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SMOOTH DENSITY FOR SOME NILPOTENT ROUGH DIFFERENTIAL EQUATIONS

YAOZHONG HU AND SAMY TINDEL

Abstract. In this note, we provide a non trivial example of differential equation driven by a fractional Brownian motion with Hurst parameter $1/3 < H < 1/2$, whose solution admits a smooth density with respect to Lebesgue's measure. The result is obtained through the use of an explicit representation of the solution when the vector fields of the equation are nilpotent, plus a Norris type lemma in the rough paths context.

1. Introduction

Let $B = (B^1, \ldots, B^d)$ be a d-dimensional fractional Brownian motion with Hurst parameter $1/3 < H < 1/2$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Remind that this means that all the component B^i of B are independent centered Gaussian processes with covariance

$$R_H(t,s) := \mathbf{E}\left[B_t^i\, B_s^i\right] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{1}$$

In particular, the paths of B are γ-Hölder continuous for all $\gamma \in (0, H)$. This paper is concerned with a class of $\mathbb{R}^m$-valued stochastic differential equations driven by B, of the form

$$dy_t = \sum_{i=1}^{d} V_i(y_t)\, dB_t^i, \quad t \in [0,T], \quad y_0 = a, \tag{2}$$

where $T > 0$ is a fixed time horizon, $a \in \mathbb{R}^m$ stands for a given initial condition and $(V_1, \ldots, V_d)$ is a family of smooth vector fields of $\mathbb{R}^m$.

Stochastic differential systems driven by fractional Brownian motion have been the object of intensive studies during the past decade, both for their theoretical interest and for the wide range of application they open, covering for instance finance [15, 32] or biophysics [20, 29] situations. The first aim in the theory has thus been to settle some reasonable tools allowing to solve equations of type (2). This has been achieved, when the Hurst parameter H of the underlying fBm is $> 1/2$, thanks to methods of fractional integration [27, 33], or simply by means of Young type integration (see e.g [14]). When one moves to more irregular cases, namely $H < 1/2$, the standard method by now in order to solve equations like (2) relies on rough paths considerations, as explained for instance in [12, 14, 22].

A second natural step in the study of fractional differential systems consists in establishing some properties about their probability law. Some substitute for the semigroup

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property governing $\mathcal{L}(y_t)$ in the Markovian case (namely when $H = 1/2$) have been given in [2, 24], in terms of asymptotic expansions in a neighborhood of $t = 0$. nSome considerable efforts have also been made in order to analyze the density of $\mathcal{L}(y_t)$ with respect to Lebesgue measure. To that respect, in the regular case $H > 1/2$ the situation is rather clear: the existence of a density is shown in [28] under some standard nondegeneracy conditions, the smoothness of the density is established in [19] under elliptic conditions on the coefficients, and this result is extended to the hypoelliptic case in [3]. In all, this set of results replicates what has been obtained for the usual Brownian motion, at the price of huge technical complications.

In the irregular case $H < 1/2$, the picture is far from being so complete. Indeed, the existence part of the density results have been thoroughly studied under elliptic and Hörmander conditions (see [6, 12] for a complete review). However, when one wishes to establish the smoothness of the density, some strong moment assumptions on the inverse of the Malliavin derivative of y_t are usually required. These moment estimates are still an important open question in the field, as well as the smoothness of density for random variables like y_t.

The current paper proposes to make a step in this direction, and we wish to prove that $\mathcal{L}(y_t)$ can be decomposed as $p_t(z)\,dz$ for a smooth function p_t in some special non trivial examples of equation (2). Namely, we will handle in the sequel the case of nilpotent vector fields $V_1, \ldots, V_d$ (see Hypothesis 4.1 for a precise description), and in this context we shall derive the following density result:

Theorem 1.1. *Suppose that the vector fields V_i, $1 = 1, 2, \ldots, d$ are smooth with all derivatives bounded, and that they are n-nilpotent in the sense that their Lie brackets of order n vanish for some positive integer n. We also assume that $V_1, \ldots, V_d$ satisfy Hörmander's hypoelliptic condition (their Lie brackets generate $\mathbb{R}^m$ at any point $x \in \mathbb{R}^m$), and that all the Lie brackets of order greater or equal to 2 are constant. Then for all $t > 0$ the probability law of the random variable y_t, defined by (2) admits a smooth density with respect to Lebesgue measure.*

Notice that the hypoelliptic assumption is quite natural in our context. Indeed, it would certainly be too restrictive to consider a family of vector fields $V_1, \ldots, V_d$ being nilpotent and elliptic at the same time. Moreover, some interesting examples of equations satisfying our standing assumptions will be given below. It should be stressed however that the basic aim of this article is to prove that smoothness of density results can be obtained for rough differential equations driven by a fractional Brownian motion in some specific situations, even if the general hypoelliptic case is still an important open problem. We refer to [4] for another case, based on skew-symmetric properties, where a similar theorem holds true.

In order to prove Theorem 1.1, two main ingredients have to be highlighted:

(i) Working under the nilpotent assumptions described above enables to use a Strichartz type representation for the solution to our equation, given in terms of a finite chaos expansion. This allows to derive some bounds for the moments of both y_t and its Malliavin derivative, which is the main missing tool on the way to smoothness of density results for rough differential equations in the general case.

(ii) With the integrability of Malliavin derivative in hand, we shall follow the standard probabilistic way to prove smoothness of density under Hörmander's conditions, for which we refer to [16, 23, 25]. To this purpose, the second main ingredient is a Norris type lemma,

which has to be extended (in the rough path context) to *controlled processes*. It should be mentioned at this point that a similar result has been proven recently (an independently) in [17].

These two ingredients will be developed in the remainder of the article.

Here is how our article is structured: Some preliminaries on rough differential equations and fractional Brownian motion are given in Section 2. Section 3 is devoted to the proof of our Norris type lemma for controlled processes in the sense of [14]. Finally, Malliavin calculus tools and their application to density results for the random variable y_t are presented at Section 4.

Notation: In the remainder of the article, c, c_1, c_2 will stand for generic positive constants which may change from line to line. We also write $a \lesssim b$ (resp. $a \asymp b$) when $a \leq c\, b$ (resp. $a = c\, b$) for a universal constant c.

2. Rough differential equations and fractional Brownian motion

Generalized integrals will be needed in the sequel in order to define and solve equations of the form (2), and also to get an equivalent of Norris lemma in our context. Though all those elements might be obtained within the landmark of usual rough paths setting [12, 22] we have chosen here to work with the algebraic integration framework, which (from our point of view) is more amenable to handy calculations.

In this section, we recall thus the main concepts of algebraic integration. Namely, we state the definition of the spaces of increments, of the operator δ, and its inverse called Λ (or sewing map). We also recall some elementary but useful algebraic relations on the spaces of increments. The interested reader is sent to [14] for a complete account on the topic, or to [8, 13] for a more detailed summary.

2.1. **Increments.**

The extended integral we deal with is based on the notion of increments, together with an elementary operator δ acting on them.

The notion of increment can be introduced in the following way: for two arbitrary real numbers $\ell_2 > \ell_1 \geq 0$, a vector space V, and an integer $k \geq 1$, we denote by $\mathcal{C}_k([\ell_1, \ell_2]; V)$ the set of continuous functions $g : [\ell_1, \ell_2]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \in \{0, \dots, k-1\}$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*([\ell_1, \ell_2]; V) = \cup_{k \geq 1} \mathcal{C}_k([\ell_1, \ell_2]; V)$. To simplify the notation, we will write $\mathcal{C}_k(V)$, if there is no ambiguity about $[\ell_1, \ell_2]$.

The operator δ is an operator acting on k-increments, and is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{3}$$

where $\hat{t}_i$ means that this particular argument is omitted. Then a fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the article, are obtained by letting $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then, for any $t, u, s \in [\ell_1, \ell_2]$, we have

$$(\delta g)_{st} = g_t - g_s \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{4}$$

Our future discussions will mainly rely on k-increments with $k = 2$ or $k = 3$, for which we will use some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu = \sup_{s,t\in[\ell_1,\ell_2]} \frac{|f_{st}|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V);\, \|f\|_\mu < \infty\}. \tag{5}$$

Using this notation, we define in a natural way $\mathcal{C}_1^\mu(V) = \{f \in \mathcal{C}_1(V);\, \|\delta f\|_\mu < \infty\}$. In the sequel, we also handle norms including supremums, of the form

$$\|f\|_{\mu,\infty} = \|f\|_\mu + \|f\|_\infty, \quad \text{and} \quad \mathcal{C}_1^{\mu,0}(V) = \{f \in \mathcal{C}_1(V);\, \|f\|_{\mu,\infty} < \infty\}. \tag{6}$$

In the same way, for $h \in \mathcal{C}_3(V)$ we set

$$\begin{aligned} \|h\|_{\gamma,\rho} &= \sup_{s,u,t\in[\ell_1,\ell_2]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho}, \\ \|h\|_\mu &= \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\, h = \sum_i h_i,\, 0 < \rho_i < \mu \right\}, \end{aligned} \tag{7}$$

where the last infimum is taken over all sequences $\{h_i,\, i \in \mathbb{N}\} \subset \mathcal{C}_3(V)$ such that $h = \sum_i h_i$ and over all choices of the numbers $\rho_i \in (0,\mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we define

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V);\, \|h\|_\mu < \infty\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \cup_{\mu>1}\mathcal{C}_3^\mu(V)$, and note that the same kind of norms can be considered on the spaces $\mathcal{ZC}_3(V)$, leading to the definition of the spaces $\mathcal{ZC}_3^\mu(V)$ and $\mathcal{ZC}_3^{1+}(V)$. In order to avoid ambiguities, we sometimes denote in the following by $\mathcal{N}[\,\cdot\,;\mathcal{C}_j^\kappa]$ the κ-Hölder norm on the space $\mathcal{C}_j$, for $j = 1,2,3$. For $\zeta \in \mathcal{C}_j(V)$, we also set $\mathcal{N}[\zeta;\mathcal{C}_j^0(V)] = \sup_{s\in[\ell_1;\ell_2]^j} \|\zeta_s\|_V$.

The invertibility of δ under Hölder regularity conditions is an essential tool for the construction of our generalized integrals, and can be summarized as follows:

Theorem 2.1 (The sewing map)**.** *Let $\mu > 1$. For any $h \in \mathcal{ZC}_3^\mu(V)$, there exists a unique $\Lambda h \in \mathcal{C}_2^\mu(V)$ such that $\delta(\Lambda h) = h$. Furthermore,*

$$\|\Lambda h\|_\mu \le \frac{1}{2-2^\mu}\,\mathcal{N}[h;\,\mathcal{C}_3^\mu(V)]. \tag{8}$$

This gives rise to a continuous linear map $\Lambda : \mathcal{ZC}_3^\mu(V) \to \mathcal{C}_2^\mu(V)$ such that $\delta\Lambda = id_{\mathcal{ZC}_3^\mu(V)}$.

Proof. The original proof of this result can be found in [14]. We refer to [8, 13] for two simplified versions.

□

The sewing map creates a first link between the structures we just introduced and the problem of integration of irregular functions:

Corollary 2.2 (Integration of small increments)**.** *For any 1-increment $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^{1+}$, set $h = (id - \Lambda\delta)g$. Then, there exists $f \in \mathcal{C}_1(V)$ such that $h = \delta f$ and*

$$\delta f_{st} = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^n g_{t_i t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$ whose mesh tends to zero. The 1-increment δf is the indefinite integral of the 1-increment g.

We also need some product rules for the operator δ. For this recall the following convention: for $g \in \mathcal{C}_n([\ell_1,\ell_2]\mathbb{R}^{l,d})$ and $h \in \mathcal{C}_m([\ell_1,\ell_2];\mathbb{R}^{d,p})$ let gh be the element of $\mathcal{C}_{n+m-1}([\ell_1,\ell_2];\mathbb{R}^{l,p})$ defined by

$$(gh)_{t_1,\dots,t_{m+n-1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{m+n-1}} \tag{9}$$

for $t_1,\dots,t_{m+n-1} \in [\ell_1,\ell_2]$. With this notation, the following elementary rule holds true:

Proposition 2.3. *Let $g \in \mathcal{C}_2([\ell_1,\ell_2];\mathbb{R}^{l,d})$ and $h \in \mathcal{C}_1([\ell_1,\ell_2];\mathbb{R}^d)$. Then gh is an element of $\mathcal{C}_2([\ell_1,\ell_2];\mathbb{R}^l)$ and $\delta(gh) = \delta g\, h - g\, \delta h$.*

2.2. **Random differential equations.** One of the main appeals of the algebraic integration theory is that differential equations driven by a γ-Hölder signal x can be defined and solved rather quickly in this setting. In the case of an Hölder exponent $\gamma > 1/3$, the required structures are just the notion of *controlled processes* and the Lévy area based on x.

Indeed, recall that we wish to consider an equation of the form

$$dy_t = \sum_{i=1}^{d} V_i(y_t)\, dx_t^i, \quad t \in [0,T], \qquad y_0 = a, \tag{10}$$

where a is a given initial condition in $\mathbb{R}^m$, x is an element of $\mathcal{C}_1^\gamma([0,T];\,\mathbb{R}^d)$, and $(V_1,\dots,V_d)$ is a family of smooth vector fields of $\mathbb{R}^m$. Then it is natural that the increments of a candidate for a solution to (10) should be controlled by the increments of x in the following way:

Definition 2.4. *Let z be a path in $\mathcal{C}_1^\kappa(\mathbb{R}^m)$ with $1/3 < \kappa \le \gamma$, and set $\delta x := \mathbf{x^1}$. We say that z is a weakly controlled path based on x if $z_0 = a$ with $a \in \mathbb{R}^m$, and $\delta z \in \mathcal{C}_2^\kappa(\mathbb{R}^m)$ has a decomposition $\delta z = \zeta \mathbf{x}^1 + r$, that is, for any $s,t \in [0,T]$,*

$$\delta z_{st} = \zeta_s^j\, \mathbf{x}_{st}^{\mathbf{1},j} + r_{st}, \tag{11}$$

where we have used the summation over repeated indices convention, and with $\zeta^1,\dots,\zeta^d \in \mathcal{C}_1^\kappa(\mathbb{R}^m)$, as well as $r \in \mathcal{C}_2^{2\kappa}(\mathbb{R}^m)$.

The space of weakly controlled paths will be denoted by $\mathcal{Q}_{\kappa,a}^x(\mathbb{R}^m)$, and a process $z \in \mathcal{Q}_{\kappa,a}^x(\mathbb{R}^m)$ can be considered in fact as a couple (z,ζ). The space $\mathcal{Q}_{\kappa,a}^x(\mathbb{R}^m)$ is endowed with a natural semi-norm given by

$$\mathcal{N}[z;\mathcal{Q}_{\kappa,a}^x(\mathbb{R}^m)] = \mathcal{N}[z;\mathcal{C}_1^\kappa(\mathbb{R}^m)] + \sum_{j=1}^{d} \mathcal{N}[\zeta^j;\mathcal{C}_1^{\kappa,0}(\mathbb{R}^m)] + \mathcal{N}[r;\mathcal{C}_2^{2\kappa}(\mathbb{R}^m)], \tag{12}$$

where the quantities $\mathcal{N}[g;\mathcal{C}_j^\kappa]$ have been defined in Section 2.1. For the Lévy area associated to x we assume the following structure:

Hypothesis 2.5. *The path $x : [0,T] \to \mathbb{R}^d$ is γ-Hölder continuous with $\frac{1}{3} < \gamma \le 1$ and admits a so-called Lévy area, that is, a process $\mathbf{x^2} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$, which satisfies $\delta \mathbf{x^2} = \mathbf{x^1} \otimes \mathbf{x^1}$, namely*

$$\delta \mathbf{x}_{sut}^{\mathbf{2},ij} = [\mathbf{x}^{\mathbf{1},i}]_{su} [\mathbf{x}^{\mathbf{1},j}]_{ut},$$

for any $s,u,t \in [0,T]$ and $i,j \in \{1,\dots,d\}$.

To illustrate the idea behind the construction of the generalized integral assume that the paths x and z are smooth and also for simplicity that $d = m = 1$. Then the Riemann-Stieltjes integral of z with respect to x is well defined and we have

$$\int_s^t z_u dx_u = z_s(x_t - x_s) + \int_s^t (z_u - z_s) dx_u = z_s \mathbf{x}^{\mathbf{1}}_{st} + \int_s^t (\delta z)_{su} dx_u$$

for $\ell_1 \le s \le t \le \ell_2$. If z admits the decomposition (11) we obtain

$$\int_s^t (\delta z)_{su} dx_u = \int_s^t \left(\zeta_s \mathbf{x}^{\mathbf{1}}_{su} + \rho_{su}\right) dx_u = \zeta_s \int_s^t \mathbf{x}^{\mathbf{1}}_{su}\, dx_u + \int_s^t \rho_{su}\, dx_u. \tag{13}$$

Moreover, if we set

$$\mathbf{x}^{\mathbf{2}}_{st} := \int_s^t \mathbf{x}^{\mathbf{1}}_{su}\, dx_u, \qquad \ell_1 \le s \le t \le \ell_2,$$

then it is quickly verified that $\mathbf{x}^{\mathbf{2}}$ is the Lévy area associated to x. Hence we can write

$$\int_s^t z_u dx_u = z_s \mathbf{x}^{\mathbf{1}}_{sz} + \zeta_s\, \mathbf{x}^{\mathbf{2}}_{st} + \int_s^t \rho_{su}\, dx_u.$$

Now recast this equation as

$$\int_s^t \rho_{su}\, dx_u = \int_s^t z_u\, dx_u - z_s \mathbf{x}^{\mathbf{1}}_{st} - \zeta_s\, \mathbf{x}^{\mathbf{2}}_{st}, \tag{14}$$

and apply the increment operator δ to both sides of this equation. For smooth paths z and x we have

$$\delta\left(\int z\, dx\right) = 0, \qquad\qquad \delta(z\, \mathbf{x}^{\mathbf{1}}) = -\delta z\, \mathbf{x}^{\mathbf{1}},$$

by Proposition 2.3 (recall also our convention (9) on products of increments). Hence applying these relations to the right hand side of (14), using the decomposition (11), the properties of the Lévy area and again Proposition 2.3, we obtain

$$\begin{aligned}\left[\delta\left(\int \rho\, dx\right)\right]_{sut} &= \delta z_{su} \mathbf{x}^{\mathbf{1}}_{ut} + \delta\zeta_{su}\, \mathbf{x}^{\mathbf{2}}_{ut} - \zeta_s\, \delta \mathbf{x}^{\mathbf{2}}_{sut} \\ &= \zeta_s \mathbf{x}^{\mathbf{1}}_{su}\, \mathbf{x}^{\mathbf{1}}_{ut} + \rho_{su}\, \mathbf{x}^{\mathbf{1}}_{ut} + \delta\zeta_{su} \mathbf{x}^{\mathbf{2}}_{ut} - \zeta_s \mathbf{x}^{\mathbf{1}}_{su}\, \mathbf{x}^{\mathbf{1}}_{ut} \\ &= \rho_{su} \mathbf{x}^{\mathbf{1}}_{ut} + \delta\zeta_{su}\, \mathbf{x}^{\mathbf{2}}_{ut}.\end{aligned}$$

Summarizing, we have derived the representation

$$\left[\delta\left(\int \rho\, dx\right)\right]_{sut} = \rho_{su} \mathbf{x}^{\mathbf{1}}_{ut} + \delta\zeta_{su}\, \mathbf{x}^{\mathbf{2}}_{ut}.$$

As we are dealing with smooth paths we have $\delta\left(\int \rho\, dx\right)$ lies into the space $\mathcal{Z}\mathcal{C}_3^{1+}$ and thus belongs to the domain of Λ due to Proposition 2.1. (Recall that $\delta\delta = 0$.) Hence, it follows

$$\int_s^t \rho_{su}\, dx_u = \Lambda_{st}\left(\rho\, \mathbf{x}^{\mathbf{1}} + \delta\zeta\, \mathbf{x}^{\mathbf{2}}\right),$$

and inserting this identity into (13) we end up with

$$\int_s^t z_u dx_u = z_s\, \mathbf{x}^{\mathbf{1}}_{st} + \zeta_s\, \mathbf{x}^{\mathbf{2}}_{st} + \Lambda_{st}\left(\rho\, \mathbf{x}^{\mathbf{1}} + \delta\zeta\, \mathbf{x}^{\mathbf{2}}\right).$$

Since in addition

$$\rho\,\mathbf{x^1} + \delta\zeta\,\mathbf{x^2} = -\delta(z\,\mathbf{x^1} + \zeta\,\mathbf{x^2}),$$

we can also write this as

$$\int z_u dx_u = (\mathrm{id} - \Lambda\delta)(z\,\mathbf{x^1} + \zeta\,\mathbf{x^2}).$$

Thus we have expressed the Riemann-Stieltjes integral of z with respect to x in terms of the sewing map Λ, the couple $(\mathbf{x^1}, \mathbf{x^2})$ and of increments of z. This can now be generalized to the non-smooth case. Note that Corollary 2.2 justifies the use of the notion of integral.

Proposition 2.6. *For fixed $\frac{1}{3} < \kappa \le \gamma$, let x be a path satisfying Hypothesis 2.5 on an arbitrary interval $[0,T]$. Furthermore, let $z \in \mathcal{Q}^x_{\kappa,\alpha}([\ell_1,\ell_2];\mathbb{R}^d)$ such that the increments of z are given by (11). Define $\hat z$ by $\hat z_{\ell_1} = \hat\alpha$ with $\hat\alpha \in \mathbb{R}$ and*

$$\delta\hat z_{st} = \left[(\mathrm{id} - \Lambda\delta)(z^i\mathbf{x}^{\mathbf{1},i} + \zeta^{ji}\mathbf{x}^{\mathbf{2},ij})\right]_{st} \tag{15}$$

for $\ell_1 \le s \le t \le \ell_2$. Then $\mathcal{J}(z^*\,dx) := \hat z$ is a well-defined element of $\mathcal{Q}^x_{\kappa,\hat\alpha}([\ell_1,\ell_2];\mathbb{R})$ and coincides with the usual Riemann integral, whenever z and x are smooth functions.

Moreover, the Hölder norm of $\mathcal{J}(z^*\,dx)$ can be estimated in terms of the Hölder norm of the integrator z. (For this and also for a proof of the above Proposition, see e.g. [14].) This allows to use a fixed point argument to obtain the existence of a unique solution for rough differential equations.

Theorem 2.7. *For fixed $\frac{1}{3} < \kappa < \gamma$, let x be a path satisfying Hypothesis 2.5 on an arbitrary interval $[0,T]$. Consider a given initial condition a in $\mathbb{R}^m$ and $(V_1,\dots,V_d)$ a family of C^3 vector fields of $\mathbb{R}^m$, bounded with bounded derivatives. Let $\|f\|_{\mu,\infty} = \|f\|_\infty + \|\delta f\|_\mu$ be the usual Hölder norm of a path $f \in \mathcal{C}_1([0,T];\mathbb{R}^l)$. Then we have:*

1. *Equation (10) admits a unique solution y in $\mathcal{Q}^x_{\kappa,a}([0,T];\mathbb{R}^m)$ for any $T>0$, and there exists a polynomial $P_T : \mathbb{R}^2 \to \mathbb{R}^+$ such that*
$$\mathcal{N}[y;\mathcal{Q}^x_{\kappa,a}([0,T];\mathbb{R}^m)] \le P_T(\|\mathbf{x^1}\|_\gamma, \|\mathbf{x^2}\|_{2\gamma}) \tag{16}$$
holds.
2. *Let $F : \mathbb{R}^m \times \mathcal{C}_1^\gamma([0,T];\mathbb{R}^d) \times \mathcal{C}_2^{2\gamma}([0,T];\mathbb{R}^{m,m}) \to \mathcal{C}_1^\gamma([0,T];\mathbb{R}^m)$ be the mapping defined by*
$$F\left(a,\mathbf{x^1},\mathbf{x^2}\right) = y,$$
where y is the unique solution of equation (10). This mapping is locally Lipschitz continuous in the following sense: Let $\tilde x$ be another driving rough path with corresponding Lévy area $\tilde{\mathbf{x}}^{\mathbf{2}}$ and $\tilde a$ be another initial condition. Moreover denote by $\tilde y$ the unique solution of the corresponding differential equation. Then, there exists an increasing function $K_T : \mathbb{R}^4 \to \mathbb{R}^+$ such that
$$\begin{aligned}\|y-\tilde y\|_{\gamma,\infty,T} \le\, & K_T(\|\mathbf{x^1}\|_\gamma, \|\tilde{\mathbf{x}}^{\mathbf{1}}\|_\gamma, \|\mathbf{x^2}\|_{2\gamma}, \|\tilde{\mathbf{x}}^{\mathbf{2}}\|_{2\gamma}) \\ & \times \left(|a-\tilde a| + \|\mathbf{x^1} - \tilde{\mathbf{x}}^{\mathbf{1}}\|_\gamma + \|\mathbf{x^2} - \tilde{\mathbf{x}}^{\mathbf{2}}\|_{2\gamma}\right)\end{aligned} \tag{17}$$
holds.

The theorem above is borrowed from [12, 14, 22], and we send the reader to these references for more details on the topic.

2.3. **Fractional Brownian motion.** We shall recall here how the abstract Theorem 2.7 applies to fractional Brownian motion. We will also give some basic notions on stochastic analysis with respect to fBm, mainly borrowed from [26], which will turn out to be useful in the sequel.

As already mentioned in the introduction, on a finite interval $[0,T]$ and for some fixed $H \in (1/3, 1/2)$, we consider $(\Omega, \mathcal{F}, \mathbf{P})$ the canonical probability space associated with fractional Brownian motion with Hurst parameter H. That is, $\Omega = \mathcal{C}_0([0,T];\mathbb{R}^d)$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbf{P}$ is the unique probability measure on Ω such that the canonical process $B = \{B_t,\ t \in [0,T]\}$ is a d-dimensional fractional Brownian motion with Hurst parameter H. Specifically, B has d independent coordinates, each one being a centered Gaussian process with covariance given by (1).

2.3.1. *Functional spaces.* Let $\mathcal{E}$ be the set of the space of d-dimensional elementary functions on $[0,T]$:

$$\mathcal{E} = \Big\{ f = (f_1, \ldots, f_d);\ f_j = \sum_{i=0}^{n_j-1} a_i^j \mathbf{1}_{[t_i^j, t_{i+1}^j)}, \quad 0 = t_0 < t_1^j < \cdots < t_{n_j-1}^j < t_{n_j}^j = T, \\ \text{for } j = 1, \ldots, d \Big\}. \quad (18)$$

We call $\mathcal{H}$ the completion of $\mathcal{E}$ with respect to the semi-inner product

$$\langle f,\, g\rangle_{\mathcal{H}} = \sum_{i=1}^d \langle f_i,\, g_i\rangle_{\mathcal{H}_i}, \quad \text{where} \quad \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}_i} := R(s,t), \quad s,t \in [0,T].$$

Then, one constructs an isometry $K_H^* : \mathcal{H} \to L^2([0,1];\mathbb{R}^d)$ such that

$$K_H^* \left(\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_d]} \right) = \left(\mathbf{1}_{[0,t_1]} K_H(t_1, \cdot), \ldots, \mathbf{1}_{[0,t_d]} K_H(t_d, \cdot) \right),$$

where the kernel K_H is given by

$$K_H(t,u) = c_H \Big[\left(\frac{u}{t}\right)^{\frac12 - H} (t-u)^{H-\frac12} \\ + \left(\frac12 - H\right) u^{\frac12 - H} \int_u^t v^{H-\frac32} (v-u)^{H-\frac12}\, dv \Big] \mathbf{1}_{\{0<u<t\}}, \quad (19)$$

with a strictly positive constant c_H, whose exact value is irrelevant for our purpose. Notice that this kernel verifies $R_H(t,s) = \int_0^{s\wedge t} K_H(t,r) K_H(s,r)\, dr$. Moreover, observe that K_H^* can be represented in the following form: for $\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathcal{H}$, we have $K_H^*\varphi = (K_H^*\varphi^1, \ldots, K_H^*\varphi^d)$, with

$$K_H^*\varphi = \left(K_H^*\varphi^1, \ldots, K_H^*\varphi^d \right), \quad \text{where} \quad [K_H^*\varphi^i]_t = d_H\, t^{1/2-H} \left[D_{T^-}^{1/2-H} \left(u^{-(1/2-H)} \varphi^i \right) \right]_t,$$

for a strictly positive constant d_H. In particular, each $\mathcal{H}_i$ is a fractional integral space of the form $\mathcal{I}_{T^-}^{1/2-H}(L^2([0,T]))$ and $\mathcal{C}_1^{1/2-H}([0,T]) \subset \mathcal{H}_i$.

2.3.2. *Malliavin derivatives.* Let us start by defining the Wiener integral with respect to B: for any element f in $\mathcal{E}$ whose expression is given as in (18), we define the Wiener integral of f with respect to B as

$$B(f) := \sum_{j=1}^{d} \sum_{i=0}^{n_j-1} a_i^j (B^j_{t^j_{i+1}} - B^j_{t^j_i}) .$$

We also denote this integral as $\int_0^T f(t)dB_t$, since it coincides with a pathwise integral with respect to B.

For $\theta : \mathbb{R} \to \mathbb{R}$, and $j \in \{1, \dots, d\}$, denote by $\theta^{[j]}$ the function with values in $\mathbb{R}^d$ having all the coordinates equal to zero except the j-th coordinate that equals to θ. It is readily seen that

$$\mathbf{E}\left[B\left(\mathbf{1}^{[j]}_{[0,s)}\right) B\left(\mathbf{1}^{[k]}_{[0,t)}\right)\right] = \delta_{j,k} R_{s,t}.$$

This definition can be extended by linearity and closure to elements of $\mathcal{H}$, and we obtain the relation

$$\mathbf{E}\left[B(f)\, B(g)\right] = \langle f, g\rangle_{\mathcal{H}},$$

valid for any couple $(f, g) \in \mathcal{H}^2$. In particular, $B(\cdot)$ defines an isometric map from $\mathcal{H}$ into a subspace of $L^2(\Omega)$. It should be pointed out that $(\Omega, \mathcal{H}, \mathbf{P})$ defines an abstract Wiener space, on which chaos decompositions can be settled. We do not develop this aspect of the theory for sake of conciseness, but we will use later the fact that all L^p norms are equivalent on finite chaos.

We can now proceed to the definition of Malliavin derivatives, for which we need an additional notation:

Notation 2.8. *For $n, p \geq 1$, a function $f \in \mathcal{C}^p(\mathbb{R}^n; \mathbb{R})$ and any tuple $(i_1, \dots i_p) \in \{1, \dots, d\}^p$, we set $\partial_{i_1 \dots i_p} f$ for $\frac{\partial^p f}{\partial x_{i_1} \dots \partial x_p}$.*

With this notation in hand, let $\mathcal{S}$ be the family of smooth functionals F of the form

$$F = f(B(h_1), \dots, B(h_n)), \tag{20}$$

where $h_1, \dots, h_n \in \mathcal{H}$, $n \geq 1$, and f is a smooth function with polynomial growth, together with all its derivatives. Then, the Malliavin derivative of such a functional F is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{i=1}^{n} \partial_i f(B(h_1), \dots, B(h_n))\, h_i.$$

For all $p > 1$, it is known that the operator D is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ (see e.g. [26, Chapter 1]). We will still denote by D the closure of this operator, whose domain is usually denoted by $\mathbb{D}^{1,p}$ and is defined as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \left(E(|F|^p) + E(\|DF\|^p_{\mathcal{H}})\right)^{\frac{1}{p}} .$$

It should also be noticed that partial Malliavin derivatives with respect to each component B^j of B will be invoked: they are defined, for a functional F of the form (20) and $j = 1, \dots, d$, as

$$D^j F = \sum_{i=1}^{n} \partial_i f(B(h_1), \dots, B(h_n)) h_i^{[j]},$$

and then extended by closure arguments again. We refer to [26, Chapter 1] again for the definition of higher derivatives and Sobolev spaces $\mathbb{D}^{k,p}$ for $k > 1$.

2.3.3. *Levy area of fBm.* There are many ways to define the Levy area $\mathbf{B^2}$ associated to fBm, and the reader is referred to [12, Chapter 15] for a complete review of these. The recent paper [11] is however of special interest for us, since it enables a direct definition of $\mathbf{B^2}$ by Wiener chaos techniques. It can be summarized in the following way:

Proposition 2.9. *Let $1/3 < H < 1/2$ be a fixed Hurst parameter. Then the fBm B belongs almost surely to any space $\mathcal{C}_1^\gamma$ for $\gamma < H$, and it gives raise to an increment $\mathbf{B^2} \in \mathcal{C}_2^{2\gamma}$ which satisfies Hypothesis 2.5. Furthermore, for any $0 \le s < t \le T$, $\mathbf{B}^{\mathbf{2}}_{st}$ is an element of the second chaos associated to B, and*

$$\mathbf{E}\left[\left|\mathbf{B}^{\mathbf{2}}_{st}\right|^p\right] \le c_p\,(t-s)^{2Hp}, \quad p \ge 1.$$

Moreover, the iterated integrals of B can be obtained as limits of Riemann type integrals. Indeed, for $k \ge 1$ and $0 \le s < t \le T$, consider the simplex

$$\mathcal{S}_k([s,t]) = \left\{(u_1, \ldots, u_k);\, s \le u_1 < \cdots < u_k \le t\right\}. \tag{21}$$

For a given partition Π of $[0,T]$, we also denote by B^Π the linearization of B based on Π. Combining the results of [11, 12], the following proposition holds true:

Proposition 2.10. *Let $k \ge 1$, and for a sequence of partitions $(\Pi_n)_{n\ge1}$, set $B^n := B^{\Pi_n}$. For $0 \le s < t \le T$ and $(i_1, \ldots, i_k) \in \{1, \ldots, d\}^k$, we consider then*

$$\mathbf{B}^{\mathbf{k},n,i_1,\ldots,i_k}_{st} = \int_{\mathcal{S}_k([s,t])} dB^{n,i_1}_{u_1} \cdots dB^{n,i_k}_{u_k},$$

understood in the Riemann sense. Then there exists a sequence of partitions $(\Pi_n)_{n\ge1}$ such that $\mathbf{B}^{\mathbf{k},n,i_1,\ldots,i_k}$ converges almost surely and in L^2, as an element of $\mathcal{C}_2^{k\gamma}$ for any $\gamma < H$, to an element called $\mathbf{B}^{\mathbf{k},i_1,\ldots,i_k}$. When $k = 1$, we obtain the increment δB of our fBm. When $k = 2$, the limit corresponds to the increment $\mathbf{B^2}$ of Proposition 2.9.

As a corollary of the previous considerations, we have the

Proposition 2.11. *Assume $1/3 < H < 1/2$. Then Theorem 2.7 applies almost surely to the fBm paths, enhanced with the Levy area $\mathbf{B^2}$. We are thus able to solve equation*

$$dy_t = \sum_{i=1}^d V_i(y_t)\, dB^i_t, \quad t \in [0,T], \qquad y_0 = a, \tag{22}$$

under the conditions of Theorem 2.7.

3. A Norris type lemma

Norris' lemma [25] is one of the basic ingredients in order to obtain smoothness of densities for solutions to stochastic differential equations under hypoelliptic conditions, and was already extended to fBm with Hurst parameter $H > 1/2$ in [3]. We shall extend in the current section this lemma to the rough paths context. A preliminary step along this direction consists of proving the following elementary lemma:

Lemma 3.1. *Let $0<\alpha<\rho<1$, and consider $b\in\mathcal{C}([0,T])$. Then for any $0<\eta<1$, we have*

$$\|b\|_{\alpha,\infty}\le C_{\alpha,\rho}\left[\eta\,\|b\|_{\rho,\infty}+\eta^{-1/(\rho-\alpha)}\|b\|_{L_1}\right], \tag{23}$$

where we recall that $\|b\|_{\alpha,\infty}$ has been defined by (6).

Proof. Recall that $\|b\|_{\alpha}$ has been defined by (5). Thus, for $0<\alpha<\rho<1$, we have

$$\|b\|_{\alpha}=\sup_{s,t}\left[\left(\frac{|\delta b_{st}|}{|t-s|^{\rho}}\right)^{\frac{\alpha}{\rho}}|\delta b_{st}|^{1-\frac{\alpha}{\rho}}\right]\le 2^{1-\frac{\alpha}{\rho}}\|b\|_{\infty}^{1-\frac{\alpha}{\rho}}\|b\|_{\rho}^{\frac{\alpha}{\rho}}.$$

Thus, for an arbitrary constant $\eta>0$, we have

$$\|b\|_{\alpha}\le C_{\alpha,\rho}\left(\eta\|b\|_{\rho}+\eta^{-\frac{\alpha}{\rho-\alpha}}\|b\|_{\infty}\right)\le C_{\alpha,\rho}\left(\eta\|b\|_{\rho}+\eta^{-\frac{\alpha}{\rho-\alpha}}\|b\|_{\infty}\right),$$

thanks to Young's inequality. Therefore for an arbitrary constant $0<\eta\le 1$

$$\|b\|_{\alpha}\le C_{\alpha,\rho}\left(\eta\|b\|_{\rho}+\eta^{-\frac{\alpha}{\rho-\alpha}}\|b\|_{\infty}\right).$$

Invoke now the interpolation inequality [3, formula (3.17)] in order to get

$$\begin{aligned}\|b\|_{\alpha}&\le C_{\alpha,\rho}\left(\eta\|b\|_{\rho}+\eta^{-\frac{\alpha}{\rho-\alpha}}\left[\gamma\|b\|_{\rho}+\gamma^{-\frac{1}{1+\alpha}}\|b\|_{L_1}\right]\right)\\&= C_{\alpha,\rho}\left[\eta\|b\|_{\rho}+\eta^{-1/(\rho-\alpha)}\|b\|_{L_1}\right],\end{aligned}$$

where we have chosen $\gamma=\eta^{\frac{\alpha}{\rho-\alpha}}$. Invoking again [3, formula (3.17)] in order to go from $\|\cdot\|_{\alpha}$ to $\|\cdot\|_{\alpha,\infty}$ norms and this finishes our proof. □

We can now turn to the announced Norris type lemma, whose proof is an adaptation of [3] to the case of controlled processes.

Proposition 3.2. *Assume B is a fractional Brownian motion with $H>1/3$. Let z be a controlled path in $\mathcal{Q}^B_{\gamma}(\mathbb{R}^m)$, with decomposition*

$$\delta z^i_{st}=\zeta^{i,j}_s\,\mathbf{B}^{\mathbf{1},j}_{st}+r^i_{st}. \tag{24}$$

We assume that $1/3<\alpha<\gamma<H$ and that the quantity $\mathbf{E}[\mathcal{N}^p[z;\,\mathcal{Q}^B_{\gamma}(\mathbb{R}^m)]]$ is finite for all $p\ge 1$. Set $\delta y_{st}=\mathcal{J}(z^*dB)$ according to Proposition 2.6. Then there exists $q>0$ such that, for every $p>0$ we can find a strictly positive constant c_p such that

$$\mathbf{P}\left(\|y\|_{\gamma,\infty}<\varepsilon,\ \text{and}\ \|z\|_{\alpha,\infty}>\varepsilon^{q}\right)<c_p\,\varepsilon^{p}.$$

Proof. In order to avoid cumbersome indices, we shall prove our result in the case of 1-dimensional processes. Generalization to the multidimensional setting is a matter of trivial considerations. We also work on the interval $[0,1]$ instead of $[0,T]$ for the sake of notational simplicity. As a last preliminary observation, note that if z admits the decomposition (24), then according to (15) we have

$$\delta y=z\,\mathbf{B}^{\mathbf{1}}+\zeta\,\mathbf{B}^{\mathbf{2}}+y^{\sharp},\quad\text{where}\quad y^{\sharp}=\Lambda\left(r\,\mathbf{B}^{\mathbf{1}}+\delta\zeta\,\mathbf{B}^{\mathbf{2}}\right).$$

Similarly to what is done in [3], we consider two time scales $\delta\ll\Delta\ll 1$. We assume moreover that $\Delta/\delta=r$ with $r\in\mathbb{N}$. We use a partition $\{t_n;\,n\le 1/\delta\}$ of $[0,1]$ with $t_n=n\delta$, so that $t_{Nr}=N\Delta$. Some increments below will then be frozen on the time scale Δ, in order to take advantage of some averaging properties of the process B.

Step1: Coarse graining on increments. Consider then n such that $(N-1)r \le n \le Nr-1$, so that $(N-1)\Delta \le t_n \le N\Delta - 1$. According to (15), we have

$$\begin{aligned} \delta y_{t_n t_{n+1}} &= z_{t_n}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}} + \zeta_{t_n}\, \mathbf{B}^{\mathbf{2}}_{t_n t_{n+1}} + y^{\sharp}_{t_n t_{n+1}} \\ &= z_{N\Delta}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}} - \delta z_{t_n, N\Delta}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}} + \zeta_{t_n}\, \mathbf{B}^{\mathbf{2}}_{t_n t_{n+1}} + y^{\sharp}_{t_n t_{n+1}}, \end{aligned}$$

where $y^{\sharp}$ is an increment in $\mathcal{C}_2^{3\gamma}$. Thus

$$z_{N\Delta}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}} = \delta y_{t_n t_{n+1}} + \delta z_{t_n, N\Delta}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}} - \zeta_{t_n}\, \mathbf{B}^{\mathbf{2}}_{t_n t_{n+1}} - y^{\sharp}_{t_n t_{n+1}}. \tag{25}$$

Set now $X_N = \sum_{n=(N-1)r}^{Nr-1} \|\mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}}\|^4$ and $Y_N = X_N^{1/4}$. Then

$$|z_{N\Delta}|^4 |X_N| = \sum_{n=(N-1)r}^{Nr-1} \left| z_{N\Delta}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}} \right|^4.$$

Furthermore, invoking relation (25), we get

$$|z_{N\Delta}\, \mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}}| \le \|y\|_\gamma \delta^\gamma + \|z\|_\gamma \|B\|_\gamma \delta^\gamma \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B}^{\mathbf{2}}\|_{2\gamma} \delta^{2\gamma} + \|y^\sharp\|_{3\gamma} \delta^{3\gamma}.$$

Raising this inequality to power 4 and summing over $(N-1)r \le n \le Nr-1$, we obtain

$$|z^4_{N\Delta}\, X_N| \le \delta^{4\gamma-1}\Delta \left(\|y\|_\gamma + \|z\|_\gamma \|B\|_\gamma \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B}^{\mathbf{2}}\|_{2\gamma} \delta^{\gamma} + \|y^\sharp\|_{3\gamma} \delta^{2\gamma} \right)^4,$$

and therefore

$$|z_{N\Delta}| Y_N \le \delta^{\gamma-1/4}\Delta^{1/4} \left(\|y\|_\gamma + \|z\|_\gamma \|B\|_\gamma \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B}^{\mathbf{2}}\|_{2\gamma} \delta^{\gamma} + \|y^\sharp\|_{3\gamma} \delta^{2\gamma} \right).$$

Sum now over N (recall that $1 \le N \le 1/\Delta$) in order to get

$$\sum_{N=1}^{1/\Delta} |z_{N\Delta}| Y_N \le \delta^{\gamma-1/4}\Delta^{-3/4} \left(\|y\|_\gamma + \|z\|_\gamma \|B\|_\gamma \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B}^{\mathbf{2}}\|_{2\gamma} \delta^{\gamma} + \|y^\sharp\|_{3\gamma} \delta^{2\gamma} \right). \tag{26}$$

Step 2: Behavior of a 4^{th} order variation. Throughout the proof, we shall use the notations $\lesssim, \asymp$ given in the introduction. For $K \ge 1$, set $\tilde{X}_K = \sum_{n=1}^K |\mathbf{B}^{\mathbf{1}}_{t_n t_{n+1}}|^4$. We shall prove that

$$\mathbf{E}[\tilde{X}_K] \asymp K\delta^{4H}, \quad \text{and} \quad \mathbf{Var}(\tilde{X}_K) \lesssim K\delta^{8H}. \tag{27}$$

Indeed, a simple scaling argument shows that $\tilde{X}_K \overset{(\mathcal{L})}{=} \delta^{4H}\, \hat{X}_K$, with $\hat{X}_K = \sum_{n=1}^K |\mathbf{B}^{\mathbf{1}}_{n,n+1}|^4$. Introduce now the k-th Hermite polynomial H_k (see [26] for a definition and properties of these objects), and notice in particular that $H_2(x) = x^2 - 1$ and $H_4(x) = x^4 - 6x^2 + 3$. This enables to decompose $\hat{X}_K$ as

$$\hat{X}_K = \sum_{n=1}^K \left[H_4(\mathbf{B}^{\mathbf{1}}_{n,n+1}) + 6 H_2(\mathbf{B}^{\mathbf{1}}_{n,n+1}) \right] + 3K. \tag{28}$$

Recall now that for a centered Gaussian vector (U, V) in $\mathbb{R}^2$ such that $\mathbf{E}[U^2] = \mathbf{E}[V^2] = 1$ we have

$$\mathbf{E}[H_k(U)] = 0, \quad \text{and} \quad \mathbf{E}\left[H_k(U)\, H_l(V)\right] = k!\, (\mathbf{E}[U\, V])^k\, \mathbf{1}_{(k=l)}. \tag{29}$$

Plugging this identity in (28), this immediately yields $\mathbf{E}[\hat{X}_K] = 3K$, which is our first assertion in (27). In addition, the second part of (29) entails

$$\mathbf{Var}(\hat{X}_K) = 2 \sum_{n_1,n_2=1}^{K} \left(12\, \alpha_{n_1,n_2}^4 + \alpha_{n_1,n_2}^2\right) := 2S_K,$$

where we have set

$$\alpha_{n_1,n_2} = \mathbf{E}[\mathbf{B}^{\mathbf{1}}_{n_1,n_1+1}\, \mathbf{B}^{\mathbf{1}}_{n_2,n_2+1}] = \frac{1}{2}\left[|n_2-n_1+1|^{2H} + |n_2-n_1-1|^{2H} - 2|n_2-n_1|^{2H}\right].$$

Summarizing, we have obtained that

$$\mathbf{Var}(\tilde{X}_K) = \delta^{8H}\, \mathbf{Var}(\hat{X}_K) = 2\, \delta^{8H}\, S_K. \tag{30}$$

We will now prove that $S_K \lesssim K$. Indeed, write first

$$S_K = \sum_{1\le n_1\le K} \left(12\, \alpha_{n_1,n_1}^4 + \alpha_{n_1,n_1}^2\right) + 2 \sum_{1\le n_1<n_2\le K} \left(12\, \alpha_{n_1,n_2}^4 + \alpha_{n_1,n_2}^2\right) := S_K^1 + 2S_K^2.$$

Then, since $\alpha_{n_1,n_1} = 1$, it is readily checked that $S_K^1 \lesssim K$. Moreover, the term S_K^2 can be decomposed into

$$S_K^2 = \sum_{1\le n_1\le K-1} \left(12\, \alpha_{n_1,n_1+1}^4 + \alpha_{n_1,n_1+1}^2\right) + \sum_{1\le n_1,n_2\le K,\, n_2-n_1\ge 2} \left(12\, \alpha_{n_1,n_2}^4 + \alpha_{n_1,n_2}^2\right) := S_K^{21} + S_K^{22}.$$

Notice now that $\alpha_{n_1,n_1+1} = -[1-2^{-2(H-1/2)}]$, which easily yields $S_K^{21} \le c_H K$.

As far as S_K^{22} is concerned, write for $n_2 - n_1 \ge 2$

$$\alpha_{n_1,n_2} = H(2H-1)\int_0^1 dr \int_{-r}^{r} |(n_2-n_1)+u|^{2H-2}\, du,$$

which immediately yields $\alpha_{n_1,n_2} \lesssim |(n_2-n_1)-1|^{2H-2}$. Thus

$$\begin{aligned} S_K^{22} &\lesssim \sum_{n_1=1}^{K-2} \sum_{n_2=n_1+2}^{K} |(n_2-n_1)-1|^{8H-8} + 3|(n_2-n_1)-1|^{4H-4} \\ &\lesssim \sum_{n_1=1}^{K-2} \sum_{m=1}^{K-n_1+1} m^{4H-4} \le K \sum_{m=1}^{\infty} m^{4H-4} \lesssim K, \end{aligned}$$

since $\sum_{m=1}^{\infty} m^{4H-4}$ is finite whenever $H < 3/4$.

Gathering our bounds on S_K^1, S_K^{21} and S_K^{22}, we obtain $S_K \lesssim K$, and plugging this bound into (30), we end up with $\mathbf{Var}(\tilde{X}_K) \lesssim K\, \delta^{8H}$, which is our claim. This finishes the proof of (27).

Step 3: Concentration inequalities for Y_N. Let us recall that X_N is in the 4^{th} chaos of the fBm B. Hence, a result by Borell [5] entails

$$\mathbf{P}\left(\frac{|X_N - \mathbf{E}[X_N]|}{[\mathbf{Var}(X_N)]^{1/2}} \ge u\right) \le c_1 e^{-c_2 u^{1/2}}, \qquad u \ge 0,$$

for two universal constant $c_1, c_2 > 0$. With (27) in hand, this yields

$$\mathbf{P}\left(|X_N - 3\Delta\delta^{4H-1}| \ge \Delta^{1/2}\delta^{4H-1/2}u\right) \le c_1 e^{-c_2 u^{1/2}}, \qquad u \ge 0. \tag{31}$$

We now wish to produce a concentration inequality for $Y_N = X_N^{1/4}$. Since X_N is a small random quantity of order $\Delta\delta^{4H-1}$, let us use the inequality

$$|b^{1/4} - a^{1/4}| \lesssim \xi^{-3/4}|b-a|, \quad \text{where} \quad \xi \in (a \wedge b,\, a \vee b).$$

Apply this with $\xi = \frac{3}{2}\Delta\delta^{4H-1}$ in order to get

$$\mathbf{P}\left(|Y_N - 3^{1/4}\Delta^{1/4}\delta^{H-1/4}| \geq c\,\Delta^{-3/4}\delta^{-3(4H-1)/4}\Delta^{1/2}\delta^{4H-1/2}u\right) \leq A_1 + A_2, \tag{32}$$

with

$$\begin{aligned} A_1 &= \mathbf{P}\left(|X_N - 3\Delta\delta^{4H-1}| \geq c\,\Delta^{1/2}\delta^{4H-1/2}u,\, X_N \geq \frac{3}{2}\Delta\delta^{4H-1}\right) \\ A_2 &= \mathbf{P}\left(X_N \leq \frac{3}{2}\Delta\delta^{4H-1}\right). \end{aligned}$$

Furthermore, a straightforward application of (31) gives

$$A_1 \leq c_1 e^{-c_2 u^{1/2}}, \quad \text{and} \quad A_2 \leq c_1 e^{-c_2(\Delta/\delta)^{1/4}}.$$

Plugging these inequalities into (32), we end up with the following concentration inequality for Y_N.

$$\mathbf{P}\left(|Y_N - 3^{1/4}\Delta^{1/4}\delta^{H-1/4}| \geq c\,\Delta^{-1/4}\delta^{H+1/4}u\right) \leq c_1 e^{-c_2 u^{1/2}} + c_1 e^{-c_2(\Delta/\delta)^{1/4}}. \tag{33}$$

We shall thus retain the fact that Y_N i a random quantity of order $3^{1/4}\Delta^{1/4}\delta^{H-1/4}$, with fluctuations of order $\Delta^{-1/4}\delta^{H+1/4}$:

$$\left|Y_N - 3^{1/4}\Delta^{1/4}\delta^{H-1/4}\right| \approx \delta^{H+1/4}\Delta^{-1/4}. \tag{34}$$

Step 4: Use of the interpolation inequality. Start again from equation (26). One would like to have an approximation of the L^1 norm of z appearing on the left hand side of this inequality, that is one would like to replace Y_N by Δ. To this purpose, replace first Y_N by its approximation $\Delta^{1/4}\delta^{H-1/4}$ from the last step. This yields an inequality of the form:

$$\begin{aligned} \Delta^{1/4}\delta^{H-1/4}\sum_{N=1}^{1/\Delta}|z_{N\Delta}| &\leq \|z\|_\infty \sum_{N=1}^{1/\Delta}\left|Y_N - 3^{1/4}\Delta^{1/4}\delta^{H-1/4}\right| \\ &\quad + \delta^{\gamma-1/4}\Delta^{-3/4}\Big(\|y\|_\gamma + \|z\|_\gamma\|B\|_\gamma\Delta^\gamma + \|\zeta\|_\infty\|\mathbf{B^2}\|_{2\gamma}\delta^\gamma + \|y^\sharp\|_{3\gamma}\delta^{2\gamma}\Big). \end{aligned}$$

Rescale this inequality in order to get Δ multiplying on the left hand side, which gives:

$$\begin{aligned} &\Delta\sum_{N=1}^{1/\Delta}|z_{N\Delta}| \\ &\quad \leq \|z\|_\infty\, R_{\delta,\Delta} + \delta^{-(H-\gamma)}\Big(\|y\|_\gamma + \|z\|_\gamma\|B\|_\gamma\Delta^\gamma + \|\zeta\|_\infty\|\mathbf{B^2}\|_{2\gamma}\delta^\gamma + \|y^\sharp\|_{3\gamma}\delta^{2\gamma}\Big), \end{aligned} \tag{35}$$

where

$$R_{\delta,\Delta} := \delta^{-(H-1/4)}\Delta^{3/4}\sum_{N=1}^{1/\Delta}|Y_N - 3^{1/4}\Delta^{1/4}\delta^{H-1/4}|. \tag{36}$$

Furthermore, it is well known that $|\Delta \sum_{N=1}^{1/\Delta} |z_{N\Delta}| - \|z\|_{L^1}| \leq \|z\|_\gamma \Delta^\gamma$, so that we can recast (35) into

$$\|z\|_{L^1} \leq \|z\|_\gamma \Delta^\gamma + \|z\|_\infty R_{\delta,\Delta} + \delta^{-(H-\gamma)} \Big(\|y\|_\gamma + \|z\|_\gamma \|B\|_\gamma \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B^2}\|_{2\gamma} \delta^\gamma + \|y^\sharp\|_{3\gamma} \delta^{2\gamma} \Big). \quad (37)$$

Recall that we take $0 < \alpha < \gamma < H$ and set $\nu_H := 1/(\gamma - \alpha)$. According to (23) we have

$$\|z\|_{\alpha,\infty} \lesssim \eta \|z\|_{\gamma,\infty} + \eta^{-\nu_H} \|z\|_{L_1},$$

for any (small enough) constant η, and plugging (37) into this last relation we obtain

$$\|z\|_{\alpha,\infty} \lesssim \eta^{-\nu_H} \Delta^\gamma \|z\|_{\gamma,\infty} + \eta^{-\nu_H} \|z\|_\infty R_{\delta,\Delta} + \eta \|z\|_{\gamma,\infty} + \eta^{-\nu_H} \Big(\|y\|_{\gamma,\infty} \delta^{-(H-\gamma)} + \|z\|_{\gamma,\infty} \|B\|_\gamma \delta^{-(H-\gamma)} \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B^2}\|_{2\gamma} \delta^{2\gamma - H} + \|y^\sharp\|_{3\gamma} \delta^{3\gamma - H} \Big).$$

Defining $\hat\epsilon$ as $\hat\epsilon = H - \gamma$, we get

$$\|z\|_{\alpha,\infty} \lesssim \eta^{-\nu_H} \Delta^\gamma \|z\|_{\gamma,\infty} + \eta^{-\nu_H} \|z\|_\infty R_{\delta,\Delta} + \eta \|z\|_{\gamma,\infty} + \eta^{-\nu_H} \Big(\|y\|_{\gamma,\infty} \delta^{-\hat\epsilon} + \|z\|_{\gamma,\infty} \|B\|_\gamma \delta^{-\hat\epsilon} \Delta^\gamma + \|\zeta\|_\infty \|\mathbf{B^2}\|_{2\gamma} \delta^{\gamma - \hat\epsilon} + \|y^\sharp\|_{3\gamma} \delta^{2\gamma - \hat\epsilon} \Big). \quad (38)$$

Step 4: Tuning the parameters. Recall that we have chosen $0 \ll \delta \ll \Delta \ll 1$. We express this fact in terms of powers of ε, by taking $\delta = \varepsilon^\mu$ and $\Delta = \varepsilon^\lambda$, with $\mu > \lambda > 0$. We also choose η of the form $\eta = \varepsilon^{\tau/\nu_H}$. We shall now see how to choose λ, μ, τ conveniently: write (38) as

$$\|z\|_{\alpha,\infty} \lesssim \varepsilon^{-\tau + \lambda\gamma} \|z\|_{\gamma,\infty} + \varepsilon^{-\tau} \|z\|_\infty R_{\delta,\Delta} + \varepsilon^{\tau/\nu_H} \|z\|_{\gamma,\infty} + \|y\|_{\gamma,\infty} \varepsilon^{-\tau - \hat\epsilon\mu} + \|z\|_{\gamma,\infty} \|B\|_\gamma \varepsilon^{-\tau - \hat\epsilon\mu + \lambda\gamma} + \|\zeta\|_\infty \|\mathbf{B^2}\|_{2\gamma} \varepsilon^{-\tau + \mu(\gamma - \hat\epsilon)} + \|y^\sharp\|_{3\gamma} \varepsilon^{-\tau + \mu(2\gamma - \hat\epsilon)}. \quad (39)$$

In order to be able to bound z when y is assumed to satisfy $\|y\|_{\gamma,\infty} \leq \varepsilon$, the coefficients in the right hand side of (39) should fulfill the following conditions:

- The coefficient in front of $\|y\|_{\gamma,\infty}$ should be smaller than ε^{-1}.
- The other coefficients should be $\ll 1$.

Looking at the exponents in (39), assuming that $\hat\epsilon$ is arbitrarily small and letting for the moment $R_{\delta,\Delta}$ apart, this imposes the following relations:

$$\lambda\gamma > \tau, \quad \text{and} \quad 0 < \tau < 1. \quad (40)$$

Let us go back now to the evaluation of $R_{\delta,\Delta}$, given by expression (36), with the order of magnitude of $|Y_N - \Delta^{1/4} \delta^{H-1/4}|$ given by (34). Therefore

$$R_{\delta,\Delta} \approx \delta^{-(H-1/4)} \Delta^{3/4} \Delta^{-1} \delta^{H+1/8} \Delta^{-1/4} = \delta^{1/2} \Delta^{-1/2}.$$

Expressing this in terms of powers of ε, we end up with

$$\eta^{-\nu_H} R_{\delta,\Delta} \approx \varepsilon^\kappa, \quad \text{with} \quad \kappa = \frac{\mu - \lambda}{2} - \tau.$$

If we wish this remainder term to be small, this adds the condition

$$\mu - \lambda > 2\tau, \quad (41)$$

which can be fulfilled easily. From now on, we shall assume that both (40) and (41) are satisfied.

Step 5: Conclusion. Recall that we wish to study the probability $\mathbf{P}(\|y\|_{\gamma,\infty} < \varepsilon$, and $\|z\|_{\alpha,\infty} > \varepsilon^q)$. This quantity is obviously bounded by $B_1 + B_2$, where

$$\begin{aligned} B_1 &= \mathbf{P}\left(\|y\|_{\gamma,\infty} < \varepsilon,\ \|z\|_{\alpha,\infty} > \varepsilon^q,\ |R_{\delta,\Delta}| \le c\,\Delta^{-1/4}\delta^{H+1/4}(\Delta/\delta)^{\tilde\varepsilon}\right) \\ B_2 &= \mathbf{P}\left(|R_{\delta,\Delta}| \ge c\,\Delta^{-1/4}\delta^{H+1/4}(\Delta/\delta)^{\tilde\varepsilon}\right), \end{aligned}$$

where $\tilde\varepsilon$ is an arbitrary small positive constant. Furthermore, inequalities (33) and (36) yield, for any $p \ge 1$,

$$B_2 \le c_1 e^{-c_2(\Delta/\delta)^{\tilde\varepsilon/2}} + c_1 e^{-c_2(\Delta/\delta)^{1/4}} \le c_{p,\lambda,\mu,\tilde\varepsilon}\,\varepsilon^p,$$

where we have used the fact that $\delta/\Delta = \varepsilon^{\mu-\lambda}$.

We can now bound B_1: notice that according to (39), if we are working on

$$\left(|R_{\delta,\Delta}| \le c\,\Delta^{-1/4}\delta^{H+1/4}(\Delta/\delta)^{\tilde\varepsilon}\right) \cap \left(\|y\|_{\gamma,\infty} < \varepsilon\right),$$

then there exists a $\rho > 0$ such that

$$\|z\|_{\alpha,\infty} \lesssim \varepsilon^\rho\left[1 + \mathcal{N}^2[z; \mathcal{Q}_\gamma^B(\mathbb{R}^m)] + \|B\|_\gamma^2 + \|\mathbf{B^2}\|_{2\gamma}^2\right].$$

Moreover, recall that $\mathcal{N}[z; \mathcal{Q}_\gamma^B(\mathbb{R}^m)]$ is assumed to be an L^r random variable for all $r \ge 1$, while $\|B\|_\gamma$ and $\|\mathbf{B^2}\|_{2\gamma}$ are also elements of L^r, since they can be bounded by a finite chaos random variable. Thus Tchebychev inequality can be applied here, which entails

$$B_1 \lesssim \left(1 + \mathbf{E}\left[\mathcal{N}^{2l}[z; \mathcal{Q}_\gamma^B(\mathbb{R}^m)] + \|B\|_\gamma^{2l} + \|\mathbf{B^2}\|_{2\gamma}^{2l}\right]\right)\varepsilon^{l(\rho-q)},$$

for an arbitrary $l \ge 1$. It is now sufficient to choose $q < \rho$ and l large enough so that $l(\rho - q) = p$ to conclude the proof, by putting together our bounds on B_1 and B_2. □

4. Malliavin calculus for solutions to fractional SDEs

This section is the core of our paper, where we derive smoothness of density for the solution to (22). We first recall some classical notions on representations of solutions to SDEs, and then move to Malliavin calculus considerations.

4.1. Representation of solutions to SDEs.

The first representations results for solutions to SDEs in terms of the driving vector fields can be traced back to the seminal work of Chen [7]. They have then been deeply analyzed in [18, 30], and also lye at the basis of the rough path theory [22]. We have chosen here to present these formulas according to [1], which is a recent and didactically useful account on the topic.

Recall that we are considering a d-dimensional fBm B with $1/3 < H < 1/2$. According to Section 2.3, this allows to construct some increments $\mathbf{B^k}$ out of B which can be seen as limits of Riemann iterated integrals over the simplex $\mathcal{S}_k([s,t])$, as recalled at Proposition 2.10. Furthermore, one can solve equation (22) under the conditions of Theorem 2.7.

Let us introduce some additional notation: let $\mathcal{V}$ be the space of smooth bounded vector fields over $\mathbb{R}^m$. If $V \in \mathcal{V}$, the vector field $\exp(V) \in \mathcal{V}$ is defined by the relation $[\exp(V)](\xi) = \Psi_1(\xi)$, where $\{\Psi_t(\xi);\, t \ge 0\}$ is solution to the ordinary differential equation

$$\partial_t \Psi_t(\xi) = V(\Psi_t(\xi)), \quad \Psi_0(\xi) = \xi. \tag{42}$$

The aim of Chen-Strichartz formula is to express the solution y_t to equation (22), for an arbitrary $t \in [0,T]$, as $y_t = [\exp(Z_t)](a)$ for a certain $Z_t \in \mathcal{V}$.

To this purpose, let us give some more classical notations on vector fields: if $V, W \in \mathcal{V}$, then the vector field $[V, W] \in \mathcal{V}$ (called Lie bracket of V and W) is defined by

$$[V, W]^i = V^l \partial_{x_l} W^i - W^l \partial_{x_l} V^i.$$

Notice that this notion is usually introduced though the interpretation of $\mathcal{V}$ as a set of first order differential operators. A Lie bracket of order k can also be defined inductively for $k \geq 2$ by setting

$$[U_1 \cdots U_k] = [[U_1 \cdots U_{k-1}]\,,\, U_k]\,,$$

for $U_1 \ldots U_k \in \mathcal{V}$. With this notation in hand, our main assumption on the vector fields $V_1, \ldots, V_d$ governing equation (22) is the following:

Hypothesis 4.1. *The vector fields $V_1, \ldots, V_d$ are n-nilpotent for some given positive integer n. Namely, for any $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$, we have $[V_{i_1} \cdots V_{i_n}] = 0$.*

We are now ready to state our formulation of Strichartz' identity, for which we need two last notations: for $k \geq 1$, we call $\mathfrak{S}_k$ the set of permutations of $\{1, \ldots, k\}$. Moreover, for $\sigma \in \mathfrak{S}_k$, write $e(\sigma)$ for the quantity $\mathrm{Card}(\{j \in \{1, \ldots, k-1\};\, \sigma(j) > \sigma(j+1)\})$. Then the following formula is proven e.g. in [1, 18, 30]:

Proposition 4.2. *Under the hypothesis of Theorem 2.7, let y be the solution to equation (22). Assume Hypothesis 4.1 holds true, and consider $t \in [0,T]$. Then $y_t = [\exp(Z_t)](a)$, where Z_t can be expressed as follows:*

$$Z_t = \sum_{k=1}^{n-1} \sum_{i_1, \ldots, i_k = 1}^{d} \mathbf{V}_{i_1, \ldots, i_k}\, \psi_t^{i_1, \ldots, i_k}, \quad \text{with} \quad \psi_t^{i_1, \ldots, i_k} = \sum_{\sigma \in \mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \mathbf{B}_{0t}^{\mathbf{k}, i_{\tau(1)}, \ldots, i_{\tau(k)}},$$

where we have set $\tau = \sigma^{-1}$ and $\mathbf{V}_{i_1, \ldots, i_k} = [V_{i_1} \cdots V_{i_k}]$ in the formula above.

As a warmup to the computations below, we prove now that one can extend our inequality (16) thanks to Strichartz representation, covering the case of unbounded vector fields with bounded derivatives:

Proposition 4.3. *Suppose Hypothesis 4.1 holds true, and that the smooth vector fields V_i, $i = 1, 2, \cdots, d$ have bounded derivative. Assume moreover that all the Lie brackets of order greater or equal to 2 are bounded vector fields. Then the solution y of equation (22) admits moments of any order. Namely, for any $m > 1$ and any $T \in (0, \infty)$,*

$$\mathbf{E}\left[\sup_{0 \leq t \leq T} |y_t|^m\right] < \infty\,. \tag{43}$$

Proof. One can restate Proposition 4.2 as follows: for any $t \leq T$, the random variable y_t can be expressed as $y_t = \phi_1$, where $\phi_s^t := \phi_s : \mathbb{R}^d \to \mathbb{R}^d$ satisfies (for t fixed)

$$\partial_s \phi_s = \sum_{k=1}^{n-1} \sum_{i_1, \ldots, i_k = 1}^{d} \psi_t^{i_1, \ldots, i_k} \mathbf{V}_{i_1, \ldots, i_k}(\phi_s)\,, \quad 0 \leq s \leq 1\,, \quad \phi_0 = a\,.$$

Let us separate the first order integrals in this equation, in order to get

$$\partial_s\phi_s = \sum_{i=1}^d V_i(\phi_s)B_t^i + \sum_{k=2}^{n-1} \sum_{i_1,\dots,i_k=1}^d \psi_t^{i_1,\dots,i_k}\mathbf{V}_{i_1,\dots,i_k}(\phi_s)\,, \quad \phi_0 = a\,. \tag{44}$$

Since V_i, $i=1,2,\cdots,d$, have bounded derivatives and since all the Lie brackets of order greater or equal to 2 are bounded we see that

$$\begin{aligned}|\partial_s\phi_s| &\le \sum_{i=1}^d |V_i(\phi_s)|\,|B_t^i| + \sum_{k=2}^{n-1} \sum_{i_1,\dots,i_k=1}^d |\psi_t^{i_1,\dots,i_k}||\mathbf{V}_{i_1,\dots,i_k}(\phi_s)| \\ &\le c_1\,|\phi_s| \sum_{i=1}^d \sup_{0\le t\le T}|B_t^i| + c_2 \sum_{k=1}^{n-1} \sum_{i_1,\dots,i_k=1}^d \sup_{0\le t\le T}|\psi_t^{i_1,\dots,i_k}|\,.\end{aligned}$$

Thus by Gronwall's lemma, we have

$$|\phi_s| \le c_2\left(\sum_{k=1}^{n-1} \sum_{i_1,\dots,i_k=1}^d \sup_{0\le t\le T}|\psi_t^{i_1,\dots,i_k}|\right)\exp\left\{c_1\sum_{i=1}^d \sup_{0\le t\le T}|B_t^i|\right\}\,.$$

This inequality holds true for all $0\le s\le 1$. Thus

$$\sup_{0\le t\le T}|y_t| \le c_2\left(\sum_{k=1}^{n-1} \sum_{i_1,\dots,i_k=1}^d \sup_{0\le t\le T}|\psi_t^{i_1,\dots,i_k}|\right)\exp\left\{c_1\sum_{i=1}^d \sup_{0\le t\le T}|B_t^i|\right\}\,,$$

which implies (43). □

4.2. **Malliavin derivative.** This subsection is devoted to enhance our Proposition 4.3, and prove that the Malliavin derivative of y_t has also bounded moments of any order in our particular nilpotent situation. Notice once again that the boundedness of moments of the Malliavin derivative is still an open problem for rough differential equations in the general case. We refer to Section 2.3.2 for notations on Malliavin calculus.

Theorem 4.4. *Let the vector fields V_i, $1=1,2,\cdots,d$ be smooth with all derivatives bounded, satisfying Hypothesis 4.1. Assume that all the Lie brackets of order greater or equal to 2 are constants. Then the Malliavin derivative y_t has moments of any order. More precisely, for any $q>1$ and $T\in(0,\infty)$,*

$$\mathbf{E}\left[\sup_{0\le u\le t\le T}|D_u y_t|^q\right] < \infty\,. \tag{45}$$

Proof. Go back to our representation (44), which can easily be differentiated in the Malliavin calculus sense in order to obtain

$$\begin{aligned}\partial_s D_u\phi_s &= \sum_{i=1}^d \nabla V_i(\phi_s)B_t^i\,D_u\,\phi_s + \sum_{k=2}^{n-1} \sum_{i_1,\dots,i_k=1}^d \psi_t^{i_1,\dots,i_k}\,\nabla\mathbf{V}_{i_1,\dots,i_k}(\phi_s)\,D_u\phi_s \\ &\quad + \sum_{i=1}^d V_i(\phi_s)\mathbf{1}_{[0,t)}^{[i]}(u) + \sum_{k=2}^{n-1} \sum_{i_1,\dots,i_k=1}^d D_u\psi_t^{i_1,\dots,i_k}\mathbf{V}_{i_1,\dots,i_k}(\phi_s)\,,\end{aligned}$$

where we have set $\nabla\mathbf{V}_{i_1,\dots,i_k}$ for the (matrix valued) gradient of $\mathbf{V}_{i_1,\dots,i_k}$, and where we recall that the notation $\mathbf{1}_{[0,t)}^{[i]}$ has been introduced at Section 2.3.2.

Since we assume that all the Lie brackets of order greater or equal to 2 of the vector fields V_i are constant vector fields, it is easily checked that

$$\partial_s D_u \phi_s = \sum_{i=1}^d \nabla V_i(\phi_s) B_t^i D_u \phi_s + \sum_{i=1}^d V_i(\phi_s) \mathbf{1}_{[0,t)}^{[i]}(u) + \sum_{k=2}^{n-1} \sum_{i_1,\ldots,i_k=1}^d D_u \psi_t^{i_1,\ldots,i_k} \mathbf{V}_{i_1,\ldots,i_k}(\phi_s)\,. \tag{46}$$

Therefore there exist two positive constants c_1, c_2 such that

$$\begin{aligned} |\partial_s D_u \phi_s| &\leq c_1 \sum_{i=1}^d |B_t^i||D_u \phi_s| \\ &\quad + c_2 \sum_{i=1}^d [1 + |\phi_s|]\, \mathbf{1}_{[0,t)}^{[i]}(u) + \sum_{k=2}^{n-1} \sum_{i_1,\ldots,i_k=1}^d \left|D_u \psi_t^{i_1,\ldots,i_k}\right| |\mathbf{V}_{i_1,\ldots,i_k}(\phi_s)|\,. \end{aligned}$$

By Gronwall's lemma we obtain

$$\begin{aligned} \sup_{0\leq s\leq 1, 0\leq u\leq t\leq T} |D_u \phi_s| \leq c_2 \exp\left\{ c_1 \sum_{i=1}^d \sup_{0\leq t\leq T} |B_t^i| \right\} \times \left\{ \sum_{i=1}^d \left[1 + \sup_{0\leq s\leq 1} |\phi_s| \right] \right. \\ \left. \times \left(1 + \sup_{0\leq t\leq T} \sum_{k=2}^{n-1} \sum_{i_1,\ldots,i_k=1}^d \sup_{0\leq s\leq 1, 0\leq u\leq t\leq T} \left|D_u \psi_t^{i_1,\ldots,i_k}\right| |\mathbf{V}_{i_1,\ldots,i_k}(\phi_s)| \right) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} \sup_{0\leq u\leq t\leq T} |D_u y_t| \leq c_2 \exp\left\{ c_1 \sum_{i=1}^d \sup_{0\leq t\leq T} |B_t^i| \right\} \\ \times \left\{ \sum_{i=1}^d \left[1 + \sup_{0\leq t\leq T} |y_t| \right] \left(1 + \sum_{k=2}^{n-1} \sum_{i_1,\ldots,i_k=1}^d \sup_{0\leq s\leq 1, 0\leq u\leq t\leq T} \left|D_u \psi_t^{i_1,\ldots,i_k}\right| \right) \right\}, \end{aligned}$$

which ends the proof easily by boundedness of moments for y_t, $D_u \psi_t^{i_1,\ldots,i_k}$ and B_t^i.

□

Example 4.5. A classical example of nilpotent vector fields in $\mathbb{R}^3$ is due to Yamato [31]. Let us check that this example fullfils our standing assumptions. Indeed, the example provided in [31] is the following:

$$A_1 = 0\,, \quad A_2 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}\,, \quad \text{and} \quad A_3 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}\,.$$

Then

$$[A_2, A_3] \;=\; -4\frac{\partial}{\partial x_3}\,, \quad [[A_2, A_3], A_2] = [[A_2, A_3], A_3] = 0\,.$$

It is thus readily checked that the conditions of Theorem 4.4 are met for these vector fields. Moreover, in this particular case the solution to equation (22) is explicit and we have

$$y_1^1 = y_1 + B_t^2, \qquad y_t^2 = y_2 + B_t^3, \qquad y_t^3 = y_3 + 2\left(\mathbf{B}_{0t}^{\mathbf{2},32} - \mathbf{B}_{0t}^{\mathbf{2},23}\right)\,,$$

if the solution starts from the initial condition (y_1, y_2, y_3). Interestingly enough, though the solution is explicit here, the smoothness of the density of y_t is not immediate and we recover here the results of [9].

4.3. **Stochastic flows.** The probabilistic proof of the smoothness of density for diffusion processes originally given by Malliavin [23] heavily relies on stochastic flows methods and their relationship with stochastic derivatives. We now establish those relations for SDEs driven by a fractional Brownian motion.

To this aim, denote by $y^{s,a}$ the solution to equation (22) starting from the initial condition $y_s = a$ at time s:

$$dy_t^{s,a} = \sum_{i=1}^{d} V_i(y_t^{s,a})\, dB_t^i, \quad t \in [s,T]\,, \qquad y_s^{s,a} = a\,. \tag{47}$$

The above equation gives rise to a family of smooth nonlinear mappings $\Phi_{s,t} : \mathbb{R}^m \to \mathbb{R}^m$, $0 \le s \le t \le T$, determined by $\Phi_{s,t}(a) := y_t^{s,a}$, and the family $\{\Phi_{s,t}; \le s \le t \le T\}$ has the following flow property (we refer e.g. to [12] for the properties of flows driven by rough paths quoted below):

$$\Phi_{s,t} = \Phi_{u,t} \circ \Phi_{s,u}\,, \quad \forall\, 0 \le s \le u \le t \le T\,.$$

Let $J_{s,t}$ denote the gradient of the nonlinear mapping $\Phi_{s,t}$ with respect to the initial condition. Then the family $\{J_{s,t}; \le s \le t \le T\}$ also satisfies the relation $J_{s,t} = J_{u,t} J_{s,u}$ for $0 \le s \le u \le t \le T$. In addition, the map $J_{s,t}$ is invertible, and we have $J_{s,t} = J_{0,t} J_{0,s}^{-1}$. The equation followed by $J_{0,t}$ is obtained by differentiating formally equation (47) with respect to the initial value a, which yields

$$dJ_{0,t} = \sum_{i=1}^{d} \nabla V_i(y_t) J_{0,t}\, dB_t^i\,, \quad J_{0,0} = I\,.$$

By applying the rules of differential calculus for rough paths, we also get that $J_{0,t}^{-1}$ is solution to the following equation:

$$dJ_{0,t}^{-1} = -\sum_{i=1}^{d} \nabla V_i(y_t) J_{0,t}^{-1}\, dB_t^i\,, \quad J_{0,0}^{-1} = I\,. \tag{48}$$

We have thus ended up with two linear equations for the derivatives of the flow. In our nilpotent case, we are able thus able to bound these derivatives along the same lines as for Theorem 4.4:

Theorem 4.6. *Let the vector fields V_i, $1 = 1, 2, \ldots, d$ be smooth with all derivatives bounded, satisfying Hypothesis 4.1. Assume that all the Lie brackets of order greater or equal to 2 are constant. Then the Jacobian $J_{0,t}$ and its inverse $J_{0,t}^{-1}$ have moments of any order: for any $q \ge 1$ and $T \in (0,\infty)$,*

$$\mathbf{E}\left[\sup_{0\le t\le T} |J_{0,t}|^q\right] < \infty, \quad \textit{and} \quad \mathbf{E}\left[\sup_{0\le t\le T} \left|J_{0,t}^{-1}\right|^q\right] < \infty\,. \tag{49}$$

Proof. As mentioned in the proof of Proposition 4.3, one can write $\Phi_{0,t}(a) = \exp(Z_t)(a) = \phi_1(a)$, where $\phi_1(a)$ satisfies (44). Thus if we introduce $\tilde{J}_s = \nabla \phi_s$, then $J_{0,t} = \tilde{J}_1$ where $\tilde{J}_s$

satisfies

$$\partial_s \tilde{J}_s = \sum_{i=1}^{d} B_t^i \, \nabla V_i(\phi_s) \, \tilde{J}_s + \sum_{k=2}^{n-1} \sum_{i_1,\dots,i_k=1}^{d} \psi_t^{i_1,\dots,i_k} \, \nabla \mathbf{V}_{i_1,\dots,i_k}(\phi_s) \, \tilde{J}_s \,, \qquad \tilde{J}_0 = I \,.$$

The first part of (49) is thus proved following the steps of Proposition 4.3.

As far as the second part of (49) is concerned, observe that $J_{0,t}^{-1} = \bar{J}_1$, where $\bar{J}_s$ is inverse of $\tilde{J}_s$. It is clear that $\bar{J}_s$ satisfies

$$\partial_s \bar{J}_s = -\sum_{i=1}^{d} \nabla V_i(\phi_s) \bar{J}_s B_t^i - \sum_{k=2}^{n-1} \sum_{i_1,\dots,i_k=1}^{d} \psi_t^{i_1,\dots,i_k} \nabla \mathbf{V}_{i_1,\dots,i_k}(\phi_s) \bar{J}_s \,, \qquad \tilde{J}_0 = I \,.$$

Once again, the methodology of Proposition 4.3 easily yields our claim.

□

Corollary 4.7. *Under the same assumptions as in Theorem 4.6, the following holds true:*
(i) *For any $0 < \gamma < H$ and $q \geq 1$ we have*

$$\mathbf{E}\left[\|Dy_t\|_{\gamma,\infty}^q\right] + \mathbf{E}\left[\|J_{0,\cdot}\|_{\gamma,\infty}^q\right] + \mathbf{E}\left[\|J_{0,\cdot}^{-1}\|_{\gamma,\infty}^q\right] < c_{T,q}, \tag{50}$$

for a finite constant $c_{T,q}$.
(ii) *As a consequence, inequality (50) also holds true when the $\|\cdot\|_{\gamma,\infty}$ norms are replaced by norms in $\mathcal{H}$, where $\mathcal{H}$ has been defined at Section 2.3.1.*
(iii) *For any smooth bounded vector field U on $\mathbb{R}^m$ and $t \in [0,T]$, set $Z_t^U := \langle J_{0,t}^{-1} U(y_t), \, \eta \rangle$. Then Z^U is a controlled process, and satisfies the inequality $\mathbf{E}[\mathcal{N}^q[Z^U; \, \mathcal{Q}^\gamma(\mathbb{R}^m)]] \leq c_{T,q}$ for any $q \geq 1$ and a finite constant $c_{T,q}$.*

Proof. Going back to equation (46), it is readily checked that all the terms $u \mapsto D_u \psi_t^{i_1,\dots,i_k}$ are $\mathcal{C}_1^\gamma$-Hölder continuous on $[0,t]$ for any $\gamma < H$, since the elements $\psi_t^{i_1,\dots,i_k}$ are nice multiple integrals with respect to B. Moreover, we have

$$\mathbf{E}\left[\|D\psi_t^{i_1,\dots,i_k}\|_{\gamma,\infty}^q\right] < \infty,$$

for any $m \geq 1$. This easily yields $\mathbf{E}[\|Dy_t^{i_1,\dots,i_k}\|_{\gamma,\infty}^q] < \infty$ by a standard application of Gronwall's lemma, as in the proof of Theorem 4.4.

Our second assertion stems from the fact that one can choose $1/2 - H < \gamma < H$, since $H > 1/3$. For such γ we have $\mathcal{C}_1^\gamma \subset \mathcal{H}$, which ends the proof.

Finally, our claim (iii) derives from the fact that the equation governing Z^U is of the following form:

$$Z_t^U = \langle \eta, \, U(a) \rangle + \sum_{j=1}^{d} \int_0^t Z_s^{[U,V_j]} dB_s^j. \tag{51}$$

The process Z^U can thus be decomposed as a controlled process as in Section 2.2, and since we already have estimates for $J_{0,t}^{-1}$ and y_t, the bound on $\mathbf{E}[\mathcal{N}^q[Z^U; \, \mathcal{Q}^\gamma(\mathbb{R}^m)]]$ follows easily.

□

4.4. **Proof of Theorem 1.1.** As mentioned in the introduction, once we have shown the integrability of the Malliavin derivative and proved a Norris type lemma, the proof of our main theorem goes along classical lines. We have chosen to follow here the exposition of [16], to which we refer for further details.

Step 1: Reduction to a lower bound on Hölder norms. Recall that the process Z^U has been defined for any smooth vector field U in Corollary 4.7. For any $p \geq 1$, our first goal is to reduce our problem to the existence of a constant c_p such that

$$\mathbf{P}\left(\|Z^{V_k}\|_{\alpha,\infty} \leq \varepsilon\right) \leq c_p \varepsilon^p, \tag{52}$$

for $1 \leq k \leq d$, a given $\alpha \in (1/3, 1/2)$, all $\varepsilon \in (0,1)$ and where we observe that all the norms below are understood as norms on $[0,T]$.

Indeed, according to [16, Relation (4.9)] the smoothness of density can be obtained from the estimate

$$\mathbf{P}\left(\|Z^{V_k}\|_{\mathcal{H}} \leq \varepsilon\right) \leq c_p \varepsilon^p,$$

where we recall that $\mathcal{H}$ has been defined at Section 2.3.1. Furthermore, we have

$$\mathbf{P}\left(\|Z^{V_k}\|_{\mathcal{H}} \leq \varepsilon\right) \leq \mathbf{P}\left(\|Z^{V_k}\|_{L^2} \leq \varepsilon\right) \leq \mathbf{P}\left(\|Z^{V_k}\|_{L^1} \leq \varepsilon\right).$$

It is thus sufficient for our purposes to check

$$\mathbf{P}\left(\|Z^{V_k}\|_{L^1} \leq \varepsilon\right) \leq c_p \varepsilon^p. \tag{53}$$

In order to go from (53) to (52), let us use our interpolation bound (23) in the following form: for any $0 < \eta < 1$ and $0 < \alpha < \rho < H$ we have

$$\|b\|_{L^1} \geq \eta^{1/(\rho-\alpha)} \left(\|b\|_{\alpha,\infty} - C_{\alpha,\rho}\eta\, \|b\|_{\rho,\infty}\right).$$

Take now $\delta \in (0,1)$ to be fixed later on and $\eta^{1(\rho-\alpha)} = \varepsilon^{1-\delta}$, that is $\eta = \varepsilon^{(\rho-\alpha)(1-\delta)}$. Then

$$\mathbf{P}\left(\|Z^{V_k}\|_{L^1} \leq \varepsilon\right) \leq \mathbf{P}\left(\|Z^{V_k}\|_{\alpha,\infty} \leq 2\varepsilon^{\delta}\right) + R, \quad \text{where} \quad R = \mathbf{P}\left(\|Z^{V_k}\|_{\rho,\infty} \leq \frac{1}{4c\,\varepsilon^{\nu}}\right), \tag{54}$$

with $\nu = \rho - \alpha - (1-(\rho-\alpha))\delta$. Choose now δ small enough, so that $\nu > 0$. Since $\|Z^{V_k}\|_{\rho,\infty}$ admits moments of any order according to Corollary 4.7, it is easily checked that R can be made smaller than any quantity of the form $c_q \varepsilon^q$. It is thus sufficient to prove (52) in order to get the smoothness of density for y_t.

Step 2: An iterative procedure. For $l \geq 1$ and $x \in \mathbb{R}^m$, let $\mathcal{V}_l(x)$ be the vector space generated by the Lie brackets of order l of our vector fields $V_1, \ldots, V_d$ at point x:

$$\mathcal{V}_l(x) = \text{Span}\left\{[V_{k_1} \cdots V_{k_j}](x);\ j \leq l,\ 1 \leq k_1, \ldots, k_j \leq d\right\}.$$

We assume that the vector fields are ℓ-hypoelliptic for a given $\ell > 0$, which can be read as $\mathcal{V}_\ell(x) = \mathbb{R}^m$ for any $x \in \mathbb{R}^m$. In order to start our induction procedure, we set $\alpha_1 = \alpha$, so that we have to prove $\mathbf{P}(\|Z^{V_k}\|_{\alpha_1,\infty} \leq \varepsilon) \leq c_p \varepsilon^p$.

Recall that Z^{V_k} satisfies the relation

$$Z_t^{V_k} = \langle \eta,\, V_k(a)\rangle + \sum_{j=1}^{d} \int_0^t Z_s^{[V_k, V_j]} dB_s^j.$$

Thus Proposition 3.2 asserts that for any $1/3 < \alpha_2 < \alpha_1 < H$ there exists $q_2 > 0$ such that

$$\mathbf{P}\left(\left(\|Z^{V_k}\|_{\alpha_1,\infty} \leq \varepsilon\right) \cap \left(\cup_{j=1}^{d}\left(\|Z^{[V_{k_1},V_{k_2}]}\|_{\alpha_2,\infty} > \varepsilon^{q_2}\right)\right)\right) \leq c_p \varepsilon^p.$$

Relation (52) is thus implied by

$$\mathbf{P}\left(\left(\|Z^{V_k}\|_{\alpha_1,\infty}\le\varepsilon\right)\cap\left(\cap_{j=1}^{d}\left(\|Z^{[V_{k_1},V_{k_2}]}\|_{\alpha_2,\infty}\le\varepsilon^{q_2}\right)\right)\right)\le c_p\varepsilon^p.$$

Iterating this procedure we end up with the following claim to prove: $\mathcal{B}_\ell(\varepsilon)\le c_p\varepsilon^p$ for all $\varepsilon\in(0,1)$, with

$$\mathcal{B}_\ell(\varepsilon)=\mathbf{P}\left(\|Z^{V_k}\|_{\alpha_1,\infty}\le\varepsilon,\ \|Z^{[V_{k_1},V_{k_2}]}\|_{\alpha_2,\infty}\le\varepsilon^{q_2},\ldots,\|Z^{[V_{k_1}\cdots V_{k_\ell}]}\|_{\alpha_\ell,\infty}\le\varepsilon^{q_\ell}\right),$$

where the intersection above extends to all possible combinations $1\le k_1,\ldots,k_\ell\le d$, and where $1/3<\alpha_\ell<\cdots<\alpha_1<H$.

Going back now to the very definition of Z^U as $Z^U_t=\langle J^{-1}_{0,t}U(y_t),\,\eta\rangle$, it is readily checked that

$$\mathcal{B}_\ell(\varepsilon)\le\mathbf{P}\left(\langle\eta,\,V_{k_1}(a)\rangle\le\varepsilon,\ldots,\langle\eta,\,[V_{k_1}\cdots V_{k_\ell}](a)\rangle\le\varepsilon^{q_\ell}\right).$$

Owing to the fact that $\mathcal{V}_\ell(a)=\mathbb{R}^m$, we thus have $\mathcal{B}_\ell(\varepsilon)=0$ for ε small enough, which ends the proof.

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