

The resolution of Diophantine equations according to Bhāskara and a justification of the *cakravāla* by Kṛṣṇadaivajña

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- 1 The *kuṭṭaka*: $au + c = bv$
- 2 The *vargaprakṛti*: $px^2 + k = y^2$
- 3 The *cakravāla*
- 4 Kṛṣṇa's *upapatti*

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The *kuṭṭaka*

$$au + c = bv$$

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bhājyo hāraḥ kṣepakaś cāpavartyaḥ
 kena apy ādau saṁbhava kuṭṭakārthaṁ |
 yena cchinnau bhājyahārau na tena
 kṣepaś cet tad duṣṭam uddiṣṭam eva ||

Firstly, the **dividend**, the **divisor** and the **additive** must be simplified, when possible, by some [number] for the *kuṭṭaka*. If the number by which the **dividend** and the **divisor** are divided does not [divide] the additive, the [problem] is impossible.

$$a\mathbf{u} + c = b\mathbf{v}$$

bhājyo hāraḥ kṣepakaś cāpavartyaḥ
 kena apy ādau saṁbhavē **kuṭṭakārthaṁ** |
 yena cchinnau bhājyahārau na teṇa
 kṣepaś cet tad duṣṭaṁ uddiṣṭaṁ eva ||

Firstly, the **dividend**, the **divisor** and the **additive** must be simplified, when possible, by some [number] for the **kuṭṭaka**. If the number by which the **dividend** and the **divisor** are divided does not [divide] the additive, the [problem] is impossible.

or: multiplier (*guṇa*)

$$au + c = bv$$

bhājyo hāraḥ kṣepakaś cāpavartyaḥ
 kena apy ādau saṁbhavē kuṭṭakārthaṁ |
 yena cchinnau bhājyahārau na tena
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Firstly, the **dividend**, the **divisor** and the **additive** must be simplified, when possible, by some [number] for the *kuṭṭaka*. If the number by which the **dividend** and the **divisor** are divided does not [divide] the additive, the [problem] is impossible.

$$au + c = bv$$

bhājyo **hārah** kṣepakaś cāpavartyaḥ
 kena apy ādau saṁbhava kuṭṭakārthaṁ |
 yena cchinnau bhājyahārau na tena
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$$au + c = bv$$

bhājyo hāraḥ **kṣepakaś** cāpavartyaḥ
 kena apy ādau saṁbhavē kuṭṭakārthaṁ |
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Firstly, the **dividend**, the **divisor** and the **additive** must be simplified, when possible, by some [number] for the *kuṭṭaka*. If the number by which the **dividend** and the **divisor** are divided does not [divide] the additive, the [problem] is impossible.

v is the **quotient**

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatiha rūpam |

One will divide mutually these reduced **dividend** and **divisor**,
until unity will be in the place of the **dividend**.

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatiha rūpam |

One will **divide mutually** these reduced **dividend** and **divisor**,
until unity will be in the place of the **dividend**.

After having divided one number by another number, one will
divide the other by the remainder of this division. . .
(*Com. Kriyākramakarī*)

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatiha rūpam |

One will divide mutually these reduced **dividend** and **divisor**,
until unity will be **in the place of** the **dividend**.

$$\begin{array}{r} 17 \\ 15 \end{array}$$

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatīha rūpam |

One will divide mutually these reduced **dividend** and **divisor**,
until unity will be **in the place of** the **dividend**.

$$1 \quad \left| \begin{array}{r} 2 \\ 15 \end{array} \right|$$

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatīha rūpam |

One will divide mutually these reduced **dividend** and **divisor**,
until unity will be in the place of the **dividend**.

$$17u + 5 = 15v$$

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatīha rūpam |

One will divide mutually these reduced **dividend** and **divisor**,
until unity will be in the place of the **dividend**.

$$17 = 1 \times 15 + 2$$

$$17u + 5 = 15v$$

mitho bhajet tau dṛḍhabhājyahārau
yāvad vibhājyē bhavatīha rūpam |

One will divide mutually these reduced **dividend** and **divisor**,
until unity will be in the place of the **dividend**.

$$17u + 5 = 15v$$

$$17 = 1 \times 15 + 2$$

$$15 = 7 \times 2 + 1$$

phalāny adho 'dhas tad adho niveśyaḥ
kṣepas tathānte kham. . .

The quotients [will have to be placed] one under the other;
the additive must be placed below them, then zero at the last
place.

phalāny adho 'dhas tad adho niveśyaḥ
kṣepas tathānte kham. . .

The quotients [will have to be placed] one under the other;
the additive must be placed below them, then zero at the last
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$$17u + 5 = 15v$$

phalāṇy adho 'dhas tad adho niveśyaḥ
kṣepas tathānte kham. . .

The quotients [will have to be placed] one under the other;
the additive must be placed below them, then zero at the last
place.

$$17u + 5 = 15v$$

1
7

phalāṇy adho 'dhas tad adho niveśyaḥ
kṣepas tathānte kham. . .

The quotients [will have to be placed] one under the other;
the additive must be placed below them, then zero at the last
place.

$$17u + 5 = 15v$$

1
7
5

phalāṇy adho 'dhas tad adho niveśyaḥ
kṣepas tathānte kham. . .

The quotients [will have to be placed] one under the other;
the additive must be placed below them, **then zero at the last place.**

$$17u + 5 = 15v$$

1
7
5
0

... upāntimena ||
 svordhve hate 'ntyena yute tad antyaṃ
 tyajen muhuḥ syād iti rāśiyugmam |

The last but one having multiplied the [number] which is above and the last [number] having been added, one will remove this last; repeat the operation until there will be a couple of numbers.

... upāntimena ||
 svordhve hate 'ntyena yute tad antyaṃ
 tyajen muhuḥ syād iti rāśiyugmam |

The last but one having multiplied the [number] which is above and the last [number] having been added, one will remove this last; repeat the operation until there will be a couple of numbers.

$$17u + 5 = 15v$$

1
7
5
0

... upāntimena ||
 svordhve hate 'ntyena yute tad antyaṃ
 tyajen muhuḥ syād iti rāśiyugmam |

The last but one having multiplied the [number] which is above and the last [number] having been added, one will remove this last; repeat the operation until there will be a couple of numbers.

$$17u + 5 = 15v$$

$$\begin{array}{rcl} 1 & & 1 \\ 7 & 7 \times 5 + 0 = & 35 \\ 5 & & 5 \\ 0 & & \end{array}$$

... upāntimena ||
 svordhve hate 'ntyena yute tad antyaṃ
 tyajen muhuḥ syād iti rāśiyugmam |

The last but one having multiplied the [number] which is above and the last [number] having been added, one will remove this last; repeat the operation until there will be a couple of numbers.

$$17u + 5 = 15v$$

1		1	$1 \times 35 + 5 = 40$
7	$7 \times 5 + 0 = 35$		35
5	5		
0			

ūrdhvo vibhājyena dṛdhena taṣṭaḥ
phalaṃ guṇaḥ syād aparo hareṇa ||

The uppermost, being reduced to the remainder by the **dividend**, will be the **quotient**; the other, [reduced to the remainder] by the **divisor**, will be the **multiplier**.

ūrdhvo vibhājyena dṛḍhena taṣṭaḥ
phalaṃ guṇaḥ syād aparo hareṇa ||

The uppermost, being **reduced to the remainder** by the **dividend**, will be the **quotient**; the other, [reduced to the remainder] by the **divisor**, will be the **multiplier**.

When the remainder only is needed in a division, the quotient being useless, the conventionally agreed word “*taṣṭa*” is used.
(*Com. Sūryadāsa*)

ūrdhvo vibhājyena dṛḍhena taṣṭaḥ
phalaṃ guṇaḥ syād aparo hareṇa ||

The uppermost, being reduced to the remainder by the **dividend**, will be the **quotient**; the other, [reduced to the remainder] by the **divisor**, will be the **multiplier**.

$$17u + 5 = 15v$$

1	1	$1 \times 35 + 5 = 40$
7	$7 \times 5 + 0 = 35$	35
5	5	
0		

ūrdhvo vibhājyena dṛdhena taṣṭaḥ
phalaṃ guṇaḥ syād aparo hareṇa ||

The uppermost, being reduced to the remainder by the **dividend**, will be the **quotient**; the other, [reduced to the remainder] by the **divisor**, will be the **multiplier**.

$$17u + 5 = 15v$$

$$\begin{array}{rclcl}
 1 & & 1 & 1 \times 35 + 5 = 40 & = 17 \times 2 + 6 \\
 7 & 7 \times 5 + 0 = 35 & & 35 & = 15 \times 2 + 5 \\
 5 & & 5 & & \\
 0 & & & &
 \end{array}$$

ūrdhvo vibhājyena dṛḍhena taṣṭaḥ
phalaṃ guṇaḥ syād aparo hareṇa ||

The uppermost, being reduced to the remainder by the **dividend**, will be the **quotient**; the other, [reduced to the remainder] by the **divisor**, will be the **multiplier**.

$$17u + 5 = 15v$$

$$\begin{array}{r} 1 \\ 7 \\ 5 \\ 0 \end{array} \quad \begin{array}{r} 1 \\ 7 \\ 5 \\ 0 \end{array} \quad \begin{array}{l} 1 \times 35 + 5 = 40 \\ 7 \times 5 + 0 = 35 \\ 5 \\ 0 \end{array} \quad \begin{array}{l} 1 \times 35 + 5 = 40 \\ 35 \\ 35 \end{array} \quad \begin{array}{l} = 17 \times 2 + 6 \\ = 15 \times 2 + 5 \end{array}$$

$$17 \times 5 + 5 = 15 \times 6$$

iṣṭāhatasvasvahareṇa yukte
te vā bhavetām bahudhā guṇāptī ||

There will be many **multipliers** and **quotients** if they are added to their respective simplifier multiplied by an arbitrary number.

iṣṭāhataśvasvāhareṇa yukte
te vā bhavetāṃ bahudhā guṇāptī ||

There will be many **multipliers** and **quotients** if they are added to their respective simplifier multiplied by an arbitrary number.

If (u_0, v_0) is a solution of: $au + c = bv$, then:

$(u_t, v_t) = (u_0 + bt, v_0 + at)$ will be also a solution for any integer t .

The *vargaprakṛti*

$$px^2 + k = y^2$$

iṣṭaṃ hrasvaṃ tasya vargaḥ prakṛtyā
 kṣuṇṇo yukto varjito vā sa yeṇa |
 mūlaṃ dādyāt kṣepakaṃ taṃ dhanarṇaṃ
 mūlaṃ tac ca jyeṣṭhamūlaṃ vadanti ||

Let an assumed [number], the least root; its square is multiplied by the *prakṛti*; the additive is this [number], positive or negative, by which this [square multiplied by the *prakṛti*] is increased or decreased to produce a root and [mathematicians] call this root, the greatest root.

iṣṭaṃ hrasvaṃ tasya vargaḥ prakṛtyā
 kṣuṇṇo yukto varjito vā sa yeṇa |
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$$px_1^2 \pm k_1 = y_1^2$$

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Let an assumed [number], the least root; its square is multiplied by the *prakṛti*; the additive is this [number], positive or negative, by which this [square multiplied by the *prakṛti*] is increased or decreased to produce a root and [mathematicians] call this root, the greatest root.

$$px_1^2 \pm k_1 = y_1^2$$

$$[x_1, y_1; k_1]$$

The *bhāvanā*: arrangement

hrasvajyeṣṭhakṣepakān nyasya teṣāṃ
 tāt anyān vādho niveśya krameṇa |
 sādhyāny ebhyo bhāvanābhīr bahūni
 mūlāny eṣāṃ bhāvanā procyate 'taḥ ||

Having set down a least and a greatest roots and an additive, then successively placed under them, the same ones or others, many roots can be calculated by compositions (*bhāvanā*); that is why the composition is taught.

The *bhāvanā*: arrangement

hrasvajyeṣṭhakṣepakān nyasya teṣāṃ
 tām anyān vādho niveśya krameṇa |
 sādhyāny ebhyo bhāvanābhīr bahūni
 mūlāny eṣāṃ bhāvanā procyate 'taḥ ||

Having set down a least and a greatest roots and an additive, then successively placed under them, the same ones or others, many roots can be calculated by compositions (*bhāvanā*); that is why the composition is taught.

$$x_1 \quad y_1 \quad k_1$$

The *bhāvanā*: arrangement

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Having set down a least and a greatest roots and an additive, then successively placed under them, **the same ones** or others, many roots can be calculated by compositions (*bhāvanā*); that is why the composition is taught.

$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ x_1 & y_1 & k_1 \end{array}$$

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Having set down a least and a greatest roots and an additive, then successively placed under them, the same ones **or others**, many roots can be calculated by compositions (*bhāvanā*); that is why the composition is taught.

$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ x_2 & y_2 & k_2 \end{array}$$

The *bhāvanā*: calculation

vajrābhyāso jyeṣṭhalaghvos tadaikyam
 hrasvam laghvor āhatiś ca prakṛtyā |
 kṣuṇṇā jyeṣṭhābhyāsayug jyeṣṭhamūlam
 tatrābhyāsaḥ kṣepayoḥ kṣepakaḥ syāt ||

Given the cross products of the greatest and least roots, their sum is a least root. And the product of the least roots, multiplied by the *prakṛti* and added to the product of the greatest, is a greatest root. The product of the additives will be an additive.

The *bhāvanā*: calculation

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$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ x_2 & y_2 & k_2 \end{array}$$

- ▶ $x_3 = x_1 \cdot y_2 + x_2 \cdot y_1$
- ▶ $y_3 = p x_1 \cdot x_2 + y_1 \cdot y_2$
- ▶ $k_3 = k_1 \cdot k_2$

The *bhāvanā*: calculation

vajrābhyāso jyeṣṭhalaghvos tadaikyam
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Given the cross products of the greatest and least roots, their sum is a least root. And the product of the least roots, multiplied by the *prakṛti* and added to the product of the greatest, is a greatest root. **The product of the additives will be an additive.**

$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ x_2 & y_2 & k_2 \end{array}$$

- ▶ $x_3 = x_1 \cdot y_2 + x_2 \cdot y_1$
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$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ x_2 & y_2 & k_2 \end{array}$$

$$\triangleright x_3 = x_1 \cdot y_2 + x_2 \cdot y_1$$

$$[x_3, y_3; k_3]$$

$$\triangleright y_3 = p x_1 \cdot x_2 + y_1 \cdot y_2$$

$$\triangleright k_3 = k_1 \cdot k_2$$

The *bhāvanā*: calculation

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$$\begin{array}{ccc} x_1 & y_1 & k_1 \\ x_2 & y_2 & k_2 \end{array}$$

- ▶ $x_3 = x_1 \cdot y_2 + x_2 \cdot y_1$
- ▶ $y_3 = p x_1 \cdot x_2 + y_1 \cdot y_2$
- ▶ $k_3 = k_1 \cdot k_2$

$$\begin{array}{l} [x_3, y_3; k_3] \\ p x_3^2 + k_3 = y_3^2 \end{array}$$

iṣṭavargahr̥taḥ kṣepaḥ
 kṣepaḥ syād iṣṭabhājite |
 mūle te sto 'thavā kṣepaḥ
 kṣuṇṇaḥ kṣuṇṇē tadā pade ||

The additive divided by the square of an assumed number will be the additive; the two roots divided by this assumed number, are the [roots]; or the additive multiplied is the additive; in this case the roots are multiplied.

iṣṭavargahr̥taḥ kṣepaḥ
 kṣepaḥ syād iṣṭabhājite |
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The additive divided by the square of an assumed number will be the additive; the two roots divided by this assumed number, are the [roots]; or the additive multiplied is the additive; in this case the roots are multiplied.

$$px_1^2 + k_1 = y_1^2 \quad k_1 = y_1^2 - px_1^2$$

iṣṭavargahr̥taḥ kṣepaḥ
 kṣepaḥ syād iṣṭabhājite |
 mūle te sto 'thavā kṣepaḥ
 kṣuṇṇaḥ kṣuṇṇe tadā pade ||

The additive divided by the square of an assumed number will be the additive; the two roots divided by this assumed number, are the [roots]; or the additive multiplied is the additive; in this case the roots are multiplied.

$$px_1^2 + k_1 = y_1^2 \quad k_1 = y_1^2 - px_1^2$$

$$\text{If } d^2 | k_1 \quad k_1 = d^2 k'_1$$

$$d^2 k'_1 = y_1^2 - px_1^2$$

$$k'_1 = \left(\frac{y_1}{d}\right)^2 - p\left(\frac{x_1}{d}\right)^2$$

An example

...
 ekādaśaguṇaḥ ko vā vargaḥ saikaḥ kṛtiḥ sakhe ||

Which square multiplied by eleven and increased by one is a square, O my friend!

An example

...
 ekādaśaguṇaḥ ko vā vargaḥ saikaḥ kṛtiḥ sakhe ||

Which square multiplied by eleven and increased by one is a square, O my friend!

1. Choose a suitable triplet: $[1, 3; -2] \quad 11 \times 1^2 - 2 = 3^2$

An example

...
 ekādaśaguṇaḥ ko vā vargaḥ saikaḥ kṛtiḥ sakhe ||

Which square multiplied by eleven and increased by one is a square, O my friend!

1. Choose a suitable triplet:

$$[1, 3; -2] \quad 11 \times 1^2 - 2 = 3^2$$

2. The *bhāvanā*: $\begin{array}{ccc} 1 & 3 & -2 \\ 1 & 3 & -2 \end{array}$

$$[6, 20; 4] \quad 11 \times 6^2 + 4 = 20^2$$

An example

...
 ekādaśaguṇaḥ ko vā vargaḥ saikaḥ kṛtiḥ sakhe ||

Which square multiplied by eleven and increased by one is a square, O my friend!

1. Choose a suitable triplet: $[1, 3; -2]$ $11 \times 1^2 - 2 = 3^2$
2. The *bhāvanā*: $\begin{matrix} 1 & 3 & -2 \\ 1 & 3 & -2 \end{matrix}$ $[6, 20; 4]$ $11 \times 6^2 + 4 = 20^2$
3. "The additive divided by the square...": $[3, 10; 1]$ $11 \times 3^2 + 1 = 10^2$

An example

...
 ekādaśaguṇaḥ ko vā vargaḥ saikaḥ kṛtiḥ sakhe ||

Which square multiplied by eleven and increased by one is a square, O my friend!

1. Choose a suitable triplet: $[1, 3; -2]$ $11 \times 1^2 - 2 = 3^2$
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3. "The additive divided by the square...": $[3, 10; 1]$ $11 \times 3^2 + 1 = 10^2$
4. The *bhāvanā*: $\begin{array}{ccc} 3 & 10 & 1 \\ 3 & 10 & 1 \end{array}$ $[60, 199; 1]$ $11 \times 60^2 + 1 = 199^2$

An example

...
 ekādaśaguṇaḥ ko vā vargaḥ saikaḥ kṛtiḥ sakhe ||

Which square multiplied by eleven and increased by one is a square, O my friend!

1. Choose a suitable triplet: $[1, 3; -2]$ $11 \times 1^2 - 2 = 3^2$
2. The *bhāvanā*: $\begin{array}{ccc} 1 & 3 & -2 \\ 1 & 3 & -2 \end{array}$ $[6, 20; 4]$ $11 \times 6^2 + 4 = 20^2$
3. "The additive divided by the square...": $[3, 10; 1]$ $11 \times 3^2 + 1 = 10^2$
4. The *bhāvanā*: $\begin{array}{ccc} 3 & 10 & 1 \\ 3 & 10 & 1 \end{array}$ $[60, 199; 1]$ $11 \times 60^2 + 1 = 199^2$
5. Evaṃ ānantyam |

The *cakravāla*

hrasvajyeṣṭhapadakṣepān
 bhājyāprakṣepabhājakān |
 kṛtvā kalpyo guṇas tatra
 tathā prakṛtitaś cyute ||
 guṇavarge prakṛtyone
 'thavālpam śeṣakam yathā |

Having made the least and greatest roots and the additive, a **dividend**, an additive and a **divisor**, a **multiplier** must be produced so that in this procedure the remainder will be small when the square of the **multiplier** is removed from the prakṛti, or when it is decreased by the *prakṛti*;

hrasvajyeṣṭhapaḍakṣepān
 bhājyāprakṣepabhājakān |
 kṛtvā kalpyo guṇas tatra
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- ▶ Let the equation: $px_1^2 + k_1 = y_1^2$
- ▶ Find a solution $(u, v) = (\alpha, \beta)$ of the *kuṭṭaka*: $x_1 u + y_1 = k_1 v$
- ▶ So that $|\alpha^2 - p|$ will be "small"

hrasvajyeṣṭhapaḍakṣepān
 bhājyāprakṣepabhājakān |
 kṛtvā kalpyo guṇas tatra
 tathā prakṛtitaś cyute ||
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- ▶ Find a solution $(u, v) = (\alpha, \beta)$ of the *kuṭṭaka*: $x_1 u + y_1 = k_1 v$
- ▶ So that $|\alpha^2 - p|$ will be "small"

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padam̐ hrasvaṃ
 tato jyeṣṭham ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder,

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padam̐ hrasvaṃ
 tato jyeṣṭham ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder,

► $\alpha^2 - p$

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padaṃ hrasvaṃ
 tato jyeṣṭhaṃ ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder, divided by the additive is an additive,

$$\blacktriangleright \frac{\alpha^2 - p}{k_1} = k_2$$

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padaṃ hrasvaṃ
 tato jyeṣṭham ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder, divided by the additive is an additive,
 reversed if there was subtraction from the *prakṛti*.

$$\blacktriangleright \frac{\alpha^2 - p}{k_1} = k_2 \quad \left(-\frac{p - \alpha^2}{k_1} = k_2 \right)$$

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padaṃ hrasvaṃ
 tato jyeṣṭham ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder, divided by the additive is an additive, reversed if there was subtraction from the *prakṛti*. The **quotient** associated to the **multiplier** is a least root,

$$\begin{aligned} \blacktriangleright \quad \frac{\alpha^2 - p}{k_1} = k_2 \quad & \left(-\frac{p - \alpha^2}{k_1} = k_2 \right) \\ \blacktriangleright \quad \beta = \frac{\alpha x_1 + y_1}{k_1} = x_2 \quad & (x_1 \alpha + y_1 = k_1 \beta) \end{aligned}$$

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padaṃ hrasvaṃ
 tato jyeṣṭhaṃ ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder, divided by the additive is an additive, reversed if there was subtraction from the *prakṛti*. The **quotient** associated to the **multiplier** is a least root, whence a greatest root.

$$\begin{aligned} \blacktriangleright \quad \frac{\alpha^2 - p}{k_1} = k_2 \quad & \left(-\frac{p - \alpha^2}{k_1} = k_2 \right) \\ \blacktriangleright \quad \beta = \frac{\alpha x_1 + y_1}{k_1} = x_2 \quad & (x_1 \alpha + y_1 = k_1 \beta) \\ \blacktriangleright \quad p x_2^2 + k_2 = y_2^2 \end{aligned}$$

tat tu kṣepahr̥taṃ kṣepo
 vyāstaḥ prakṛtitaś cyute |
 guṇalabdhiḥ padaṃ hrasvaṃ
 tato jyeṣṭhaṃ ato 'sakṛt |
 tyaktvā pūrvapadakṣepāc
 cakravālam idaṃ jaguḥ ||

This remainder, divided by the additive is an additive, reversed if there was subtraction from the *prakṛti*. The **quotient** associated to the **multiplier** is a least root, whence a greatest root. Putting them aside severally from the previous roots and additives, [mathematicians] call this [procedure] the *cakravāla*.

- ▶ $\frac{\alpha^2 - p}{k_1} = k_2 \quad \left(-\frac{p - \alpha^2}{k_1} = k_2 \right)$
- ▶ $\beta = \frac{\alpha x_1 + y_1}{k_1} = x_2 \quad (x_1 \alpha + y_1 = k_1 \beta)$
- ▶ $p x_2^2 + k_2 = y_2^2$

caturdvyekayutāṁ evaṁ abhinne bhavataḥ pade |
 caturdvikṣepamūlābhyāṁ rūpakṣepārthabhāvanā ||

If the additive is four, two or one, the two roots are certainly integers; starting from two roots associated with the additives four and two, a composition the goal of which is the additive one [must to be carried out].

caturdvyekayutāḥ evaṁ abhinne bhavataḥ pade |
 caturdvikṣepamūlābhyāṁ rūpakṣepārthabhāvanā ||

If the additive is four, two or one, the two roots are certainly integers; starting from two roots associated with the additives four and two, a composition the goal of which is the additive one [must to be carried out].

$$px_1^2 + 2 = y_1^2$$

caturdvyekayutāḥ evaṁ abhinne bhavataḥ pade |
caturdvikṣepamūlābhyāṁ rūpakṣepārthabhāvanā ||

If the additive is four, two or one, the two roots are certainly integers; starting from two roots associated with the additives four and two, a composition the goal of which is the additive one [must to be carried out].

$$px_1^2 + 2 = y_1^2$$

► The *bhāvanā*: $\begin{array}{ccc} x_1 & y_1 & 2 \\ x_1 & y_1 & 2 \end{array}$

caturdvyekayutāḥ evaṁ abhinne bhavataḥ pade |
caturdvikṣepamūlābhyāṁ rūpakṣepārthabhāvanā ||

If the additive is four, two or one, the two roots are certainly integers; starting from two roots associated with the additives four and two, a composition the goal of which is the additive one [must to be carried out].

$$px_1^2 + 2 = y_1^2$$

► The *bhāvanā*:

x_1	y_1	2
x_1	y_1	2

► $x_2 = 2x_1y_1$

$$y_2 = px_1^2 + y_1^2$$

$$k_2 = 4$$

caturdvyekayutāṁ evaṁ abhinne bhavataḥ pade |
caturdvikṣepamūlābhyāṁ rūpakṣepārthabhāvanā ||

If the additive is four, two or one, the two roots are certainly integers; starting from two roots associated with the additives four and two, a composition the goal of which is the additive one [must to be carried out].

$$p x_1^2 + 2 = y_1^2$$

► The *bhāvanā*:
$$\begin{array}{ccc} x_1 & y_1 & 2 \\ x_1 & y_1 & 2 \end{array}$$

► $x_2 = 2x_1y_1$

$$y_2 = p x_1^2 + y_1^2$$

$$k_2 = 4$$

► “The additive divided by the square of an assumed number. . .”

$$p\left(\frac{x_2}{2}\right)^2 + \frac{4}{4} = \left(\frac{y_2}{2}\right)^2$$

caturdvyekayutāḥ evaṁ abhinne bhavataḥ pade |
caturdvikṣepamūlābhyāṁ rūpakṣepārthabhāvanā ||

If the additive is four, two or one, the two roots are certainly integers; starting from two roots associated with the additives four and two, a composition the goal of which is the additive one [must to be carried out].

$$p x_1^2 + 2 = y_1^2$$

► The *bhāvanā*:
$$\begin{array}{ccc} x_1 & y_1 & 2 \\ x_1 & y_1 & 2 \end{array}$$

► $x_2 = 2 x_1 y_1$ even

$$y_2 = p x_1^2 + y_1^2 = 2 y_1^2 - 2 \quad \text{even}$$

$$k_2 = 4$$

► “The additive divided by the square of an assumed number. . .”

$$p \left(\frac{x_2}{2} \right)^2 + \frac{4}{4} = \left(\frac{y_2}{2} \right)^2$$

Kṛṣṇa's *upapatti*

Kṛṣṇa's notations for operations

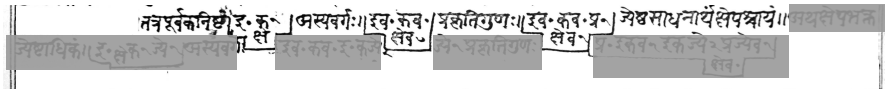
निष्ठेसाज्यं॥ ज्येष्ठपदक्षेपं॥ क्षेपंहरन्चप्रकल्प्यगुणातीताधिते॥ पूर्वगुणगुणितं कनिष्ठं क्षेपसक्तं सत्कनिष्ठं भवतीति स्थितं॥ इदानीं तु गुणगुणितं कनिष्ठं ज्येष्ठयुतं क्षेपसक्तं सत्॥ कनिष्ठं स्यात्॥ तस्माज्ज्येष्ठक्षेपसक्तं कनिष्ठं अधिकं जाते॥ एवेति प्रकृतिगुणे कनिष्ठवर्गे किमधिकं भवतीति विचार्यते॥ तत्र पूर्वकनिष्ठौ इ. क॥ अस्यवर्गः॥ इव. कव. प्रकृतिगुणः॥ इव. कव. प्र. ज्येष्ठसाधनार्थं क्षेपश्चायं॥ अथक्षेपसक्तं ज्येष्ठाधिकं॥ इ. क॥ ज्ये. अस्यवर्गः॥ इव. कव. इ. क॥ ज्ये. प्रकृतिगुणः॥ इव. प्र. इ. कव. इ. क॥ ज्ये. प्रज्येव॥ अत्रोत्तरं उभयथासाध्यते॥ कनिष्ठवर्गः प्रकृतिगुणः क्षेपयुतो ज्येष्ठवर्गो भवतीति जातः कव. प्र. क्षे. अयं प्रकृतिगुणः कव. इव. प्र. प्रक्षे. एव जाते॥ प्र. इव. कव. प्र. इ. क॥ ज्ये. कव. प्र. प्र. क्षे. तस्मादत्राधिकं॥ प्र. इ. क॥ ज्ये. कव. प्रक्षे. अतः प्रकृतिगुणे कनिष्ठवर्गे एतावाप्सितं स्यात्॥ ज्येष्ठार्थं पूर्वयुक्त्या गुणवर्गः क्षेपसक्तः॥ क्षेपणी यो भवति॥ तदर्थमधिकं खंडं यंकते॥ प्र. इ. क॥ ज्ये. कव. प्र. इदमेकं अन्यदिदं अस्मिन्साज्यसाजकयोः क्षेपेणापवर्ते जाते॥ प्र. अनेनाधिकेन क्षेपसक्ता प्रकृतिः स्यात्॥ क्षेपणीयस्तक्षेपसक्तः॥ कृतिगुणवर्गः॥ तदत्र गुणवर्गे प्रकृत्योरंतरालमपि क्षेपसक्तं क्षेप्यं॥ तथा सति क्षेपसक्तो गुणवर्ग एव क्षिप्तो भवेत्॥ अत उक्तं तथा प्रकृतितस्म्युते गुणवर्गे प्रकृत्यौ

Kṛṣṇa's notations for operations

निष्ठेमाज्यं॥ ज्येष्ठपदंक्षेपं॥ क्षेपंहरन्चप्रकल्प्यगुणातीताधिते॥ पूर्वगुणगुणितं कनिष्ठं क्षेपसक्तं सत्कनिष्ठं भवतीति स्थितं॥ इदानीं तु गुणगुणितं कनिष्ठं ज्येष्ठयुतं क्षेपसक्तं सत्॥ कनिष्ठं स्यात्॥ तस्माज्ज्येष्ठं क्षेपसक्तं कनिष्ठं अधिकं जाते॥ एवेति प्रकृतिगुणे कनिष्ठवर्गे किं स अधिकं भवतीति विचार्यते॥ तत्र पूर्वकनिष्ठौ इ. क॥ अस्यवर्गः॥ इव. कव. प्रकृतिगुणः॥ इव. कव. प्र. ज्येष्ठसाधनार्थं क्षेपश्चायं॥ अथक्षेपसक्तं ज्येष्ठाधिकं॥ इ. क॥ ज्ये. अस्यवर्गः॥ इव. कव. इ. क॥ ज्ये. प्रकृतिगुणः॥ इव. प्र. इ. कव. इ. क॥ ज्ये. प्रज्येव॥ अत्रोत्तरं उभयथासाध्यते॥ कनिष्ठवर्गः प्रकृतिगुणः क्षेपयुतो ज्येष्ठवर्गो भवतीति जातः कव. प्र. क्षे. अयं प्रकृतिगुणः कव. इव. प्रव. प्रक्षे. एवजाते॥ प्र. इव. कव. प्र. इ. क॥ ज्ये. कव. प्रव. प्र. क्षे. तस्मादत्राधिकं॥ प्र. इ. क॥ ज्ये. कव. प्रक्षे. अतः प्रकृतिगुणे कनिष्ठवर्गे एतावास्ति तं स्यात्॥ ज्येष्ठार्थं पूर्वयुक्त्या गुणवर्गः क्षेपसक्तः॥ क्षेपणी यो भवति॥ तदर्थं स अधिकं खंडे यं कृते॥ प्र. इ. क॥ ज्ये. कव. प्रव. इदमेकं अन्यदिदं अस्मिन्माज्यमाजकयोः क्षेपेणापवर्ते जाते॥ प्र. अनेनाधिकेन क्षेपसक्ता प्रकृतिः स्यात्॥ क्षेपणीयस्तक्षेपसक्तः॥ इति गुणवर्गः॥ तदत्र गुणवर्गे प्रकृत्योरंतरालमपि क्षेपसक्तं क्षेप्यं॥ इति तथा सति क्षेपसक्तो गुणवर्ग एव क्षिप्तो भवेत्॥ अत उक्तं तथा प्रकृतित्थ्युते गुणवर्गे प्रकृत्यो

तत्र पूर्वकनिष्ठौ इ. क. अस्यवर्गः॥ इव. कव. प्रहतिगुणः॥ इव. कव. प्र. ज्येष्ठसाधनार्थे सेपश्चायं॥ अथ सेपमत्र
 ज्येष्ठाधिकं॥ इ. क. ज्ये. अस्यवर्गः॥ इव. कव. इ. क. ज्ये. ज्ये. प्रहतिगुणः॥ इव. प्र. इ. क. ज्ये. प्र. ज्ये. व.





tatra pūrvakaniṣṭhaṁ $\boxed{i \bullet ka \ 1}$ asya vargaḥ $\boxed{iva \bullet kava \ 1}$ prakṛtiguṇaḥ $\boxed{iva \bullet kava \bullet pra \ 1}$ jyēṣṭhasādhārthaṁ kṣepaś
cāyaṁ $\boxed{\boxed{iva \ 1}$
 $\boxed{kṣe \ 1}$

Now, the preceding least root is $\frac{\alpha \cdot x_1}{k}$; its square $\frac{\alpha^2 \cdot x_1^2}{k^2}$ multiplied by the *prakṛti* is: $\frac{\alpha^2 \cdot x_1^2 \cdot p}{k^2}$ and, in order to
calculate the greatest root, the additive is this one: $\left[\frac{\alpha^2}{k} \right]$

तत्र पूर्वकनिष्ठौ इ. क. अस्यवर्गेः। इव. कव. प्रकृतिगुणः। इव. कव. प्र. ज्येष्ठसाधनायैष पञ्चायः। अथसेषमत्र
 ज्येष्ठाधिकं। इ. क. ज्ये. अस्यवर्गे. इव. कव. इ. क. ज्ये. ज्ये. प्रकृतिगुणः. इव. प्र. इ. क. व. इ. क. ज्ये. प्र. ज्ये. व.

tatra pūrvakaniṣṭhaṃ $\begin{bmatrix} i \bullet ka \ 1 \\ kṣe \end{bmatrix}$ asya vargaḥ $\begin{bmatrix} i \bullet kava \\ kṣeva \ 1 \end{bmatrix}$ prakṛtiguṇaḥ $\begin{bmatrix} i \bullet kava \bullet pra \ 1 \\ kṣeva \ 1 \end{bmatrix}$ jyeṣṭhasādhārthaṃ kṣepaś

cāyaṃ $\begin{bmatrix} i \bullet 1 \\ kṣe \ 1 \end{bmatrix}$

atha kṣepabhaktajyeṣṭhādhikam [kaniṣṭhaṃ] $\begin{bmatrix} i \bullet ka \ 1 \ jye \ 1 \\ kṣe \ 1 \end{bmatrix}$ asya vargaḥ $\begin{bmatrix} i \bullet kava \ 1 \ i \bullet ka \bullet jye \ 2 \ jye \ 1 \\ kṣeva \ 1 \end{bmatrix}$

prakṛtiguṇaḥ $\begin{bmatrix} pra \bullet i \bullet kava \ 1 \ pra \bullet i \bullet ka \bullet jye \ 2 \ pra \bullet jye \ 1 \\ kṣeva \ 1 \end{bmatrix}$

Now, the preceding least root is $\frac{\alpha \cdot x_1}{k}$; its square $\frac{\alpha^2 \cdot x_1^2}{k^2}$ multiplied by the *prakṛti* is: $\frac{\alpha^2 \cdot x_1^2 \cdot p}{k^2}$ and, in order to calculate the greatest root, the additive is this one: $\left[\frac{\alpha^2}{k} \right]$

Then, [the least root], added to the greatest one divided by the additive is: $\frac{\alpha \cdot x_1 + y_1}{k} \left(= \frac{\alpha \cdot x_1}{k} + \frac{y_1}{k} \right)$; its

square, $\frac{\alpha^2 \cdot x_1^2 + 2\alpha \cdot x_1 \cdot y_1 + y_1^2}{k^2}$, multiplied by the *prakṛti* is: $\frac{p \cdot \alpha^2 \cdot x_1^2 + 2p \cdot \alpha \cdot x_1 \cdot y_1 + p \cdot y_1^2}{k^2}$

A preliminary study

iṣṭavargahr̥taḥ kṣepaḥ
 kṣepaḥ syād iṣṭabhājite |
 mūle te sto 'thavā kṣepaḥ
 kṣuṇṇaḥ kṣuṇṇē tadā pade ||

The additive divided by the square of an assumed number will be the additive; the two roots divided by this assumed number, are the [roots]; or the additive multiplied is the additive; in this case the roots are multiplied.

If the least root is multiplied by an assumed number, according to this rule, the additive also must be multiplied by the square of this assumed number:

iṣṭavargahr̥taḥ kṣepaḥ
 kṣepaḥ syād iṣṭabhājite |
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The additive divided by the square of an assumed number will be the additive; the two roots divided by this assumed number, are the [roots]; or the additive multiplied is the additive; in this case the roots are multiplied.

If the least root is multiplied by an assumed number, according to this rule, the additive also must be multiplied by the square of this assumed number:

New least root: αx_1 new additive: $\alpha^2 k_1$

Now, [keeping the preceding root and additive]

Least root: αx_1 additive: $\alpha^2 k_1$

Now, [keeping the preceding root and additive]

Least root: αx_1 additive: $\alpha^2 k_1$

Now, [keeping the preceding root and additive] and an assumed number being chosen equal to the additive (k_1), according to the same rule, a new least root and additive are produced:

$$x_2 = \frac{\alpha x_1}{k_1} \quad k_2 = \frac{\alpha^2 k_1}{k_1^2}$$

Least root: αx_1 additive: $\alpha^2 k_1$

Now, [keeping the preceding root and additive] and an assumed number being chosen equal to the additive (k_1), according to the same rule, a new least root and additive are produced:

$$x_2 = \frac{\alpha x_1}{k_1} \quad k_2 = \frac{\alpha^2 k_1}{k_1^2}$$

After reduction, the additive is:

$$k_2 = \frac{\alpha^2}{k_1}$$

Introducing the *kuṭṭaka*

In these conditions,

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In these conditions, one must choose the assumed number so that once multiplied by the least root and reduced by the additive, there will be no remainder.

Introducing the *kuṭṭaka*

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In these conditions, one must choose the assumed number so that once multiplied by the least root and reduced by the additive, there will be no remainder.

For this purpose, a **multiplier** and a **quotient** must be calculated, putting down the [first] least root (x_1) as the **dividend**, the additive (k_1) as the **divisor** and without [*kuṭṭaka*]-additive.

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$$x_1 u = k_1 v$$

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$$x_1 u = k_1 v$$

Let (α, β) be a solution of the equation: $x_1 u = k_1 v$: $\alpha = k_1 t$ and $\beta = x_1 t$

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- ▶ Then the **quotient** $(\beta = \frac{\alpha x_1}{k_1} = x_2)$ will be a least root
- ▶ The square of the **multiplier**, divided by the previous additive, will be the new additive $(\frac{\alpha^2}{k_1} = k_2)$
- ▶ The [first] greatest root, multiplied by the **multiplier** and divided by the additive, will be a greatest root $(\frac{\alpha y_1}{k_1} = y_2)$

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- ▶ The square of the **multiplier**, divided by the previous additive, will be the new additive $(\frac{\alpha^2}{k_1} = k_2)$
- ▶ The [first] greatest root, multiplied by the **multiplier** and divided by the additive, will be a greatest root $(\frac{\alpha y_1}{k_1} = y_2)$

$$p x_2^2 + k_2 = y_2^2 \quad p \left(\frac{\alpha x_1}{k_1} \right)^2 + \frac{\alpha^2}{k_1} = \frac{\alpha^2}{k_1^2} (p x_1^2 + k_1) = \left(\frac{\alpha y_1}{k_1} \right)^2 = y_2^2$$

But the Master endeavoured [to calculate] differently, because in the [preceding calculation], the additive is too large.

But the Master endeavoured [to calculate] differently, because in the [preceding calculation], the additive is too large.

A **multiplier** and a **quotient** are calculated putting the least root (x_1) as the **dividend**, the greatest root (y_1) as the [*kuṭṭaka*]-additive and the additive (k_1) as the **divisor**:

But the Master endeavoured [to calculate] differently, because in the [preceding calculation], the additive is too large.

A **multiplier** and a **quotient** are calculated putting the least root (x_1) as the **dividend**, the greatest root (y_1) as the [kuṭṭaka]-additive and the additive (k_1) as the **divisor**:

$$x_1 u + y_1 = k_1 v$$

Let (α, β) be a solution to the *kuṭṭaka* $x_1 u + y_1 = k_1 v$

Preliminary study Bhāskara's rule

Least root	$\frac{\alpha x_1}{k_1}$	$\frac{\alpha x_1 + y_1}{k_1}$
Additive	$\frac{\alpha^2}{k_1}$	$\frac{\alpha^2 - p}{k_1}$

Let (α, β) be a solution to the *kuṭṭaka* $x_1 u + y_1 = k_1 v$

Preliminary study Bhāskara's rule

Least root	$\frac{\alpha x_1}{k_1}$	$\frac{\alpha x_1 + y_1}{k_1}$
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As the new least root is produced (by Bhāskara's rule) with the addition of the greatest root divided by the additive $\left(\frac{\alpha x_1}{k_1} + \frac{y_1}{k_1}\right)$, let see what is added to the square of the least root multiplied by the *prakṛti*.

Let (α, β) be a solution to the *kuṭṭaka* $x_1 u + y_1 = k_1 v$

Preliminary study Bhāskara's rule

Least root	$\frac{\alpha x_1}{k_1}$	$\frac{\alpha x_1 + y_1}{k_1}$
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As the new least root is produced (by Bhāskara's rule) with the addition of the greatest root divided by the additive $\left(\frac{\alpha x_1}{k_1} + \frac{y_1}{k_1}\right)$, let see what is added to the square of the least root multiplied by the *prakṛti*.

$$p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 - p\left(\frac{\alpha x_1}{k_1}\right)^2$$

The new least root,

$$\frac{\alpha x_1 + y_1}{k_1}$$

The new least root, its square

$$\frac{\alpha x_1 + y_1}{k_1}$$

$$\frac{\alpha^2 x_1^2 + 2\alpha x_1 y_1 + y_1^2}{k_1^2}$$

The new least root, its square is multiplied by the *prakṛti*.

$$\frac{\alpha x_1 + y_1}{k_1}$$

$$\frac{\alpha^2 x_1^2 + 2\alpha x_1 y_1 + y_1^2}{k_1^2}$$

$$\frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + py_1^2}{k_1^2}$$

The new least root, its square is multiplied by the *prakṛti*.
Now, the last term is calculated in another way

$$\frac{\alpha x_1 + y_1}{k_1}$$

$$\frac{\alpha^2 x_1^2 + 2\alpha x_1 y_1 + y_1^2}{k_1^2}$$

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The new least root, its square is multiplied by the *prakṛti*.
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$$\frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + py_1^2}{k_1^2} = \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + p(px_1^2 + k_1)}{k_1^2}$$

The new least root, its square is multiplied by the *prakṛti*.
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$$\frac{\alpha x_1 + y_1}{k_1}$$

$$\frac{\alpha^2 x_1^2 + 2\alpha x_1 y_1 + y_1^2}{k_1^2}$$

$$\begin{aligned} \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + py_1^2}{k_1^2} &= \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + p(px_1^2 + k_1)}{k_1^2} \\ &= \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + p^2 x_1^2 + pk_1}{k_1^2} \end{aligned}$$

The new least root, its square is multiplied by the *prakṛti*.
 Now, the last term is calculated in another way
 Whence, this number is added [to the previous square of the
 least root multiplied by the *prakṛti*]

$$\frac{\alpha x_1 + y_1}{k_1}$$

$$\frac{\alpha^2 x_1^2 + 2\alpha x_1 y_1 + y_1^2}{k_1^2}$$

$$\begin{aligned} \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + py_1^2}{k_1^2} &= \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + p(p x_1^2 + k_1)}{k_1^2} \\ &= \frac{p\alpha^2 x_1^2 + 2p\alpha x_1 y_1 + p^2 x_1^2 + pk_1}{k_1^2} \end{aligned}$$

Now, according to the previous reasoning, the square of the assumed number, divided by the additive ($\frac{\alpha^2}{k_1}$), must be added (as the additive).

Now, according to the previous reasoning, the square of the assumed number, divided by the additive ($\frac{\alpha^2}{k_1}$), must be added (as the additive).

For this purpose two parts are made with the added number [that we have just calculated].

$$\frac{2p\alpha x_1 y_1 + p^2 x_1^2 + p k_1}{k_1^2} = \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{p}{k_1}$$

In Bhāskara's rule, the difference between the square of the **multiplier** and the *prakṛti*, divided by the additive must be added,

$$\frac{2pax_1y_1 + p^2x_1^2 + pk_1}{k_1^2} = \frac{2pax_1y_1 + p^2x_1^2}{k_1^2} + \frac{p}{k_1}$$

In Bhāskara's rule, the difference between the square of the **multiplier** and the *prakṛti*, divided by the additive must be added, because, if we do so, only the square of the **multiplier** divided by the additive is added.

$$\frac{2pax_1y_1 + p^2x_1^2 + pk_1}{k_1^2} = \frac{2pax_1y_1 + p^2x_1^2}{k_1^2} + \frac{p}{k_1}$$

$$\frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}$$

In this rule, if the square of the **multiplier** is superior to the *prakṛti*, then the difference between the square of the **multiplier** and the *prakṛti*, divided by the additive, must be added.

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$$\text{If } \alpha^2 - p \geq 0, \quad \text{then} \quad \frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}$$

In this rule, if the square of the **multiplier** is superior to the *prakṛti*, then the difference between the square of the **multiplier** and the *prakṛti*, divided by the additive, must be added. But when the square of **multiplier** is inferior... the difference must be subtracted.

$$\text{If } \alpha^2 - p \geq 0, \quad \text{then} \quad \frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}$$

In this rule, if the square of the **multiplier** is superior to the *prakṛti*, then the difference between the square of the **multiplier** and the *prakṛti*, divided by the additive, must be added.

But when the square of **multiplier** is inferior... the difference must be subtracted.

$$\text{If } \alpha^2 - p \geq 0, \quad \text{then} \quad \frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}$$

$$\text{If } \alpha^2 - p \leq 0, \quad \text{then} \quad \frac{p}{k_1} + \left(- \frac{p - \alpha^2}{k_1} \right) = \frac{\alpha^2}{k_1}$$

In this rule, if the square of the multiplier is superior to the *prakṛti*, then the difference between the square of the multiplier and the *prakṛti*, divided by the additive, must be added.

But when the square of multiplier is inferior... the difference must be subtracted.

This why it has been said: "... reversed if there was subtraction from the *prakṛti*."

$$\text{If } \alpha^2 - p \geq 0, \quad \text{then} \quad \frac{p}{k_1} + \frac{\alpha^2 - p}{k_1} = \frac{\alpha^2}{k_1}$$

$$\text{If } \alpha^2 - p \leq 0, \quad \text{then} \quad \frac{p}{k_1} + \left(- \frac{p - \alpha^2}{k_1} \right) = \frac{\alpha^2}{k_1}$$

Now, this number

$$\frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2}$$

Now, according to the previous reasoning, the square of the assumed number, divided by the additive ($\frac{\alpha^2}{k_1}$), must be added (as the additive).

For this purpose two parts are made with the added number [that we have just calculated].

$$\frac{2p\alpha x_1 y_1 + p^2 x_1^2 + pk_1}{k_1^2} = \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{p}{k_1}$$

Now, this number is surely added to the square of the greatest root

$$\frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2}$$

Now, this number is surely added to the square of the greatest root

$$p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 = \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{p}{k_1}$$

Now, this number is surely added to the square of the greatest root

$$p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} = \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{p}{k_1} + \frac{\alpha^2 - p}{k_1}$$

Now, this number is surely added to the square of the greatest root

$$p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} = \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1}$$

Now, this number is surely added to the square of the greatest root

$$\begin{aligned}
 p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} &= \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1} \\
 &= \boxed{p\left(\frac{\alpha x_1}{k_1}\right)^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \boxed{+ \frac{\alpha^2}{k_1}}
 \end{aligned}$$

Now, this number is surely added to the square of the greatest root

$$\begin{aligned}
 p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} &= \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1} \\
 &= \boxed{p\left(\frac{\alpha x_1}{k_1}\right)^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \boxed{+ \frac{\alpha^2}{k_1}} \\
 &= \left(\frac{\alpha y_1}{k_1}\right)^2 + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2}
 \end{aligned}$$

Once the addition is done, the square root can be calculated using: “Having taken the roots from the squares. . .”

$$\begin{aligned}
 p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} &= \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1} \\
 &= \boxed{p\left(\frac{\alpha x_1}{k_1}\right)^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \boxed{+ \frac{\alpha^2}{k_1}} \\
 &= \left(\frac{\alpha y_1}{k_1}\right)^2 + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \\
 &= \frac{\alpha^2 y_1^2 + 2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2}
 \end{aligned}$$

kṛtibhya ādāya padāni teṣāṃ
dvayor dvayoś cābhihatim dvinighnīm |
śeṣāt tyajed rūpapadaṃ gṛhītvā
cet santi rūpāṇi tathaivā śeṣam ||

Having taken the roots from the squares, one will subtract from the remainder the double of the product of those roots two by two; in case of integers, after taking their root, [proceed] in like manner with the residue.

Once the addition is done, the square root can be calculated using: “Having taken the roots from the squares. . .”

$$\begin{aligned}
 p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} &= \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1} \\
 &= \boxed{p\left(\frac{\alpha x_1}{k_1}\right)^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \boxed{+ \frac{\alpha^2}{k_1}} \\
 &= \left(\frac{\alpha y_1}{k_1}\right)^2 + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \\
 &= \frac{\alpha^2 y_1^2 + 2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \\
 &= \left(\frac{\alpha y_1 + p x_1}{k_1}\right)^2
 \end{aligned}$$

So, this number is also the square of the greatest root

$$\begin{aligned}
 p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} &= \frac{p\alpha^2 x_1^2}{k_1^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} + \frac{\alpha^2}{k_1} \\
 &= \boxed{p\left(\frac{\alpha x_1}{k_1}\right)^2} + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \boxed{+ \frac{\alpha^2}{k_1}} \\
 &= \left(\frac{\alpha y_1}{k_1}\right)^2 + \frac{2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \\
 &= \frac{\alpha^2 y_1^2 + 2p\alpha x_1 y_1 + p^2 x_1^2}{k_1^2} \\
 &= \left(\frac{\alpha y_1 + p x_1}{k_1}\right)^2
 \end{aligned}$$

In this [calculation], even if we do not take the *kuṭṭaka* into consideration, roots are acquired by the force of the assumed number; however the *kuṭṭaka* is carried out in order to find integral roots.

In this [calculation], even if we do not take the *kuṭṭaka* into consideration, roots are acquired by the force of the assumed number; however the *kuṭṭaka* is carried out in order to find integral roots.

So is justified: “Having made the least and greatest roots and the additive, a **dividend**, an additive and a **divisor**”.

$$p\left(\frac{\alpha x_1 + y_1}{k_1}\right)^2 + \frac{\alpha^2 - p}{k_1} = \left(\frac{\alpha y_1 + p x_1}{k_1}\right)^2$$

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$\frac{\alpha^2 - p}{k_1}$ is also an integer:

$$\begin{aligned} (\alpha^2 - p)x_1^2 &= \alpha^2 x_1^2 - p x_1^2 \\ &= \alpha^2 x_1^2 - y_1^2 + k_1 \quad (\text{because } p x_1^2 = y_1^2 - k_1) \\ &= (\alpha x_1 + y_1)(\alpha x_1 - y_1) + k_1 \end{aligned}$$

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$$67x^2 + 1 = y^2$$

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- ▶ Choose $t = -2$: $u_1 = u_0 - 2 \times -3 = 7 \quad v_1 = -3 - 2 = -5$

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The additive is: $k_2 = -\frac{p - u_1^2}{k_1} = -\frac{18}{-3} = 6$

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The greatest root is: $y_2 = 41$ $67 \times (-5)^2 + 6 = 41^2$

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The greatest root is: $y_3 = 90 \quad 67 \times 11^2 - 7 = 90^2$

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$$u_0 = 9 \quad v_0 = -27$$

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The least root is: $x_4 = -27$

The greatest root is: $y_4 = 221 \quad 67 \times 27^2 - 2 = 221^2$

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$$67x^2 - 2 = y^2 \quad \text{with triplet:} \quad [x_4, y_4; k_4] = [27, 221; -2]$$

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- Use the *bhāvanā*:
- | | | |
|----|-----|----|
| 27 | 221 | -2 |
| 27 | 221 | -2 |

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- Use the *bhāvanā*:
- | | | |
|----|-----|----|
| 27 | 221 | -2 |
| 27 | 221 | -2 |

The least root is: $x_5 = 2 \times 27 \times 221 = 11934$

An example

$$67x^2 - 2 = y^2 \quad \text{with triplet: } [x_4, y_4; k_4] = [27, 221; -2]$$

- Use the *bhāvanā*:
- | | | |
|----|-----|----|
| 27 | 221 | -2 |
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The least root is: $x_5 = 2 \times 27 \times 221 = 11934$

The greatest root is: $y_5 = 67 \times 27^2 + 221^2 = 97684$

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- | | | |
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| 27 | 221 | -2 |
| 27 | 221 | -2 |

The least root is: $x_5 = 2 \times 27 \times 221 = 11934$

The greatest root is: $y_5 = 67 \times 27^2 + 221^2 = 97684$

The additive is: $k_5 = (-2)^2 = 4$

An example

$$67x^2 - 2 = y^2 \quad \text{with triplet: } [x_4, y_4; k_4] = [27, 221; -2]$$

- Use the *bhāvanā*: $\begin{array}{ccc} 27 & 221 & -2 \\ 27 & 221 & -2 \end{array}$

The least root is: $x_5 = 2 \times 27 \times 221 = 11934$

The greatest root is: $y_5 = 67 \times 27^2 + 221^2 = 97684$

The additive is: $k_5 = (-2)^2 = 4$

- Use the rule: “The additive divided by the square of an assumed number”:

$$67 \times 5967^2 + 1 = 48842^2$$