# COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS OF DEGREE 4 

GRÉGORY BERHUY


#### Abstract

In this paper, we prove a result of Rost, which describes the cohomological invariants of central simple algebras of degree 4 with values in $H^{*}\left(-, \mu_{2}\right)$ when the base field contains a square root of -1 .


## Introduction

In [5], Rost, Serre and Tignol defined a cohomological invariant of central simple algebras of degree 4 with values in $H^{4}\left(-, \mu_{2}\right)$ when the base field contains a square root of -1 . On the other hand, taking the Brauer class of the tensor square of a central simple algebra of degree 4 yields a cohomological invariant with values in $H^{2}\left(-, \mu_{2}\right)$. In this paper, we prove a result of Rost (unpublished), which asserts that these two invariants are essentially the only ones (see Section 4 for a more precise statement).
This paper is organized as follows. After proving somme preliminary results in Section 1, we determine the cohomological invariants of multicyclic algebras in Section 2. Section 3 is devoted to the construction of a generic central simple algebra of degree 4. Finally, in Section 4, we prove that a cohomological invariant which vanishes on cyclic algebras and biquaternion algebras is identically zero (which another result due to Rost). As a corollary, we obtain a complete description of cohomological invariants of central simple algebras of degree 4 .
The proofs of all the results of this paper rely heavily on the use of valuations and residue maps. We let the reader refer to [2] for the basic definitions and results on these topics.

We are grateful to Alexander Merkurjev for providing us his private lecture notes on Rost's theorem.

## 1. Preliminaries

Let $F$ be a field of characteristic different from 2. We will denote by Fields ${ }_{F}$ and $\mathbf{A l g}_{F}$ the category of fields extensions of $F$ and the category of commutative $F$-algebras respectively.
The category of sets will be denoted by Sets, and by Rings the category of of commutative rings.
For any field extension $K / F$, we will denote by $H^{*}(K)$ the cohomology ring of $K$ with coefficients in $\mu_{2}$. We then get a functor $H^{*}:$ Fields ${ }_{F} \longrightarrow$ Rings.

If $K / F$ is a field extension, then $H^{*}(K)$ carries a natural structure of a $H^{*}(F)$ module, given by the external law

$$
\begin{aligned}
H^{*}(F) \times H^{*}(K) & \longrightarrow H^{*}(K) \\
(a, \xi) & \longmapsto a \cdot \xi=\operatorname{Res}_{K / F}(a) \cup \xi
\end{aligned}
$$

We start with some results on the cohomology of rational extensions.
Let $n \geq 1$ be an integer, let $t_{1}, \ldots, t_{n}$ be algebraically independent indeterminates over $k$, and let $K_{n}=k\left(t_{1}, \ldots, t_{n}\right)$. Let us denote by $\mathcal{P}_{n}$ the set of subsets of $\{1, \ldots, n\}$. If $I=\left\{i_{1}, \ldots, i_{r}\right\} \in \mathcal{P}_{n}$, we set

$$
\left(\mathbf{t}_{I}\right)=\left(t_{i_{1}}\right) \cup \cdots \cup\left(t_{i_{r}}\right) .
$$

Notice that this definition does not depend on the numbering of the elements of $I$, since the cup-product in $H^{*}(K)$ is commutative.
If $I=\emptyset$, then $\left(\mathbf{t}_{I}\right)=1$ (and thus $a \cdot\left(\mathbf{t}_{I}\right)=\operatorname{Res}_{K / k}(a)$ in this case).
Lemma 1.1. The cohomology classes $\left(\mathbf{t}_{I}\right), I \in \mathcal{P}_{n}$ are linearly independent over $H^{*}(F)$.

Proof. We proceed by induction on $n$. Assume that $n=1$, and let $a_{0}, a_{1} \in H^{*}(F)$ such that $a_{0} \cdot 1+a_{1} \cdot\left(t_{1}\right)=0$. Taking residues with respect to the valuation $v_{t_{1}}$ shows that $a_{1}=0$. Then multiplying by $\left(-t_{1}\right)$ and taking residues shows that $a_{0}=0$.
Assume now that the result is proved for $n \geq 1$, and let us prove it for $n+1$ indeterminates. Let $a_{I} \in H^{*}(F), I \in \mathcal{P}_{n+1}$ such that $\sum_{I} a_{I} \cdot\left(\mathbf{t}_{I}\right)=0$. Then we have

$$
\sum_{J} a_{J} \cdot\left(\mathbf{t}_{J}\right)+\sum_{J} a_{J \cup\{n+1\}} \cdot\left(\mathbf{t}_{J}\right) \cup\left(t_{n+1}\right)=0
$$

where $J$ describes $\mathcal{P}_{n}$. Reasoning as in the case $n=1$, we get

$$
\sum_{J} a_{J} \cdot\left(\mathbf{t}_{J}\right)=\sum_{J} a_{J \cup\{n+1\}} \cdot\left(\mathbf{t}_{J}\right)=0
$$

and by induction we have $a_{J}=a_{J \cup\{n+1\}}=0$ for all $J \in \mathcal{P}_{n}$, that is $a_{I}=0$ for all $I \in \mathcal{P}_{n+1}$. This concludes the proof.
Proposition 1.2. Let $\xi \in H^{*}\left(K_{n}\right)$ be a cohomology class which is unramified at every discrete $k$-valuation $v \neq v_{t_{i}}, i=1, \ldots, n$. Then there exist unique cohomology classes $a_{I} \in H^{*}(F)$ such that $\xi=\sum_{I} a_{I} \cdot\left(\mathbf{t}_{I}\right)$.

Proof. We prove it by induction on $n$. Assume first that $n=1$, and let $a_{1} \in H^{*}(F)$ be the residue of $\xi$ with respect to the $t_{1}$-adic valuation. Then $\xi-\operatorname{Res}_{K / F}\left(a_{1}\right) \cup\left(t_{1}\right)$ is unramified at every discrete $F$-valuation $v \neq v_{t_{1}}$ by assumption on $\xi$ and because $a_{1}$ is constant. By choice of $a_{1}$, it is also unramified at $v_{t_{1}}$, so $\xi-\operatorname{Res}_{K_{1} / F}\left(a_{1}\right) \cup\left(t_{1}\right)$ is a constant class, so we may finally write

$$
\xi=\operatorname{Res}_{K_{1} / F}\left(a_{0}\right)+\operatorname{Res}_{K_{1} / F}\left(a_{1}\right) \cup\left(t_{1}\right)=a_{0} \cdot 1+a_{1} \cdot\left(t_{1}\right)
$$

Now assume that the result is proved for $n \geq 1$, and let $\xi \in H^{*}\left(K_{n+1}\right)$ be a cohomology class which is unramified at every discrete $F$-valuation $v \neq v_{t_{i}}, i=$ $1, \ldots, n+1$. In particular, it is unramified at every discrete $K_{n}$-valuation $v \neq v_{t_{n+1}}$.

By the case $n=1$, there exist $b_{0}, b_{1} \in H^{*}\left(K_{n}\right)$ such that $\xi=b_{0} \cdot 1+b_{1} \cdot\left(t_{n+1}\right)$. Let $v$ be a discrete $F$-valuation on $K_{n}, v \neq v_{t_{i}}, i=1, \ldots, n$, and let $w$ the $F$ valuation on $K_{n+1}$ extending $v$ such that $w\left(t_{n+1}\right)=0$. Since $\xi$ is unramified at $v$, taking residues shows that $\operatorname{Res}_{\kappa(w) / \kappa(v)}\left(r_{v}\left(b_{1}\right)\right)=0$. Now $\kappa(w)=\kappa(v)\left(t_{n+1}\right)$, so $\operatorname{Res}_{\kappa(w) / \kappa(v)}$ is injective and $r_{v}\left(b_{1}\right)=0$. Moreover, $\xi \cup\left(-t_{n+1}\right)$ is also unramified at $v$, and reasoning as before shows that $r_{v}\left(b_{0}\right)=0$. Hence $b_{0}$ and $b_{1}$ are unramified at any discrete $F$-valuation $v$ on $K_{n}, v \neq v_{t_{i}}, i=1, \ldots, n$. Now use the induction hypothesis to conclude that $\xi=\sum_{I} a_{I} \cdot\left(\mathbf{t}_{I}\right)$ for some $a_{I} \in H^{*}(F)$.

The uniqueness of the classes $a_{I}$ comes from the previous lemma. This completes the proof.

Definition 1.3. Let F: Fields ${ }_{F} \longrightarrow$ Sets be a covariant functor. A cohomological invariant of $\mathbf{F}$ over $F$ is a natural transformation $\mathbf{F} \longrightarrow H^{*}$ of functors Fields ${ }_{F} \longrightarrow$ Sets.

Cohomological invariants of $\mathbf{F}$ form an $H^{*}(F)$-module, that we will denote by $\operatorname{Inv}\left(\mathbf{F}, H^{*}\right)$.

We will need in the sequel the notion of a classifying pair for a functor $\mathbf{F}$.
Definition 1.4. Let $\mathbf{F}: \mathbf{A l g}_{F} \longrightarrow$ Sets be a covariant functor. Let $R$ be an $F$-algebra, and let $a \in \mathbf{F}(R)$. We say that the pair $(R, a)$ is classifying for $\mathbf{F}$, if the following conditions hold:
(1)The ring $R$ is a noetherian domain;
(2)For every field extension $L / F$, and every $a^{\prime} \in \mathbf{F}(L)$, there exists a maximal ideal $\mathfrak{m}$ of $R$ and a morphism of $F$-algebras $f: R \longrightarrow L$ such that $\operatorname{ker}(f) \supset \mathfrak{m}$ and $\mathbf{F}(f)$ maps $a$ onto $a^{\prime}$.

The Specialization Theorem (cf. [2, Theorem 12.2]) then immediately gives:
Lemma 1.5. Let $\mathbf{F}: \mathbf{A l g}_{F} \longrightarrow$ Sets be a subfunctor of $H^{1}(-, G)$, where $G$ is an algebraic group over $F$. Assume that $(R, a)$ is classifying for $\mathbf{F}$, and let $\alpha, \beta \in$ $\operatorname{Inv}\left(\mathbf{F}, H^{*}\right)$. Let $K$ be the quotient field of $R$. If $\alpha\left(a_{K}\right)=\beta\left(a_{K}\right)$, then $\alpha=\beta$.

## 2. InVariants of multicyclic algebras

For all integers $r, n \geq 1$, we will denote by $\mathbf{C S A}_{n}: \mathbf{A l g}_{F} \longrightarrow$ Sets the functor of isomorphism classes of Azumaya algebras algebras of rank $n$.
Let $R$ be a commutative ring. Assume that $n \nmid \operatorname{char}(R)$ and that $R$ contains a primitive $n$-th root of 1 , that we will denote by $\zeta_{n}$. For all $u, v \in R^{\times}$, the $R$-algebra $\{u, v\}_{n, R}$ generated by two elements $e$ and $f$ subject to the relations

$$
e^{n}=u, f^{n}=v, f e=\zeta_{n} e f
$$

is an Azumaya $R$-algebra of rank $n$, called a symbol algebra.
We will denote by $\mathbf{M S}_{n, r}$ the subfunctor of $\mathbf{C S A}_{n^{r}}$ of isomorphism classes of tensor products of $r$ symbols algebras of degree $n$.
We will denote by 1 the constant cohomological invariant of $\mathbf{C S A}_{n}$ (where $\mathbf{C S A}_{n}$ is now viewed as a functor from Fields ${ }_{F}$ to Sets). If $n=2 m$, then for every field
extension $K / F$ and every central simple $K$-algebra $A$ of exponent dividing $n$, the class $m[A]$ is killed by 2 , and therefore defines a cohomology class of $H^{2}(K)$ via the usual isomorphism. This defines a cohomological invariant of $\mathbf{C S A}_{n}$, as well as a cohomological invariant of $\mathbf{M S}_{n, r}$ for all $r \geq 1$, that we will denote by $f_{m}$ in both cases. In particular, we have

$$
f_{m}\left(\left\{a_{1}, b_{1}\right\}_{n, L} \otimes_{F} \cdots \otimes_{F}\left\{a_{r}, b_{r}\right\}_{n, L}\right)=\left(a_{1}\right) \cup\left(b_{1}\right)+\ldots+\left(a_{r}\right) \cup\left(b_{r}\right)
$$

for every field extension $L / F$ and all $a_{i}, b_{i} \in L^{\times}$.
If $A$ is a central simple $F$-algebra, the trace form of $A$ is the quadratic form

$$
q_{A}: \begin{aligned}
& A \longrightarrow F \\
& a \longmapsto \operatorname{Trd}_{A}\left(a^{2}\right) .
\end{aligned}
$$

Lemma 2.1. Assume that $n \nmid \operatorname{char}(F)$ and that $\mu_{n} \in F$. Assume also that $n=2 m$.
For all $u, v \in F^{\times}$, the trace form of $\{u, v\}_{n, F}$ is Witt-equivalent to

$$
\left\langle n, n u, n v,(-1)^{m} n u v\right\rangle .
$$

Proof. One may check that $\operatorname{Trd}_{A}\left(e^{i} f^{j}\right)=0$ if $(i, j) \neq(0,0)$. It easily follows that the subspaces

$$
F \cdot 1, F \cdot e^{m}, F \cdot f^{m}, F \cdot e^{m} f^{m} \text { and } F \cdot e^{i} f^{j} \oplus F \cdot e^{n-i} f^{m-j},
$$

where $0 \leq i \leq j \leq m,(i, j) \neq(0,0),(0, m),(m, 0),(m, m)$ are mutually orthogonal. Moreover, the $2 m^{2}-2$ planes above are hyperbolic since $e^{i} f^{j}$ is isotropic. The result then follows from the fact that the reduced traces of $1, e^{m}, f^{m}$ and $e^{m} f^{m}$ are respectively $n, n u, n v$ and $(-1)^{m} n u v$. This concludes the proof.

Assume that $n \nmid \operatorname{char}(F)$ and that $\mu_{n} \in F$. Moreover, assume that $-1 \in F^{\times 2}$ (this condition is automatically satisfied if $n \equiv 0[4]$ ).
If $A=\left\{u_{1}, v_{1}\right\}_{n, F} \otimes_{F} \cdots \otimes_{F}\left\{u_{r}, v_{r}\right\}_{n, F}$, it follows from the previous lemma that $q_{A}$ is Witt-equivalent to $\left\langle n^{r}\right\rangle\left\langle\left\langle u_{1}, v_{1}, \ldots, u_{r}, v_{r}\right\rangle\right\rangle$. Therefore, $q_{A} \in I^{2 r} F$, and we have

$$
e_{2 r}\left(q_{A}\right)=\left(u_{1}\right) \cup\left(v_{1}\right) \cup \cdots \cup\left(u_{r}\right) \cup\left(v_{r}\right) .
$$

This defines an element of $\operatorname{Inv}\left(\mathbf{M S}_{n, r}, H^{*}\right)$ that we denote by $e_{2 r}$.
Proposition 2.2. Let $F$ be a field of characteristic different from 2, and let $n=$ $2 m$.
(1)Assume that $n \nmid \operatorname{char}(F)$, and that $\zeta_{n} \in F$, where $\zeta_{n}$ is a primitive $n^{\text {th }}$-root of 1 . Then $\operatorname{Inv}\left(\mathbf{M S}_{n, 1}, H^{*}\right)$ is a free $H^{*}(F)$-module with basis $1, f_{m}$.
(2)Assume moreover that $-1 \in F^{\times 2}$. Then for all $r \geq 2, \operatorname{Inv}\left(\mathbf{M S}_{n, r}, H^{*}\right)$ is a free $H^{*}(F)$-module with basis $1, f_{m}, e_{2 r}$.

Proof. We first prove (1). Let $\alpha \in \operatorname{Inv}\left(\mathbf{M S}_{n, 1}, H^{*}\right)$. Let $t_{1}, t_{2}$ be two indeterminates over $F$, and let $K=F\left(t_{1}, t_{2}\right)$. Set $A=\left\{t_{1}, t_{2}\right\}_{n, K}$, and let $v \neq v_{t_{i}}, i=1,2$ be a discrete $F$-valuation. By assumption on $v, t_{1}, t_{2} \in \mathcal{O}_{v}^{\times}$, and we have $A \simeq$ $\left\{t_{1}, t_{2}\right\}_{n, \mathcal{O}_{v}} \otimes_{\mathcal{O}_{v}} K$. It follows from [2, Theorem 11.7] that the class $\alpha_{K}\left(\left\{t_{1}, t_{2}\right\}_{n, K}\right)$ is unramified at $v$. By Proposition 1.2, there exist $a_{0}, a_{1}, a_{2}, a_{3} \in H^{*}(F)$ such that

$$
\alpha_{K}\left(\left\{t_{1}, t_{2}\right\}_{n, K}\right)=a_{0} \cdot 1+a_{1} \cdot\left(t_{1}\right)+a_{2} \cdot\left(t_{2}\right)+a_{3} \cdot\left(t_{1}\right) \cup\left(t_{2}\right) .
$$

Now let $v$ be the $\left(t_{1}-1\right)$-adic valuation on $F\left(t_{2}\right)$. Applying [2, Theorem 11.7] shows by specialization that

$$
\alpha_{K}\left(\left\{1, t_{2}\right\}_{n, F\left(t_{2}\right)}\right)=a_{0} \cdot 1+a_{2} \cdot\left(t_{2}\right)
$$

that is

$$
\alpha_{K}\left(\mathrm{M}_{n}\left(F\left(t_{2}\right)\right)\right)=a_{0} \cdot 1+a_{2} \cdot\left(t_{2}\right)
$$

Now $\mathrm{M}_{n}\left(F\left(t_{2}\right)\right)$ is unramified at $v_{t_{2}}$, and therefore, so is $\alpha_{K}\left(\mathrm{M}_{n}\left(F\left(t_{2}\right)\right)\right)$. Taking residues then yields $a_{2}=0$. Similar arguments show that $a_{1}=0$, and thus

$$
\alpha_{K}\left(\left\{t_{1}, t_{2}\right\}_{n, K}\right)=a_{0} \cdot 1+a_{3} \cdot\left(t_{1}\right) \cup\left(t_{2}\right)
$$

Let $R=F\left[t_{1}, t_{1}^{-1}, t_{2}, t_{2}^{-1}\right]$. It is clear that the pair $\left(R,\left\{t_{1}, t_{2}\right\}_{n, R}\right)$ is classifying for $\mathbf{M S}_{n, 1}$. The previous equality shows that the invariants $\alpha$ and $a_{0} \cdot 1+a_{3} \cdot f_{m}$ coincide on a classifying pair, hence they are equal by Lemma 1.5.

Moreover, if $a_{0} \cdot 1+a_{3} \cdot f_{m}=0$, then applying this equality to $\left\{t_{1}, t_{2}\right\}_{n, K}$ shows that $a_{0} \cdot 1+a_{3} \cdot\left(t_{1}\right) \cup\left(t_{2}\right)$. Lemma 1.1 then yields $a_{0}=a_{3}=0$. This proves (1).
Assume now that $-1 \in F^{\times 2}$. The fact that $1, f_{m}, e_{2 r}$ are linearly independent over $H^{*}(F)$ if $r \geq 2$ easily comes from Lemma 1.1. Let us prove by induction on $r$ that $1, f_{m}$ and $e_{2 r}$ span $\operatorname{Inv}\left(\mathbf{M S}_{n, r}, H^{*}\right)$ for all $r \geq 1$ and every base field $F$. For $r=1$, this follows from (1). Assume that the result is proved for $r \geq 1$, and let $\alpha \in \operatorname{Inv}\left(\mathbf{M S}_{n, r+1}, H^{*}\right)$. Let $K / F$ be a field extension, and let $A$ be a symbol algebra over $K$.

Any field extension $L / K$ yields a field extension $L / F$. The maps

$$
\begin{aligned}
& B \longrightarrow \mathbf{M S}_{n, r}(L) \\
& \beta_{L}: \\
& H^{*}(L) \longmapsto \alpha_{L}\left(B \otimes_{L} A_{L}\right)
\end{aligned}
$$

fit together into a cohomological invariant of $\mathbf{M S}_{n, r}$. By induction, there exists $\alpha_{i, K}(A) \in H^{*}(K)$ such that

$$
\beta=\alpha_{0, K}(A) \cdot 1+\alpha_{1, K}(A) \cdot f_{m}+\alpha_{2, K}(A) \cdot e_{2 r}
$$

where $\alpha_{2, K}(A)=0$ if $r=1$.
By linear independence of $1, f_{m}, e_{2 r}$ (resp. of $1, f_{m}$ if $r=1$ ), these classes are unique. It easily follows that $\alpha_{i, K}(A)$ only depends on the isomorphism class of A. Moreover, it follows from the uniqueness of $\alpha_{i, K}(A)$ and the fact that $\alpha$ is a cohomological invariant that the maps $\alpha_{i, K}$ fit together into a cohomological invariant $\alpha_{i} \in \operatorname{Inv}\left(\mathbf{M S}_{n, 1}, H^{*}\right)$.

By (1), we may write $\alpha_{i}=a_{i} \cdot 1+b_{i} \cdot f_{m}$, for some $a_{i}, b_{i} \in H^{*}(F)$ (with $a_{2}=b_{2}=0$ if $r=1$ ). Taking $K=F\left(u_{1}, v_{1}, \ldots, u_{r+1}, v_{r+1}\right)$, where $u_{i}, v_{i}$ are indeterminates, $B=\left\{u_{1}, v_{1}\right\}_{n, K} \otimes_{K} \cdots \otimes_{K}\left\{u_{r}, v_{r}\right\}_{n, K}$ and $A=\left\{u_{r+1}, v_{r+1}\right\}_{n, K}$, we get

$$
\begin{gathered}
\alpha_{K}\left(\left\{u_{1}, v_{1}\right\}_{n, K} \otimes_{K} \cdots \otimes_{K}\left\{u_{r+1}, v_{r+1}\right\}_{n, K}\right)=a_{0} \cdot 1+b_{0} \cdot\left(u_{r+1}\right) \cup\left(v_{r+1}\right)+ \\
\left(a_{1} \cdot 1+b_{1} \cdot\left(u_{r+1}\right) \cup\left(v_{r+1}\right)\right) \cup \sum_{i=1}^{r}\left(u_{i}\right) \cup\left(v_{i}\right)+\left(a_{2} \cdot 1+b_{2} \cdot\left(u_{r+1}\right) \cup\left(v_{r+1}\right)\right) \cup \bigcup_{i=1}^{r}\left(u_{i}\right) \cup\left(v_{i}\right) .
\end{gathered}
$$

The $F$-automorphism of $K$ which exchanges $u_{1}$ and $u_{r+1}, v_{1}$ and $v_{r+1}$, and leaves invariant the other indeterminates induces maps $\mathbf{M S} \mathbf{S}_{n, r+1}(K) \longrightarrow \mathbf{M S}_{n, r+1}(K)$
and $H^{*}(K) \longrightarrow H^{*}(K)$. Notice that the first one maps the isomorphism class of $\left\{u_{1}, v_{1}\right\}_{n, K} \otimes_{K} \cdots \otimes_{K}\left\{u_{r+1}, v_{r+1}\right\}_{n, K}$ onto itself. Since $\alpha$ commutes with induced maps, we then get

$$
\begin{gathered}
\alpha_{K}\left(\left\{u_{1}, v_{1}\right\}_{n, K} \otimes_{K} \cdots \otimes_{K}\left\{u_{r+1}, v_{r+1}\right\}_{n, K}\right)=a_{0} \cdot 1+b_{0} \cdot\left(u_{1}\right) \cup\left(v_{1}\right)+ \\
\left(a_{1} \cdot 1+b_{1} \cdot\left(u_{1}\right) \cup\left(v_{1}\right)\right) \cup \sum_{i=2}^{r+1}\left(u_{i}\right) \cup\left(v_{i}\right)+\left(a_{2} \cdot 1+b_{2} \cdot\left(u_{1}\right) \cup\left(v_{1}\right)\right) \cup \bigcup_{i=2}^{r+1}\left(u_{i}\right) \cup\left(v_{i}\right) .
\end{gathered}
$$

Comparing with the previous equality and using Lemma 1.1, we get that $b_{0}=a_{1}$, and $b_{1}=a_{2}=0$. Therefore, $\left\{u_{1}, v_{1}\right\}_{n, K} \otimes_{K} \cdots \otimes_{K}\left\{u_{r+1}, v_{r+1}\right\}_{n, K}$ has same image by $\alpha$ and by $a_{0} \cdot 1+a_{1} \cdot f_{m}+b_{2} \cdot e_{2(r+1)}$. Now let $R=F\left[u_{i}, u_{i}^{-1}, v_{i}, v_{i}^{-1}, i=\right.$ $1, \ldots, r+1]$. The pair ( $R,\left\{u_{1}, v_{1}\right\}_{n, R} \otimes_{R} \cdots \otimes_{R}\left\{u_{r+1}, v_{r+1}\right\}_{n, R}$ ) is classifying for $\mathbf{M} \mathbf{S}_{n, r+1}$. It follows from the previous considerations that the invariants $\alpha$ and $a_{0} \cdot 1+a_{1} \cdot f_{m}+b_{2} \cdot e_{2(r+1)}$ are equal. This finishes the proof by induction.

## 3. Central simple algebras of degree 4

We start this section by giving a parametrization of central simple algebras of degree 4.

Let $R$ be a commutative ring such that $2 \in R^{\times}$. If $a \in R^{\times}$, we will denote by $R[\sqrt{a}]$ the étale quadratic $R$-algebra $R[X] /\left(X^{2}-a\right)$.
Let $L$ be a biquadratic étale $R$-algebra, that is an $R$-algebra $L$ generated by two elements $\alpha, \beta$ subject to the relations

$$
\alpha^{2}=d, \beta^{2}=d^{\prime}, \alpha \beta=\beta \alpha
$$

for some $d, d^{\prime} \in R^{\times}$. Such an algebra will be denoted by $F\left[\sqrt{d}, \sqrt{d^{\prime}}\right]$.
The group $G$ of automorphisms of the $R$-algebra $L=F\left[\sqrt{d}, \sqrt{d^{\prime}}\right]$ is isomorphic to $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}$, generated by the automorphisms $\sigma$ and $\tau$ which are uniquely determined by the following relations:

$$
\sigma(\alpha)=\alpha, \sigma(\beta)=-\beta, \tau(\alpha)=-\alpha, \tau(\beta)=\beta
$$

Let $a \in R[\sqrt{d}]^{\times}, b \in R\left[\sqrt{d^{\prime}}\right]^{\times}, c \in R\left[\sqrt{d d^{\prime}}\right]^{\times}$, and set

$$
n_{a}=N_{R[\sqrt{d}] / R}(a), n_{b}=N_{R\left[\sqrt{d^{\prime}}\right] / R}(b), n_{c}=N_{R\left[\sqrt{d d^{\prime}}\right] / R}(c)
$$

Notice that $n_{a}, n_{b}, n_{c} \in R^{\times}$and that we have

$$
n_{a}=a \tau(a), n_{b}=b \sigma(b), n_{c}=c \sigma(c)=c \tau(c) .
$$

Assume that $n_{c}=n_{a} n_{b}$, and let $1, e_{\sigma}, e_{\tau}, e_{\sigma \tau}$ be the canonical basis of the $L$-vector space $\operatorname{Map}(G, L)$. We define a product law on $\operatorname{Map}(G, L)$ by imposing the relations

$$
\begin{gathered}
e_{\sigma}^{2}=a, e_{\tau}^{2}=b, e_{\sigma \tau}^{2}=c, e_{\sigma} e_{\tau}=e_{\sigma \tau}, \\
e_{\sigma} \lambda=\sigma(\lambda) e_{\sigma}, e_{\tau} \lambda=\tau(\lambda) e_{\tau}, \text { for all } \lambda \in L,
\end{gathered}
$$

and extending by distributivity.
By definition, the elements $e_{\rho}, \rho \in G$ are invertible with respect to this product law. Notice that the missing products $e_{\rho} e_{\rho^{\prime}}, \rho, \rho^{\prime} \in G$ may be obtained using the
relations above, so that the product law on $\operatorname{Map}(G, L)$ is completely determined. For example, we have

$$
e_{\tau} e_{\sigma \tau}=e_{\sigma}^{-1} e_{\sigma} e_{\tau} e_{\sigma \tau}=e_{\sigma}^{-1}\left(e_{\sigma \tau}\right)^{2}=a^{-1} e_{\sigma} c=a^{-1} \sigma(c) e_{\sigma} .
$$

One can check that we obtain an Azumaya $R$-algebra of rank 4, that we will denote by $(a, b, c, L / R)$. Notice that if we set $e=e_{\sigma}$ and $f=e_{\tau}$, then we have $(a, b, c, L / R)=L \oplus L e \oplus L f \oplus L e f$, and

$$
e^{2}=a, f^{2}=b,(e f)^{2}=c, e \lambda=\sigma(\lambda) e, f \lambda=\tau(\lambda) f, \text { for all } \lambda \in L
$$

Notice for later use that we have $f e=a^{-1} \sigma\left(c b^{-1}\right) e f$, since

$$
f e=e^{-1}(e f)^{2} f^{-1}=a^{-1} c b^{-1} f=a^{-1} \sigma\left(c b^{-1}\right) e f .
$$

Lemma 3.1. Let $F$ be a field of characteristic different from 2. Then every central simple $F$-algebra of degree 4 is isomorphic to some $(a, b, c, L / F)$. Moreover, one may assume that $a, b, c \notin F^{\times}$.

Proof. Assume first that $A$ is a division $F$-algebra. By a theorem of Albert (see [1, Theorem 11.9] for example), $A$ contains a biquadratic étale $F$-algebra $L$. By Skolem-Noether's theorem, $\sigma$ and $\tau$ extend to inner automorphisms $\operatorname{Int}(e)$ and $\operatorname{Int}(f)$, for some $e, f \in A^{\times}$. This rewrites

$$
e \lambda=\sigma(\lambda) e, f \lambda=\tau(\lambda) f, \text { for all } \lambda \in L
$$

Since $\operatorname{Int}(e)_{\left.\right|_{L}}=\sigma$, we have

$$
\operatorname{Int}\left(e^{2}\right)_{\left.\right|_{L}}=\left(\operatorname{Int}(e)_{\left.\right|_{L}}\right)^{2}=\sigma^{2}=\operatorname{Id}_{L},
$$

so $a=e^{2} \in C_{A}(L)=L$ (since $L$ is a maximal subfield of $A$ ). Since $e \in A^{\times}$, we have $a \in L^{\times}$. Similarly, $b=f^{2} \in L^{\times}$and $c=(e f)^{2} \in L^{\times}$.
Associativity of the product law of $A$ implies that

$$
\sigma(a)=a, \tau(b)=b, \sigma \tau(c)=c, n_{c}=n_{a} n_{b}
$$

Indeed, we have

$$
e^{3}=e e^{2}=e a=\sigma(a) e=e^{2} e=a e
$$

and since $e$ is invertible, we get $\sigma(a)=a$. Similar arguments show that $\tau(b)=b$ and $\sigma \tau(c)=c$. We also have

$$
\begin{aligned}
b e & =f^{2} e \\
& =f(f e) \\
& =f a^{-1} \sigma\left(c b^{-1}\right) e f \\
& =\tau\left(a^{-1} \sigma\left(c b^{-1}\right)\right) f(e f) \\
& =c \tau(a)^{-1} \sigma(b)^{-1}(f e) f \\
& =c \tau(a)^{-1} \sigma(b)^{-1} a^{-1} \sigma\left(c b^{-1}\right)(e f) f \\
& =c \tau(a)^{-1} \sigma(b)^{-1} a^{-1} \sigma(c) \sigma(b)^{-1} e b \\
& =c \tau(a)^{-1} \sigma(b)^{-1} \sigma(c) a^{-1} \sigma(b)^{-1} \sigma(b) e \\
& =n_{c} n_{a}^{-1} \sigma(b)^{-1} e
\end{aligned}
$$

Since $e$ is invertible, we get $b=n_{c} n_{a}^{-1} \sigma(b)^{-1}$, that is $n_{c}=n_{a} n_{b}$. The proof of the linear independence of $1, e, f, e f$ over $L$ is straightforward and left to the reader. Hence $A$ contains a subalgebra isomorphic to $(a, b, c, L / F)$, and therefore $A \simeq(a, b, c, L / F)$ since they have same dimension over $F$.

Assume now that $A$ is not a division algebra. Then $A \simeq \mathrm{M}_{2}(Q)$, where $Q=\left(d, d^{\prime}\right)$ is a (not necessarily division) quaternion $F$-algebra.
Let $L=F\left[\sqrt{d}, \sqrt{d^{\prime}}\right]$, and let us consider the $F$-algebra $(d, 1, d, L / F)$, where $\alpha^{2}=$ $d, \beta^{2}=d^{\prime}$ and $\beta \alpha=\alpha \beta$. In particular, we have

$$
e^{2}=d, f^{2}=1,(e f)^{2}=d, e \alpha=\alpha e, e \beta=-\beta e, f \alpha=-\alpha f, f \beta=\beta f
$$

Notice that we also have $f e=d \sigma\left(d^{-1}\right) e f=e f$.
Set $i=e, j=\beta f, i^{\prime}=f, j^{\prime}=\alpha e$. It is straightforward to check that the $F$-algebra $B$ generated by $i$ and $j$ is isomorphic to $Q$, and that the $F$-algebra $B^{\prime}$ generated by $i^{\prime}$ and $j^{\prime}$ is isomorphic to $\left(1, d^{2}\right) \simeq \mathrm{M}_{2}(F)$. Moreover one may check $B^{\prime}$ is contained is the centralizer of $B$. Since $B^{\prime}$ is a central simple $F$-algebra, we get

$$
(d, 1,1, L / F) \simeq B \otimes_{F} B^{\prime} \simeq Q \otimes_{F} \mathrm{M}_{2}(F) \simeq \mathrm{M}_{2}(Q) \simeq A
$$

We now prove the last part. Write $A \simeq(a, b, c, L / F)$, for some biquadratic étale $F$-algebra $L=F\left[\sqrt{d}, \sqrt{d^{\prime}}\right]$. Replacing $f$ by $(1+\alpha+\beta) f$, one may assume that $b \notin F^{\times}$. Indeed, if $b \in F^{\times}$, we then have

$$
[(1+\alpha+\beta) f]^{2}=(1+\alpha+\beta)(1-\alpha+\beta) f^{2}=b\left(1+d^{\prime}-d+2 \beta\right) \notin F^{\times}
$$

Write $a=a_{0}+a_{1} \alpha$. Notice that, for all $s \in F^{\times}$, we have

$$
\begin{aligned}
{[(s+\alpha+\beta) e]^{2} } & =\left(s^{2}+d-d^{\prime}+2 s \alpha\right)\left(a_{0}+a_{1} \alpha\right) \\
& =\left(a_{0}\left(s^{2}+d^{\prime}-d\right) 2 s a_{1} d\right)+\left(a_{1}\left(s^{2}+d-d^{\prime}\right)+2 a_{0} s\right) \alpha
\end{aligned}
$$

and

$$
[(s+\alpha+\beta) e f]^{2}=(s+\alpha+\beta)(s-\alpha-\beta) c=c\left(s^{2}-d-d^{\prime}-2 s \alpha \beta\right)
$$

Thus, after replacing $e$ by $(s+\alpha+\beta) e$ for a suitable $s \in F^{\times}$, one may also assume that $a, c \notin F^{\times}$. This concludes the proof.

Remark 3.2. For any biquadratic $F$-algebra $L$, we have $\mathrm{M}_{4}(F) \simeq(1,1,1, L / F)$.
Indeed, the subalgebra $B$ generated by $e$ and $\beta$ is isomorphic to $\left(1, d^{\prime}\right) \simeq \mathrm{M}_{2}(F)$, and its centralizer $C$ is the subalgebra generated by $f$ and $\alpha e$, which is isomorphic to $\left(1, d^{2}\right) \simeq \mathrm{M}_{2}(F)$. Since $B$ is a central simple $F$-algebra, we have

$$
(1,1,1, L / F) \simeq B \otimes_{F} C \simeq \mathrm{M}_{2}(F) \otimes_{F} \mathrm{M}_{2}(F) \simeq \mathrm{M}_{4}(F)
$$

Proposition 3.3. Let $F$ be a field of characteristic different from 2, and let $A=$ $(a, b, c, L / F)$. Then

$$
2[A]=\left[\left(d^{\prime}, n_{c}\right)\right] \in \operatorname{Br}(F)
$$

Proof. Recall from [4] that for every central simple $F$-algebra of degree $n$, we have

$$
\operatorname{det}\left(q_{A}\right)=\operatorname{det}\left(q_{\mathrm{M}_{n}(F)}\right), w_{2}\left(q_{A}\right)=w_{2}\left(q_{\mathrm{M}_{n}(F)}\right)+\frac{n(n-1)}{2}[A]
$$

Since $q_{A}$ and $q_{\mathrm{M}_{n}(F)}$ have same dimension and determinant, the quadratic form $q_{A} \perp-q_{\mathrm{M}_{n}(F)}$ lies in $I^{2}(F)$, and the second equality may be rewritten as

$$
c\left(q_{A} \perp-q_{\mathrm{M}_{n}(F)}\right)+\frac{n(n-1)}{2}[A]
$$

where $c$ denotes the Clifford invariant. In particular, if $n=4$, we have

$$
c\left(q_{A} \perp-q_{\mathrm{M}_{n}(F)}\right)=6[A]=2[A] .
$$

Let us introduce some notation. Let $E$ be an $F$-algebra. For $u \in E^{\times}$, we denote by $q_{E, u}$ the quadratic form

$$
q_{E, u}: \begin{aligned}
& E \longrightarrow F \\
& x \longmapsto \operatorname{Tr}_{E / F}\left(u x^{2}\right) .
\end{aligned}
$$

If $u=1$, we denote it by $q_{E}$.
Let $\rho$ be an $F$-automorphism of $L$ satisfying $\rho^{2}=\operatorname{Id}_{E}$. If $u \in E^{\times}, \rho(u)=u$, we denote by $q_{E, \rho, u}$ the quadratic form

$$
q_{E, \rho, u}: \begin{aligned}
& E \longrightarrow F \\
& x \longmapsto \operatorname{Tr}_{E / F}(u x \rho(x))
\end{aligned}
$$

If $u=1$, we denote it by $q_{E, \rho}$.
Finally, if $\alpha, \beta \in F^{\times}$, we set $\langle\langle\alpha, \beta\rangle\rangle=\langle 1,-\alpha\rangle \otimes\langle 1,-\beta\rangle$.
Now assume that $A=(a, b, u, L / F)$. It is easy to check that we have

$$
\operatorname{Trd}_{A}(x+y e+z f+t e f)=\operatorname{Tr}_{L / F}(x), \text { for all } x, y, z, t, \in L
$$

This implies that the subspaces $L, L e, L f$ and Lef are mutually orthogonal. It follows from the previous observation that we have

$$
q_{A} \simeq q_{L} \perp q_{L, \sigma, a} \perp q_{L, \tau, b} \perp q_{L, \sigma \tau, c}
$$

Let $\rho$ be an $F$-automorphism of $L$, and assume that $\rho \neq \operatorname{Id}_{L}$, so we may write $L^{\langle\rho\rangle}=F[\sqrt{\Delta}]$, and $L=L^{\langle\rho\rangle}\left[\sqrt{\Delta^{\prime}}\right]$. Notice that $u x \rho(x) \in L^{\langle\rho\rangle}$, so that we have $\operatorname{Tr}_{L / F}(u x \rho(x))=2 \operatorname{Tr}_{L\langle\rho\rangle / F}(u x \rho(x))$.
Now if $x=x_{0}+x_{1} \alpha^{\prime}$ where $\alpha^{\prime 2}=\Delta^{\prime}$ and $x_{i} \in L^{\langle\rho\rangle}$, we have $x \rho(x)=x_{0}^{2}-\Delta^{\prime} x_{1}^{2}$, and therefore

$$
q_{L, \rho, u} \simeq\left\langle 2,-2 \Delta^{\prime}\right\rangle \otimes q_{L^{\langle\rho\rangle}, u}
$$

Write $u=u_{0}+u_{1} \sqrt{\Delta}, u_{i} \in F$, and set $n_{u}=u_{0}^{2}-\Delta u_{1}^{2}$. The representative matrix of the previous quadratic form in the basis $1, \sqrt{\Delta}$ is

$$
\left(\begin{array}{cc}
2 u_{0} & 2 u_{1} \Delta \\
2 u_{1} \Delta & 2 u_{0} \Delta
\end{array}\right) .
$$

If $u_{0}=0$, this 2-dimensional quadratic form is isotropic, hence hyperbolic. If $u_{0} \neq 0$, this form then represents $2 u_{0}$ and has determinant $4 n_{u} \Delta$, so it is isomorphic to $\left\langle 2 u_{0}\right\rangle \otimes\left\langle 1, n_{u} \Delta\right\rangle$. We then get

$$
q_{L, \rho, u} \simeq\left\langle u_{0}\right\rangle \otimes\left\langle\left\langle-n_{u} \Delta, \Delta^{\prime}\right\rangle\right\rangle
$$

where this form has to be understood as the hyperbolic form of dimension 4 if $u_{0}=0$. Taking $u=1$, we get

$$
q_{L, \rho} \simeq\left\langle\left\langle-\Delta, \Delta^{\prime}\right\rangle\right\rangle
$$

and thus

$$
q_{L, \rho, u} \perp-q_{L, \rho} \simeq\left\langle 1,-\Delta^{\prime}\right\rangle \otimes\left\langle u_{0}, u_{0} n_{u} \Delta, 1, \Delta\right\rangle
$$

Since $c\left(\varphi \otimes \varphi^{\prime}\right)=\left[\left(\operatorname{disc}(\varphi), \operatorname{disc}\left(\varphi^{\prime}\right)\right)\right]$ for every even-dimensional forms $\varphi, \varphi^{\prime}$, we get

$$
c\left(q_{L, \rho, u} \perp-q_{L, \rho}\right)=\left[\left(\Delta^{\prime}, n_{u}\right)\right] .
$$

Remark 3.2 shows that $q_{A} \perp-q_{\mathrm{M}_{4}(F)}$ is Witt-equivalent to

$$
\left(q_{L, \sigma, a} \perp-q_{L, \sigma}\right) \perp\left(q_{L, \tau, b} \perp-q_{L, \tau}\right) \perp\left(q_{L, \sigma \tau, c} \perp-q_{L, \sigma \tau}\right) .
$$

We then finally get

$$
c\left(q_{A} \perp-q_{\mathrm{M}_{4}(F)}\right)=\left[\left(d^{\prime}, n_{a}\right)\right]+\left[\left(d, n_{b}\right)\right]+\left[\left(d, n_{c}\right)\right] .
$$

Since the quaternion algebras $\left(d, n_{a}\right)$ and $\left(d^{\prime}, n_{b}\right)$ are split, we get

$$
\begin{aligned}
c\left(q_{A} \perp-q_{\mathrm{M}_{4}(F)}\right) & =\left[\left(d d^{\prime}, n_{a}\right)\right]+\left[\left(d d^{\prime}, n_{b}\right)\right]+\left[\left(d, n_{c}\right)\right] \\
& =\left[\left(d d^{\prime}, n_{a} n_{b}\right)\right]+\left[\left(d, n_{c}\right)\right] \\
& =\left[\left(d d^{\prime}, n_{c}\right)\right]+\left[\left(d, n_{c}\right)\right] \\
& =\left[\left(d^{\prime}, n_{c}\right)\right] .
\end{aligned}
$$

This concludes the proof.
We now define a classifying pair for $\mathbf{C S A}_{4}$. Let us consider the affine variety $\mathcal{V} \subset \mathbb{A}^{6}$ defined by the equation

$$
x^{2}-u y^{2}-v z^{2}+u v t^{2}+u v=0 .
$$

This is a rational variety, with coordinate ring

$$
F[\mathcal{V}]=F[X, Y, Z, T, U, V] /\left(X^{2}-U Y^{2}-V Z^{2}+U V T^{2}+U V\right)
$$

Let us denote by $x, y, z, t, u, v$ the images of $X, Y, Z, T, U, V$ in $F[\mathcal{V}]$ respectively. Notice that $u$ and $v$ are algebraically independent over $F$. Indeed, we have a surjective $F$-algebra morphism

$$
\begin{aligned}
F[\mathcal{V}] & \longrightarrow F[V, Z] \\
P(x, y, z, t, u, v) & \longmapsto P\left(0, i, Z, i, V Z^{2}, V\right)
\end{aligned}
$$

which maps $u$ and $v$ onto $V Z^{2}$ and $V$. Since $V Z^{2}$ and $V$ are algebraically independent over $F$, so are $u$ and $v$.

Now let us consider the open subset $\mathcal{U}$ of $\mathcal{V}$ defined by the equations

$$
u \neq 0, v \neq 0, t \neq 0, y^{2}-v t^{2} \neq 0, z^{2}-u \neq 0
$$

Then $\mathcal{U}$ is also a rational variety, whose coordinate ring $F[\mathcal{U}]$ is the localization of $F[x, y, z, t, u, v]$ at $u, v, t, y^{2}-v t^{2}$ and $z^{2}-u$. Set $R_{0}=F[\mathcal{U}]$. Then $R_{0}$ is a noetherian ring, whose quotient field $F_{0}$ is a rational extension of $F$. Moreover, $u, v \in R_{0}$ are algebraically independent over $F$. In particular, there exists a transcendence basis of $F_{0} / F$ containing $u$ and $v$.

Set $w=u\left(y^{2}-v t^{2}\right) \in R_{0}^{\times}$and let $L_{0}=R_{0}[\sqrt{u}, \sqrt{u w}]$. We now define three elements $a_{0}, b_{0}, c_{0} \in L_{0}^{\times}$by

$$
a_{0}=z+\sqrt{u} \in R_{0}[\sqrt{u}]^{\times}, b_{0}=\frac{y}{t}+\frac{\sqrt{u w}}{t u} \in R_{0}[\sqrt{u w}]^{\times}, c_{0}=x+\sqrt{w} \in R_{0}[\sqrt{w}]^{\times} .
$$

Let us check that $n_{a_{0}} n_{b_{0}}=n_{c_{0}}$. We have

$$
n_{a_{0}} n_{b_{0}}=\frac{\left(z^{2}-u\right)\left(y^{2} u^{2}-u w\right)}{t^{2} u^{2}}=\frac{\left(z^{2}-u\right)\left(y^{2} u-w\right)}{t^{2} u}
$$

Now $y^{2} u-w=u v t^{2}$, and thus $n_{a_{0}} n_{b_{0}}=\left(z^{2}-u\right) v$. Now $z^{2} v-u v=x^{2}-u y^{2}+u v t^{2}=$ $x^{2}-w=n_{c_{0}}$, and we are done. Hence we may consider the Azumaya algebra

$$
\mathcal{A}_{0}=\left(a_{0}, b_{0}, c_{0}, L_{0} / R_{0}\right)
$$

Lemma 3.4. The pair $\left(R_{0}, \mathcal{A}_{0}\right)$ is a classifying pair for $\mathbf{C S A}_{4}$.
Proof. Let $E / F$ be a field extension, and let $A$ be a central simple $E$-algebra of degree 4. By Lemma 3.1, we have $A \simeq(a, b, c, L / E)$, with $a, b, c \notin E^{\times}$. Write $L=E\left[\sqrt{d}, \sqrt{d^{\prime}}\right]$, with generators $\alpha, \beta$. Write

$$
a=\lambda_{0}+\lambda_{1} \alpha, b=\mu_{0}+\mu_{1} \beta, c=\gamma_{0}+\gamma_{1} \alpha \beta
$$

We set

$$
\bar{x}=\gamma_{0}, \bar{y}=\frac{\mu_{0} \gamma_{1}}{\lambda_{1} \mu_{1}}, \bar{z}=\lambda_{0}, \bar{t}=\frac{\gamma_{1}}{\lambda_{1} \mu_{1}}, \bar{u}=\lambda_{1}^{2} d, \bar{v}=\mu_{0}^{2}-\mu_{1}^{2} d^{\prime}
$$

Using the equality

$$
\left(\lambda_{0}^{2}-\lambda_{1}^{2} d\right)\left(\mu_{0}^{2}-\mu_{1}^{2} d\right)=\left(\gamma_{0}^{2}-\gamma_{1}^{2} d\right)
$$

and the fact that $\lambda_{1}, \mu_{1}, \gamma_{1} \in F^{\times}$, one may check that $\mathbf{p}=(\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{u}, \bar{v}) \in \mathcal{U}(E)$. Then evaluation at $\mathbf{p}$ yields an $F$-algebra morphism $f: R_{0} \longrightarrow E$ such that

$$
\mathcal{A}_{0} \otimes_{R_{0}} E \simeq A
$$

Now it suffices to notice that $\operatorname{ker}(f)=(x-\bar{x}, y-\bar{y}, z-\bar{z}, t-\bar{t}, u-\bar{u}, v-\bar{v})$ is a maximal ideal to conclude.

## 4. Cohomological invariants of $\mathbf{C S A}_{4}$

We now prove the following theorem, due to Rost (unpublished):
Theorem 4.1 (Rost). Let $F$ be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$. Then the map

$$
\operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right) \longrightarrow \operatorname{Inv}\left(\mathbf{M S}_{2,2}, H^{*}\right) \times \operatorname{Inv}\left(\mathbf{M S}_{4,1}, H^{*}\right)
$$

is injective. In other words, a cohomological invariant which is zero on biquaternion algebras and cyclic algebras is identically zero.

Proof. Let $\alpha \in \operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right)$. Let $\left(R_{0}, A_{0}\right)$ be the classifying pair of $\mathbf{C S A}_{4}$ defined in the previous section, and let $F_{0}$ be the quotient field of $R_{0}$. Finally, set

$$
A_{0}=\mathcal{A}_{0} \otimes_{R_{0}} F_{0}=\left(z+\sqrt{u}, \frac{y}{t}+\frac{\sqrt{u w}}{t u}, x+\sqrt{w}, F_{0}[\sqrt{u}, \sqrt{u w}] / F_{0}\right)
$$

Let $v$ be a discrete $F$-valuation on $F_{0}$. If $v \neq v_{\pi}$, with $\pi=u, v, t, y^{2}-v t^{2}, z^{2}-u$, then $u, v, w, t \in \mathcal{O}_{v}^{\times}$and $z+\sqrt{u}, \frac{y}{t}+\frac{\sqrt{u w}}{t u}, x+\sqrt{w} \in \mathcal{O}_{v}[\sqrt{u}, \sqrt{u w}]^{\times}$. We then get that

$$
A_{0} \simeq\left(z+\sqrt{u}, \frac{y}{t}+\frac{\sqrt{u w}}{t u}, x+\sqrt{w}, \mathcal{O}_{v}[\sqrt{u}, \sqrt{u w}] / \mathcal{O}_{v}\right) \otimes_{\mathcal{O}_{v}} F_{0}
$$

and therefore $\alpha_{F_{0}}\left(A_{0}\right)$ is unramified at $v$ by [2, Theorem 11.7].
Assume now that $v=v_{\pi}$, with $\pi=t, z^{2}-u$ or $y^{2}-t v^{2}$. Then $u, v \in \mathcal{O}_{v}^{\times}$, and thus we have

$$
(u) \cup(v)=j_{v}((\bar{u}) \cup(\bar{v}))
$$

where $j_{v}: H^{2}(\kappa(v)) \hookrightarrow H^{2}\left(\left(F_{0}\right)_{v}\right)$ is the canonical injection. Now in all three cases, the quadratic form $\langle 1,-\bar{u},-\bar{v}, \overline{u v}\rangle$ is isotropic over $\kappa(v)$. It is clear if $\pi=z^{2}-u$ or $y^{2}-v t^{2}$, and follows from the equalities

$$
0=x^{2}-u y^{2}-v z^{2}+u v t^{2}+u v=x^{2}-u y^{2}-v z^{2}+u v \quad \bmod t
$$

if $\pi=t$. Thus $(\bar{u}) \cup(\bar{v})=0$, and therefore $(u) \cup(v)=0$ as well. It follows that the quaternion algebra $(u, v)_{F_{0}}$ is split over $\left(F_{0}\right)_{v}$. Hence, by Proposition 3.3, 2[ $A_{0}$ ] splits over $\left(F_{0}\right)_{v}$. By a theorem of Albert, $A_{0} \otimes_{F_{0}}\left(F_{0}\right)_{v}$ is a biquaternion algebra. By assumption, we get

$$
\operatorname{Res}_{\left(F_{0}\right)_{v} / F_{0}}\left(\alpha_{F_{0}}\left(A_{0}\right)\right)=\alpha_{\left(F_{0}\right)_{v}}\left(A_{0} \otimes_{F_{0}}\left(F_{0}\right)_{v}\right)=0
$$

Consequently, $\partial_{v}\left(\alpha_{F_{0}}\left(A_{0}\right)\right)=0$ and $\alpha_{F_{0}}\left(A_{0}\right)$ is once again unramified at $v$.
Finally, $\alpha_{F_{0}}\left(A_{0}\right)$ is unramified at every discrete $F$-valuation $v \neq v_{u}, v_{v}$. In particular, it is unramified at every $F_{1}$-valuation, where $F_{1}=F(u, v)$. Since there exists a transcendence basis of $F_{0} / F$ containing $u$ and $v$, the field extension $F_{0} / F_{1}$ is rational, and we deduce that $\alpha_{F_{0}}\left(A_{0}\right)=\operatorname{Res}_{F_{0} / F_{1}}(\beta)$, for some $\beta \in H^{*}\left(F_{1}\right)$. Let $v_{1}$ be a discrete $F$-valuation on $F_{0}$, which different from the $u$-adic and the $v$-adic valuations. Let us show that $\beta$ is unramified at $v_{1}$.

Let $u, v, t_{1}, t_{2}, t_{3}$ be a transcendence basis of $F_{0} / F$, and let us extend $v_{1}$ to a discrete $F$-valuation on $F_{0}$ by setting

$$
v_{F_{1}}=v_{1}, v\left(t_{i}\right)=0, i=1,2,3
$$

Then the corresponding ramification index is 1 and $\kappa\left(v_{1}\right)=\kappa(v)\left(t_{1}, t_{2}, t_{3}\right)$, so that $\kappa(v) / \kappa\left(v_{1}\right)$ is a rational extension. Since $\alpha_{F_{0}}\left(A_{0}\right)$ is unramified at $v$, we get

$$
0=\partial_{v}\left(\alpha_{F_{0}}\left(A_{0}\right)\right)=\operatorname{Res}_{\kappa(v) / \kappa\left(v_{1}\right)}\left(\partial_{v_{1}}(\beta)\right)
$$

Since $\kappa(v) / \kappa\left(v_{1}\right)$ is rational, the map $\operatorname{Res}_{\kappa(v) / \kappa\left(v_{1}\right)}$ is injective, and we get that $\beta$ is unramified at $v_{1}$. By Proposition 1.2, we have

$$
\beta=a_{0} \cdot 1+a_{1} \cdot(u)+a_{2} \cdot(v)+a_{3} \cdot(u) \cup(v), a_{i} \in H^{*}(F) .
$$

Hence we get

$$
\alpha_{F_{0}}\left(A_{0}\right)=a_{0} \cdot 1+a_{1} \cdot \operatorname{Res}_{F_{0} / F_{1}}((u))+a_{2} \cdot \operatorname{Res}_{F_{0} / F_{1}}((v))+a_{3} \cdot \operatorname{Res}_{F_{0} / F_{1}}((u) \cup(v))
$$

Let us consider the $(v-1)$-adic valuation on $F_{0}$, and let $F_{2}$ its residue field. Since $A_{0}$ is unramified at this valuation, one may specialize $A_{0}$ to a central simple $F_{2^{-}}$ algebra $B$. Then $2\left[A_{0}\right]$ specializes to $2[B]$. But $2\left[A_{0}\right]=[(u, v)]$ specializes to 0 , and therefore $B$ is a biquaternion algebra. Then $\alpha_{F_{2}}(B)=0$ by assumption, and therefore

$$
a_{0} \cdot 1+a_{1} \cdot \operatorname{Res}_{F_{2} / F_{1}}((u))=0 .
$$

In other words, $\operatorname{Res}_{F_{2} / F_{1}}\left(a_{0} \cdot 1+a_{1} \cdot(u)\right)=0$. Since $F_{1} \subset F_{2} \subset F_{0}$, and $F_{0} / F_{1}$ is rational, the map $\operatorname{Res}_{F_{2} / F_{0}}$ is injective and thus

$$
a_{0} \cdot 1+a_{1} \cdot(u)=0 \text { in } H^{*}\left(F_{1}\right) .
$$

By Lemma 1.1, we get $a_{0}=a_{1}=0$. Considering the $u-1$-adic valuation shows that $a_{2}=0$ in a similar way, so that

$$
\alpha_{F_{0}}\left(A_{0}\right)=a_{3} \cdot \operatorname{Res}_{F_{0} / F_{1}}((u) \cup(v)) .
$$

Since $\operatorname{Res}_{F_{0} / F_{1}}((u) \cup(v))$ corresponds to $[(u, v)]=2\left[A_{0}\right]$ in $\operatorname{Br}_{2}\left(F_{0}\right)$ via the usual isomorphism, the previous equality rewrites

$$
\alpha_{F_{0}}\left(A_{0}\right)=a_{3} \cdot f_{2, F_{0}}\left(A_{0}\right)
$$

By Lemma 1.5, we get $\alpha=a_{3} \cdot f_{2}$. By assumption on $\alpha$, we then have

$$
0=\alpha_{F_{1}}\left(\{u, v\}_{4, F_{1}}\right)=a_{3} \cdot(u) \cup(v) \in H^{*}\left(F_{1}\right)
$$

By Lemma 1.1, we get $a_{3}=0$, and thus $\alpha=0$. This concludes the proof.

We now describe the cohomological invariants of $\mathbf{C S A}_{4}$. Let $F$ be a field of characteristic different from 2 . Assume that $-1 \in F^{\times 2}$, and let $K / F$ be a field extension. By [5], for every central simple $K$-algebra $A$ of degree 4 , there exists a 2-fold Pfister form $q_{2}$ and a 4 -fold Pfister form $q_{4}$ such that

$$
q_{A} \sim q_{2}+q_{4} \in W(K)
$$

Taking the cohomology class $e_{4}\left(q_{4}\right) \in H^{4}(K)$ yields a cohomological invariant of $\operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right)$. This invariant restricts to zero on cyclic algebras, and its restriction to $\mathbf{M S}_{2,2}$ obviously coincide with the invariant $e_{4}$ defined in Section 2. Therefore, we still denote this invariant by $e_{4}$.

Corollary 4.2. Let $F$ be a field of characteristic different from 2. Assume that $-1 \in F^{\times 2}$. Then $\operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right)$ is a free $H^{*}(F)$-module with basis $1, f_{2}, e_{4}$.

Proof. We first prove that $1, f_{2}, e_{4}$ are linearly independent. Assume that we have

$$
a_{0} \cdot 1+a_{1} \cdot f_{2}+a_{2} \cdot e_{4}=0 \text { for some } a_{i} \in H^{*}(F)
$$

Since the restriction of $e_{4}$ to $\mathbf{M S}_{4,1}$ is zero, it follows that $a_{0}+a_{1} \cdot f_{2}=0 \in$ $\operatorname{Inv}\left(\mathbf{M S}_{4,1}, H^{*}\right)$. By Proposition 2.2 (1), we get $a_{0}=a_{1}=0$. Thus, we get $a_{2} \cdot e_{4}=0 \in \operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right)$. In particular, $a_{2} \cdot e_{4}=0 \in \operatorname{Inv}\left(\mathbf{M S}_{2,2}, H^{*}\right)$. By Proposition 2.2 (2), we get $a_{3}=0$.

We now prove that $1, f_{2}, e_{4}$ span $\operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right)$ as an $H^{*}(F)$-module. Let $\alpha \in$ $\operatorname{Inv}\left(\mathbf{C S A}_{4}, H^{*}\right)$. By Proposition 2.2, we have

$$
\alpha_{\mathrm{MS}_{2,2}}=a_{0} \cdot 1+a_{1} \cdot f_{2}+a_{2} \cdot e_{4}
$$

for some $a_{i} \in H^{*}(F)$, and also

$$
\alpha_{\left.\right|_{\mathrm{Ms}} ^{4,1}}=b_{0} \cdot 1+b_{1} \cdot f_{2}
$$

for some $b_{i} \in H^{*}(F)$. Since $\mathrm{M}_{4}(F) \simeq\{1,1,\}_{4, F} \simeq(1,1) \otimes_{F}(1,1)$, we may apply $\mathrm{M}_{4}(F)$ to both equalities to get $a_{0}=b_{0}$. Now we have

$$
\mathrm{M}_{2}((u, v)) \simeq(1,1) \otimes_{F(u, v)}(u, v) \simeq\left\{u^{2}, v^{2}\right\}_{4, F(u, v)}
$$

and applying $\mathrm{M}_{2}((u, v))$ to both equalities yields

$$
a_{1} \cdot(u) \cup(v)=b_{1} \cdot(u) \cup(v) \in H^{*}(F(u, v))
$$

It follows from Lemma 1.1 that $a_{1}=b_{1}$. Since $e_{4}$ is zero on $\mathbf{M S}_{4,1}$, we conclude that $\alpha$ and $a_{0} \cdot 1+a_{1} \cdot f_{2}+a_{2} \cdot e_{4}$ coincide on $\mathbf{M S} \mathbf{S}_{4,1}$ and $\mathbf{M S} \mathbf{S}_{2,2}$. By the previous theorem, we get $\alpha=a_{0} \cdot 1+a_{1} \cdot f_{2}+a_{2} \cdot e_{4}$, and this concludes the proof.

## References

[1] A.A.Albert, Structure of Algebras. A.M.S. Coll.Pub. 24, New York, A.M.S., 1939.
[2] R. S. Garibaldi, A. S. Merkurjev, J.-P. Serre, Cohomological invariants in Galois cohomology, University Lecture Series 28, A.M.S., Providence, RI, 2003.
[3] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, The book of involutions, A.M.S. Coll. Pub. 44, 1998.
[4] D. W. Lewis, J. Morales, The Hasse invariant of the trace form of a central simple algebra. Pub. Math. Besançon, Théorie des nombres, Année 1993-94, 1-6.
[5] M. Rost, J.-P. Serre, J.-P. Tignol, La forme trace d'une algbre simple centrale de degré 4. C.R. Acad. Sci. Paris, Sér. I 342 (2006), 83-87.

Email: berhuy@ujf-grenoble.fr
Address: Université Grenoble I, UFR de Mathématiques, Institut Fourier 100 rue des maths, BP 74, F-38402 St Martin d'Hères Cedex, France

