [BGT87] chapter 1). Recall the definition

Definition 7.9. *A non-negative function l is slowly varying if for all $u > 0$,*

$$\lim_{t\to+\infty} \frac{l(ut)}{l(t)} = 1.$$

Using the direct half of Karamata's theorem (see [BGT87] Proposition 1.5.8 and equation (1.5.8)) for (7.8) to hold it is enough that

$$\beta(s) = \frac{l(s)}{s} \quad \text{for some slowly varying } l. \tag{7.10}$$

Indeed if (7.10) holds, $\int_0^t s\,\beta(s)\, ds \sim t\, l(t)$ so that (7.8) is equivalent to

$$\lim_{t\to+\infty} \frac{l(t)}{\int_0^t \beta(s)\, ds} = 0\,,$$

which is exactly [BGT87] Proposition 1.5.9a.

The converse half of Karamata's theorem ([BGT87] Theorem 1.6.1) indicates that this condition is not far to be necessary too.

Furthermore, according to [BGT87] Proposition 1.5.9a. if (7.10) is satisfied, then η is slowly varying too, so that (7.6) is also satisfied. These remarks combined with the explicit value of $\mathrm{Var}_\mu(S_t)$ show that the latter is then equivalent to $4t\,\eta(t)$ at infinity.

We have obtained

Proposition 7.11. (7.7) *and* (7.6) *are both satisfied as soon as* (7.10) *is. In this situation* $\mathrm{Var}_\mu(S_t)/t$ *is equivalent to* $4\,\eta(t)$ *at infinity.*

Of course if we replace (7.7) by (7.5) we do not need the full strength of (7.10) since (7.6) is satisfied as soon as η is slowly varying.

7.5. **Study of** $(M_t^T)^2/\mathrm{Var}(S_t)$**.** Now on we shall thus take $T = t/2$ and simply denote M_t^T by M_t. In order to show that $(M_t)^2/\mathrm{Var}(S_t)$ is uniformly integrable, we can use Proposition 4.14 yielding the following :

Proposition 7.12. *If the process is reversible and strongly mixing and if* η *given in* (4.9) *is slowly varying (in particular if* (7.10) *is satisfied), then there is an equivalence between*

(1) $\dfrac{S_t}{2\sqrt{t\eta(t)}}$ *converges in distribution to a standard Gaussian law as* $t \to +\infty$,

(2) $\left(\dfrac{1}{t\eta(t)}\displaystyle\int_0^t \Gamma(g_{t/2})(X_s)\,ds\right)_{t\geq 1}$ *is uniformly integrable, where* $g_{t/2} := -\int_0^{t/2} P_s f\,ds$.

We shall say (as Denker himself said when writing his theorem) that the previous proposition is not really tractable. Indeed in general we do not know any explicit expression for the semigroup (hence for g_t). The main interest of the previous discussion is perhaps contained in the feeling that anomalous rate shall only occur when (7.10) is satisfied.

In the next section we shall even go further in explaining:

7.6. **Why is it delicate?** The previous theorem reduces the problem to show that

$$\sup_t \mathbb{E}_\mu\left[\gamma\left(\frac{1}{\mathrm{Var}(S_t)}\int_0^t \Gamma(g_{t/2})(X_s)\,ds\right)\right] < \infty.$$

The first idea is to use the convexity of γ, yielding

$$\begin{aligned}\mathbb{E}_\mu\left[\gamma\left(\frac{1}{\mathrm{Var}(S_t)}\int_0^t \Gamma(g_{t/2})(X_s)\,ds\right)\right] &\leq \frac{1}{t}\mathbb{E}_\mu\left[\int_0^t \gamma\left(\frac{1}{h(t)}\Gamma(g_{t/2})(X_s)\right)ds\right]\\ &\leq \int \gamma\left(\frac{1}{h(t)}\Gamma(g_{t/2})\right)d\mu\end{aligned}$$

so that our problem reduces to show that $\Gamma(g_t)/h(2t)$ is μ uniformly integrable, or, since we assume that η is slowly varying, that $\Gamma(g_t)/\eta(t)$ is μ uniformly integrable.

The simplest case, namely if $\nabla g_t/\sqrt{h(t)}$ is strongly convergent in $\mathbb{L}^2(\mu)$, holds if and only if $\eta(t)$ has a limit at infinity, i.e. in the Kipnis- Varadhan situation. The situation when $\eta(t)$ goes to infinity is thus more delicate.

It is so delicate that we shall see a natural generic obstruction. In what follows we assume that $\eta(t) \to +\infty$ as $t \to +\infty$.

For simplicity we consider the one dimensional situation with

$$L = \partial^2_{x^2} + \partial_x(\log p)\,\partial_x$$

p being a density of probability on $\mathbb{R}$ which is assumed to be smooth (C^∞) and everywhere positive with $p(x) \to 0$ as $x \to \infty$. $\mu(dx) = p(x)dx$ is thus a reversible measure, and we assume that the underlying diffusion process is strongly mixing.

We already know that $\int |\partial_x g_t|^2 \, d\mu \sim 4\,\eta(t)$. If $|\partial_x g_t|^2/\eta(t)$ is uniformly integrable, we may find a function $h \in \mathbb{L}^1(\mu)$ such that a sequence $|\partial_x g_{t_n}|^2/\eta(t_n)$ weakly converges to h in $\mathbb{L}^1(\mu)$. This implies that $p\,|\partial_x g_{t_n}|^2/\eta(t_n)$ converges to $p\,h = \nu$ in $\mathcal{D}'(\mathbb{R})$, the set of Schwartz distributions. Notice that $\nu \in \mathbb{L}^1(\mathbb{R})$ and satisfies $\int \nu(x)dx = 4$.

Of course we may replace f by $P_\varepsilon f$ for any $\varepsilon \geq 0$ up to an error term going to 0. Thanks to (hypo-)ellipticity we know that $P_\varepsilon f$ is C^∞, hence we may and will assume that f is C^∞, so that g_t is C^∞ too.

Accordingly the derivatives

$$\partial_x(p\,|\partial_x g_{t_n}|^2/\eta(t_n)) = \frac{p\,\partial_x g_{t_n}}{\eta(t_n)} \,\left(2\,\partial^2_{x^2} g_{t_n} + \partial_x(\log p)\,\partial_x g_{t_n}\right) \,\to\, \partial_x \nu$$

in $\mathcal{D}'(\mathbb{R})$. But

$$\partial^2_{x^2} g_{t_n} = L g_{t_n} - \partial_x(\log p)\,\partial_x g_{t_n} = f - P_{t_n} f - \partial_x(\log p)\,\partial_x g_{t_n}\,,$$

so that

$$\partial_x \nu = \lim \frac{1}{\eta(t_n)} \,\left(2\,p\,\partial_x g_{t_n}\,(f - P_{t_n} f) \,-\, \partial_x p\,(\partial_x g_{t_n})^2\right) \;=\; -\,\partial_x(\log p)\,\nu\,.$$

Indeed the first term in the limit goes to 0 in $\mathcal{D}'(\mathbb{R})$ since for a smooth φ with compact support

$$\begin{aligned}\int \varphi\,\frac{1}{\eta(t_n)}\,2\,p\,\partial_x g_{t_n}\,(f - P_{t_n} f)\,dx &\leq \|\varphi\|_\infty \frac{2}{\eta(t_n)} \|\partial_x g_{t_n}\|_{\mathbb{L}^2(\mu)} \|f - P_{t_n} f\|_{\mathbb{L}^2(\mu)} \\ &\leq \|\varphi\|_\infty \frac{4}{\sqrt{\eta(t_n)}} \left\|\frac{\partial_x g_{t_n}}{\sqrt{\eta(t_n)}}\right\|_{\mathbb{L}^2(\mu)} \|f\|_{\mathbb{L}^2(\mu)},\end{aligned}$$

and we assumed that η goes to infinity, while for the second term we know that $p\,|\partial_x g_{t_n}|^2/\eta(t_n)$ converges to ν and that $\partial_x p$ is smooth.

Hence ν solves $\partial_x \nu = -\partial_x(\log p)\,\nu$ in $\mathcal{D}'(\mathbb{R})$, i.e. $\nu = c/p$ which is not in $\mathbb{L}^1(\mathbb{R})$ unless $c = 0$ in which case $\int \nu\,dx \neq 4$. Accordingly $|\partial_x g_t|^2/\eta(t)$ cannot be uniformly integrable.

Hence, contrary to all the cases we have discussed before, anomalous rate of convergence cannot be uniquely described by the behavior of the semigroup. We need to use pathwise properties of the process. (This sentence may look strange since the semigroup uniquely determines the process, but the important word here is "path".)

In the situation of lemma 3.23 the good strategy is to use some cut-off of g as we did in the previous section, which in a sense is generic for this situation.

8. Fluctuations out of equilibrium

In this section we shall mainly discuss the CLT and FCLT out of equilibrium. But before, we shall show that in the strong mixing case (i.e. uniformly ergodic situation), the (CLT) ensures the (FCLT).

Proposition 8.1 (From CLT to FCLT)**.** *Assume that the process is strongly mixing (i.e. uniformly ergodic) and that* $\mathrm{Var}_\mu(S_t) = th(t)$ *for some slowly varying function* h*. If* (CLT) *holds under* $\mathbb{P}_\mu$ *with* $s_t^2 = \mathrm{Var}_\mu(S_t) = th(t)$ *then* (FCLT) *holds with* $s_t^2 = \mathrm{Var}_\mu(S_t) = th(t)$*.*

Proof. Since h is slowly varying, $\mathrm{Var}(S_{t/\varepsilon}) \sim th(1/\varepsilon)/\varepsilon$ as $\varepsilon \to 0$. For $0 \leq s < t$, define

$$S(s,t,\varepsilon) = \sqrt{\frac{\varepsilon}{h(1/\varepsilon)}} \int_{s/\varepsilon}^{t/\varepsilon} f(X_u)\,du.$$

To prove our statement it is thus enough to show that, for indices $0 \leq s_1 < t_1 \leq s_2 < t_2 \cdots < t_N$ the joint law of $(S(s_i,t_i,\varepsilon))_{1\leq i\leq N}$ converges to the law of a Gaussian vector with appropriate diagonal covariance matrix. Up to an easy induction procedure, we shall only give the details for $N=2$ and $0=s_1<t_1=s=s_2<t_2=t$. For $0<s<t$ and $\lambda\in\mathbb{R}$ define

$$V(\varepsilon,s,t,\lambda)=\exp\left(i\,\lambda\,S(s,t,f,\varepsilon)\right)\quad,\quad H(x,s,t,\varepsilon)=\mathbb{E}_x\left[V(\varepsilon,s,t,\lambda)\right]\,.$$

As usual we denote by $\bar H$ the centered $H-\mu(H)$.

We only have to show that

$$\lim_{\varepsilon\to 0}\mathbb{E}_\mu[V(\varepsilon,0,s,\lambda)\,V(\varepsilon,s,t,\theta)]=e^{s\,\lambda^2/2}\,e^{(t-s)\,\theta^2/2}.$$

The main difficulty here is that $t_1=s_2=s$. We introduce an auxiliary time

$$s_\varepsilon=(s/\varepsilon)-(s/\varepsilon^{\frac14}).$$

We then have

$$\begin{aligned}
&\mathbb{E}_\mu\left[V(\varepsilon,0,s,\lambda)\,V(\varepsilon,s,t,\theta)\right]=\\
=\;&\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\,V(\varepsilon,s(1-\varepsilon^{\frac34}),s,\lambda)\,V(\varepsilon,s,t,\theta)\Big]\\
=\;&\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\,V(\varepsilon,s,t,\theta)\Big]+\\
&+\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\,\Big(V(\varepsilon,s(1-\varepsilon^{\frac34}),s,\lambda)-1\Big)\,V(\varepsilon,s,t,\theta)\Big]\\
=\;&A_\varepsilon+B_\varepsilon\,.
\end{aligned}$$

Now

$$\begin{aligned}
A_\varepsilon\;=\;&\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\,P_{s/\varepsilon^{\frac34}}H(X_{s_\varepsilon},s,t,\varepsilon)\Big]\\
=\;&\mu(H(.,s,t,\varepsilon))\,\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\Big]+\\
&+\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\,P_{s/\varepsilon^{\frac34}}\,\bar H(X_{s_\varepsilon},s,t,\varepsilon)\Big]\\
=\;&\mu(H(.,s,t,\varepsilon))\mathbb{E}_\mu\left[V(\varepsilon,0,s,\lambda)\right]+\\
&+\mu(H(.,s,t,\varepsilon))\,\mathbb{E}_\mu\Big[\Big(V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)-V(\varepsilon,0,s,\lambda)\Big)\Big]+\\
&+\mathbb{E}_\mu\Big[V(\varepsilon,0,s(1-\varepsilon^{\frac34}),\lambda)\,P_{s/\varepsilon^{\frac34}}\,\bar H(X_{s_\varepsilon},s,t,\varepsilon)\Big]\\
=\;&A_{1,\varepsilon}+A_{2,\varepsilon}+A_{3,\varepsilon}\,.
\end{aligned}$$

Note that

$$\lim_{\varepsilon\to 0}\,A_{1,\varepsilon}=e^{s\,\lambda^2/2}\,e^{(t-s)\,\theta^2/2}\,,$$

according to the CLT. For the two remaining terms we have

$$(1/\sqrt{2})\,|A_{2,\varepsilon}|\;\leq\;\mathbb{E}_\mu\left[\sqrt{\frac{\varepsilon}{h(1/\varepsilon)}}\,\int_{s(1-\varepsilon^{\frac34})/\varepsilon}^{s/\varepsilon}|f|(X_u)\,du\right]\;\leq\;\sqrt{\frac{\varepsilon}{h(1/\varepsilon)}}\,\frac{s}{\varepsilon^{\frac14}}\,\mu(|f|)\,,$$

hence goes to 0 as $\varepsilon\to 0$. Similarly

$$|A_{3,\varepsilon}|\;\leq\;\mathbb{E}_\mu\left[\left|P_{s/\varepsilon^{\frac34}}\,\bar H(X_{s_\varepsilon},s,t,\varepsilon)\right|\right]\;=\;\int\left|P_{s/\varepsilon^{\frac34}}\,\bar H(.,s,t,\varepsilon)\right|\,d\mu\;\leq\;\alpha(s/\varepsilon^{\frac34})\,,$$

also goes to 0 as $\varepsilon\to 0$.

In the same way

$$(1/\sqrt{2})\,|B_\varepsilon|\;\leq\;\mathbb{E}_\mu\left[\sqrt{\frac{\varepsilon}{h(1/\varepsilon)}}\,\int_{s(1-\varepsilon^{\frac34})/\varepsilon}^{s/\varepsilon}|f|(X_u)\,du\right]\,,$$

hence goes to 0 as $\varepsilon\to 0$ exactly as $A_{2,\varepsilon}$. The proof is completed. □

Corollary 8.2. *If $\mathrm{Var}_\mu(S_t) = t\,h(t)$ for some slowly varying function h, we may replace the CLT by the FCLT in all results of section 4.2 (in particular Theorem 4.18), in Theorem 6.14 and in Proposition 7.12.*

8.1. **About the law at time t.**

Theorem 8.3. [DMT95] *Thm 5.2.c, and* [DFG09] *Thm 3.10 and Thm 3.12.*

Under the assumptions of Proposition 5.10, there exists a positive constant c such that for all x,

$$\|P_t(x,\cdot) - \mu\|_{TV} \leq cV(x)\psi(t),$$

where $\|\cdot\|_{TV}$ is the total variation distance and ψ (which goes to 0 at infinity) is defined as follows: $\psi(t) = 1/(\varphi \circ H_\varphi^{-1})(t)$ for $H_\varphi(t) = \int_1^t (1/\varphi(s))ds$, if $\lim_{u\to+\infty} \varphi'(u) = 0$ and $\psi(t) = e^{-\lambda t}$ for a well chosen $\lambda > 0$ if φ is linear.

In particular for any probability measure ν such that $V \in \mathbb{L}^1(\nu)$, if we denote by $P_t^*\nu$ the law of the process at time t starting with initial law ν,

$$\lim_{t\to+\infty} \|P_t^*\nu - \mu\|_{TV} = 0.$$

The second result is mentioned (in the case of a brownian motion with a drift) in [CGG07] and proved for a stopped diffusion in dimension one in [CCL+09] Theorem 2.3. The proof given there extends immediately to the uniformly elliptic case below thanks to the standard Gaussian estimates for the density at time t of such a diffusion, details are left to the reader

Theorem 8.4. *In the diffusion situation* (2.4)*, assume that the diffusion matrix a is uniformly elliptic and bounded. Assume in addition that the invariant measure $\mu(dx) = e^{-W(x)}\,dx$ is reversible, and that $2\Gamma(W,W)(x) - LW(x) \geq -c > -\infty$.*

Then for all $t > 0$ and all x, $P_t(x,dy) = r(t,x,y)\,\mu(dy)$ with $r(t,x,.) \in \mathbb{L}^2(\mu)$. Furthermore if $e^W \in \mathbb{L}^1(\nu)$, $P_t^*\nu(dy) = r(t,\nu,y)\,\mu(dy)$ with $r(t,\nu,.) \in \mathbb{L}^2(\mu)$.

Consequently, if the diffusion is uniformly ergodic (or strongly mixing) and if $e^W \in \mathbb{L}^1(\nu)$, we have again

$$\lim_{t\to+\infty} \|P_t^*\nu - \mu\|_{TV} = 0.$$

8.2. **Fluctuations out of equilibrium.** Let ν be a given initial distribution. A direct application of the Markov property shows that

Lemma 8.5. *Assume that*

$$\lim_{t\to+\infty} \|P_t^*\nu - \mu\|_{TV} = 0.$$

Let $u(\varepsilon) > \varepsilon$ going to 0 as ε goes to 0. For any bounded $H_1, ..., H_k$, denote $H(Z_.) = \otimes H_i(Z_{t_i})$. Then

$$\lim_{\varepsilon\to 0} \left| \mathbb{E}_\nu \left[H\left(v(\varepsilon) \int_{./u(\varepsilon)}^{./\varepsilon} f(X_s)\,ds \right) \right] - \mathbb{E}_\mu \left[H\left(v(\varepsilon) \int_{./u(\varepsilon)}^{./\varepsilon} f(X_s)\,ds \right) \right] \right| = 0\,.$$

As a consequence we immediately obtain

Theorem 8.6. *Let ν satisfying the assumptions of Theorem 8.4 or Theorem 8.3. If the FCLT holds under $\mathbb{P}_\mu$ (i.e. at equilibrium) with $v(\varepsilon) \to 0$ as $\varepsilon \to 0$ but $v(\varepsilon) \gg \varepsilon$, then it also holds under $\mathbb{P}_\nu$ (i.e out of equilibrium) provided one of the following additional assumptions is satisfied*

- *ν is absolutely continuous w.r.t. μ*
- *$\nu = \delta_x$ for μ almost all x,*
- *f is bounded.*

Proof. Choose $u(\varepsilon)$ such that $u(\varepsilon) \to 0$ as $\varepsilon \to 0$, but with $u(\varepsilon) \gg v(\varepsilon)$. We may apply the previous lemma and to conclude it is enough to show that

$$\lim_{\varepsilon \to 0} v(\varepsilon) \int_0^{t/u(\varepsilon)} f(X_s)\, ds$$

in $\mathbb{P}_\nu$ probability, which is immediate when f is bounded and follows from the almost sure ergodic theorem in the two others cases. □

Several authors have tried to obtain the FCLT started from a point i.e. under $\mathbb{P}_x$ for all x, not only for μ almost all x, see [DL01a, DL03]. Here is a result in this direction:

Theorem 8.7. *Assume that $P_t^*\nu$ is absolutely continuous with respect to μ for some $t > 0$, that the state space E is locally compact and that f is continuous. Then if the assumptions of Theorem 8.4 or Theorem 8.3 are fulfilled, then* (FCLT) *holds under $\mathbb{P}_\nu$ as soon as it holds under $\mathbb{P}_\mu$.*

Proof. Note that, if $P_t^*\nu$ is absolutely continuous w.r.t. μ, we may apply the previous theorem to the additive functional $\int_t^{\cdot/\varepsilon} f(X_s)\, ds$, i.e. we may replace 0 by some fixed t. It thus remains to control $v(\varepsilon) \int_0^t f(X_s)\, ds$ for the same fixed t. But since f is continuous, since $X_.$ is $\mathbb{P}_\nu$ almost surely continuous and E is locally compact, $\int_0^t f(X_s)\, ds$ is $\mathbb{P}_\nu$ almost surely bounded, hence goes to 0 when $\varepsilon \to 0$ once multiplied by $v(\varepsilon)$. □

Corollary 8.8. *If L given by* (2.4) *is elliptic or more generally hypoelliptic, the previous theorem applies to all initial ν satisfying the assumptions of Theorem 8.4 or Theorem 8.3. In particular it applies to $\nu = \delta_x$ for all x.*

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