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# Numerical analysis of an energy-like minimization method to solve a parabolic Cauchy problem with noisy data

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## Abstract

This paper is concerned with solving Cauchy problem for parabolic equation by minimizing an energy-like error functional and by taking into account noisy Cauchy data. After giving some fundamental results, numerical convergence analysis of the energy-like minimization method is carried out and leads to an adapted stopping criteria depending on noise rate for the minimization process. Numerical experiments are performed and confirm theoretical convergence order and the good behavior of the minimization process.

## 1 Introduction

The Cauchy problem considered here consists of solving a parabolic partial differential equation on a domain for which over-specified boundary conditions are given on a part of its boundary. It means to solve a data completion problem and identify the missing boundary conditions on the remaining part of the boundary. This kind of problem arises in many industrial, engineering or biomedical applications.

Since J.Hadamard's works [1], the Cauchy problem is known to be ill-posed and an important numerical instability may occur during its resolution process. It provides researchers with an interesting challenge to carry out numerical procedure approximating the solution of Cauchy problem in the particular case of noisy data. Much theoretical and applied works were proposed about this subject, using iterative methods [2], regularization methods [3, 4], quasi-reversibility methods [5] or minimal error methods [6, 7, 8].

In this paper, we focus on a method introduced in [9, 10, 11, 12, 13] based on minimization of an energy-like functional. More precisely, we introduce two distinct fields, each of them meeting only one of the over specified data. They are then solutions of two well-posed problems. Next, an energy-like error functional is introduced to measure the gap between these two fields. If it exists and is unique, the Cauchy problem solution is then obtained when the functional reaches its minimum. Then, the resolution of the ill-posed Cauchy problem is made by successive resolutions of well-posed problems. This method provides hopeful results. Nevertheless, as many other methods, it becomes unstable in the case of noisy data. To overcome this numerical instability, we propose an adequate stopping criteria parametrized by the noise rate inspired by of numerical convergence analysis. This analysis has been already performed for elliptic Cauchy problems in [14].

The outline of the paper is as follows. In section 2, we give the Cauchy problem and report classical theoretical results. In section 3, we formulate the Cauchy problem as a data completion problem and introduce the related minimization problem. In sections 4 and 5, we present finite element and time discretization, convergence analysis and study of noise effects for the minimization problem. We give then a priori error estimates taking into account data noise and propose a

stopping criteria to control instability of the minimization process. Finally, the numerical procedure and results are presented.

## 2 Statement of problem

We consider a Lipschitz bounded domain  $\Omega$  in  $\mathbb{R}^d$ ,  $d = 2, 3$  with  $n$  the outward unit normal to the boundary  $\Gamma = \partial\Omega$ . Assume that  $\Gamma$  is partitioned into two parts  $\Gamma_u$  (for *unknowns*) and  $\Gamma_m$  (for *measures*), of non-vanishing measure and such that  $\Gamma_u \cap \Gamma_m = \emptyset$ .

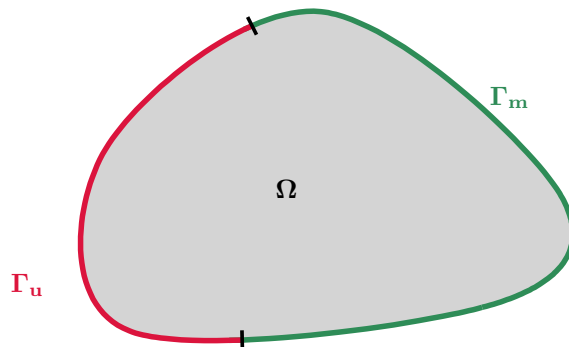


Figure 1: An example of geometry

The more common problem consists of solving the heat transfer equation in a given domain  $\Omega$  and a time interval  $[0, D]$ , assuming temperature distribution and heat flux are given over the accessible region of the boundary. We denote for  $D > 0$

$$Q = \Omega \times ]0, D[, \quad \Sigma_u = \Gamma_u \times ]0, D[, \quad \Sigma_m = \Gamma_m \times ]0, D[.$$

Given an initial temperature  $u_0$  in  $\Omega$ , a source term  $f$ , a conductivity field  $\tilde{k}$ , a density  $\rho$  and a heat capacity  $c$  in  $Q$ , a flux  $\phi$  and the corresponding temperature  $T$  on  $\Sigma_m$ , we would like to identify the corresponding flux and temperature on  $\Sigma_u$ . The nondimensionalized Cauchy problem is then written as

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (k(x)\nabla u) = f & \text{in } Q \\ k(x)\nabla u \cdot n = \phi & \text{on } \Sigma_m \\ u = T & \text{on } \Sigma_m \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \quad (1)$$

where  $k(x) = \tilde{k}(x)/\rho c$ .

A problem is well-posed in the sense of Hadamard (see [1, 15, 3]) if it fulfills the following properties : uniqueness, existence and stability of the solution. The extended Holmgren's theorem to the Sobolev spaces (see [15]) guarantees uniqueness under regularity assumptions on the solution of the Cauchy problem. The well known Cauchy-Kowalevsky theorem (see [16]) being applicable only in the case of analytic data, the existence of this solution is then caution to a verification of a compatibility condition which can hardly be explicitly formulated. This compatibility condition added to the fact that, for one fixed data, the set of compatible data is dense in the set of all data (see [17]), imply that the stability assumption is not satisfied in the sense that the dependence of the solution  $u$  of (1) on the data  $(\phi, T)$  is not continuous. Hereafter, we assume that data  $(\phi, T)$  in (1) are compatible.

*Some notations* : Let  $x$  be a generic point of  $\Omega$ . The space of squared integrable functions  $L^2(\Omega)$  is endowed by natural inner product written  $(\cdot, \cdot)_{0,\Omega}$ . The associated norm is written  $\|\cdot\|_{0,\Omega}$ . We note  $H^p(\Omega)$  the Sobolev space of functions of  $L^2(\Omega)$  for which their  $p$ -th order derivatives and lower are also in  $L^2(\Omega)$ . Its norm and semi norm are written  $\|\cdot\|_{p,\Omega}$  and  $|\cdot|_{p,\Omega}$  respectively. Moreover, let

$u = (u_1, u_2) \in (H^p(\Omega))^2$ , the semi-norm of this space is written  $\|u\|_{p,\Omega} = (|u_1|_{p,\Omega}^2 + |u_2|_{p,\Omega}^2)^{1/2}$ . Let  $\gamma \subset \Gamma$ , we define the space  $H_{0,\gamma}^1(\Omega) = \{v \in H^1(\Omega); v|_\gamma = 0\}$  and  $H_{00}^{1/2}(\gamma)$  is the space of restrictions to  $\gamma$  of the functions of  $H^{1/2}(\Omega) = \text{tr}(H^1(\Omega))$ . Its topological dual is written  $H_{00}^{-1/2}(\gamma) = (H_{00}^{1/2}(\gamma))'$ . The associated norms are written  $\|\cdot\|_{1/2,00,\gamma}$  and  $\|\cdot\|_{-1/2,00,\gamma}$  respectively and  $\langle \cdot, \cdot \rangle_{1/2,00,\gamma}$  states for the duality inner product. Now, let  $t$  be the time variable. We denote by  $L^2(0, D; F)$  the space of squared integrable functions in  $[0, D]$  with values in  $F$ , where  $F$  is a normed functional space. In the same way,  $\mathcal{C}^n(0, D; F)$  points out the space of  $n$  times continuously derivable functions in  $[0, D]$  with values in  $F$ . We write  $\mathcal{D}'(]0, D[)$  the space of distributions in  $]0, D[$ . In the sequel  $C$ , points out a positive generic constant.

### 3 Energy-like minimization method

Let  $f \in L^2(0, D; L^2(\Omega))$ ,  $k(x) \in L^\infty(\Omega)$  positive,  $\phi \in L^2(0, D; H_{00}^{-1/2}(\Gamma_m))$  and  $T \in L^2(0, D; H_{00}^{1/2}(\Gamma_m))$ . The Cauchy problem can be written as a data completion problem :

Find  $(\varphi, \xi) \in L^2(0, D; H_{00}^{-1/2}(\Gamma_u) \times H_{00}^{1/2}(\Gamma_u))$  such that there exists  $u \in L^2(0, D; H^1(\Omega))$  solution of

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (k(x)\nabla u) = f \text{ in } Q \\ u = T, \quad k(x)\nabla u \cdot n = \phi \text{ on } \Sigma_m \\ u = \xi, \quad k(x)\nabla u \cdot n = \varphi \text{ on } \Sigma_u \\ u(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases} \quad (2)$$

**Remark 1** : We note that in case  $\bar{\Gamma}_u \cap \bar{\Gamma}_m = \emptyset$ , as given in figure 2 illustrating the ring numerical tests of section 6.2, the spaces  $H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$  and  $H^{-1/2}(\Gamma_m) \times H^{1/2}(\Gamma_m)$  for unknowns and data respectively would be more appropriate. Nevertheless, the general functional framework is not restrictive because the spaces  $H_{00}^s(\Gamma_u)$  and  $H_{00}^s(\Gamma_m)$  are dense in  $H^s(\Gamma_u)$  and  $H^s(\Gamma_m)$  for  $s = \pm 1/2$ , respectively.

Following [12], we introduce now two distinct fields  $u_1$  and  $u_2$  solution of well posed problems which differ by their boundary conditions. We attribute to each of them one data on  $\Sigma_m$  and one unknown on  $\Sigma_u$ . Then, we have

$$\begin{cases} \frac{\partial u_1}{\partial t} - \nabla \cdot (k(x)\nabla u_1) = f \text{ in } Q \\ u_1 = T \text{ on } \Sigma_m \\ k(x)\nabla u_1 \cdot n = \eta \text{ on } \Sigma_u \\ u_1(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \quad (3) \quad \begin{cases} \frac{\partial u_2}{\partial t} - \nabla \cdot (k(x)\nabla u_2) = f \text{ in } Q \\ u_2 = \tau \text{ on } \Sigma_u \\ k(x)\nabla u_2 \cdot n = \phi \text{ on } \Sigma_m \\ u_2(\cdot, 0) = u_0 \text{ in } \Omega. \end{cases} \quad (4)$$

We denote  $a_i(\cdot, \cdot)$  and  $l_i(\cdot)$ ,  $i = 1, 2$  the bilinear and linear forms associated to the weak forms of the problems (3) and (4) respectively. They are given by

$$a_i(\tilde{u}_i(t), v) = \int_{\Omega} k(x)\nabla \tilde{u}_i(t)\nabla v \, dx, \quad \text{for } i = 1, 2, \quad (5)$$

$$l_1(v; t) = \int_{\Omega} f(t)v \, dx - \frac{d}{dt}(\bar{u}_1(t), v_1)_H - a_1(\bar{u}_1(t), v) + \langle \eta(t), v \rangle_{1/2,00,\Gamma_u}, \quad (6)$$

$$l_2(v; t) = \int_{\Omega} f(t)v \, dx - \frac{d}{dt}(\bar{u}_2(t), v_1)_H - a_2(\bar{u}_2(t), v) + \langle \phi(t), v \rangle_{1/2,00,\Gamma_m}, \quad (7)$$

where  $\bar{u}_1(t)$  and  $\bar{u}_2(t)$  are the lifting of the extended Dirichlet conditions  $T(t)$  and  $\tau(t)$  respectively

and  $\tilde{u}_i = u_i - \bar{u}_i$ ,  $i = 1, 2$ . We have by summation the following weak problem :

$$\begin{aligned} \text{Find } u &= (\tilde{u}_1, \tilde{u}_2) \in L^2(0, D; V) \cap \mathcal{C}^0(0, D; H) \text{ such that} \\ &\frac{d}{dt}(u(t), v)_H + a(u(t), v) = L(v; t), \quad \forall v = (v_1, v_2) \in V \text{ in } \mathcal{D}'([0, D]) \\ &u(\cdot, 0) = u_{00} = (\tilde{u}_{10}, \tilde{u}_{20}), \\ \text{with } a(u(t), v) &= a_1(\tilde{u}_1(t), v_1) + a_2(\tilde{u}_2(t), v_2), \\ \text{and } L(v; t) &= l_1(v_1; t) + l_2(v_2; t), \end{aligned} \quad (8)$$

where  $H = (L^2(\Omega))^2$  endowed with the scalar product  $(u, v)_H = ((u_1, v_1)_{0, \Omega}^2 + (u_2, v_2)_{0, \Omega}^2)^{1/2}$ ,  $V = H_{0, \Gamma_m}^1(\Omega) \times H_{0, \Gamma_u}^1(\Omega)$  and  $\|v\|_V = (\|v_1\|_{1, \Omega}^2 + \|v_2\|_{1, \Omega}^2)^{1/2}$  is the norm associated to the space  $V$ . It is easy to show that the linear form  $L(\cdot)$  is continuous and that the bilinear form  $a(\cdot, \cdot)$  is continuous and  $V$ -elliptic. Then, by a theorem due to J.L. Lions (see [18]), the weak problem (8) admits a unique solution.

We consider now the following energy-like functional :

$$E(\eta, \tau) = \int_0^D \int_{\Omega} k(x) (\nabla u_1(\eta, t) - \nabla u_2(\tau, t))^2 dx dt + \frac{1}{2} \int_{\Omega} (u_1(\eta, D) - u_2(\tau, D))^2 dx, \quad (9)$$

and the following minimization problem :

$$\left\{ \begin{array}{l} (\eta^*, \tau^*) = \underset{(\eta, \tau) \in \mathcal{U}}{\operatorname{argmin}} E(\eta, \tau), \quad \mathcal{U} = L^2(0, D; H_{00}^{-1/2}(\Gamma_u) \times H_{00}^{1/2}(\Gamma_u)), \\ \text{with } u_1 \text{ and } u_2 \text{ solutions of (3) and (4) respectively.} \end{array} \right. \quad (10)$$

Using convexity of the space  $\mathcal{U}$ , existence and uniqueness of the Cauchy problem solution in the case of compatible data, we can prove that the solution  $(\eta^*(t), \tau^*(t))$  of the minimization problem (10), if it exists and is unique, is solution of the data completion problem (*i.e.*  $(\eta^*, \tau^*) = (\varphi, \xi)$ ).

## 4 Discretization method and error estimation

Let  $X_h$  be the finite element space for which the following classical assumptions are verified :

- (i)  $\Omega$  is polyhedral domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ .
- (ii)  $\mathcal{T}_h$  is a regular triangulation of  $\bar{\Omega}$  *i.e.*  $h = \max_{K \in \mathcal{T}_h} h_K \rightarrow 0$  and  $\max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \leq c$  being a constant independent on  $h$ ,  $h_K$  the element  $K$  diameter and  $\rho_K$  the  $K$  inscribed circle diameter.
- (iii)  $\Gamma_u$  and  $\Gamma_m$  can be written exactly as union of faces of some finite elements  $K \in \mathcal{T}_h$ .
- (iv) The family  $(K, P_K, \Sigma_K)$ ,  $K \in \mathcal{T}_h$  for all  $h$  is affine-equivalent to a reference finite element  $(\hat{K}, \hat{P}, \hat{\Sigma})$  of  $\mathcal{C}^0$  regularity.
- (v) The following inclusion is satisfied :  $P_l(\hat{K}) \subset \hat{P} \subset H^1(\hat{K})$  for  $l \geq 1$ .

These assumptions imply that  $X_h \subset H^1(\Omega)$ . We define the following spaces :

$$\begin{aligned} X_{uh} &= \{v_h \in X_h; v_h|_{\Gamma_u} = 0\}, \\ X_{mh} &= \{v_h \in X_h; v_h|_{\Gamma_m} = 0\}, \end{aligned}$$

and  $V_h = X_{mh} \times X_{uh} \subset V$  the finite dimensional approximation space.

Given  $u_{0h} \in V_h$  the  $V_h$ -interpolation of  $u_{00}$ , the semi-discrete problem associated with (8) is written

$$\begin{aligned} \text{Find } u_h(t) &= (u_{1h}(t), u_{2h}(t)) \in L^2(0, D; V_h) \text{ such that} \\ &\frac{d}{dt}(u_h(t), v_h)_H + a(u_h(t), v_h) = L(v_h; t), \quad \forall v_h = (v_{1h}, v_{2h}) \in V_h \text{ in } \mathcal{D}'([0, D]) \\ &u_h(0) = u_{0h}. \end{aligned} \quad (11)$$

We now turn to the time discretization. We introduce the time step  $\Delta t$  and the time  $t_n = n\Delta t$ ,  $0 \leq n \leq N$ , and denote by  $u_h^n \in V_h$  the approximation of  $u(\cdot, t_n)$ . It gives the discrete problem based on the backward Euler scheme :

Find  $\{u_h^n \in V_h; 0 \leq n \leq N\}$  such that

$$\begin{aligned} \frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h)_H + a(u_h^{n+1}, v_h) &= L(v_h; t_{n+1}), \quad 0 \leq n \leq N-1, \quad \forall v_h \in V_h \\ u_h^0 &= u_{0h}. \end{aligned} \quad (12)$$

The same argument as for the weak problem (8) provides existence and uniqueness of the solutions of (11) and (12).

Using standard procedure described in [19], we report the following error estimate :

**Proposition 4.1** *In addition to the assumptions stated above, assume that there exists an integer  $l \geq 1$  such that the following inclusion is satisfied :*

$$H^{l+1}(\hat{K}) \subset \mathcal{C}^s(\hat{K}) \text{ with continuous injection} \quad (13)$$

where  $s$  is the maximal order of partial derivatives occurring in the definition of the set  $\hat{\Sigma}$ .

Then, if the solution  $u$  of the variational problem (8) also verifies  $u(t) \in (H^{l+1}(\Omega))^2$  for all  $t \in [0, D]$ , there exists a constant  $C$  independent on  $h$  and  $\Delta t$  such that

$$\begin{aligned} \|u(t_n) - u_h^n\|_V \leq C \left\{ h^l \left( \|u_{00}\|_{l+1, \Omega} + \left( \int_0^{t_n} \left\| \frac{du}{dt}(s) \right\|_{l+1, \Omega}^2 ds \right)^{1/2} \right) + \right. \\ \left. + \Delta t \left( \int_0^{t_n} \left\| \frac{d^2u}{dt^2}(s) \right\|_{l+1, \Omega}^2 ds \right)^{1/2} \right\}, \quad (14) \end{aligned}$$

where  $\{u_h^n \in V_h, 0 \leq n \leq N\}$  is the discrete solution.

## 5 Noisy data, error estimates and stopping criterion

### 5.1 Error estimates and data noise effects

In the case of given perturbed data, say  $(\phi^\delta, T^\delta)$ , problem (12) writes as

Find  $\{u_{h\delta}^n \in V_h; 0 \leq n \leq N\}$  such that

$$\begin{aligned} \frac{1}{\Delta t}(u_{h\delta}^{n+1} - u_{h\delta}^n, v_h)_H + a(u_{h\delta}^{n+1}, v_h) &= L^\delta(v_h; t_{n+1}), \quad 0 \leq n \leq N-1, \quad \forall v_h \in V_h \\ u_{h\delta}^0 &= u_{0h}, \end{aligned} \quad (15)$$

where  $L^\delta(\cdot)$  is given by the expression of  $L(\cdot)$  with  $(\phi, T)$  replaced by  $(\phi^\delta, T^\delta)$ .

**Proposition 5.1** *Under assumptions of proposition 4.1, if the solution  $u$  of the variational problem (8) also verify  $u(t) \in (H^{l+1}(\Omega))^2$  for all  $t \in [0, D]$ , then there exist a constant  $C$  independent on  $h$ ,  $\Delta t$  and data such that*

$$\begin{aligned} \|u(t_n) - u_{h\delta}^n\|_V \leq C \left\{ h^l \left( \|u_{00}\|_{l+1, \Omega} + \left( \int_0^{t_n} \left\| \frac{du}{dt}(s) \right\|_{l+1, \Omega}^2 ds \right)^{1/2} \right) + \right. \\ \left. + \Delta t \left( \int_0^{t_n} \left\| \frac{d^2u}{dt^2}(s) \right\|_{l+1, \Omega}^2 ds \right)^{1/2} + \sqrt{\Delta t} \left( \sum_{j=1}^n \left\| \frac{d(T - T^\delta)}{dt}(t_j) \right\|_{1/2, 0, \Gamma_m} + \frac{1}{h} \|\delta(t_j)\| \right) \right\}, \quad (16) \end{aligned}$$

where

$$\|\delta(t)\| = \left( \|T(t) - T^\delta(t)\|_{1/2, 0, \Gamma_m}^2 + \|\phi(t) - \phi^\delta(t)\|_{-1/2, 0, \Gamma_m}^2 \right)^{1/2} \quad (17)$$

is the data noise norm and  $\{u_{h\delta}^n \in V_h; 0 \leq n \leq N\}$  is the solution of the discrete problem (15) associated to the noisy Cauchy problem.

**Proof** We write  $u(t_n) - u_{h\delta}^n = u(t_n) - u_h^n + u_h^n - u_{h\delta}^n = \rho^n + \theta_\delta^n$ . An estimation for  $\|\rho^n\|_V$  is immediately given by the proposition 4.1.

From (12) and (15), we have

$$\frac{1}{\Delta t}(\theta_\delta^{n+1} - \theta_\delta^n, v_h)_H + a(\theta_\delta^{n+1}, v_h) = L^\delta(v_h; t_{n+1}) - L(v_h; t_{n+1}), \quad \forall v_h \in V_h. \quad (18)$$

There exists  $\omega_\delta^n \in V_h$  defined by

$$\begin{aligned} (\omega_\delta^n, v_h)_H &= L^\delta(v_h; t_{n+1}) - L(v_h; t_{n+1}) = \left( \frac{d(\bar{u} - \bar{u}_\delta)}{dt}(t_n), v_h \right)_H + \\ &\quad + a(\bar{u}(t_n) - \bar{u}_\delta(t_n), v_h) + \langle \phi(t_n) - \phi^\delta(t_n), v_{2h} \rangle_{-1/2, 00, \Gamma_m}, \end{aligned} \quad (19)$$

where  $\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t))$  and  $\bar{u}_\delta(t) = (\bar{u}_1^\delta(t), \bar{u}_2^\delta(t))$  which are the lifting of boundary conditions with noisy data. Using trace and lifting operators properties and inverse inequalities, we prove that

$$\|\omega_\delta^n\|_H \leq C \left( \left\| \frac{d(T - T^\delta)}{dt}(t_n) \right\|_{1/2, 00, \Gamma_m} + \frac{1}{h} \|\delta(t_n)\| \right). \quad (20)$$

Choosing  $v_h = \frac{\theta_\delta^{n+1} - \theta_\delta^n}{\Delta t}$  in (18), it gives

$$\|\theta_\delta^n\|_V^2 \leq C \Delta t \sum_{j=1}^n \|\omega_\delta^j\|_H^2. \quad (21)$$

and then, with (20),

$$\|\theta_\delta^n\|_V^2 \leq C \Delta t \sum_{j=1}^n \left( \left\| \frac{d(T - T^\delta)}{dt}(t_n) \right\|_{1/2, 00, \Gamma_m} + \frac{1}{h} \|\delta(t_n)\| \right)^2. \quad (22)$$

Therefore, using (20) and (22) we have an estimation of  $\|\theta_\delta^{n+1}\|_V$  that leads to (16).  $\blacksquare$

## 5.2 Stopping criterion for the minimization process

When noise is introduced on the Cauchy data, we observe during the optimization process that the error reaches a minimum before increasing very fast and leading to a numerical explosion. At the same time, the energy-like functional attains asymptotically a minimal threshold, which is a strictly positive constant depending on the noise. Notice that this constant vanishes in case of compatible Cauchy data. Now, the aim is to theoretically determine this threshold in order to propose a stopping criterion depending on the noise rate, which allows to stop the minimization process just before numerical explosion.

We introduce a general quadrature formula with specified nodes and weights  $(t_j, \alpha_j)$  to approximate the integral of a continuous function  $f$  on the time interval,

$$I_N(f) = \sum_{j=0}^N \alpha_j f(t_j) \sim \int_0^D f(t) dt. \quad (23)$$

The noisy discrete functional is then given by

$$E_h^\delta(\eta, \tau) = \sum_{j=0}^N \alpha_j \int_\Omega k(x) \left( \nabla u_{1h\delta}^j(\eta) - \nabla u_{2h\delta}^j(\tau) \right)^2 dx + \frac{1}{2} \int_\Omega (u_{1h\delta}^N(\eta) - u_{2h\delta}^N(\tau))^2 dx. \quad (24)$$

**Proposition 5.2** *Under assumptions of proposition 4.1, if the solution  $u$  of the variational problem (8) also verifies  $u(t) \in (H^{l+1}(\Omega))^2$  for all  $t \in [0, D]$  and if  $(\eta^*, \tau^*)$  is the solution of the minimization problem (10), then there exists a constant  $C$  independent on  $h$  and data such that*

$$E_h^\delta(\eta^*, \tau^*) \leq C \left\{ h^{2l} \sum_{n=0}^N \alpha_n \left( \|u_0\|_{l+1, \Omega} + \left( \int_0^{t_n} \left\| \frac{du}{dt}(s) \right\|_{l+1, \Omega}^2 ds \right)^{1/2} \right)^2 + \right. \\ \left. + \Delta t^2 \left( \sum_{n=0}^N \alpha_n \int_0^{t_n} \left\| \frac{d^2u}{dt^2}(s) \right\|_{l+1, \Omega}^2 ds \right) + \right. \\ \left. \Delta t \sum_{n=0}^N \alpha_n \left( \sum_{j=1}^n \left\| \frac{d(T - T^\delta)}{dt}(t_j) \right\|_{1/2, 00, \Gamma_m} + \frac{1}{h} \|\delta(t_j)\| \right)^2 \right\}. \quad (25)$$

**Proof** Let  $(\eta^*, \tau^*)$  be the solution of the minimization problem (10) with compatible Cauchy data. After some algebraic operations and taking into account the fact that  $u_1(\eta^*; t) = u_2(\tau^*; t)$ ,  $\forall t \in [0, D]$ , we can write

$$E_h^\delta(\eta^*, \tau^*) = \sum_{j=0}^N \alpha_j \int_{\Omega} k(x) \left[ \left( \nabla u_{1h\delta}^j(\eta^*) - \nabla u_1(\eta^*; t_j) \right) - \left( \nabla u_{2h\delta}^j(\tau^*) - \nabla u_2(\tau^*; t_j) \right) \right]^2 dx + \\ + \frac{1}{2} \int_{\Omega} \left[ (u_{1h\delta}^N(\eta^*) - u_1(\eta^*; t_N)) - (u_{2h\delta}^N(\tau^*) - u_2(\tau^*; t_N)) \right]^2 dx. \quad (26)$$

It follows that

$$E_h^\delta(\eta^*, \tau^*) \leq 2\|k\|_{L^\infty(\Omega)} \sum_{j=0}^N \alpha_j \left( |u_{1h\delta}^j(\eta^*) - u_1(\eta^*; t_j)|_{1, \Omega}^2 + |u_{2h\delta}^j(\tau^*) - u_2(\tau^*; t_j)|_{1, \Omega}^2 \right) + \\ + (\|u_{1h\delta}^N(\eta^*) - u_1(\eta^*; t_N)\|_{0, \Omega}^2 + \|u_{2h\delta}^N(\tau^*) - u_2(\tau^*; t_N)\|_{0, \Omega}^2), \quad (27)$$

and then, there exists a constant  $C$  that may depend on  $\Delta t$ , such that

$$E_h^\delta(\eta^*, \tau^*) \leq C \sum_{j=0}^N \alpha_j \|u_{h\delta}^j(\eta^*, \tau^*) - u(\eta^*, \tau^*; t_j)\|_V^2. \quad (28)$$

Therefore, using proposition 5.1, we derive (25).  $\blacksquare$

Nevertheless, at each time step except the last, the error semi-norm in  $V$  is majorized by the error norm in  $V$ . Then, we have to take into account this overestimation to be more precise. We prove that

$$\|u_{h\delta}^n(\eta^*, \tau^*) - u(t_n; \eta^*, \tau^*)\|_H^2 \geq \frac{1}{2} \|u_{1h\delta}^n(\eta^*) - u_{2h\delta}^n(\tau^*)\|_{0, \Omega}^2 \quad (29)$$

Hence, when the noisy discrete functional (24) reaches its minimum and for  $h$  and  $\Delta t$  sufficiently small, we have by (25) and (29),

$$E_h^\delta(\eta^*, \tau^*) \sim O(S_\delta(\eta^*, \tau^*)), \quad (30)$$

where

$$S_\delta(\eta^*, \tau^*) = \Delta t \sum_{n=0}^N \alpha_n \left( \left( \sum_{j=1}^n \left\| \frac{d(T - T^\delta)}{dt}(t_j) \right\|_{1/2, 00, \Gamma_m} + \frac{1}{h} \|\delta(t_j)\| \right)^2 - \frac{1}{2} \|u_{1h\delta}^n(\eta^*) - u_{2h\delta}^n(\tau^*)\|_{0, \Omega}^2 \right). \quad (31)$$



In order to propose a stopping criterion based on these theoretical estimates, let us denote by  $(\eta^j, \tau^j)$  the unknowns and  $E_j$  the value of the discrete noisy functional at the  $j$ -th iteration.

At first, the stopping criterion relies on verify the noisy discrete functional has reached  $S_\delta$ . Taking into account the asymptotical behavior of the functional, we want to stop the optimization algorithm when the functional variations become lower than the functional itself, and then lower than  $S_\delta$ . Moreover, the ratio  $E_j/E_{j-1} < 1$  tends to 1. Then, multiplying  $S_\delta$  by this ratio, the threshold  $S_\delta$  is weakened before the asymptote is reached by the functional. Then, a consistent stopping criterion, based on the described behavior of  $E_h^\delta(\cdot, \cdot)$  and the estimate (25), could be

$$\max\{E_j, |E_j - E_{j-1}|\} \leq \frac{E_j}{E_{j-1}} S_\delta(\eta_j, \tau_j). \quad (32)$$

## 6 Numerical issues

### 6.1 Numerical procedure

Let us describe the calculation method of the required elements for the optimization procedure, specifically the gradient of the functional. Assume that the triangulation  $\mathcal{T}_h$  of  $\Omega$  is characterized by  $n$  nodes. Let  $p$  and  $q$  denote the number of nodes on the boundaries  $\Gamma_u$  and  $\Gamma_m$  respectively and  $(\omega_i)_{1 \leq i \leq n} = (\omega_{1i}, \omega_{2i})_{1 \leq i \leq n}$  the canonical basis of  $V_h$ . We write  $X_\eta$  and  $X_\tau$  the unknowns. The vectors  $U_1^n$  and  $U_2^n$  correspond to the fields  $u_1(t_n)$  and  $u_2(t_n)$  respectively, the vector  $U_T^n$  corresponds to the Dirichlet data  $T(t_n)$ . We introduce the following notations,  $(K_1)_{kl} = a_1(\omega_{1k}, \omega_{1l})$ ,  $(K_2)_{kl} = a_2(\omega_{2k}, \omega_{2l})$ ,  $(F_1^n)_k = l_1(\omega_{1k}; t_n)$  and  $(F_2^n)_k = l_2(\omega_{2k}; t_n)$  depending on the Neumann data  $\phi(t_n)$ . The bi-linear forms being similar, we note  $K = K_1 = K_2$ .

The linear systems associated with (3) and (4) respectively are given by :

$$\left\{ \begin{array}{l} \left(\frac{M}{\Delta t} + K\right) U_1^{n+1} = F_1^{n+1} + \frac{M}{\Delta t} U_1^n \\ L_m U_1^{n+1} = U_T^{n+1} \\ U_1^0 = U_0, \end{array} \right. \quad (33) \quad \left\{ \begin{array}{l} \left(\frac{M}{\Delta t} + K\right) U_2^{n+1} = F_2^{n+1} + \frac{M}{\Delta t} U_2^n \\ L_u U_2^{n+1} = X_\tau^{n+1} \\ U_2^0 = U_0. \end{array} \right. \quad (34)$$

The functional can be written as follows :

$$E(X_\eta, X_\tau) = \frac{1}{2} \sum_{n=0}^N \alpha_n (U_1^n - U_2^n)^t K (U_1^n - U_2^n) + (U_1^N - U_2^N)^t M (U_1^N - U_2^N) \quad (35)$$

We want to calculate the functional derivatives with respect to each component with index  $i$  and at each time step  $k$  of the two unknowns  $X_\eta$  and  $X_\tau$  written  $X_\eta^{i,k}$  and  $X_\tau^{i,k}$  respectively. First, we derive the functional with respect to the unknown  $X_\eta$ .

$$\frac{\partial E}{\partial X_\eta^{i,k}}(X_\eta, X_\tau) = 2 \sum_{n=k}^N \alpha_n \left( \frac{\partial U_1^n}{\partial X_\eta^{i,k}} \right)^t K (U_1^n - U_2^n) + 2\delta_k^N \left( \frac{\partial U_1^N}{\partial X_\eta^{i,k}} \right)^t M (U_1^N - U_2^N), \quad k = 1, \dots, N. \quad (36)$$

In the sequel, we denote by  $U_{1,i,k}^n$  the derivative of  $U_1^n$  with respect to  $X_\eta^{i,k}$ . Deriving the linear system (33),  $U_{1,i,k}^{n+1}$  is solution of :

$$\left(\frac{M}{\Delta t} + K\right) U_{1,i,k}^{n+1} = \frac{\partial F_1^{n+1}}{\partial X_\eta^{i,k}} + \frac{M}{\Delta t} U_{1,i,k}^n, \quad (37)$$

$U_T^n$  and  $U_0$  being independent on  $X_\eta^{n,k}$ , their derivatives vanishe. Moreover,

$$U_{1,i,k}^n = 0 \text{ if } k > n \text{ and } \frac{\partial F_1^n}{\partial X_\eta^{i,k}} = 0 \text{ if } k \leq n.$$

We note  $\tilde{F}_{1,i}^n = \frac{\partial F_1^n}{\partial X_\eta^{i,n}} = (\delta_i^j)_{1 \leq j \leq m}$ . We have then

$$\begin{cases} \left( \frac{M}{\Delta t} + K \right) U_{1,i,k}^{n+1} = \delta_{n+1}^k \tilde{F}_{1,i}^{n+1} + \mathbb{1}_{\{k > n+1\}} \frac{M}{\Delta t} U_{1,i,k}^n \\ L_m U_{1,i,k}^{n+1} = 0 \\ U_{1,i,k}^0 = 0, \quad 1 \leq i \leq m, \quad 0 \leq n \leq N-1, \quad 1 \leq k \leq N. \end{cases} \quad (38)$$

Now, we derive the functional with respect to the unknown  $X_\tau$ .

$$\frac{\partial E}{\partial X_\tau^{i,k}}(X_\eta, X_\tau) = - \sum_{n=k}^N \alpha_n \left( \frac{\partial U_2^n}{\partial X_\tau^{i,k}} \right)^t K(U_1^n - U_2^n) - 2\delta_k^N \left( \frac{\partial U_2^N}{\partial X_\tau^{i,k}} \right)^t M(U_1^N - U_2^N), \quad k = 1, \dots, N. \quad (39)$$

In the sequel, we denote by  $U_{2,i,k}^{n+1}$  the derivative of  $U_2^n$  with respect to  $X_\tau^{i,k}$ . Deriving the linear system (34),  $U_{2,i,k}^{n+1}$  is solution of

$$\left( \frac{M}{\Delta t} + K \right) U_{2,i,k}^{n+1} = \frac{\partial F_2^{n+1}}{\partial X_\tau^{i,k}} + \frac{M}{\Delta t} U_{2,i,k}^n, \quad (40)$$

$F_2^n$  and  $U_0$  being independent on  $X_\tau^{n,k}$ , their derivatives vanish. Moreover,

$$U_{2,i,k}^n = 0 \text{ if } k > n \text{ and } \frac{\partial X_\tau^n}{\partial X_\tau^{i,k}} = 0 \text{ if } k \neq n.$$

We note  $\tilde{X}_{\tau,i}^n = \frac{\partial X_\tau^n}{\partial X_\tau^{i,n}} = (\delta_i^j)_{1 \leq j \leq m}$ . We have then

$$\begin{cases} \left( \frac{M}{\Delta t} + K \right) U_{2,i,k}^{n+1} = \mathbb{1}_{\{k > n+1\}} \frac{M}{\Delta t} U_{2,i,k}^n \\ L_u U_{2,i,k}^{n+1} = \tilde{X}_{\tau,i}^{n+1} \\ U_{2,i,k}^0 = 0, \quad 1 \leq i \leq m, \quad 0 \leq n \leq N-1, \quad 1 \leq k \leq N. \end{cases} \quad (41)$$

We consider here the case of real applications where we have only measured and noisy data  $(T^\delta, \phi^\delta)$  given with a noise rate  $0 < a < 1$ . We are then not able to calculate exactly the norm of the difference between the exact and noisy data which are involved in the stopping criterion (32). We have therefore to estimate these norm. We have :

$$T(x, t) - aT(x, t) \leq T^\delta(x, t) \leq T(x, t) + aT(x, t), \quad \forall x \in \Gamma_m \quad (42)$$

$$\iff \frac{-a}{1-a} T^\delta(x, t) \leq T(x, t) - T^\delta(x, t) \leq \frac{a}{1+a} T^\delta(x, t) \quad (43)$$

$$\text{and then } \|T(t) - T^\delta(t)\|_{1/2, 0, \Gamma_m}^2 \leq \max \left\{ \frac{a}{1-a}, \frac{a}{1+a} \right\} \|T^\delta(t)\|_{1/2, 0, \Gamma_m}^2 \quad (44)$$

Proceeding by the same way for the Neumann data and the time derivative of Dirichlet data, we have :

$$\|\delta(t)\| \leq \frac{a}{1-a} \left( \|T^\delta(t)\|_{1/2, 0, \Gamma_m}^2 + \|\phi^\delta(t)\|_{-1/2, 0, \Gamma_m}^2 \right)^{1/2} \quad (45)$$

and

$$\left\| \frac{d(T - T^\delta)}{dt}(t) \right\|_{1/2, 0, \Gamma_m} \leq \frac{a}{1-a} \left\| \frac{dT^\delta}{dt}(t) \right\|_{1/2, 0, \Gamma_m}. \quad (46)$$

The stopping criterion (32) can then be written as follows :

$$\begin{aligned} \max\{E_j, |E_j - E_{j-1}|\} &\leq \frac{E_j a^2 \Delta t}{E_{j-1} (1-a)^2} \sum_{n=1}^N \alpha_n \left( \sum_{k=1}^n \left\| \frac{dT^\delta}{dt}(t_k) \right\|_{1/2, 0, \Gamma_m} \right. \\ &\quad \left. + \frac{1}{h} \left( \|T^\delta(t_k)\|_{1/2, 0, \Gamma_m}^2 + \|\phi^\delta(t_k)\|_{-1/2, 0, \Gamma_m}^2 \right) \right)^2 \quad (47) \end{aligned}$$

## 6.2 Numerical results

We consider the following Cauchy problem on the domain  $\Omega$  given by figure (2) :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 \text{ in } \Omega \times ]0, 1[ \\ u = g \text{ on } \Gamma_m \times ]0, 1[ \\ \nabla u \cdot n = h \text{ on } \Gamma_u \times ]0, 1[ \\ u(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \quad (48)$$

where  $g$ ,  $h$  and  $u_0$  are the Cauchy data extracted from the exact solution which we intend to approximate.

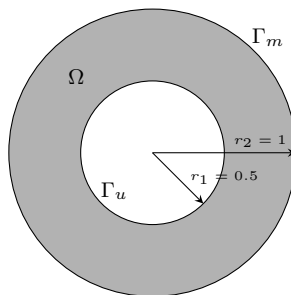


Figure 2: Ring

### 6.2.1 Axisymmetric example

The Cauchy problem is written in polar coordinates  $(\rho, \theta)$  and we suppose that its solution does not depend on the angular coordinate. The state equation of (48) is then

$$\frac{\partial u(\rho, t)}{\partial t} - \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) u(\rho, t) = 0. \quad (49)$$

An analytic solution of this equation is given by  $u(\rho, t) = e^{-t} J_0(\rho)$  where  $J_0(\cdot)$  is the Bessel function of first kind of order 0.

Figure 3 represents the finite element discretization error with respect to the maximum edge size of the mesh. We choose  $\Delta t$  sufficiently small such that time discretization is negligible. In the same way, figure 4 shows the time discretization error with respect to the time step considering  $h$  sufficiently small. These results are in agreement with theoretical error estimate (14).

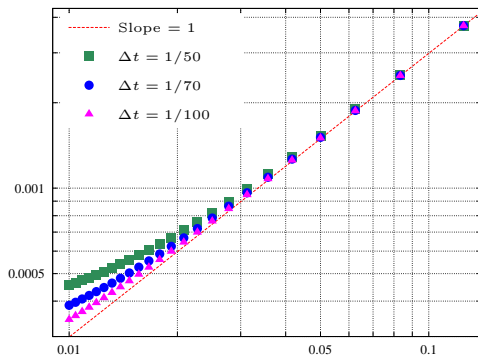


Figure 3: Evolution of  $\|u(t_N) - u_h^N\|_V$  with respect to  $h$  for different  $\Delta t$  and for the axisymmetric example.

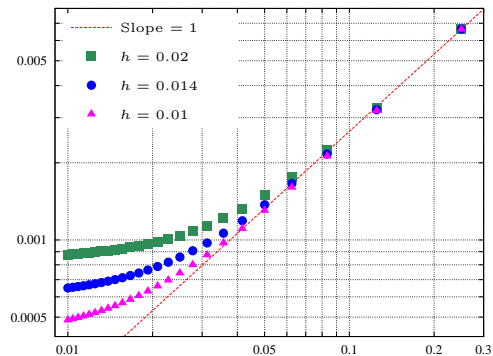


Figure 4: Evolution of  $\|u(t_N) - u_h^N\|_V$  with respect to  $\Delta t$  for different  $h$  and for the axisymmetric example.

### 6.2.2 Two dimensional example

Now, we consider the resolution of the Cauchy problem (48) in two dimensions. An analytic solution of this problem is given by  $u(x, y; t) = e^{-2t} \cos(x + y)$  which provides Cauchy data on  $\Gamma_m$ .

The figure 5 represents the discrete solution of the Cauchy problem along with the selected points  $p_1$ ,  $p_2$  and  $p_3$  eventually used to represent time evolution of the data completion problem solution. The figures 6 and 7 represent the solution and the discrete solution of the data completion problem obtained using energy-like method. We can see that the recovered temperature and heat flux are close to the exact ones.

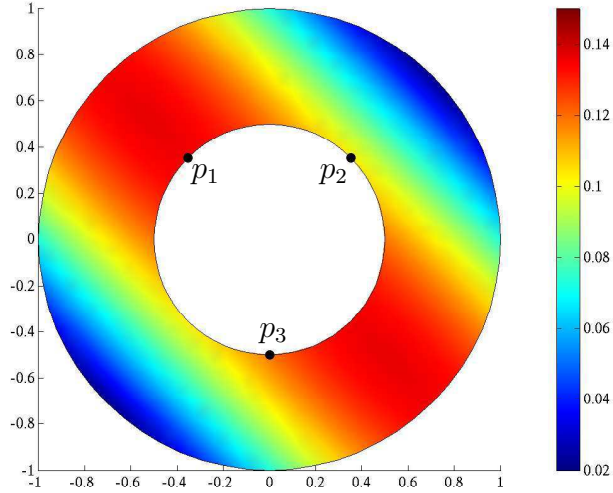


Figure 5: Exact temperature and selected points  $p_1$ ,  $p_2$  and  $p_3$ ,  $h = 0.1$ ,  $\Delta t = 0.1$ .

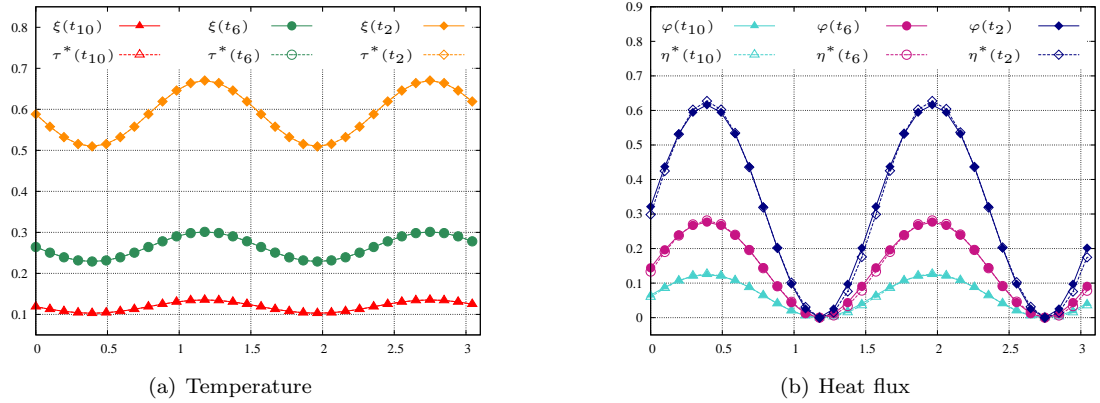


Figure 6: Exact (filled markers) and identified (empty markers) temperature and heat flux on  $\Gamma_u$  at times  $t_2 = 0.2$ ,  $t_6 = 0.6$  and  $t_{10} = 1$  for the 2D example,  $h = 0.1$ ,  $\Delta t = 0.1$ .

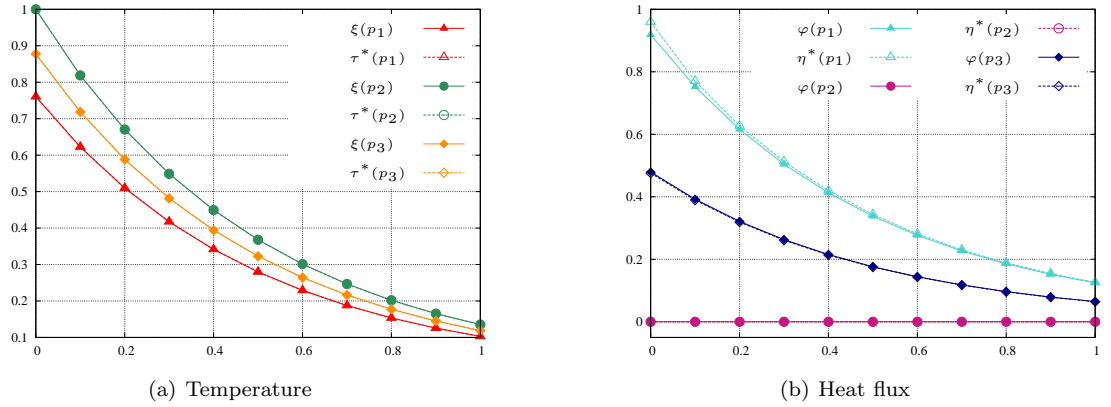


Figure 7: Time evolution of exact (filled markers) and identified (empty markers) temperature and heat flux on selected points  $p_1$ ,  $p_2$  and  $p_3$  on  $\Gamma_u$  for the 2D example,  $h = 0.1$ ,  $\Delta t = 0.1$ .

Figure 9 represents the finite element discretization error with respect to the maximum edge size of the mesh with  $\Delta t$  sufficiently small. In the same way, figure 8 shows the time discretization error with respect to the time step with  $h$  sufficiently small.

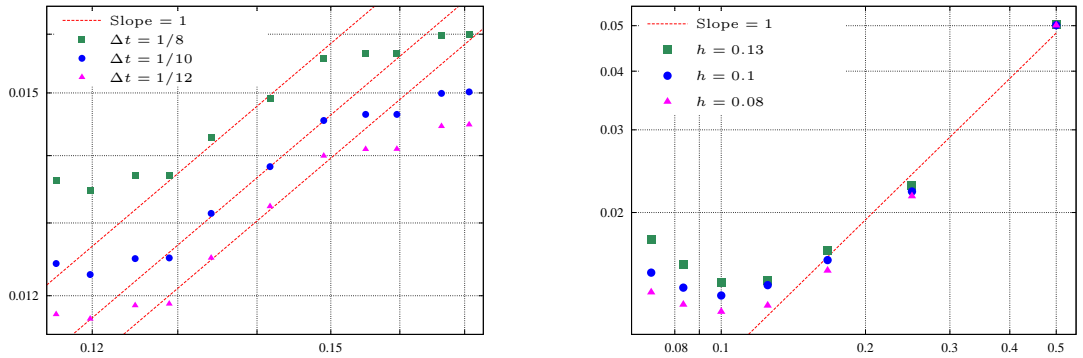


Figure 8: Evolution of  $\|u(t_N) - u_h^N\|_V$  with respect to  $\Delta t$  for different  $h$  for the 2D example.

Figure 9: Evolution of  $\|u(t_N) - u_h^N\|_V$  with respect to  $h$  for different  $\Delta t$  for the 2D example.

These results are in agreement with the theoretical error estimate (14). Nevertheless, the optimization process is significantly perturbed when the discretization step  $h$  or  $\Delta t$  are of order not similar. Indeed, since the error associated with the greatest discretization step behaves like numerical noise, the energy-like method does not provide a discrete solution with the required accuracy. Moreover, the results obtained in the axisymmetric case unambiguously confirming the theoretical estimate, this numerical noise could also be related with a mesh effect.

We introduce a Gaussian random noise on data with an amplitude depending on a rate  $a$ . The figures 10 and 11 represent the error and the energy-like functional at each iteration for different noise rates. These behaviors make it necessary to introduce a criterion to stop the optimization process before numerical explosion.

Next, we choose  $h$  and  $\Delta t$  such that discretization error is negligible in comparison to error due to noise and we observe error and functional behaviors with respect to the adequate noise measures. These noise measures correspond to the terms depending on data in the estimates (16) and (25), they are denoted by

$$m_\delta^n = \sqrt{\Delta t} \sum_{j=1}^n \left\| \frac{d(T - T^\delta)}{dt}(t_j) \right\| + \frac{1}{h} \|\delta(t_j)\| \quad \text{and} \quad M_\delta = \sum_{n=0}^N \alpha_n (m_\delta^n)^2. \quad (50)$$

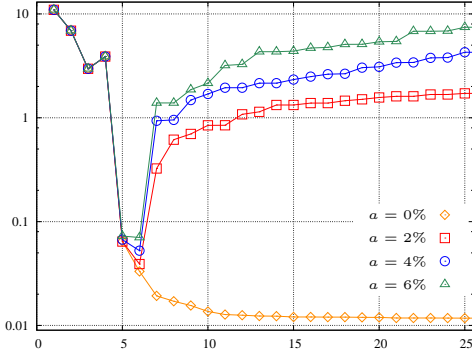


Figure 10: Evolution of  $\|u(t_N) - u_{h\delta}^N\|_V$  during the optimization procedure for different noise rates and for the 2D example.

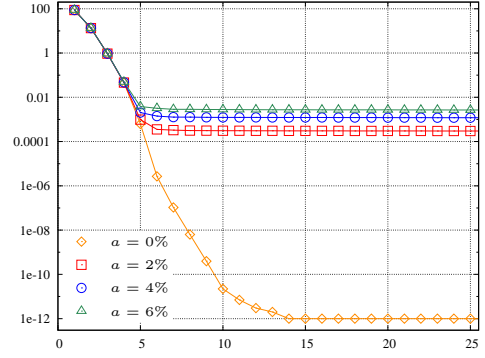


Figure 11: Evolution of  $E_h^\delta(\eta, \tau)$  during the optimization procedure for different noise rates and for the 2D example.

These results, shown on figure 13, are in agreement with the error estimates (16) and (25).

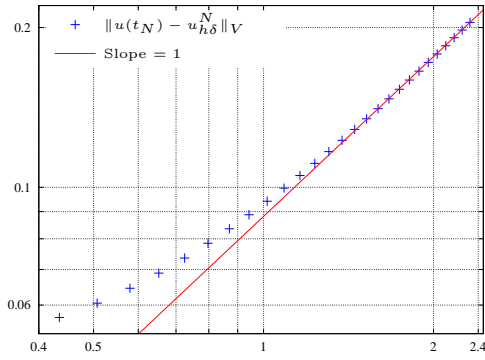


Figure 12: Evolution of  $\|u(t_N) - u_{h\delta}^N\|_V$  with respect to  $m_\delta^N$  for different noise rates and for the 2D example,  $h = 0.09$ ,  $\Delta t = 1/12$ .

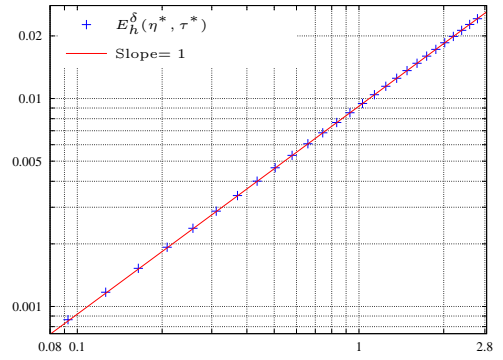


Figure 13: Evolution of  $E_h^\delta(\eta^*, \tau^*)$  with respect to  $M_\delta$  for different noise rates and for the 2D example,  $h = 0.09$ ,  $\Delta t = 1/12$ .

As illustrated by figures 14, 15 and 16, the proposed stopping criterion allows to identify a consistent solution.

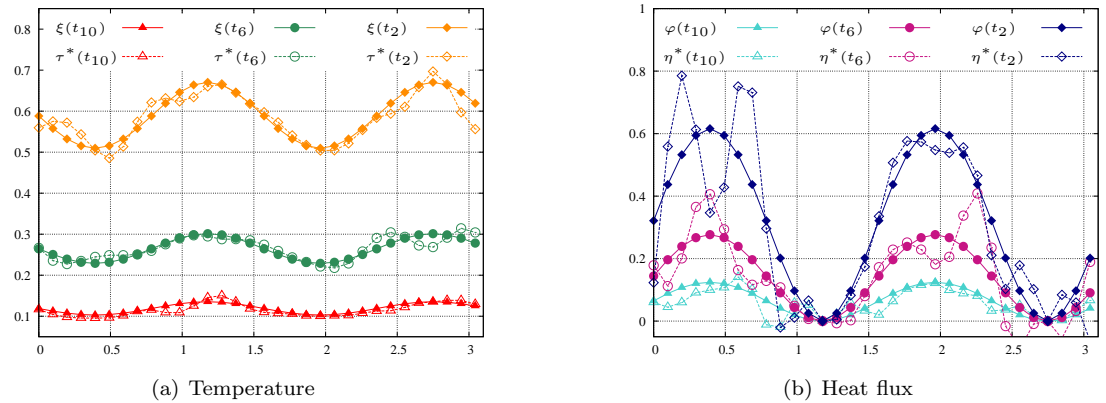


Figure 14: Exact (filled markers) and identified (empty markers) temperature and heat flux on  $\Gamma_u$  at times  $t_2 = 0.2$ ,  $t_6 = 0.6$  and  $t_{10} = 1$  for the 2D example,  $h = 0.1$ ,  $\Delta t = 0.1$ ,  $a = 5\%$ .

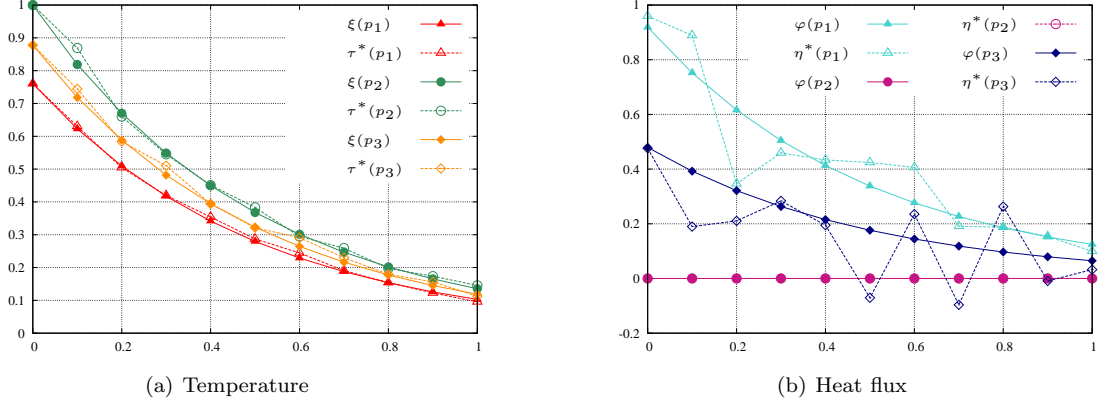


Figure 15: Time evolution of exact (filled markers) and identified (empty markers) temperature and heat flux on selected points  $p_1$ ,  $p_2$  and  $p_3$  on  $\Gamma_u$  for the 2D example,  $h = 0.1$ ,  $\Delta t = 0.1$ ,  $a = 5\%$ .

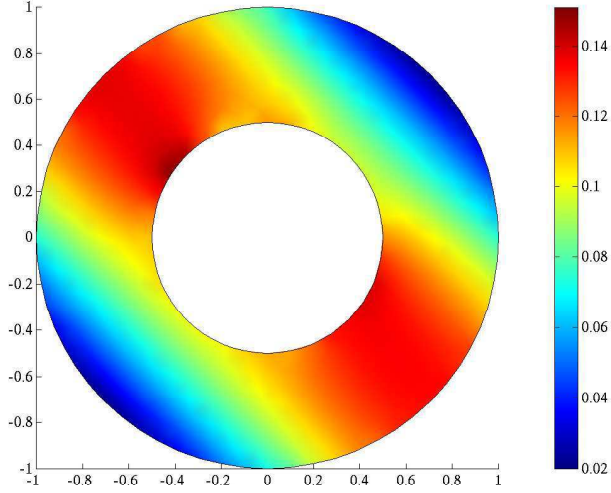


Figure 16: Identified solution for the 2D example,  $a = 5\%$ ,  $h = 0.1$ ,  $\Delta t = 0.1$ .

### 6.2.3 Stratified inner fluid problem

Now, we explore the efficiency of proposed stopping criterion on the stratified inner fluid problem already studied in [9, 14]. We consider therefore the reconstruction of temperature and flux in a pipeline of infinite length. This application arises in several industrial processes. Indeed, knowledge of temperature on internal wall of a pipeline is necessary for controlling the material safety : stratified inner fluid generates mechanical stresses, which may cause damages such as cracks. We assume that the temperature does not depend on the longitudinal coordinate. We consider then the following problem on the geometry defined by figure 17 :

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = 0 & \text{in } \Omega \\ k \nabla u \cdot n + \alpha u = 20 & \text{on } \Gamma_m \\ k \nabla u(x, t) \cdot n + \alpha u(x, t) = 250 \cdot \mathbb{1}_{\{\Gamma_{u, up}(t)\}}(x) + 50 \cdot \mathbb{1}_{\{\Gamma_{u, lo}(t)\}}(x) & \text{on } \Gamma_u \\ u(\cdot, 0) = u_0 \end{cases} \quad (51)$$

where  $k = 17 \text{ W.m}^{-1}.\text{C}^{-1}$  is the constant thermal conductivity,  $\alpha = 12$  on  $\Gamma_m$  and 1000 on  $\Gamma_u$  is the Fourier coefficient. The radius of the inner and outer circles in figure 17 are the same as in figure 2. The boundary  $\Gamma_u$  is partitioned into two parts, the lower arc  $\Gamma_{u, lo}(t) = \{(x, y) \in \Gamma_u; y < y_s(t)\}$  and the upper arc  $\Gamma_{u, up}(t) = \{(x, y) \in \Gamma_u; y \geq y_s(t)\}$ . The angle  $\theta(t)$  evolves linearly from 0 to  $\pi$

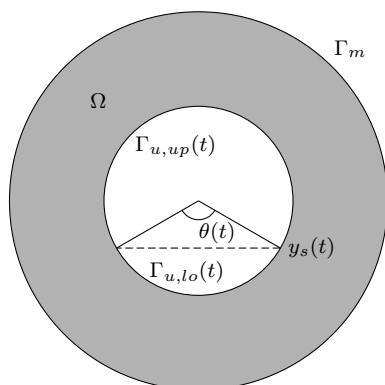


Figure 17: Stratified ring

with time. Therefore, the upper and lower parts of  $\Gamma_u$ , and then temperature on  $\Gamma_u$ , depend on  $t$ . The initial condition  $u_0$  is the stationary solution of the problem (51) with  $\theta = 0$ .

The Cauchy data are generated by solving the forward problem defined by (51). Then, a random noise with a rate of  $a = 5\%$  is introduced on Dirichlet data while we assume that the flux is exactly known on  $\Gamma_m$ . The results presented here are obtained using the proposed stopping criterion. The figure 18 represents the temperature field that has to be identified along with the selected points  $p_1$ ,  $p_2$  and  $p_3$  used to represent time evolution of the data completion problem solution. On one hand, the figure 19 shows the recovered temperature and heat flux in comparison to the data completion problem solution given by numerical resolution of (51) at times  $t_2 = 0.2$ ,  $t_6 = 0.6$  and  $t_{10} = 1$ . On the other hand, the figure 20 shows the time evolution of the recovered temperature and heat flux on in comparison to time evolution of the data completion problem solution on points  $p_1$ ,  $p_2$ ,  $p_3$ . The figure 21 represents the identified temperature field relative to figure 18. Notice that reconstructed field is close to the field to recover given by . Finally, the figure 22 represents the solution of the generic optimization algorithm, a numerical explosion without the proposed stopping criteria is then clearly observed.

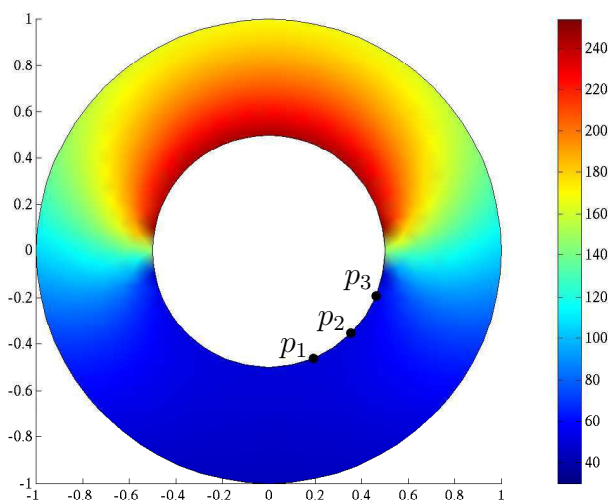


Figure 18: Exact temperature and selected points  $p_1$ ,  $p_2$  and  $p_3$  for the stratified inner fluid example at time  $t = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.1$ .



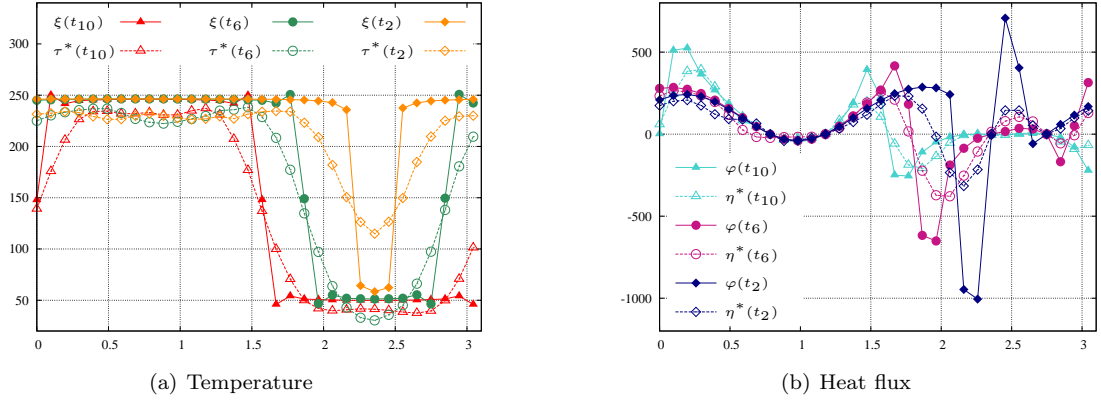


Figure 19: Exact (filled markers) and identified (empty markers) temperature and heat flux on  $\Gamma_u$  at times  $t_2$ ,  $t_6$  and  $t_{10}$  for the stratified inner fluid example,  $h = 0.1$ ,  $\Delta t = 0.1$ ,  $a = 5\%$ .

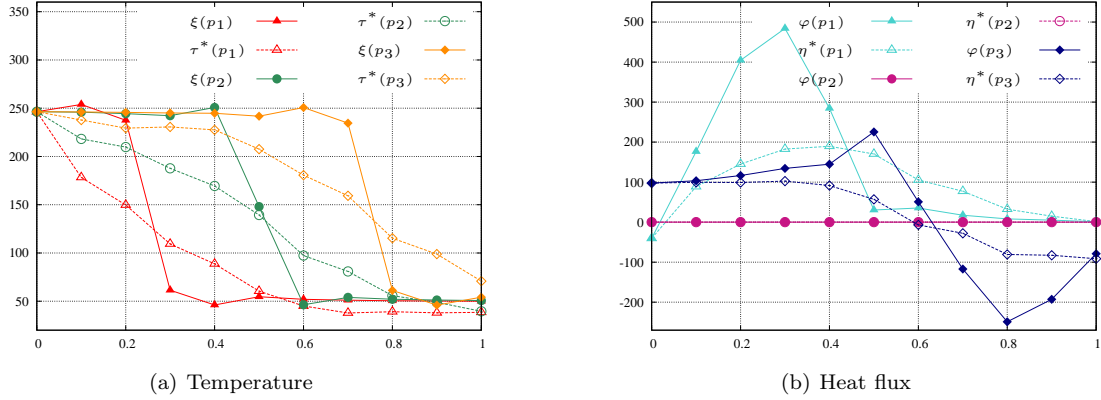


Figure 20: Time evolution of exact (filled markers) and identified (empty markers) temperature and heat flux on selected points  $p_1$ ,  $p_2$  and  $p_3$  on  $\Gamma_u$  (cf. figure 18) for the stratified inner fluid example,  $h = 0.1$ ,  $\Delta t = 0.1$ ,  $a = 5\%$ .

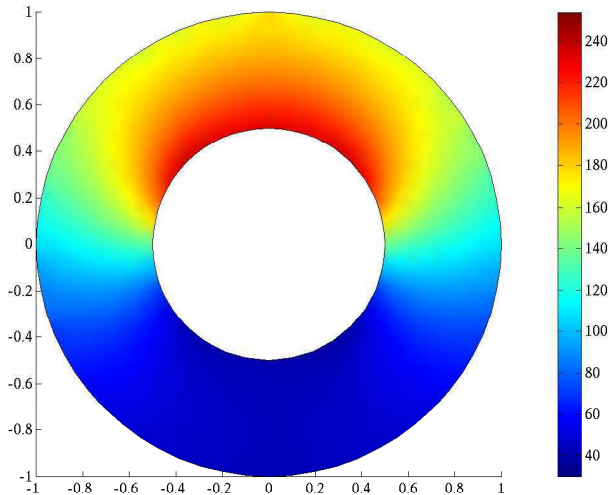


Figure 21: Identified solution for the stratified inner fluid example using the proposed stopping criterion at time  $t = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.1$ ,  $a = 5\%$ .

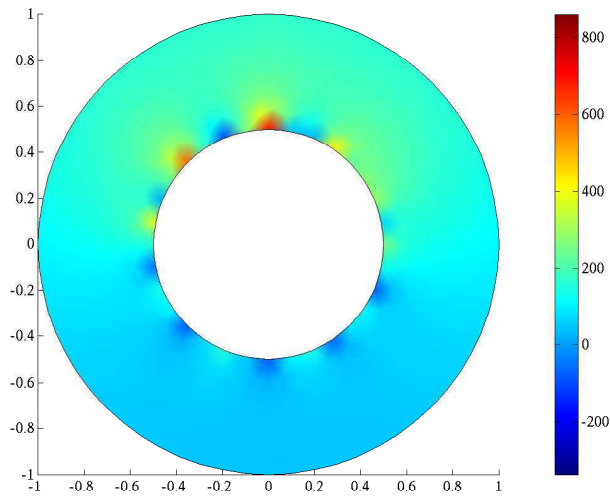


Figure 22: Identified solution for the stratified inner fluid example without using the proposed stopping criterion at time  $t = 1$ ,  $h = 0.1$ ,  $\Delta t = 0.1$ ,  $a = 5\%$ .

## 7 Conclusion

In this work, we stated Cauchy problem as a minimization of an energy-like functional and presented classical theoretical results. Then, we gave the finite element discretization and performed numerical convergence analysis. We derived a priori error estimates taking into account noisy data effects. Then we proposed a stopping criteria depending on the noise rate in order to control numerical instability of the minimization process due to noisy data. We proposed a numerical procedure, performed numerical experiments and confirmed theoretical error estimates. Finally, we illustrated robustness and efficiency of the proposed stopping criteria, especially in the case of singular data. It would be desirable to couple this approach with regularization procedure.

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