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# PERTURBATIONS OF DIAGONAL MATRICES BY BAND RANDOM MATRICES

FLORENT BENAYCH-GEORGES AND NATHANAËL ENRIQUEZ

ABSTRACT. We exhibit an explicit formula for the spectral density of a (large) random matrix which is a diagonal matrix whose spectral density converges, perturbed by the addition of a symmetric matrix with Gaussian entries and a given (small) limiting variance profile.

## 1. PERTURBATION OF THE SPECTRAL DENSITY OF A LARGE DIAGONAL MATRIX

In this paper, we consider the spectral measure of a random matrix  $D_n^\varepsilon$  defined by  $D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n$ , for  $D_n$  a deterministic diagonal matrix whose spectral measure converges and  $X_n$  an Hermitian or real symmetric matrix whose entries are Gaussian independent variables, with a limiting variance profile (such matrices are called *band matrices*). We give a first order Taylor expansion, as  $\varepsilon \rightarrow 0$ , of the limit spectral density, as  $n \rightarrow \infty$ , of  $D_n^\varepsilon$ .

The proof is elementary and based on a formula given in [12] for the Cauchy transform of the limit spectral distribution of  $D_n^\varepsilon$  as  $n \rightarrow \infty$ .

For each  $n$ , we consider an Hermitian or real symmetric random matrix  $X_n = [x_{i,j}^n]_{i,j=1}^n$  and a real diagonal matrix  $D_n = \text{diag}(a_n(1), \dots, a_n(n))$ . We suppose that:

- (a) the entries  $x_{i,j}^n$  of  $X_n$  are independent (up to symmetry), centered, Gaussian with variance denoted by  $\sigma_n^2(i, j)$ ,
- (b) for a certain bounded function  $\sigma$  defined on  $[0, 1] \times [0, 1]$  and a certain bounded real function  $f$  defined on  $[0, 1]$ , we have, in the  $L^\infty$  topology,  
$$\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) \xrightarrow{n \rightarrow \infty} \sigma^2(x, y) \quad \text{and} \quad a_n(\lfloor nx \rfloor) \xrightarrow{n \rightarrow \infty} f(x),$$
- (c) the set of discontinuities of the function  $\sigma$  is closed and intersects a finite number of times any vertical line of the square  $[0, 1]^2$ .

For  $\varepsilon \geq 0$ , let us define, for all  $n$ ,

$$D_n^\varepsilon = D_n + \sqrt{\frac{\varepsilon}{n}}X_n.$$

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It is known, from Shlyakhtenko in [12, Th. 4.3] (see also [2], which also provides a fluctuation result), that as  $n$  tends to infinity, the spectral distribution of  $D_n^\varepsilon$  tends to a limit  $\mu_\varepsilon$  with Cauchy transform

$$C_\varepsilon(z) = \int_{x=0}^1 C_\varepsilon(x, z) dx,$$

where  $C_\varepsilon(x, \cdot)$  is defined by the fact that it is analytic, maps the upper half-plane  $\mathbb{C}^+$  into the lower one  $\mathbb{C}^-$ , and satisfies the relation

$$(1) \quad C_\varepsilon(x, z) = \frac{1}{z - f(x) - \varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy}.$$

Our goal here is to understand  $\mu_\varepsilon - \mu$  for small values of  $\varepsilon$ . Let us introduce the set  $\mathcal{T}$  of test functions we shall use here. We define

$$\mathcal{T} = \left\{ t \mapsto \frac{1}{z - t}; z \in \mathbb{C}^+ \right\}.$$

Let us now define the *Hilbert transform*, denoted by  $H[u]$ , of a function  $u$ :

$$H[u](s) := \text{p. v.} \int_{t \in \mathbb{R}} \frac{u(t)}{s - t} dt = \int_{y \in \mathbb{R}} \frac{u(s - y) - u(s)}{y} dy.$$

Before stating our main result, let us make some assumptions on the functions  $\sigma$  and  $f$ :

- (d) the push-forward  $\mu$  of the uniform measure on  $[0, 1]$  by the function  $f$  has a density  $\rho$  with respect to the Lebesgue measure on  $\mathbb{R}$ ,
- (e) there exists a symmetric function  $\tau(\cdot, \cdot)$  such that for all  $x, y$ ,  $\sigma^2(x, y) = \tau(f(x), f(y))$ ,
- (f) there exist  $\eta_0 > 0, \alpha > 0$  and  $C < \infty$  such that for almost all  $s \in \mathbb{R}$ , for all  $t \in [s - \eta_0, s + \eta_0]$ ,  $|\tau(s, t)\rho(t) - \tau(s, s)\rho(s)| \leq C|t - s|^\alpha$ .

Note that by hypothesis (f) and by the boundedness of the function  $f$ , the function

$$s \mapsto \rho(s)H[\tau(s, \cdot)\rho(\cdot)](s)$$

is well defined and compactly supported.

**Theorem 1.** *Under the hypotheses (a) to (f), as  $\varepsilon \rightarrow 0$ , for all  $g \in \mathcal{T}$ ,*

$$\int g(s) d\mu_\varepsilon(s) = \int g(s) d\mu(s) - \varepsilon \int g'(s) F(s) ds + o(\varepsilon),$$

with  $F(s) := -\rho(s)H[\tau(s, \cdot)\rho(\cdot)](s)$ .

*As a consequence, if the function  $F(\cdot)$  has bounded variations, then*

$$\mu_\varepsilon = \mu + \varepsilon dF + o(\varepsilon).$$

**Remark 1.** *Roughly speaking, this theorem states that*

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\text{spectral law}(D_n^\varepsilon) - \text{spectral law}(D_n)}{\varepsilon} = dF.$$

*It would be interesting to let  $\varepsilon$  and  $n$  tend to 0 and  $\infty$  together, and to find out the adequate rate of convergence to get a deterministic limit or non degenerated fluctuations. We are working on this question.*

**Remark 2.** *This result provides an analogue, for our random matrix model, of the following formula about real random variables (valid when  $Y$  is centered and independent of  $X$ ):*

$$\text{density}_{X+\sqrt{\varepsilon}Y}(s) = \text{density}_X(s) + \varepsilon \frac{\mathbb{E}[Y^2]}{2} \text{density}_X''(s) + o(\varepsilon).$$

**Remark 3.** *In the case where  $X_n$  is a GUE or GOE matrix, the limiting spectral distribution of  $D_n^\varepsilon$  as  $n \rightarrow \infty$  is the free convolution of the limiting spectral distribution of  $D_n$  with a semi-circle distribution. Several papers are devoted to the study of qualitative properties (like regularity) of the free convolution (see [8, 7, 4, 3, 6]). Besides, it has recently been proved that type-B free probability theory allows to give Taylor expansions, for small values of  $t$ , of the moments of  $\mu_t \boxplus \nu_t$  for two time-dependent probability measures  $\mu_t$  and  $\nu_t$  (see [5, 10, 9]). Our work differs from the ones mentioned above by the fact that we allow to perturb  $D_n$  by any band matrix, but also by the fact that it is focused on the density and not on the moments, giving an explicit formula rather than qualitative properties.*

**Proof.** For all  $z \in \mathbb{C}^+$ , we have

$$(2) \quad |C_\varepsilon(x, z)| \leq \frac{1}{\Im z}.$$

Indeed, for all  $y, z$  such that  $z \in \mathbb{C}^+$ ,  $C_\varepsilon(y, z) \in \mathbb{C}^-$ . As a consequence, the imaginary part of the denominator of the right hand term of (1) is larger than  $\Im(z)$ .

Hence by (1) and (2), as  $\varepsilon \rightarrow 0$ ,  $C_\varepsilon(x, z) \rightarrow \frac{1}{z-f(x)}$  uniformly in  $x$ .

From what precedes,

$$\begin{aligned} C_\varepsilon(x, z) - \frac{1}{z-f(x)} &= \frac{\varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy}{(z-f(x) - \varepsilon \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy)(z-f(x))} \\ &= \varepsilon \frac{1}{(z-f(x))^2} \int_{y=0}^1 \sigma^2(x, y) C_\varepsilon(y, z) dy + o(\varepsilon) \\ &= \varepsilon \frac{1}{(z-f(x))^2} \int_{y=0}^1 \frac{\sigma^2(x, y)}{z-f(y)} dy + o(\varepsilon) \end{aligned}$$

where each  $o(\varepsilon)$  is uniform in  $x \in [0, 1]$ .

But for all  $a \neq b$ ,  $\frac{1}{(z-a)^2(z-b)} = \frac{1}{(a-b)^2} \left( \frac{1}{z-b} - \frac{1}{z-a} - \frac{b-a}{(z-a)^2} \right)$ , hence since the Lebesgue measure of the set  $\{y \in [0, 1]; f(y) = f(x)\}$  is null, we have

$$\frac{1}{(z-f(x))^2} \int_{y=0}^1 \frac{\sigma^2(x, y)}{z-f(y)} dy = \int_{y=0}^1 \frac{\sigma^2(x, y)}{(f(x)-f(y))^2} \left( \frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2} \right) dy.$$

As a consequence, it follows by an integration in  $x \in [0, 1]$  that

$$C_\varepsilon(z) - C(z) = \varepsilon \int_{x=0}^1 \int_{y=0}^1 \frac{\sigma^2(x, y)}{(f(x)-f(y))^2} \left( \frac{1}{z-f(y)} - \frac{1}{z-f(x)} - \frac{f(y)-f(x)}{(z-f(x))^2} \right) dy dx + o(\varepsilon),$$

where  $C(\cdot)$  is the Cauchy transform of  $\mu$ .

Let us now recall that the push-forward of the uniform law on  $[0, 1]$  by  $f$  is the measure  $\rho(x)dx$  and that  $\sigma^2(x, y)$  can be rewritten  $\sigma^2(x, y) = \tau(f(x), f(y))$ . Hence

$$C_\varepsilon(z) - C(z) = \varepsilon \int_{s \in \mathbb{R}} \int_{t \in \mathbb{R}} \left\{ \frac{1}{z-t} - \frac{1}{z-s} - \frac{1}{(z-s)^2} (t-s) \right\} \frac{\tau(s, t)}{(s-t)^2} \rho(s) \rho(t) dt ds + o(\varepsilon).$$

This allows us to write that for any test function  $g \in \mathcal{T}$ ,

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(g) - \mu(g)}{\varepsilon} = \Lambda(g),$$

where

$$\Lambda(g) = \int_{(s,t) \in \mathbb{R}^2} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) dt ds.$$

Note that by the Taylor-Lagrange formula, for all  $s, t$ ,

$$\left| \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) \right| \leq \frac{\rho(s)\rho(t) \times \|\tau(\cdot, \cdot)\|_{L^\infty} \|g''\|_{L^\infty}}{2},$$

so that, since  $\rho$  is a density, by dominated convergence,

$$\Lambda(g) = \lim_{\eta \rightarrow 0} \int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} \{g(t) - g(s) - g'(s)(t-s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) ds dt.$$

But by symmetry, for all  $\eta > 0$ ,

$$\int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} \{g(t) - g(s)\} \frac{\tau(s,t)}{(t-s)^2} \rho(s)\rho(t) ds dt = 0.$$

As a consequence,  $\Lambda(g) = \lim_{\eta \rightarrow 0} \Lambda_\eta(g)$ , with

$$\Lambda_\eta(g) := \int_{\substack{(s,t) \in \mathbb{R}^2 \\ |s-t| > \eta}} g'(s) \frac{\tau(s,t)}{s-t} \rho(s)\rho(t) ds dt.$$

Let us prove that almost all  $s \in \mathbb{R}$ ,  $\lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt$  exists and that

$$\Lambda(g) = \int_{s \in \mathbb{R}} g'(s) \left( \lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \right) ds.$$

For  $\eta > 0$  and  $s \in \mathbb{R}$ , set

$$\theta_\eta(s) := \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt.$$

Set also  $M := \|f\|_{L^\infty}$ . Then the support of the function  $\rho$  is contained in  $[-M, M]$ , and so does the support of the function  $\theta_\eta$ , for any  $\eta > 0$ . For almost all  $s \in [-M, M]$ ,  $\lim_{\eta \rightarrow 0} \theta_\eta(s)$  exists by the formula

$$\theta_\eta(s) = \int_{t \in [s-2M, s-\eta] \cup [s+\eta, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t) - \tau(s,s)\rho(s)\rho(s)}{s-t} dt$$

and by Hypothesis (f). Moreover, for  $\eta_0$  as in Hypothesis (f),

$$\begin{aligned} |\theta_\eta(s)| &\leq 2C\rho(s) \int_{t=s+\eta}^{s+\eta_0} (s-t)^{\alpha-1} dt + \int_{t \in [s-2M, s-\eta_0] \cup [s+\eta_0, s+2M]} \frac{\tau(s,t)\rho(s)\rho(t)}{s-t} dt \\ &\leq \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{1}{\eta_0} \int_{t \in \mathbb{R}} \tau(s,t)\rho(s)\rho(t) ds dt \\ &\leq \frac{2C\rho(s)}{\alpha} (\eta_0)^\alpha + \frac{\|\tau(\cdot, \cdot)\|_{L^\infty}}{\eta_0} \rho(s). \end{aligned}$$

Hence by dominated convergence,  $\int_{s \in \mathbb{R}} g'(s) \lim_{\eta \rightarrow 0} \theta_\eta(s) ds = \lim_{\eta \rightarrow 0} \int_{s \in \mathbb{R}} g'(s) \theta_\eta(s) ds$ , i.e.

$$\Lambda(g) = \int_{s \in \mathbb{R}} g'(s) \left( \lim_{\eta \rightarrow 0} \int_{\substack{t \in \mathbb{R} \\ |s-t| > \eta}} \frac{\tau(s,t) \rho(s) \rho(t)}{s-t} dt \right) ds.$$

□

## 2. EXAMPLES

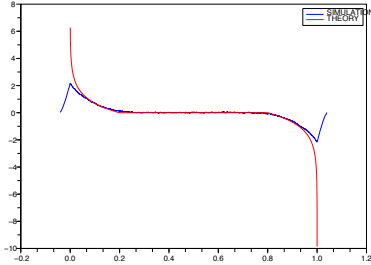
**2.1. Perturbation of a uniform distribution by a standard band matrix.** Let us consider the case where  $f(x) = x$  (so that  $\mu$  is the uniform distribution on  $[0, 1]$ ) and  $\sigma^2(x, y) = \mathbb{1}_{|y-x| \leq \ell}$ , where  $\ell$  is a fixed parameter in  $[0, 1]$  (the width of the band). In this case,  $\tau(\cdot, \cdot) = \sigma^2(\cdot, \cdot)$  and

$$F(s) = \mathbb{1}_{(0,1)}(s) \log \left( \frac{\ell \wedge (1-s)}{\ell \wedge s} \right).$$

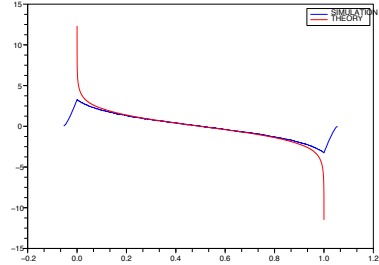
For small values of  $\varepsilon$  and large values of  $n$ , the density  $\rho_\varepsilon$  of the eigenvalue distribution  $\mu_\varepsilon$  of  $D_n^\varepsilon$  is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon) = \mathbb{1}_{(0,1)}(s) - \varepsilon \left( \frac{\mathbb{1}_{(0,\ell)}(s)}{s} + \frac{\mathbb{1}_{(1-\ell,1)}(s)}{1-s} \right) + o(\varepsilon),$$

which means that the additive perturbation  $\sqrt{\frac{\varepsilon}{n}} X_n$  alters the spectrum of  $D_n$  essentially by decreasing the amount of extreme eigenvalues. This phenomenon is illustrated by Figure 1 (where we plotted the cumulative distribution functions rather than the densities for visual reasons).



(a) Case where  $n = 4.10^3$ ,  $\varepsilon = 10^{-2}$ , with width  $\ell = 0.2$



(b) Case where  $n = 4.10^3$ ,  $\varepsilon = 10^{-2}$ , with width  $\ell = 0.9$

**FIGURE 1. Perturbation of a uniform distribution by a standard band matrix:** plot of the functions  $F(\cdot)$  and  $\frac{F_{D_n^\varepsilon}(\cdot) - F_{D_n}(\cdot)}{\varepsilon}$  (with  $F_{D_n^\varepsilon}(\cdot)$  and  $F_{D_n}(\cdot)$  the cumulative eigenvalue distribution functions of  $D_n^\varepsilon$  and  $D_n$ ) for different values of  $\ell$ .

**2.2. Perturbation of the triangular pulse distribution by a GOE matrix.** Let us consider the case where  $\rho(x) = (1 - |x|) \mathbb{1}_{[-1,1]}(x)$  and  $\sigma^2 \equiv 1$  (what follows can be adapted to the case  $\sigma^2(x, y) = \mathbb{1}_{|y-x| \leq \ell}$ , but the formulas are a bit heavy). In this case, thanks to the formula (9.6) of  $H[\rho(\cdot)]$  given p. 509 of [11], we get

$$F(s) = (1 - |s|) \mathbb{1}_{[-1,1]}(s) \{ (1 - s) \log(1 - s) - (1 + s) \log(1 + s) + 2s \log |s| \}.$$

For small values of  $\varepsilon$  and large values of  $n$ , the density  $\rho_\varepsilon$  of the eigenvalue distribution  $\mu_\varepsilon$  of  $D_n^\varepsilon$  is approximately

$$\rho_\varepsilon(s) = \rho(s) + \varepsilon \frac{\partial}{\partial s} F(s) + o(\varepsilon),$$

which implies that the additive perturbation  $\sqrt{\frac{\varepsilon}{n}} X_n$  alters the spectrum of  $D_n$  by increasing the amount of eigenvalues in  $[-1, -0.5] \cup [0.5, 1]$  and decreasing the amount of eigenvalues around zero. This phenomenon is illustrated by Figure 2.

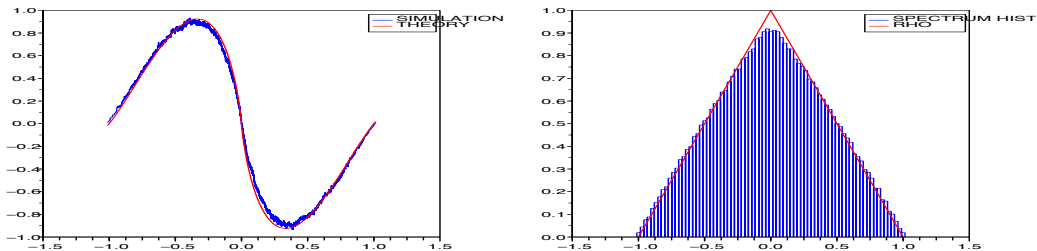


FIGURE 2. **Perturbation of the triangular pulse distribution by a GOE matrix:** *Left:* plot of the functions  $F(\cdot)$  and  $\frac{F_{D_n^\varepsilon}(\cdot) - F_{D_n}(\cdot)}{\varepsilon}$  (with  $F_{D_n^\varepsilon}(\cdot)$  and  $F_{D_n}(\cdot)$  the cumulative eigenvalue distribution functions of  $D_n^\varepsilon$  and  $D_n$ ). *Right:* plot of the eigenvalues histogram of  $D_n^\varepsilon$  and of the spectral density  $\rho$  of  $D_n$ . On the right figure, the (infinitesimal) increase of eigenvalues with respect to  $\rho$  on  $[-1, -0.5] \cup [0.5, 1]$  and the (infinitesimal) decrease around zero can be observed, in agreement with the fact that, as the left figure shows,  $F' \gg 0$  on (approximately)  $[-1, -0.5] \cup [0.5, 1]$  and  $F' \ll 0$  around zero. Both figures were made with the same simulation ( $n = 6.10^3$  and  $\varepsilon = 10^{-2}$ ).

**2.3. Free convolution with a semi-circular distribution and complex Burger's equation.** Let us consider the case where  $\sigma^2 \equiv 1$ , which happens for example if the matrix  $X_n$  is taken in the Gaussian Orthogonal Ensemble. In this case, by the theory of free probability developed by Dan Voiculescu (see e.g. [13] or [1, Cor 5.4.11 (ii)]), for all  $t \geq 0$ ,

$$\mu_t = \mu \boxplus \lambda_t,$$

where  $\lambda_t$  is the *semi-circular distribution with variance  $t$* , i.e. the distribution with support  $[-2\sqrt{t}, 2\sqrt{t}]$  and density  $\frac{1}{2\pi t} \sqrt{4t - x^2}$ . In this case, we know by the work of Biane [8, Cor. 2] that for all  $t > 0$ ,  $\mu_t$  admits a density  $\rho_t$ . By the implicit function theorem, and the formula given in [8, Cor. 2], one easily sees that the function  $(s, t) \mapsto \rho_t(s)$  is regular. Then, by Theorem 1 and the fact that the linear span of  $\mathcal{T}$  is dense in the set of continuous functions on the real line with null limit at infinity, one easily recovers the following PDE, which is a kind of projection on the real axis of the imaginary part of complex Burger's equation given in [8, Intro.]

$$(3) \quad \begin{cases} \frac{\partial}{\partial t} \rho_t(s) + \frac{\partial}{\partial s} \{ \rho_t(s) H[\rho_t(\cdot)](s) \} = 0, \\ \rho_0(s) = \rho(s). \end{cases}$$

For example, if  $\mu = \lambda_c$  for a certain  $c > 0$ , then by the semi-group property of the semi-circle distribution [1, Ex. 5.3.26], for all  $t \geq 0$ ,  $\mu_t = \lambda_{c+t}$  and  $\rho_t(s) = \frac{1}{2\pi(c+t)} \sqrt{4(c+t) - s^2}$ . One can

then verify (3), using the formula (9.21) of  $H[\rho_t(\cdot)]$  given p. 511 of [11].

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