



HAL
open science

Perturbations of diagonal matrices by band random matrices

Florent Benaych-Georges, Nathanael Enriquez

► **To cite this version:**

Florent Benaych-Georges, Nathanael Enriquez. Perturbations of diagonal matrices by band random matrices. 2011. hal-00584577v2

HAL Id: hal-00584577

<https://hal.science/hal-00584577v2>

Submitted on 27 Apr 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PERTURBATIONS OF DIAGONAL MATRICES BY BAND RANDOM MATRICES

FLORENT BENAYCH-GEORGES AND NATHANAËL ENRIQUEZ

ABSTRACT. We exhibit an explicit formula for the spectral density of a (large) random matrix which is a diagonal matrix whose spectral density converges, perturbed by the addition of a symmetric matrix with Gaussian entries and a given (small) limiting variance profile.

1. PERTURBATION OF THE SPECTRAL DENSITY OF A LARGE DIAGONAL MATRIX

In this paper, we consider the spectral measure of a random matrix D_n^ε defined by $D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n$, for D_n a deterministic diagonal matrix whose spectral measure converges and X_n an Hermitian or real symmetric matrix whose entries are Gaussian independent variables, with a limiting variance profile (such matrices are called *band matrices*). We give a first order Taylor expansion, as $\varepsilon \rightarrow 0$, of the limit spectral density, as $n \rightarrow \infty$, of D_n^ε .

The proof is elementary and based on a formula given in [12] for the Cauchy transform of the limit spectral distribution of D_n^ε as $n \rightarrow \infty$.

For each n , we consider an Hermitian or real symmetric random matrix $X_n = [x_{i,j}^n]_{i,j=1}^n$ and a real diagonal matrix $D_n = \text{diag}(a_n(1), \dots, a_n(n))$. We suppose that:

- (a) the entries $x_{i,j}^n$ of X_n are independent (up to symmetry), centered, Gaussian with variance denoted by $\sigma_n^2(i, j)$,
- (b) for a certain bounded function σ defined on $[0, 1] \times [0, 1]$ and a certain bounded real function f defined on $[0, 1]$, we have, in the L^∞ topology,
$$\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) \xrightarrow{n \rightarrow \infty} \sigma^2(x, y) \quad \text{and} \quad a_n(\lfloor nx \rfloor) \xrightarrow{n \rightarrow \infty} f(x),$$
- (c) the set of discontinuities of the function σ is closed and intersects a finite number of times any vertical line of the square $[0, 1]^2$.

For $\varepsilon \geq 0$, let us define, for all n ,

$$D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n.$$

Date: April 27, 2011.

2000 Mathematics Subject Classification. 15A52, 46L54.

Key words and phrases. Random matrices, band matrices, Hilbert transform, spectral density.

This work was partially supported by the *Agence Nationale de la Recherche* grant ANR-08-BLAN-0311-03.

It is known, from Shlyakhtenko in [12, Th. 4.3] (see also [2], which also provides a fluctuation result), that as n tends to infinity, the spectral distribution of D_n^ε tends to a limit μ_ε with Cauchy transform

$$C_\varepsilon(z) = \int_{x=0}^1 C_\varepsilon(x, z) dx,$$

where $C_\varepsilon(x, \cdot)$ is defined by the fact that it is analytic, maps the upper half-plane \mathbb{C}^+ into the lower one \mathbb{C}^- , and satisfies the relation

$$(1) \quad C_\varepsilon(x, z) = \frac{1}{z - f(x) - \varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy}.$$

Our goal here is to understand $\mu_\varepsilon - \mu$ for small values of ε . Let us introduce the set \mathcal{T} of test functions we shall use here. We define

$$\mathcal{T} = \left\{ t \mapsto \frac{1}{z - t}; z \in \mathbb{C}^+ \right\}.$$

Let us now define the *Hilbert transform*, denoted by $H[u]$, of a function u :

$$H[u](s) := \text{p. v.} \int_{t \in \mathbb{R}} \frac{u(t)}{s - t} dt = \int_{y \in \mathbb{R}} \frac{u(s - y) - u(s)}{y} dy.$$

Before stating our main result, let us make some assumptions on the functions σ and f :

- (d) the push-forward μ of the uniform measure on $[0, 1]$ by the function f has a density ρ with respect to the Lebesgue measure on \mathbb{R} ,
- (e) there exists a symmetric function $\tau(\cdot, \cdot)$ such that for all x, y , $\sigma^2(x, y) = \tau(f(x), f(y))$,
- (f) there exist $\eta_0 > 0, \alpha > 0$ and $C < \infty$ such that for almost all $s \in \mathbb{R}$, for all $t \in [s - \eta_0, s + \eta_0]$, $|\tau(s, t)\rho(t) - \tau(s, s)\rho(s)| \leq C|t - s|^\alpha$.

Note that by hypothesis (f) and by the boundedness of the function f , the function

$$s \mapsto \rho(s)H[\tau(s, \cdot)\rho(\cdot)](s)$$

is well defined and compactly supported.

Theorem 1. *Under the hypotheses (a) to (f), as $\varepsilon \rightarrow 0$, for all $g \in \mathcal{T}$,*

$$\int g(s) d\mu_\varepsilon(s) = \int g(s) d\mu(s) - \varepsilon \int g'(s) F(s) ds + o(\varepsilon),$$

with $F(s) := -\rho(s)H[\tau(s, \cdot)\rho(\cdot)](s)$.

As a consequence, if the function $F(\cdot)$ has bounded variations, then

$$\mu_\varepsilon = \mu + \varepsilon dF + o(\varepsilon).$$

Remark 1. *Roughly speaking, this theorem states that*

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\text{spectral law}(D_n^\varepsilon) - \text{spectral law}(D_n)}{\varepsilon} = dF.$$

It would be interesting to let ε and n tend to 0 and ∞ together, and to find out the adequate rate of convergence to get a deterministic limit or non degenerated fluctuations. We are working on this question.

Remark 2. *This result provides an analogue, for our random matrix model, of the following formula about real random variables (valid when Y is centered and independent of X):*

$$\text{density}_{X+\sqrt{\varepsilon}Y}(s) = \text{density}_X(s) + \varepsilon \frac{\mathbb{E}[Y^2]}{2} \text{density}_X''(s) + o(\varepsilon).$$

Remark 3. *In the case where X_n is a GUE or GOE matrix, the limiting spectral distribution of D_n^ε as $n \rightarrow \infty$ is the free convolution of the limiting spectral distribution of D_n with a semi-circle distribution. Several papers are devoted to the study of qualitative properties (like regularity) of the free convolution (see [8, 7, 4, 3, 6]). Besides, it has recently been proved that type-B free probability theory allows to give Taylor expansions, for small values of t , of the moments of $\mu_t \boxplus \nu_t$ for two time-dependent probability measures μ_t and ν_t (see [5, 10, 9]). Our work differs from the ones mentioned above by the fact that we allow to perturb D_n by any band matrix, but also by the fact that it is focused on the density and not on the moments, giving an explicit formula rather than qualitative properties.*

Proof. For all $z \in \mathbb{C}^+$, we have

$$(2) \quad |C_\varepsilon(x, z)| \leq \frac{1}{\Im z}.$$

Indeed, for all y, z such that $z \in \mathbb{C}^+$, $C_\varepsilon(y, z) \in \mathbb{C}^-$. As a consequence, the imaginary part of the denominator of the right hand term of (1) is larger than $\Im(z)$.

Hence by (1) and (2), as $\varepsilon \rightarrow 0$, $C_\varepsilon(x, z) \rightarrow \frac{1}{z-f(x)}$ uniformly in x .

From what precedes,

$$\begin{aligned} C_\varepsilon(x, z) - \frac{1}{z-f(x)} &= \frac{\varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy}{(z-f(x) - \varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy)(z-f(x))} \\ &= \varepsilon \frac{1}{(z-f(x))^2} \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy + o(\varepsilon) \\ &= \varepsilon \frac{1}{(z-f(x))^2} \int_{y=0}^1 \frac{\sigma^2(x, y)}{z-f(y)} dy + o(\varepsilon) \end{aligned}$$

where each $o(\varepsilon)$ is uniform in $x \in [0, 1]$.

But for all $a \neq b$, $\frac{1}{(z-a)^2(z-b)} = \frac{1}{(a-b)^2} \left(\frac{1}{z-b} - \frac{1}{z-a} - \frac{b-a}{(z-a)^2} \right)$, hence since the Lebesgue measure of the set $\{y \in [0, 1]; f(y) = f(x)\}$ is null, we have

$$\frac{1}{(z-f(x))^2} \int_{y=0}^1 \frac{\sigma^2(x, y)}{z-f(y)} dy = \int_{y=0}^1 \frac{\sigma^2(x, y)}{(f(x)-f(y))^2} \left(\frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2} \right) dy.$$

As a consequence, it follows by an integration in $x \in [0, 1]$ that

$$C_\varepsilon(z) - C(z) = \varepsilon \int_{x=0}^1 \int_{y=0}^1 \frac{\sigma^2(x, y)}{(f(x)-f(y))^2} \left(\frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2} \right) dy dx + o(\varepsilon),$$

where $C(\cdot)$ is the Cauchy transform of μ .

Let us now recall that the push-forward of the uniform law on $[0, 1]$ by f is the measure $\rho(x)dx$ and that $\sigma^2(x, y)$ can be rewritten $\sigma^2(x, y) = \tau(f(x), f(y))$. Hence

$$C_\varepsilon(z) - C(z) = \varepsilon \int_{s \in \mathbb{R}} \int_{t \in \mathbb{R}} \left\{ \frac{1}{z-t} - \frac{1}{z-s} - \frac{1}{(z-s)^2} (t-s) \right\} \frac{\tau(s, t)}{(s-t)^2} \rho(s) \rho(t) dt ds + o(\varepsilon).$$

This allows us to write that for any test function $g \in \mathcal{T}$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(g) - \mu(g)}{\varepsilon} = \Lambda(g),$$

where

$$\Lambda(g) = \int_{(s,t) \in \mathbb{R}^2} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) dt ds.$$

Note that by the Taylor-Lagrange formula, for all s, t ,

$$\left| \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \right| \leq \frac{\rho(s)\rho(t) \times \|\tau(\cdot, \cdot)\|_{L^\infty} \|g''\|_{L^\infty}}{2},$$

so that, since ρ is a density, by dominated convergence,

$$\Lambda(g) = \lim_{\eta \rightarrow 0} \int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) ds dt.$$

But by symmetry, for all $\eta > 0$,

$$\int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} \{g(t) - g(s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) ds dt = 0.$$

As a consequence, $\Lambda(g) = \lim_{\eta \rightarrow 0} \Lambda_\eta(g)$, with

$$\Lambda_\eta(g) := \int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} g'(s) \frac{\tau(s,t)}{s-t} \rho(s)\rho(t) ds dt.$$

Let us prove that almost all $s \in \mathbb{R}$, $\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt$ exists and that

$$\Lambda(g) = \int_{s \in \mathbb{R}} g'(s) \left(\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \right) ds.$$

For $\eta > 0$ and $s \in \mathbb{R}$, set

$$\theta_\eta(s) := \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt.$$

Set also $M := \|f\|_{L^\infty}$. Then the support of the function ρ is contained in $[-M, M]$, and so does the support of the function θ_η , for any $\eta > 0$. For almost all $s \in [-M, M]$, $\lim_{\eta \rightarrow 0} \theta_\eta(s)$ exists by the formula

$$\theta_\eta(s) = \int_{t \in [s-2M, s-\eta] \cup [s+\eta, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t) - \tau(s,s)\rho(s)\rho(s)}{s-t} dt$$

and by Hypothesis (f). Moreover, for η_0 as in Hypothesis (f),

$$\begin{aligned} |\theta_\eta(s)| &\leq 2C\rho(s) \int_{t=s+\eta}^{s+\eta_0} (s-t)^{\alpha-1} dt + \int_{t \in [s-2M, s-\eta_0] \cup [s+\eta_0, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \\ &\leq \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{1}{\eta_0} \int_{t \in \mathbb{R}} \tau(s,t)\rho(s)\rho(t) ds dt \\ &\leq \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{\|\tau(\cdot, \cdot)\|_{L^\infty}}{\eta_0} \rho(s). \end{aligned}$$

Hence by dominated convergence, $\int_{s \in \mathbb{R}} g'(s) \lim_{\eta \rightarrow 0} \theta_\eta(s) ds = \lim_{\eta \rightarrow 0} \int_{s \in \mathbb{R}} g'(s) \theta_\eta(s) ds$, i.e.

$$\Lambda(g) = \int_{s \in \mathbb{R}} g'(s) \left(\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s, t) \rho(s) \rho(t)}{s-t} dt \right) ds.$$

□

2. EXAMPLES

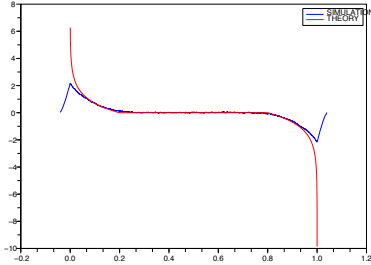
2.1. Perturbation of a uniform distribution by a standard band matrix. Let us consider the case where $f(x) = x$ (so that μ is the uniform distribution on $[0, 1]$) and $\sigma^2(x, y) = \mathbb{1}_{|y-x| \leq \ell}$, where ℓ is a fixed parameter in $[0, 1]$ (the width of the band). In this case, $\tau(\cdot, \cdot) = \sigma^2(\cdot, \cdot)$ and

$$F(s) = \mathbb{1}_{(0,1)}(s) \log \left(\frac{\ell \wedge (1-s)}{\ell \wedge s} \right).$$

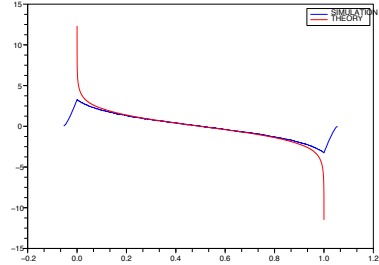
For small values of ε and large values of n , the density ρ_ε of the eigenvalue distribution μ_ε of D_n^ε is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon) = \mathbb{1}_{(0,1)}(s) - \varepsilon \left(\frac{\mathbb{1}_{(0,\ell)}(s)}{s} + \frac{\mathbb{1}_{(1-\ell,1)}(s)}{1-s} \right) + o(\varepsilon),$$

which means that the additive perturbation $\sqrt{\frac{\varepsilon}{n}} X_n$ alters the spectrum of D_n essentially by decreasing the amount of extreme eigenvalues. This phenomenon is illustrated by Figure 1 (where we plotted the cumulative distribution functions rather than the densities for visual reasons).



(a) Case where $n = 4.10^3$, $\varepsilon = 10^{-2}$, with width $\ell = 0.2$



(b) Case where $n = 4.10^3$, $\varepsilon = 10^{-2}$, with width $\ell = 0.9$

FIGURE 1. Perturbation of a uniform distribution by a standard band matrix: plot of the functions $F(\cdot)$ and $\frac{F_{D_n^\varepsilon}(\cdot) - F_{D_n}(\cdot)}{\varepsilon}$ (with $F_{D_n^\varepsilon}(\cdot)$ and $F_{D_n}(\cdot)$ the cumulative eigenvalue distribution functions of D_n^ε and D_n) for different values of ℓ .

2.2. Perturbation of the triangular pulse distribution by a GOE matrix. Let us consider the case where $\rho(x) = (1 - |x|) \mathbb{1}_{[-1,1]}(x)$ and $\sigma^2 \equiv 1$ (what follows can be adapted to the case $\sigma^2(x, y) = \mathbb{1}_{|y-x| \leq \ell}$, but the formulas are a bit heavy). In this case, thanks to the formula (9.6) of $H[\rho(\cdot)]$ given p. 509 of [11], we get

$$F(s) = (1 - |s|) \mathbb{1}_{[-1,1]}(s) \{ (1 - s) \log(1 - s) - (1 + s) \log(1 + s) + 2s \log |s| \}.$$

For small values of ε and large values of n , the density ρ_ε of the eigenvalue distribution μ_ε of D_n^ε is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon),$$

which implies that the additive perturbation $\sqrt{\frac{\varepsilon}{n}} X_n$ alters the spectrum of D_n by increasing the amount of eigenvalues in $[-1, -0.5] \cup [0.5, 1]$ and decreasing the amount of eigenvalues around zero. This phenomenon is illustrated by Figure 2.

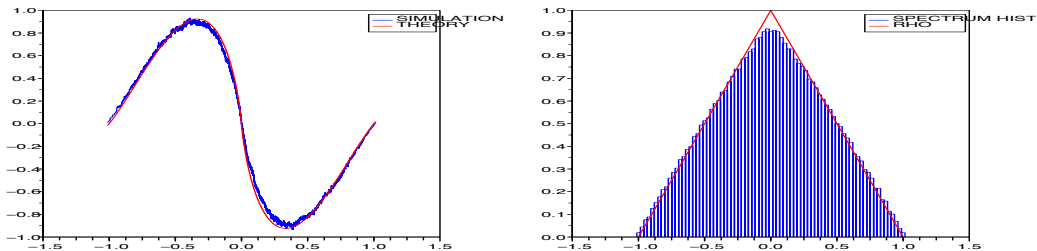


FIGURE 2. **Perturbation of the triangular pulse distribution by a GOE matrix:** *Left:* plot of the functions $F(\cdot)$ and $\frac{F_{D_n^\varepsilon}(\cdot) - F_{D_n}(\cdot)}{\varepsilon}$ (with $F_{D_n^\varepsilon}(\cdot)$ and $F_{D_n}(\cdot)$ the cumulative eigenvalue distribution functions of D_n^ε and D_n). *Right:* plot of the eigenvalues histogram of D_n^ε and of the spectral density ρ of D_n . On the right figure, the (infinitesimal) increase of eigenvalues with respect to ρ on $[-1, -0.5] \cup [0.5, 1]$ and the (infinitesimal) decrease around zero can be observed, in agreement with the fact that, as the left figure shows, $F' \gg 0$ on (approximately) $[-1, -0.5] \cup [0.5, 1]$ and $F' \ll 0$ around zero. Both figures were made with the same simulation ($n = 6.10^3$ and $\varepsilon = 10^{-2}$).

2.3. Free convolution with a semi-circular distribution and complex Burger's equation. Let us consider the case where $\sigma^2 \equiv 1$, which happens for example if the matrix X_n is taken in the Gaussian Orthogonal Ensemble. In this case, by the theory of free probability developed by Dan Voiculescu (see e.g. [13] or [1, Cor 5.4.11 (ii)]), for all $t \geq 0$,

$$\mu_t = \mu \boxplus \lambda_t,$$

where λ_t is the *semi-circular distribution with variance t* , i.e. the distribution with support $[-2\sqrt{t}, 2\sqrt{t}]$ and density $\frac{1}{2\pi t} \sqrt{4t - x^2}$. In this case, we know by the work of Biane [8, Cor. 2] that for all $t > 0$, μ_t admits a density ρ_t . By the implicit function theorem, and the formula given in [8, Cor. 2], one easily sees that the function $(s, t) \mapsto \rho_t(s)$ is regular. Then, by Theorem 1 and the fact that the linear span of \mathcal{T} is dense in the set of continuous functions on the real line with null limit at infinity, one easily recovers the following PDE, which is a kind of projection on the real axis of the imaginary part of complex Burger's equation given in [8, Intro.]

$$(3) \quad \begin{cases} \frac{\partial}{\partial t} \rho_t(s) + \frac{\partial}{\partial s} \{ \rho_t(s) H[\rho_t(\cdot)](s) \} = 0, \\ \rho_0(s) = \rho(s). \end{cases}$$

For example, if $\mu = \lambda_c$ for a certain $c > 0$, then by the semi-group property of the semi-circle distribution [1, Ex. 5.3.26], for all $t \geq 0$, $\mu_t = \lambda_{c+t}$ and $\rho_t(s) = \frac{1}{2\pi(c+t)} \sqrt{4(c+t) - s^2}$. One can

then verify (3), using the formula (9.21) of $H[\rho_t(\cdot)]$ given p. 511 of [11].

Acknowledgements. It is a pleasure to thank Guy David for his useful advices about the Hilbert transform.

REFERENCES

- [1] G. Anderson, A. Guionnet, O. Zeitouni. *An Introduction to Random Matrices*. Cambridge studies in advanced mathematics, 118 (2009).
- [2] G. Anderson, O. Zeitouni. *A CLT for a band matrix model*. Probab. Theory Related Fields 134 (2005), 283–338.
- [3] S.T. Belinschi. *A note on regularity for free convolutions*, Ann. Inst. H. Poincaré Probab. Statist. 42 (2006), no. 5, 635–648.
- [4] S.T. Belinschi. *The Lebesgue decomposition of the free additive convolution of two probability distributions*, Probab. Theory Related Fields 142 (2008), no. 1–2, 125–150.
- [5] S.T. Belinschi, D. Shlyakhtenko. *Free probability of type B: analytic aspects and applications*, to appear in American J. Math.
- [6] S.T. Belinschi, F. Benaych-Georges, A. Guionnet. *Regularization by Free Additive Convolution, Square and Rectangular Cases*. Complex Analysis and Operator Theory. Vol. 3, no. 3 (2009) 611–660.
- [7] H. Bercovici and D. Voiculescu. *Regularity questions for free convolution*, Nonselfadjoint operator algebras, operator theory, and related topics, 37–47, Oper. Theory Adv. Appl. 104, Birkhäuser, Basel, 1998.
- [8] P. Biane. *On the Free convolution by a semi-circular distribution*. Indiana University Mathematics Journal, Vol. 46, (1997), 705–718.
- [9] M. Février. *Higher order infinitesimal freeness*, arXiv.
- [10] M. Février, A. Nica. *Infinitesimal non-crossing cumulants and free probability of type B*, J. Funct. Anal. 258, 2983–2023 (2010).
- [11] F. W. King. *Hilbert transforms. Vol. 2*. Encyclopedia of Mathematics and its Applications, 125. Cambridge University Press, Cambridge, 2009
- [12] D. Shlyakhtenko. *Random Gaussian band matrices and freeness with amalgamation*. Internat. Math. Res. Notices 1996, no. 20, 1013–1025.
- [13] D. Voiculescu. *Limit laws for random matrices and free products*. Inventiones Mathematicae, 104 (1991) 201–220.

FLORENT BENAYCH-GEORGES, LPMA, UPMC UNIV PARIS 6, CASE COURIER 188, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE AND CMAP, ÉCOLE POLYTECHNIQUE, ROUTE DE SACLAY, 91128 PALAISEAU CEDEX, FRANCE.

E-mail address: florent.benaych@upmc.fr

NATHANAËL ENRIQUEZ, MODAL'X, 200 AVENUE DE LA RÉPUBLIQUE, 92001, NANTERRE AND LPMA, CASE COURIER 188, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE.

E-mail address: nenriquez@u-paris10.fr