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# ONSAGER-MACHLUP FUNCTIONAL FOR UNIFORMLY ELLIPTIC TIME-INHOMOGENEOUS DIFFUSION

KOLÉHÈ A. COULIBALY-PASQUIER

ABSTRACT. In this paper we will make the computation of the Onsager-Machlup functional of an inhomogeneous uniformly elliptic diffusion process. This functional will have formally the same picture as in the homogeneous case, the only difference come from the infinitesimal variation of the volume. For example in the Ricci flow case, we find some functional which is not so far to the  $\mathcal{L}_0$  distance used by Lott to study this flow [6]. We finish by a application to small ball probability for weighted sup norm, for inhomogeneous diffusion.

## 1. INTRODUCTION

Let  $M$  be a  $n$ -dimensional manifold, and a inhomogeneous uniformly elliptic operator  $L_t$  over  $M$ . It is always possible to put a time dependent family of metrics  $g(t)$  over  $M$  such that

$$(1.1) \quad L_t = \frac{1}{2}\Delta_t + Z(t),$$

where  $\Delta_t$  is a Laplace Beltrami operator for a metric  $g(t)$  and  $Z(t, \cdot)$  is a time dependent vector field over  $M$ . Let  $X_t(x_0)$  a  $L_t$ -diffusion process on  $M$  starting at point  $x_0$ , an example of such a diffusion could be the  $g(t)$ -MB introduced in [2], when the family of metrics  $g(t)$  come from the Ricci flow.

Let  $d(t, x, y)$  be the Riemannian distance on  $M$  according to the metric  $g(t)$ . Consider a smooth curve  $\varphi : [0, T] \rightarrow M$ , such that  $\varphi(0) = x_0$ , we are interested in the asymptotic equivalent as  $\epsilon$  goes to zero of the following probability

$$\mathbb{P}_{x_0}[\forall t \in [0, T] \quad d(t, X_t, \varphi(t)) \leq \epsilon].$$

This asymptotic will depend on the product of two terms. The first one is a decreasing function of  $\epsilon$  that does not depend on the curve and the geometries (except the dimension), and a second term that depends on the geometries around the curve  $\varphi$ . This second term is expressed as a certain Lagrangian, and maximizing this term could be interpreted as finding the most probable path for the diffusion. This term is historically called the Onsager-Machlup functional of the diffusion  $X_t$ .

This computation will be made using the same technique as in the paper of Takahashi and Watanabe [7]. We will also use the non singular drift introduced by Hara. Using this drift, Hara and Takahashi in [4] have made a substantial simplification of the previous proof of Onsager-Machlup

functional. We propose a time dependent parallel transport along a curve according to a family of metrics and use it to make the computation of the Onsager-Machlup functional in the inhomogeneous case.

This is the main result of the paper, here in the following theorem

**Theorem 1.1.** *Let  $X_t(x_0)$  be a  $L_t$  diffusion process starting at point  $x_0$ , where  $L_t = \frac{1}{2}\Delta_t + Z(t, \cdot)$ , then we have the following asymptotic:*

$$\mathbb{P}_{x_0}[\forall t \in [0, T] \quad d(t, X_t, \varphi(t)) \leq \epsilon] \sim_{\epsilon \downarrow 0} C \exp\left\{-\frac{\lambda_1 T}{\epsilon^2}\right\} \exp\left\{-\int_0^T H(t, \varphi, \dot{\varphi})\right\},$$

where  $H$  is a time dependent function on the tangent bundle defined for  $v \in T_x M$  as:

$$\begin{aligned} H(t, x, v) &= \frac{1}{2}\|Z(t, x) - v\|_{g(t)}^2 + \frac{1}{2}\operatorname{div}_{g(t)}(Z)(t, x) - \frac{1}{12}R_{g(t)}(x) \\ &\quad + \frac{1}{4}\operatorname{trace}_{g(t)}(\dot{g}(t)). \end{aligned}$$

Here  $C$ ,  $\lambda_1$  are explicit constants,  $\operatorname{div}_{g(t)}$  and  $R_{g(t)}$  are respectively the divergence operator and the scalar curvature with respect to the metric  $g(t)$ .

The paper will be organized as follows: In the first section we will give a parallel transport along a curve according to a family of metrics. We will use this parallel transport to get a Fermi coordinate around the smooth curve  $\varphi$ . We will also give some local development of certain tensor that we need in the sequel.

To be self contained, in the second section we will quickly expose some probabilistic lemmas all of them are clearly exposed in the paper of Capitaine [1], so we will keep the same notation. In this paper [1] the case of different norms are investigated. In the literature the case of non smooth functions  $\varphi$  are also investigated. But we will not discuss these cases in this paper.

In the three last sections we will expose the proof of the theorem and give some related results. The case of  $g(t)$ -BM, when the family of metrics  $g(t)$  comes from the Ricci flow will be investigated as an application of the theorem. The resulting Lagrangian gives a notion of “space-time distance” which presents many similarities with the  $\mathcal{L}_0$ -distance used in the theory of Ricci flow.

## 2. PARALLEL TRANSPORT ALONG A CURVE, AND FERMI COORDINATE

Let  $\varphi : [0, T] \rightarrow M$  be smooth curve. And consider that the manifold  $M$  is endowed with a  $C^1$ -family of metrics  $g(t)_{t \in [0, T]}$ . This family of metrics produces a time dependent family of Levi-Civita connexions that we will write  $\nabla^t$ .

If  $A$  is a bilinear form over a vector space  $E$ ,  $v, w \in E$  and we have a scalar product  $\langle \cdot, \cdot \rangle_{g(t)}$  over  $E$  then we define  $A^{\#g(t)}(v) \in E$  such that  $\langle A^{\#g(t)}v, w \rangle = A(v, w)$ .

**Proposition 2.1.** *Let  $(e_1, e_2, \dots, e_n)$  be an orthonormal basis of  $T_{\varphi(0)}M$  for the metric  $g(0)$ . Let  $\tau_t e_i$  be the solutions of the following first order equation on  $TM$  above the curve  $\varphi$ :*

$$\begin{cases} \nabla_{\dot{\varphi}(t)}^t \tau_t e_i = -\frac{1}{2} \dot{g}(t)^{\#g(t)}(\tau_t e_i) \\ \tau_0 e_i = e_i. \end{cases}$$

*Then  $(\tau_t e_1, \tau_t e_2, \dots, \tau_t e_n)$  is an orthonormal basis of  $T_{\varphi(t)}M$  for the metric  $g(t)$ .*

*Proof.* We have just to compute quantity like:

$$\begin{aligned} \frac{d}{dt} \langle \tau_t e_i, \tau_t e_j \rangle_{g(t)} &= \nabla^t g(t)(\tau_t e_i, \tau_t e_j) + \langle \nabla^t \tau_t e_i, \tau_t e_j \rangle_{g(t)} + \langle \tau_t e_i, \nabla^t \tau_t e_j \rangle_{g(t)} \\ &\quad + \dot{g}(t)(\tau_t e_i, \tau_t e_j) \\ &= -\frac{1}{2} \langle \dot{g}(t)^{\#g(t)}(\tau_t e_i), \tau_t e_j \rangle_{g(t)} - \frac{1}{2} \langle \tau_t e_i, \dot{g}(t)^{\#g(t)}(\tau_t e_j) \rangle_{g(t)} \\ &\quad + \dot{g}(t)(\tau_t e_i, \tau_t e_j) \\ &= 0. \end{aligned}$$

□

We are now able to write the Fermi coordinate around a curve. Let  $\varphi : [0, T] \rightarrow M$  be a smooth curve and let  $\tau$  be the parallel transport above  $\varphi$  in the sense of (2.1), where we have fix a  $g(0)$  orthonormal basis  $(e_1, \dots, e_n)$  of  $T_{\varphi(0)}M$ . Consider the map

$$\begin{aligned} \Psi : U \subset [0, T] \times \mathbb{R}^n &\longrightarrow V \subset [0, T] \times M \\ (t, v_1, \dots, v_n) &\longmapsto (t, \exp_{\varphi(t)}^t(\tau_t \sum_1^n v_i e_i)). \end{aligned}$$

Where  $\exp_x^t$  means the exponential map for the metric  $g(t)$ . The map  $\Psi$  is clearly a diffeomorphism on some neighborhood  $U$  of  $[0, T] \times 0$ , and let  $V = \Psi(U)$ . Remark that for each fixed  $t$ , the map  $\Psi(t, \cdot)$  is the normal coordinate for the metric  $g(t)$  around the point  $\varphi(t)$ .

Let  $X_t(x_0)$  be a  $L_t$ -diffusion that starts at the point  $x_0$ , where  $L_t$  is a time dependent operator as is (1.1). Using this Fermi coordinate, the time-dependent norm in the theorem 1.1 will be translate in term of Euclidean one, but the generator will be changed by the pull back by  $\Psi$  of the operator  $L_t$ . The generator of  $(t, X_t)$  is  $\partial_t + \frac{1}{2} \Delta_t + Z(t, \cdot)$  and we will compute the generator of  $\Psi^{-1}(t, X_t)$ , or more precisely its local development.

$$\Psi^*(\partial_t + \frac{1}{2} \Delta_t + Z(t, \cdot)) = \frac{\tilde{\partial}}{\partial t} + \frac{1}{2} \tilde{\Delta}_t + \tilde{Z}(t, \cdot).$$

The second term in the right hand side is computed in [2] as :

$$\tilde{\Delta}_t = g^{ij}(\Psi(t, \cdot)) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{1}{2} g^{kl}(\Psi(t, \cdot)) \Gamma_{kl}^i(\Psi(t, \cdot)) \frac{\partial}{\partial x_i},$$

where  $(x_1^t, \dots, x_n^t)$ ,  $g_{ij}(\Psi(t, \cdot))$ ,  $g^{ij}(\Psi(t, \cdot))$ , and  $\Gamma_{kl}^i(\Psi(t, \cdot))$  are respectively the normal coordinate at the point  $\varphi(t)$  for the metric  $g(t)$  with respect to the vector basis  $(\tau_t e_1, \dots, \tau_t e_n)$ , the coefficient of metric  $g(t)$  in this basis, its inverse, and the Christoffel symbols of the Levi-Civita connexion of the metric  $g(t)$  in this basis. Clearly we have

$$\tilde{Z}(t, \cdot) = \sum_{i=1}^n Z^i(t, \cdot) \frac{\partial}{\partial x_i}$$

where  $Z^i(t, \cdot) = \langle Z(\Psi(t, \cdot)), \frac{\partial}{\partial x_i} |_{\Psi(t, \cdot)} \rangle_{g(t)}$ .

For a point  $(t, x) \in [0, T] \times M$  in the neighborhood  $V$  of  $\{(t, \varphi(t)), t \in [0, T]\}$  in which  $\Psi$  induce a diffeomorphism, we write  $\Psi^{-1}(t, x) = (t, x_1^t, \dots, x_n^t) \in [0, T] \times \mathbb{R}^n$ .

We have to compute  $\frac{\partial}{\partial t} |_{(t, x)} = \sum_{i=1}^n a_i(t, x) \frac{\partial}{\partial x_i} + a_0(t, x) \frac{\partial}{\partial t}$ , clearly we have  $a_i(t_0, x) = \frac{\partial}{\partial t} |_{t_0} (x_i^t) |_{\Psi(t_0, x)}$ . We will compute this term using the equality :

$$\frac{\partial}{\partial t} (\exp^{g(t)}(\varphi(t), \sum_{i=1}^n \tau_t e_i x_i^t)) = 0,$$

where, for  $v \in T_x M$ ,  $\exp^{g(t)}(x, v)$  is the exponential map for the metric  $g(t)$  at the point  $x$ . The three propositions below will be used to compute  $a_i(t, x) = \frac{\partial}{\partial t} (x_i^t) |_{\Psi(t, x)}$ .

**Proposition 2.2.** *Let  $v \in T_x M$  then*

$$\frac{\partial}{\partial t} |_{t_0} \exp^{g(t)}(x, v) = O(\|v\|_{g(t_0)}^2)$$

*Proof.* Let  $x_i(t, s)$  be the coordinate of the geodesic  $\exp^{g(t)}(x, s.v)$  in the normal coordinate system centered at  $\varphi(t_0)$  with respect to the metric  $g(t_0)$ , we will write  $\dot{x}(t, s)$  for  $\frac{\partial}{\partial s} x(t, s)$ . The usual equation for the geodesic gives:

$$\begin{aligned} \frac{\partial}{\partial t} |_{t_0} x_i(t, s) &= -\frac{\partial}{\partial t} |_{t_0} \left[ \int_0^s du \int_0^u dl \Gamma_{jk}^i(t, x(t, l)) \dot{x}_j(t, l) \dot{x}_k(t, l) \right] \\ &= -\int_0^s du \int_0^u dl \left( \frac{\partial}{\partial t} |_{t_0} \Gamma_{jk}^i(t, x(t_0, l)) \dot{x}_j(t_0, l) \dot{x}_k(t_0, l) \right. \\ &\quad + \langle d\Gamma_{jk}^i(t_0, \cdot), \frac{\partial}{\partial t} |_{t_0} x(t, l) \rangle \dot{x}_j(t_0, l) \dot{x}_k(t_0, l) \\ &\quad \left. + 2\Gamma_{jk}^i(t_0, x(t_0, l)) \frac{\partial}{\partial t} |_{t_0} (\dot{x}_j(t, l)) \dot{x}_k(t_0, l) \right). \end{aligned}$$

Note that  $\| \dot{x}(t_0, s) \|_{g(t_0)}^2 = \| v \|_{g(t_0)}^2$ ,  $\Gamma_{jk}^i(t_0, x) = O(\| x \|_{g(t_0)})$ , also  $|\frac{\partial}{\partial t} \Gamma_{jk}^i(t, \cdot)|$  and  $\| d\Gamma_{jk}^i(t, \cdot) \|$  are bounded by some constant  $C$ , in a neighborhood  $V$  of  $\{(t, \varphi(t)), t \in [0, T]\}$ , hence :

$$\begin{aligned} \frac{\partial}{\partial t|_{t_0}} x(t, s) &:= \frac{\partial}{\partial t|_{t_0}} (x_1(t, s), \dots, x_n(t, s)) \\ &= O(\|v\|_{g(t_0)}^2) + \int_0^s dl O(\|v\|_{g(t_0)}^2) \frac{\partial}{\partial t|_{t_0}} x(t, l) \\ &\quad + \int_0^s du \int_0^u dl O(\|v\|_{g(t_0)}^2) \frac{\partial}{\partial t|_{t_0}} x(t, l). \end{aligned}$$

By Gronwall's Lemma we deduce that :

$$\| \frac{\partial}{\partial t|_{t_0}} x(t, 1) \| = O(\|v\|_{g(t_0)}^2)$$

□

**Proposition 2.3.** *Let  $(x_1(t), \dots, x_n(t))$  be the coordinate of  $\exp_{g(t_0)}(\varphi(t), \sum_{i=1}^n \tau_t e_i x_i^t)$  in the normal coordinate system at the point  $\varphi(t_0)$  for the metric  $g(t_0)$ , and  $\partial_i$  are the associated vectors field (it is a short notation for  $\frac{\partial}{\partial x_i^t}$ ). Then*

$$\begin{aligned} \frac{\partial}{\partial t|_{t_0}} x_i(t) &= \frac{\partial}{\partial t|_{t_0}} x_i^t - \frac{1}{2} \frac{\partial}{\partial t|_{t_0}} (g(t))_{\varphi(t_0)} (\partial_i, \sum_{j=1}^n x_j^{t_0} \partial_j) + \langle \frac{\partial}{\partial t|_{t_0}} \varphi(t), \partial_i \rangle_{g(t_0)} \\ &\quad + O(\| x^{t_0} \|^2). \end{aligned}$$

*Proof.* As in the proof of the above proposition, we write the geodesic  $\exp_{g(t_0)}(\varphi(t), s, \sum_{i=1}^n \tau_t e_i x_i^t)$  in normal coordinate, it usually satisfy :

$$\begin{cases} \ddot{x}_i(t, s) = - \sum_{jk} \Gamma_{jk}^i(t_0, x(t, s)) \dot{x}_j(t, s) \dot{x}_k(t, s) \\ \dot{x}_i(t, 0) = \langle \sum_{l=1}^n \tau_t e_l x_l^t, \partial_i|_{\varphi(t)} \rangle_{g(t_0)} \\ x_i(t, 0) = \varphi(t)^i \end{cases}$$

We have :

$$\begin{aligned} \frac{\partial}{\partial t|_{t_0}} x_i(t, s) &= - \int_0^s du \int_0^u dl \sum_{jk} \frac{\partial}{\partial t|_{t_0}} \left[ \Gamma_{jk}^i(t_0, x(t, l)) \dot{x}_j(t, l) \dot{x}_k(t, l) \right] \\ &\quad + s \frac{\partial}{\partial t|_{t_0}} \dot{x}_i(t, 0) + \frac{\partial}{\partial t|_{t_0}} x_i(t, 0). \end{aligned}$$

After the same computation as before, we could write this equality in a matrix form, using again the Gronwall's lemma we deduce that  $\frac{\partial}{\partial t|_{t_0}} x_i(t, s)$  is bounded for  $s \in [0, 1]$ , so the first term in the previous computation is a  $O(\|x^{t_0}\|^2)$ . Hence we have

$$\begin{aligned} \frac{\partial}{\partial t|_{t_0}} x_i(t, 1) &= O(\|x^{t_0}\|_{g(t_0)}^2) \\ &+ \frac{\partial}{\partial t|_{t_0}} \left\langle \sum_{l=1}^n \tau_t e_l x_l^t, \partial_{i|\varphi(t)} \right\rangle_{g(t_0)} + \left\langle \frac{\partial}{\partial t|_{t_0}} \varphi(t), \partial_{i|\varphi(t_0)} \right\rangle_{g(t_0)}. \end{aligned}$$

Remark that  $\partial_{i|\varphi(t_0)} = \tau_{t_0} e_i$  (only for  $t_0$ ), so:

$$\begin{aligned} \frac{\partial}{\partial t|_{t_0}} x_i(t, 1) &= O(\|x^{t_0}\|_{g(t_0)}^2) + \frac{\partial}{\partial t|_{t_0}} x_l^t \delta_i^l \\ &+ \sum_{l=1}^n x_l^{t_0} \frac{\partial}{\partial t|_{t_0}} \langle \tau_t e_l, \partial_{i|\varphi(t)} \rangle_{g(t_0)} + \left\langle \frac{\partial}{\partial t|_{t_0}} \varphi(t), \partial_{i|\varphi(t_0)} \right\rangle_{g(t_0)}. \end{aligned}$$

By the construction of the parallel transport  $\tau$  we have :

$$\begin{aligned} \frac{\partial}{\partial t|_{t_0}} \langle \tau_t e_l, \partial_{i|\varphi(t)} \rangle_{g(t_0)} &= \langle \nabla^{t_0} \tau_t e_l, \partial_{i|\varphi(t_0)} \rangle_{g(t_0)} + \langle \tau_{t_0} e_l, \nabla^{t_0} \partial_i \rangle_{g(t_0)} \\ &= -\frac{1}{2} \frac{\partial}{\partial t|_{t_0}} (g(t)) (\tau_{t_0} e_l, \partial_{i|\varphi(t_0)}) \\ &= -\frac{1}{2} \frac{\partial}{\partial t|_{t_0}} (g(t)) (\partial_{l|\varphi(t_0)}, \partial_{i|\varphi(t_0)}) \end{aligned}$$

and the last term of the right hand side of the first equality vanishes because  $\partial_i$  comes from a normal coordinate for the metric  $g(t_0)$ . And the result follows.  $\square$

**Proposition 2.4.**

$$\frac{\partial}{\partial t|_{t_0}} x_j^t = \frac{1}{2} \frac{\partial}{\partial t|_{t_0}} (g(t))_{\varphi(t_0)} (\partial_i, \sum_{j=1}^n x_j^{t_0} \partial_j) - \left\langle \frac{\partial}{\partial t|_{t_0}} \varphi(t), \partial_i \right\rangle_{g(t_0)} + O(\|x^{t_0}\|^2)$$

*Proof.* Recall that:

$$\frac{\partial}{\partial t} (\exp^{g(t)} (\varphi(t), \sum_{i=1}^n \tau_t e_i x_i^t)) = 0,$$

and the two propositions above compute the first term of the previous equation.  $\square$

We get the Taylor series of the generator :

$$\begin{aligned}
\frac{\tilde{\partial}}{\partial t} + \tilde{L}_t &:= \Psi^*(\partial_t + \frac{1}{2}\Delta_t + Z(t, \cdot))|_{(t,x)} \\
&= \frac{\tilde{\partial}}{\partial t} + \sum_{i,j=1}^n g^{ij}(t,x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^n \tilde{b}^i(t,x) \frac{\partial}{\partial x_i} \\
&= \frac{\partial}{\partial t} + \sum_{i,j=1}^n \left( \frac{1}{2} \dot{g}(t) \left( \frac{\partial}{\partial x_i^t}, \frac{\partial}{\partial x_j^t} \right) x_j - \dot{\varphi}(t)^i \right) \frac{\partial}{\partial x_i} \\
&\quad - \frac{1}{2} \sum_{k,l,i=1}^n g^{kl}(t,x) \Gamma_{kl}^i(t,x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n g^{ij}(t,x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \\
&\quad + \sum_{i=1}^n Z^i(t,x) \frac{\partial}{\partial x_i} + O(\|x\|^2),
\end{aligned}$$

where  $g^{ij}(t,x)$  are the metric  $g(t)$  in the normal coordinate  $(x_1^t, \dots, x_n^t)$  evaluated at the point  $\Psi(t,x)$ , also  $\Gamma_{ij}^k(t,x)$  are the Christoffel symbols in this coordinate at the point  $\Psi(t,x)$ ,  $\dot{\varphi}^i(t)$  and  $Z^i(t,x)$  are the coordinate of the corresponding vector in this normal coordinate.

**Remark 2.5.** We have no time dependence such as  $O(\|\cdot\|_{g(t)})$  because all the metrics are equivalent on  $U$ .

### 3. TRANSFER TO $\mathbb{R}^n$ , AND PROBABILISTIC LEMMA

Let  $X(t)$  be a  $L_t$  diffusion, let  $\tilde{T} = \inf\{t \in [0, T], s.t. (t, X(t)) \notin V\}$  and we define  $\tilde{X}(t)$  a process in  $\mathbb{R}^n$  such that  $(t \wedge \tilde{T}, \tilde{X}(t)) = \Psi^{-1}(t \wedge \tilde{T}, X(t \wedge \tilde{T}))$ . Then for a small  $\epsilon$  we have :

$$\mathbb{P}_{x_0} \left[ \sup_{t \in [0, T]} d(t, X(t), \varphi(t)) \leq \epsilon \right] = \mathbb{P}_0 \left[ \sup_{t \in [0, T]} \|\tilde{X}(t)\| \leq \epsilon \right].$$

Clearly  $(t, \tilde{X}(t))$  is a  $\frac{\tilde{\partial}}{\partial t} + \tilde{L}_t$  diffusion, so for a  $\mathbb{R}^n$ -valued Brownian motion  $\tilde{B}$ ,  $\tilde{X}(t)$  is a solution of the following Itô stochastic differential equation :

$$\begin{cases} d\tilde{X}^i(t) = \sum_{j=1}^n \sqrt{g^{ij}}(t, \tilde{X}(t)) d\tilde{B}_t^j + \tilde{b}^i(t, \tilde{X}(t)) dt \\ \tilde{X}(0) = 0 \end{cases}$$

Where  $\sqrt{g^{ij}}(t,x)$  is the square root of the metric  $g(t)$  in the coordinate  $(x_1^t, \dots, x_n^t)$  at the point  $\Psi(t,x)$  (we take the same notation for  $\Gamma$ ) and



$$\begin{aligned}\tilde{b}^i(t, x) &= -\dot{\varphi}^i(t) - \frac{1}{2} \sum_{kl} g^{kl}(t, x) \Gamma_{kl}^i(t, x) \\ &+ \frac{1}{2} \dot{g}(t)|_{\varphi(t)} \left( \frac{\partial}{\partial x_i^t}, \sum_{j=1}^n x^j \frac{\partial}{\partial x_j^t} \right) + Z^i(t, x) + O(\|x\|^2).\end{aligned}$$

We will quickly describe the Hara Besselizing drift. We have the following equality which essentially comes from the fact that the coordinate is normal, and Gauss Lemma. For all  $i \in [1..n]$  we have :

$$\begin{aligned}\sum_{j=1}^m g^{ij}(t, x) x_j &= x_i, \\ \sum_{j=1}^m \sqrt{g}^{ij}(t, x) x_j &= x_i.\end{aligned}$$

Let us define the Hara drift :

$$\gamma^i(t, x) = \frac{1}{2} \sum_{j=1}^n \frac{\partial g^{ij}}{\partial x_j}(t, x).$$

It satisfies the following useful equation :

$$\sum_{i=1}^n (1 - g^{ii}(t, x)) = 2 \sum_{j=1}^n \gamma^j(t, x) x_j.$$

Let write  $\tilde{\sigma}_{ij}(t, \cdot) = \sqrt{g}^{ij}(t, x)$  the unique square root of the metric  $g(t)$  in the normal coordinate. We recall the equation of  $\tilde{X}(t)$  :

$$(3.1) \quad \begin{cases} d\tilde{X}(t) = \tilde{\sigma}(t, \tilde{X}(t)) d\tilde{B}t + \tilde{b}(t, \tilde{X}(t)) dt \\ \tilde{X}(0) = 0 \end{cases}$$

We now define the process  $Y(t)$  as a solution of the following Itô equation

$$(3.2) \quad \begin{cases} dY(t) = \tilde{\sigma}_{ij}(t, Y(t)) d\tilde{B}_t^i + \gamma(t, Y(t)) dt \\ Y(0) = 0. \end{cases}$$

Using Itô formula and the definition of the vector field  $\gamma$  we get :

$$d \| Y(t) \|^2 = 2 \sum_{k=1}^n Y^k(t) d\tilde{B}_t^k + n dt.$$

Let  $B(t) = \sum_{k=1}^n \int_0^t \frac{Y^k(s)}{\|Y(s)\|} d\tilde{B}_s^k$ , using Lévy's Theorem, it is a one dimensional Brownian motion in the filtration generated by  $\tilde{B}$  and

$$d \| Y(t) \|^2 = 2 \| Y(t) \| dB_t + n dt,$$

so  $\|Y(t)\|$  is a  $n$  dimensional Bessel process. Let us define a Girsanov's transform, such that after a change of probability, the process  $Y(t)$  become equal in law to the process  $\tilde{X}(t)$ . Let

$$N_t = \int_0^t \langle \tilde{\sigma}^{-1}(t, Y(t))(\tilde{b}(t, Y(t)) - \gamma(t, Y(t))), d\tilde{B}_t \rangle,$$

$$M_t = \exp(N_t - \frac{1}{2}\langle N \rangle_t)$$

$$\mathbb{Q} = M_T \cdot \mathbb{P},$$

by Girsanov's Theorem,  $(Y, \mathbb{Q})$  is a solution of (3.1). By the uniqueness in law we have :

$$(3.3) \quad \mathbb{P}_0[\sup_{t \in [0, T]} \|\tilde{X}(t)\| \leq \epsilon] = \mathbb{Q}[\sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon]$$

$$= \mathbb{E}_{\mathbb{P}}[M_T; \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon]$$

$$= \mathbb{E}_{\mathbb{P}}[M_T \mid \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon] \mathbb{P}[\sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon].$$

The term  $\mathbb{P}[\sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon]$ , is easily controled by a stopping time argument. So the problem of finding the Onsager Machlup functional become a study of the comportment of conditioned exponential martingale, as in the paper [7]. Let us rewrite the last term in equation (3.3) as :

$$(3.4) \quad \mathbb{E}_{\mathbb{P}}[\exp\left(\sum_{i,j=1}^n \int_0^T \sqrt{g_{ij}}(t, Y(t)) \delta^j(t, Y(t)) d\tilde{B}_t^i - \frac{1}{2} \sum_{i,j=1}^n \int_0^T g_{ij}(t, Y(t)) \delta^i(t, Y(t)) \delta^j(t, Y(t)) dt\right) \mid \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon].$$

Where we write  $\delta^i(t, x) = \tilde{b}^i(t, x) - \gamma^i(t, x)$ .

**Remark 3.1.** From the Lemma 1 in [1] it is enough to control the exponential momentum one by one in the following sense.

Let us recall briefly this lemma :

**Lemma 3.2** ([5],[1]). *Let  $I_1, \dots, I_n$  be  $n$  random variables,  $\{A_\epsilon\}_{0 < \epsilon}$  a family of events, and  $a_1, \dots, a_n$  be real numbers. If, for every real number  $c$  and every  $1 \leq i \leq n$ ,*

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(cI_i) \mid A_\epsilon] \leq \exp(ca_i),$$

then,

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}(\exp(\sum_{i=1}^n I_i) \mid A_\epsilon) = \exp(\sum_{i=1}^n a_i).$$

We recall Cartan's Theorem concerning Taylor series of metric and curvature in normal coordinate. Note that in this case, all the metrics  $g(t)$  are equivalent. We have :

$$g_{ij}(t, x) = \delta_i^j - \frac{1}{3} \sum_{kl} R_{iklj}(t, 0) x_k x_l + O(\|x\|^3),$$

where  $R_{iklj}(t, 0)$  are the components of the Riemannian curvature tensor, for the metric  $g(t)$  in normal coordinate centered at the point  $\varphi(t)$ . So we deduce the following equality :

$$g^{ij}(t, x) = \delta_i^j + O(\|x\|^2),$$

$$\gamma^i(t, x) = -\frac{1}{6} \sum_{j=1}^n R_{ij}(t, 0) x_j + O(\|x\|^2),$$

where  $R_{ij}(t, 0)$  are the component of the Ricci curvature tensor, for the metric  $g(t)$  in normal coordinate, at the point  $\varphi(t)$ . Using the definition of the Christoffel symbol we have,

$$\begin{aligned} \Gamma_{ij}^k(t, x) &= \frac{1}{2} \left( \frac{\partial}{\partial x_i} g_{jk}(t, x) + \frac{\partial}{\partial x_j} g_{ik}(t, x) - \frac{\partial}{\partial x_k} g_{ij}(t, x) \right) \\ (3.5) \quad &= -\frac{1}{3} \sum_{l=1}^n (R_{jlik}(t, 0) + R_{iljk}(t, 0)) x_l + O(\|x\|^2). \end{aligned}$$

So we obtain,

$$-\frac{1}{2} \sum_{i,j=1}^n g^{ij}(t, x) \Gamma_{ij}^k(t, x) = -\frac{1}{3} \sum_{l=1}^n R_{lk}(t, 0) x_l + O(\|x\|^2),$$

and thus,

$$\begin{aligned} (3.6) \quad \delta^i(t, x) &= -\dot{\varphi}^i(t) + \sum_{j=1}^n \left( \frac{1}{2} \dot{g}_{ij}(t, 0) - \frac{1}{6} R_{ij}(t, 0) \right) x_j \\ &\quad + Z^i(t, x) + O(\|x\|^2), \\ &= -\dot{\varphi}^i(t) + Z^i(t, 0) + \sum_{j=1}^n \left( \frac{1}{2} \dot{g}_{ij}(t, 0) - \frac{1}{6} R_{ij}(t, 0) + \frac{\partial}{\partial x_j} Z^i(t, 0) \right) x_j \\ &\quad + O(\|x\|^2), \end{aligned}$$

where  $\dot{g}_{ij}(t, 0) = \dot{g}(t) \left( \frac{\partial}{\partial x_i} |_{\varphi(t)}, \frac{\partial}{\partial x_j} |_{\varphi(t)} \right)$ .

## 4. PROOF OF THE THEOREM

According to Lemma 3.2 we will separately compute the terms in (3.4). The easiest to compute is the drift term :

$$\begin{aligned}
(4.1) \quad & \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp\{-\frac{c}{2} \int_0^T g_{ij}(t, Y(t)) \delta^i(t, Y(t)) \delta^j(t, Y(t)) dt\} \mid \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon] \\
& \leq \lim_{\epsilon \rightarrow 0} \exp[-\frac{c}{2} \int_0^T \delta_i^j(-\dot{\varphi}^i(t) + Z^i(t, 0))^2 + O(\epsilon) dt] \\
& \leq \exp(-\frac{c}{2} \int_0^T \delta_i^j(-\dot{\varphi}^i(t) + Z^i(t, 0))^2 dt),
\end{aligned}$$

where we have used in the second inequality the fact that the  $O(\epsilon)$  is uniform in  $t$  according to the uniform equivalence of the family of metrics  $\{g(t)\}_{t \in [0, T]}$ .

In order to control the first term in (3.4) we will use the following Theorem [4].

**Theorem 4.1** ([4]). *Let  $\alpha$  be a one form on  $[0, T] \times \mathbb{R}^n$ , which does not depend on  $dt$  and  $Y(t)$  be a diffusion process in  $\mathbb{R}^n$  whose radial part is a Bessel process, and*

$$\left\langle \int_0^\cdot Y^i dY^j - Y^j dY^i, \|Y\| \cdot \right\rangle = 0 \quad \forall i, j.$$

*Then the following estimate holds for the stochastic line integral  $\int_{*d(t, Y_t)} \alpha$  (in the sens of Stratonovich integration of a one form along a process):*

$$\mathbb{E}[\exp(\int_{*d(t, Y_t)} \alpha) \mid \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon] = O(\epsilon).$$

**Remark 4.2.** The proof of this Theorem is based on the Stokes theorem witch is transferred to stochastic Stokes theorem using Stratonovich integral, and Kunita-Watanabe theorem on orthogonal martingale theorem.

To use the above Theorem we have to write the first term of (3.4) in term of Stratonovich integral of a one form along a Bessel radial part process. Using equation (3.2) we get :

$$d\tilde{B}_t^i = \sum_{j=1}^n \tilde{\sigma}_{ij}^{-1}(t, Y(t)) dY_t^j - \sum_{j=1}^n \tilde{\sigma}_{ij}^{-1} \gamma^j(t, Y(t)) dt,$$

$$\begin{aligned}
(4.2) \quad & \sum_{i,j=1}^n \int_0^T \sqrt{g_{ij}}(t, Y(t)) \delta^j(t, Y(t)) d\tilde{B}_t^i \\
&= \sum_{i,j=1}^n \int_0^T g_{ij}(t, Y(t)) \delta^j(t, Y(t)) dY_t^i - \int_0^T g_{ij}(t, Y(t)) \delta^j(t, Y(t)) \gamma^i(t, Y(t)) dt \\
&= \sum_{i,j=1}^n \int_0^T g_{ij}(t, Y(t)) \delta^j(t, Y(t)) * dY_t^i \\
&\quad - \frac{1}{2} \sum_{i,j=1}^n \int_0^T \langle d(g_{ij}(t, Y(t)) \delta^j(t, Y(t))), dY^i \rangle_t - \int_0^T g_{ij}(t, Y(t)) \delta^j(t, Y(t)) \gamma^i(t, Y(t)) dt.
\end{aligned}$$

Where  $*d$  is the Stratonovich differential.

**Proposition 4.3.** *Let the event  $A_\epsilon$  be written as  $\{\sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon\}$ . The following equalities hold, for all  $i, j \in [1..n]$  and  $c \in \mathbb{R}$ :*

i)

$$\mathbb{E}[\exp(c \int_0^T \sum_{i,j=1}^n g_{ij}(t, Y(t)) \delta^j(t, Y(t)) * dY_t^i) \mid A_\epsilon] = O(\epsilon)$$

ii)

$$\begin{aligned}
& \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-\frac{c}{2} \int_0^T \langle d(g_{ij}(t, Y(t)) \delta^j(t, Y(t))), dY^i \rangle_t) \mid A_\epsilon] \\
& \leq \exp(-\frac{c}{2} \int_0^T \delta_i^j g_{ij}(t, 0) \{ \frac{1}{2} \dot{g}_{ij}(t, 0) - \frac{1}{6} R_{ij}(t, 0) + \frac{\partial}{\partial x_j} Z^i(t, 0) \} dt)
\end{aligned}$$

iii)

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-c \int_0^T g_{ij}(t, Y(t)) \delta^j(t, Y(t)) \gamma^i(t, Y(t)) dt) \mid A_\epsilon] = 1$$

*Proof.* i) Let  $\alpha = c \sum_{i,j=1}^n g_{ij}(t, x) \delta^j(t, x) dx^i$  in the neighborhood  $U \subset [0, T] \times \mathbb{R}^n$ , and extend it in all the space. The asymptotic in the proposition is a direct consequence of Theorem 4.1.

ii) Using Itô formula, equation (3.2) leads to

$$\begin{aligned}
& \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-\frac{c}{2} \int_0^T \langle d(g_{ij}(t, Y(t))\delta^j(t, Y(t))), dY^i \rangle_t) \mid A_\epsilon] \\
&= \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-\frac{c}{2} \int_0^T \sum_{l=1}^n \frac{\partial}{\partial x_l} (g_{ij}(t, \cdot)\delta^j(t, \cdot))(Y(t)) dY^l dY^i) \mid A_\epsilon] \\
&= \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-\frac{c}{2} \int_0^T \sum_{l=1}^n \frac{\partial}{\partial x_l} (g_{ij}(t, \cdot)\delta^j(t, \cdot))(Y(t)) g_{il}(t, Y(t)) dt) \mid A_\epsilon] \\
&\leq \exp(-\frac{c}{2} \delta_i^j \int_0^T g_{ij}(t, 0) (\frac{1}{2} \dot{g}_{ii}(t, 0) - \frac{1}{6} R_{ii}(t, 0) + \frac{\partial}{\partial x_i} Z^i(t, 0)) dt).
\end{aligned}$$

In the last computation we used the Taylor expansion that we compute in the last section.

iii) In a similar way we have :

$$\begin{aligned}
(4.3) \quad & \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-c \int_0^T g_{ij}(t, Y(t)) \delta^j(t, Y(t)) \gamma^i(t, Y(t)) dt) \mid A_\epsilon] \\
&= \limsup_{\epsilon \rightarrow 0} \mathbb{E}[\exp(-c \int_0^T O(\|Y(t)\|) dt) \mid A_\epsilon] \\
&= 1.
\end{aligned}$$

□

*Proof.* Theorem 1.1

Putting all things together, Lemma 3.2 , (3.3), (3.4), (4.1) and Proposition 4.3, we get :

$$\begin{aligned}
(4.4) \quad & \lim_{\epsilon \rightarrow 0} \mathbb{E}_{\mathbb{P}}[M_T \mid \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon] \\
&= \exp \left( \int_0^T \left\{ -\frac{1}{2} \|Z(t, \varphi(t)) - \dot{\varphi}(t)\|_{g(t)}^2 - \frac{1}{4} (\text{Tr}_{g(t)}(\dot{g}(t)))_{\varphi(t)} \right. \right. \\
&\quad \left. \left. + \frac{1}{12} R(t, \varphi(t)) - \frac{1}{2} \text{div}_{g(t)} Z(t, \varphi(t)) \right\} dt \right) \\
&= \exp \left( - \int_0^T H(t, \varphi(t), \dot{\varphi}(t)) dt \right)
\end{aligned}$$

The second term in (3.3) is clearly given by the scaling property of Brownian motion:

$$\mathbb{P}_0 \left[ \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon \right] = \mathbb{P}_0 \left[ \tau_1^n(B) > \frac{T}{\epsilon^2} \right],$$

where  $\tau_1^n(B)$  is the hitting time of the ball of radius 1; and  $B$  is a  $n$  dimensional Brownian motion. With standard argument of stopping time, Dirichlet problem and spectral Theorem we get the following :

$$\mathbb{P}_0 \left[ \sup_{t \in [0, T]} \|Y(t)\| \leq \epsilon \right] \sim_{\epsilon \rightarrow 0} C \exp(-\lambda_1 \frac{T}{\epsilon^2}),$$

where  $\lambda_1$  is the first eigenvalue of Laplace operator  $(-\frac{1}{2}\Delta_{\mathbb{R}^n})$  in the unit ball in  $\mathbb{R}^n$  with Dirichlet's boundary condition and  $C$  is also an explicit constant that only depends on the dimension.  $\square$

(4.5)

### 5. SMALL DISCUSSION ABOUT THE MOST PROBABLE PATH

In this section we will use Theorem 1.1, and we will give the equation of the "most likely" curve. We keep the same notations as before.

Let  $\varphi$  and  $\psi$  be two curves in  $M$  such that  $\varphi(0) = \psi(0)$  and  $\varphi(T) = \psi(T)$ , in the same way as in Theorem 1.1 we have :

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{\mathbb{P}_{x_0}[\sup_{t \in [0, T]} d(t, X(t), \varphi(t)) \leq \epsilon]}{\mathbb{P}_{x_0}[\sup_{t \in [0, T]} d(t, X(t), \psi(t)) \leq \epsilon]} \\ &= \frac{\exp\left(-\int_0^T H(t, \varphi(t), \dot{\varphi}(t)) dt\right)}{\exp\left(-\int_0^T H(t, \psi(t), \dot{\psi}(t)) dt\right)}. \end{aligned}$$

Let us compute the equation of the curve which is critical for the functional :

$$\varphi \mapsto \int_0^T H(t, \varphi(t), \dot{\varphi}(t)) dt,$$

when the end point is also fixed. In the next proposition we will compute the equation of this curve in a particular case of  $g(t)$  – *Brownian motion* (see [2]), the general case could be easily deduced by the same computation.

**Proposition 5.1.** *Let  $X_t$  be a  $L_t := \frac{1}{2}\Delta_t$  diffusion, where  $\Delta_t$  is the Laplace operator with respect to a family of metric  $g(t)$  that come from the Ricci flow ( $\partial_t g(t) = \alpha \text{Ric}_{g(t)}$ ) as in [2]. Then the critical curve for the functional:  $E : \varphi \mapsto \int_0^T H(t, \varphi(t), \dot{\varphi}(t)) dt$  satisfy the following second order differential equation:*

$$\nabla_{\partial_t}^t \dot{\varphi}(t) + \alpha \text{Ric}^{\#g(t)}(\dot{\varphi}(t)) + \frac{1-3\alpha}{12} \nabla^t R_t(\varphi(t)) = 0$$

*Proof.* Let  $\varphi$  be a critical curve for  $E$  and let  $\exp$  be the exponential map according to some fixed metric, then for all vector field  $V$  over  $\varphi$  such that  $V(0) = V(T) = 0$ , we have:

$$\frac{\partial}{\partial s} \Big|_{s=0} E[t \mapsto \exp_{\varphi(t)}(sV(t))] = 0.$$

Let us recall that in this situation  $Z(t, \cdot) = 0$ , so

$$H(t, x, v) = \frac{1}{2} \|v\|_{g(t)}^2 - \frac{1-3\alpha}{12} R_{g(t)}(x).$$

Let us write the variation of the curve  $\varphi$  as  $\varphi_V(t, s) := \exp_{\varphi(t)}(sV(t))$ , and  $\dot{\varphi}_V(t, s) := \frac{\partial}{\partial t}\varphi_V(t, s)$  then the above equation becomes:

$$\begin{aligned} & \frac{\partial}{\partial s}\Big|_{s=0} \int_0^T \frac{1}{2} \|\dot{\varphi}_V(t, s)\|_{g(t)}^2 - \frac{1-3\alpha}{12} R_{g(t)}(\varphi_V(t, s)) = 0 \\ &= \int_0^T \langle \dot{\varphi}_V(t, 0), \nabla_{\partial_s}^t \dot{\varphi}_V(t, 0) \rangle_{g(t)} \\ & \quad - \frac{1-3\alpha}{12} \langle \nabla^t R_{g(t)}(\varphi_V(t, 0)), \frac{\partial}{\partial s}\Big|_{s=0} \varphi_V(t, s) \rangle_{g(t)} dt \\ &= \int_0^T \langle \dot{\varphi}(t), \nabla_{\partial_s}^t \dot{\varphi}_V(t, s) \rangle_{g(t)} - \frac{1-3\alpha}{12} \langle \nabla^t R_{g(t)}(\varphi(t)), V(t) \rangle_{g(t)} dt. \end{aligned}$$

Since  $\partial_t$  and  $\partial_s$  commute, and the connection  $\nabla^t$  is torsion free we have  $\nabla_{\partial_s}^t \dot{\varphi}_V(t, s) = \nabla_{\partial_t}^t \frac{\partial}{\partial s} \varphi_V(t, s)$ . So the characterization of the critical curve become for all vector field  $V$  such that  $V(0) = V(T) = 0$  :

$$(5.1) \quad \int_0^T \langle \dot{\varphi}(t), \nabla_{\partial_t}^t V(t) \rangle_{g(t)} - \frac{1-3\alpha}{12} \langle \nabla^t R_{g(t)}(\varphi(t)), V(t) \rangle_{g(t)} dt = 0.$$

By direct computations,

$$\begin{aligned} \partial_t \langle \dot{\varphi}(t), V(t) \rangle_{g(t)} &= \langle \nabla_{\partial_t}^t \dot{\varphi}(t), V(t) \rangle_{g(t)} + \langle \dot{\varphi}(t), \nabla_{\partial_t}^t V(t) \rangle_{g(t)} \\ & \quad + \dot{g}(t) \left( \dot{\varphi}(t), V(t) \right), \end{aligned}$$

and by the final condition of the vector field  $V(0) = V(T) = 0$  we have:

$$\int_0^T \partial_t \left( \langle \dot{\varphi}(t), \nabla_{\partial_t}^t V(t) \rangle_{g(t)} \right) dt = 0.$$

Hence equation (5.1) becomes, for all  $V$  vector field such that  $V(0) = V(T) = 0$  :

$$\int_0^T \langle \nabla_{\partial_t}^t \dot{\varphi}(t), V(t) \rangle_{g(t)} + \langle \alpha \operatorname{Ric}^{\#g(t)}(\dot{\varphi}(t)), V(t) \rangle + \frac{1-3\alpha}{12} \langle \nabla^t R_{g(t)}(\varphi(t)), V(t) \rangle_{g(t)} dt = 0.$$

We conclude that  $\varphi$  is a critical value of  $E$  if and only if it satisfies:

$$\nabla_{\partial_t}^t \dot{\varphi}(t) + \alpha \operatorname{Ric}^{\#g(t)}(\dot{\varphi}(t)) + \frac{1-3\alpha}{12} \nabla^t R_{g(t)}(\varphi(t)) = 0$$

□

**Remark 5.2.** The choice of  $\alpha = \frac{1}{3}$  for the speed of the backward Ricci flow, produces a simplification of the expression above and makes the functional  $E$  positive for all time.

**Remark 5.3.** In the similar way let  $X_t$  be a  $L_t := \frac{1}{2}\Delta_t$  diffusion, where  $\Delta_t$  is the Laplace operator with respect to a family of metric  $g(t)$  then the



$E$ -critical curve  $\varphi$  satisfy:

$$\nabla_{\partial_t}^t \dot{\varphi}(t) + \dot{g}(t)^{\#g(t)}(\dot{\varphi}(t)) + \frac{1}{12} \nabla^t R_{g(t)}(\varphi(t)) - \frac{1}{4} \nabla^t (\text{Tr}_{g(t)} \dot{g}(t))(\varphi(t)) = 0.$$

We could also use this formula for the Brownian motion that come from the mean curvature flow as in [3], and compute the most probable path for this inhomogeneous diffusion. We could use this result to compute the most probable path for the degenerated diffusion  $Z(t)$  (see Remark 2.9 of [3]).

## 6. SMALL BALL PROPERTIES OF INHOMOGENEOUS DIFFUSIONS FOR WEIGHTED SUP NORM

Let  $X_t(x)$  be a  $L_t = \frac{1}{2} \Delta_t + Z(t)$  diffusion, with the same notation as in the introduction. Let  $f \in C^1([0, T])$  which we assume to be a positive function on  $[0, T]$ , we want to estimate the following probability

$$\mathbb{P}_{x_0}[\forall t \in [0, T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)],$$

when  $\epsilon$  is closed to 0.

### Proposition 6.1.

$$\begin{aligned} & \mathbb{P}_{x_0}[\forall t \in [0, T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)] \sim_{\epsilon \downarrow 0} \\ & C \exp\left\{-\frac{\lambda_1 \int_0^T \frac{1}{f^2(s)} ds}{\epsilon^2}\right\} \exp\left\{-\int_0^T \tilde{H}(t, \varphi, \dot{\varphi})\right\} \end{aligned}$$

where

$$\begin{aligned} \tilde{H}(t, x, v) = & \|Z(t, x) - v\|_{g(t)}^2 + \frac{1}{2} \text{div}_{g(t)}(Z)(t, x) - \frac{1}{12} R_{g(t)}(x) \\ & + \frac{1}{4} f^{-2}(t) \text{trace}_{g(t)}(\dot{g}(t)) - \frac{1}{2} n(f'(t)) f^{-3}(t). \end{aligned}$$

*Proof.* Let  $\tilde{g}(t) = \frac{1}{f^2(t)} g(t)$ , and  $\tilde{d}(t, \cdot, \cdot)$  the associated distance. Then the probability that we have to estimate is

$$\mathbb{P}_{x_0}[\forall t \in [0, T] \quad \tilde{d}(t, X_t, \varphi(t)) \leq \epsilon].$$

Now after a change of time we will transform  $X$  the  $L_t$  diffusion to a  $\tilde{L}(t)$  diffusion, in order to use Theorem 1.1. Let

$$\delta(t) = \left( \int_0^t \frac{1}{f^2(s)} ds \right)^{-1} (t),$$

and let  $\tilde{X}(t) := X_{\delta(t)}$ , then  $\tilde{X}$  become a  $\tilde{L}(t)$  diffusion, where

$$\tilde{L}(t) := \frac{1}{2} \Delta_{\tilde{g}(\delta(t))} + f^2(\delta(t)) Z(\delta(t), \cdot).$$

We deduce that:

$$\begin{aligned}
& \mathbb{P}_{x_0}[\forall t \in [0, T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)] = \\
& \mathbb{P}_{x_0}[\forall t \in [0, T] \quad \tilde{d}(t, X_t, \varphi(t)) \leq \epsilon] = \\
& \mathbb{P}_{x_0}[\forall t \in [0, \delta^{-1}(T)] \quad \tilde{d}(\delta(t), \tilde{X}_t, \varphi(\delta(t))) \leq \epsilon] = \\
& \sim_{\epsilon \downarrow 0} C \exp\left\{-\frac{\lambda_1 \delta^{-1}(T)}{\epsilon^2}\right\} \exp\left\{-\int_0^{\delta^{-1}(T)} H(\delta(t), \varphi(\delta(t)), \dot{\delta}(t)\dot{\varphi}(\delta(t))) dt\right\},
\end{aligned}$$

where in the last line we have used Theorem 1.1, also the Lagrangian  $H$  in the last equation is related to the diffusion  $\tilde{X}$ . After a change of variable we get the proposition.  $\square$

We directly deduce the following small ball estimate:

**Corollary 6.2.**

$$\epsilon^2 \log\{\mathbb{P}_{x_0}[\forall t \in [0, T] \quad d(t, X_t, \varphi(t)) \leq \epsilon f(t)]\} \xrightarrow{\epsilon \rightarrow 0} -\lambda_1 \int_0^T \frac{1}{f^2(s)} ds$$

#### REFERENCES

- [1] Mireille Capitaine. On the Onsager-Machlup functional for elliptic diffusion processes. In *Séminaire de Probabilités, XXXIV*, volume 1729 of *Lecture Notes in Math.*, pages 313–328. Springer, Berlin, 2000.
- [2] A. K. Coulibaly. Brownian motion with respect to time-changing riemannian metrics, applications to ricci flow. To appear A.I.H.P.
- [3] A. K. Coulibaly. Some stochastic process without birth, linked to the mean curvature flow. To appear A.O.P.
- [4] Keisuke Hara and Yoichiro Takahashi. Lagrangian for pinned diffusion process. In *Itô's stochastic calculus and probability theory*, pages 117–128. Springer, Tokyo, 1996.
- [5] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.
- [6] John Lott. Optimal transport and perelman's reduced volume. arXiv:0804.0343v2.
- [7] Y. Takahashi and S. Watanabe. The probability functionals (Onsager-Machlup functions) of diffusion processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 433–463. Springer, Berlin, 1981.

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