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CONFORMAL GALILEI GROUPS, VERONESE CURVES, AND NEWTON-HOOKE SPACETIMES

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Abstract

Finite-dimensional nonrelativistic conformal Lie algebras spanned by polynomial vector fields of Galilei spacetime arise if the dynamical exponent is $z = 2/N$ with $N = 1, 2, \dots$. Their underlying group structure and matrix representation are constructed (up to a covering) by means of the Veronese map of degree N . Suitable quotients of the *conformal Galilei groups* provide us with Newton-Hooke nonrelativistic spacetimes with quantized negative cosmological constant $\Lambda = -Nd$, where d the dimension of space.

Keywords: Schrödinger algebra, conformal Galilei algebras and Galilei groups, Newton-Cartan theory, Veronese maps, Newton-Hooke spacetimes, cosmological constant.

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1 Introduction

“Newton-Hooke” spacetimes provide solutions of the nonrelativistic gravitational field equations with nonvanishing cosmological constant. They can be viewed as deformations of their Galilean counterparts, to which they reduce when the cosmological constant is turned off. Remarkably, they may play a role in cosmology [1, 2, 3]. They also can be obtained as nonrelativistic limits of the de Sitter or anti-de Sitter solutions of Einstein’s equations [1, 2, 3, 4].

Another way of constructing these nonrelativistic spacetimes is to first contract the (anti-)de Sitter group to yield the “Newton-Hooke” group(s), and then factor out the homogeneous part of the latter [3].

On the other hand, various conformal extensions of the Lie algebra of Galilean symmetries have attracted much recent attention [5, 6, 7, 8], and one may wonder about their group structure and associated homogeneous spacetimes.

This paper is devoted to studying this question.

The most common, and historically first, of such extensions, referred to as the *Schrödinger group* [9, 10, 11], has been first discovered in classical mechanics [12], and then also for the heat equation [13], before being forgotten for almost one hundred years and then rediscovered as the maximal group of symmetries of the free Schrödinger equation [14, 8]. In addition to those of the Galilei group, it has two more generators given by their spacetime action $(\mathbf{x}, t) \mapsto (\mathbf{x}^*, t^*)$, namely *dilations*

$$\mathbf{x}^* = a \mathbf{x}, \quad t^* = a^2 t, \quad (1.1)$$

with $a \in \mathbb{R}^*$, and *expansions* (also called *inversions*), viz.,

$$\mathbf{x}^* = \Omega(t) \mathbf{x}, \quad t^* = \Omega(t) t, \quad \text{where} \quad \Omega(t) = \frac{1}{ct + 1}, \quad (1.2)$$

with $c \in \mathbb{R}$. These transformations span, along with Galilean time-translations: $\mathbf{x}^* = \mathbf{x}, t^* = t + b$, with $b \in \mathbb{R}$, the unimodular group $\text{SL}(2, \mathbb{R})$. Note that the *dynamical exponent* is $z = 2$; see (1.1). Schrödinger symmetry typically arises for massive systems, as it combines with the one-parameter central extension of the Galilei group.

The *Conformal Galilei* (CG) symmetry algebra¹ [6, 7] was first found, and then discarded, by Barut in his attempt to derive the by then newly (re)discovered Schrödinger symmetry by contraction from the relativistic conformal Lie algebra [15]. At the group level, this new symmetry also features an $\text{SL}(2, \mathbb{R})$ subgroup generated by time-translations augmented with modified dilations

$$\mathbf{x}^* = a \mathbf{x}, \quad t^* = at, \quad (1.3)$$

and expansions

$$\mathbf{x}^* = \Omega^2(t) \mathbf{x}, \quad t^* = \Omega(t) t, \quad (1.4)$$

with the same parameters and factor $\Omega(t)$ as above.

This second type of nonrelativistic conformal symmetry has dynamical exponent is $z = 1$, and also contains *accelerations*

$$\mathbf{x}^* = \mathbf{x} + \mathbf{B}_2 t^2, \quad t^* = t, \quad (1.5)$$

where $\mathbf{B}_2 \in \mathbb{R}^d$ (our notation will be justified below, see (4.11) and (4.13)). Moreover, this second type of conformal extension only allows for a *vanishing mass* [6]. It is rather difficult therefore to find physical systems which exhibit this kind of symmetry [16].

¹Henkel [5] refers to it as to “Alt₁”.

Both types of nonrelativistic symmetries have been related to the geometric “Newton-Cartan” structure of nonrelativistic spacetime [14, 17, 18, 19, 8].

Now, as recognized by Negro et al. [19], and by Henkel [20, 5], both infinitesimal Schrödinger and CG symmetry belong to a much larger, generally infinite dimensional, class of Lie algebras with *arbitrary*, possibly even *fractional, dynamical exponent* z ; their “conformal nonrelativistic algebra” [19] is, however finite dimensional for the particular values

$$z = \frac{2}{N}, \quad N = 1, 2, \dots \quad (1.6)$$

The terminology is justified by that, for all z as in (1.6), the algebra has an $\mathfrak{sl}(2, \mathbb{R})$ Lie subalgebra, highlighted by the dilation generator

$$X = \frac{1}{z} \mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}} + t \frac{\partial}{\partial t}. \quad (1.7)$$

Taking into account rotations, boosts, and translations yields, for $z = 2$, the Schrödinger algebra; the CG algebra is obtained, for $z = 1$, after incorporating also accelerations.

Let us emphasize that, for general z , the results known so far and summarized here above only concern *Lie algebras*. However, for both $N = 1$ (Schrödinger) and $N = 2$ (CG), the infinitesimal action integrates to a *Lie group* action. Our first new result the derivation of the *global group structure* for all N as in (1.6).

A crucial observation for our purposes is the following: owing to the factor $\Omega(t) = (ct + 1)^{-1}$ in (1.2) and (1.4), *neither* Schrödinger, nor Conformal Galilei transformations are *globally well-defined* over ordinary Galilean spacetime. As explained in Sections 2.2 and 4, Galilei spacetime should be replaced by a “better one”. Our investigations in Section 5 show indeed that the proper arena where our conformal Galilei symmetry groups act is in fact provided by Newton-Hooke spacetimes with quantized negative cosmological constant.

2 Nonrelativistic spacetimes

The *Galilei spacetime* is the affine space modeled on \mathbb{R}^{d+1} , endowed with its canonical flat affine connection Γ , and a *Galilei structure* (γ, θ) defined by a pair of (covariantly) constant tensor fields: (i) a spatial “metric”, viz., the degenerate twice-symmetric contravariant tensor field

$$\gamma = \sum_{i=1}^d \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i} \quad (2.1)$$

expressed in an affine coordinate system $(x^1, \dots, x^d, x^{d+1})$, and (ii) a “clock” provided by the 1-form

$$\theta = dt \tag{2.2}$$

where $t = x^{d+1}$ is an affine coordinate of the time axis, $T \cong \mathbb{R}$ [22, 23, 24, 25]. Notice that θ spans $\ker(\gamma)$.

Generalized Galilei structures consist therefore of triples (M, γ, θ) where M is a smooth $(d + 1)$ -dimensional spacetime manifold, γ a twice-symmetric contravariant tensor field of M whose kernel is spanned by a nowhere vanishing closed 1-form θ , the Galilei “clock”. Due to the lack of a canonical affine connection on a Galilei structure, one is compelled to introduce *Newton-Cartan* (NC) structures as quadruples $(M, \gamma, \theta, \Gamma)$ where (M, γ, θ) is a Galilei structure, and Γ a symmetric affine connection compatible with (γ, θ) . Upon introducing field equations (i) imposing extra symmetries to the Riemann curvature tensor, and (ii) relating the Ricci tensor to the mass-density, ρ , of the sources and the cosmological constant, Λ , viz.,

$$\text{Ric} = (4\pi G\rho - \Lambda)\theta \otimes \theta, \tag{2.3}$$

the connection Γ is interpreted as the gravitational field in a purely geometric generalization of Newtonian gravitation theory [22, 23, 24]. See [10] for a formulation of Newton-Cartan theory in a Kaluza-Klein type (“Bargmann”) framework.

2.1 The Galilei group and its Lie algebra

The group of automorphisms of $(\mathbb{R}^{d+1}, \gamma, \theta, \Gamma)$ is the *Galilei group*, $\text{Gal}(d)$; it consists of all diffeomorphisms $g \in \text{Diff}(\mathbb{R}^{d+1})$ such that

$$g_*\gamma = \gamma, \quad g_*\theta = \theta, \quad g_*\Gamma = \Gamma. \tag{2.4}$$

This is the group of symmetries that governs nonrelativistic physics in d spatial dimensions. It clearly consists of $(d + 2) \times (d + 2)$ matrices of the form [21]

$$g = \begin{pmatrix} A & \mathbf{B}_1 & \mathbf{B}_0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \text{Gal}(d) \tag{2.5}$$

where $A \in O(d)$, and $\mathbf{B}_0, \mathbf{B}_1 \in \mathbb{R}^d$ stand respectively for a space translation and a boost, and $b \in \mathbb{R}$ is a time translation.

The (affine) action of $\text{Gal}(d)$ on spacetime $\mathbb{R}^d \times \mathbb{R}$ reads

$$g_{\mathbb{R}^{d+1}} : \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} A\mathbf{x} + \mathbf{B}_1 t + \mathbf{B}_0 \\ t + b \\ 1 \end{pmatrix}. \tag{2.6}$$

Because of the above definitions (2.4), Galilei transformations (2.6) preserve the clock (2.2), i.e., they are thus necessarily orthochronous.

Infinitesimal Galilei transformations thus form a Lie algebra, $\mathfrak{gal}(d)$, spanned by all vector fields $X \in \text{Vect}(\mathbb{R}^{d+1})$ such that

$$L_X \gamma = 0, \quad L_X \theta = 0, \quad L_X \Gamma = 0 \quad (2.7)$$

(see [23, 17] for a generalization to (curved) NC structures); these vector field read

$$X = (\omega_j^i x^j + \beta_1^i t + \beta_0^i) \frac{\partial}{\partial x^i} + \varepsilon \frac{\partial}{\partial t}, \quad (2.8)$$

where $\omega \in \mathfrak{so}(d)$, $\beta_0, \beta_1 \in \mathbb{R}^d$, and $\varepsilon \in \mathbb{R}$. Latin indices run in the range $1, \dots, d$, and Einstein's summation convention is assumed throughout this article.

The Lie algebra $\mathfrak{gal}(d)$ admits the faithful $(d+2)$ -dimensional (anti-)representation $X \mapsto Z$ where,

$$Z = \begin{pmatrix} \omega & \beta_1 & \beta_0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gal}(d) \quad (2.9)$$

with the above notation.

2.2 The Schrödinger group and its Lie algebra

Let us first discuss the *Schrödinger group*, $\text{Sch}(d)$, which includes, in addition to the standard Galilei generators, those of the projective group, $\text{PSL}(2, \mathbb{R})$, of the time axis. Up to a quotient that we will make more precise later on, the Schrödinger group will be defined as the matrix group whose typical element reads [26, 17]

$$g = \begin{pmatrix} A & \mathbf{B}_1 & \mathbf{B}_0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \in \text{Sch}(d), \quad (2.10)$$

where $A \in O(d)$, $\mathbf{B}_0, \mathbf{B}_1 \in \mathbb{R}^d$, and $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. The *projective* “action” of $g \in \text{Sch}(d)$ on spacetime $\mathbb{R}^d \times \mathbb{R}$ takes the form

$$\mathfrak{g}_{\mathbb{R}^{d+1}} : \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{A\mathbf{x} + \mathbf{B}_1 t + \mathbf{B}_0}{ct + d} \\ \frac{at + b}{ct + d} \\ 1 \end{pmatrix} \quad (2.11)$$

defined on the *open subset of spacetime* where $ct + d \neq 0$.

It is an easy matter to check that the action (2.11) is consistent with the one presented in the introduction; Schrödinger dilations (1.1) correspond to $b = 0, c = 0$, and expansions (1.2) to $a = 1, b = 0, d = 1$.

The group structure is $\text{Sch}(d) = (\text{O}(d) \times \text{SL}(2, \mathbb{R})) \ltimes (\mathbb{R}^d \times \mathbb{R}^d)$.

Now, in order to guarantee a well-behaved action of this group on spacetime, one must demand that *time be compactified*, viz., $T \cong \mathbb{RP}^1$. In fact, the Schrödinger group does *not* act on “ordinary” Galilei spacetime, but rather on the *Möbius manifold*

$$M = (\mathbb{R}^d \times (\mathbb{R}^2 \setminus \{0\})) / \mathbb{R}^* \quad (2.12)$$

fibered above the projective line, \mathbb{RP}^1 , see [17]. Note that (2.12) can be recovered by factoring out the homogeneous subgroup generated by rotations, expansions, dilations, and boosts,²

$$M = \text{Sch}(d)/H \quad \text{where} \quad H = (\text{O}(d) \times \text{Aff}(1, \mathbb{R})) \ltimes \mathbb{R}^d. \quad (2.13)$$

Note that, unlike to conformally compactified Minkowski spacetime $(S^d \times S^1)/\mathbb{Z}_2$, only time, not space, is compactified here,

$$M \cong (\mathbb{R}^d \times S^1)/\mathbb{Z}_2. \quad (2.14)$$

It will be shown in Section 5 that the Möbius manifold carries a nonrelativistic Newton-Cartan structure; it is, in fact a Newton-Hooke spacetime with cosmological constant $\Lambda = -d$, minus the dimension of space; see (5.28).

It has been emphasized [17, 10, 8] that the Schrödinger group can, indeed, be defined in a geometric way — namely in the NC framework —, i.e., as the group, $\text{Sch}(d)$, of all (locally defined) diffeomorphisms g of \mathbb{R}^{d+1} such that

$$g_*(\gamma \otimes \theta) = \gamma \otimes \theta \quad \& \quad g \in \text{Proj}(\mathbb{R}^{d+1}, \Gamma), \quad (2.15)$$

where $\text{Proj}(\mathbb{R}^{d+1}, \Gamma)$ denotes the set of all projective transformations of spacetime, namely of all (local) diffeomorphisms which permute the geodesics of $(\mathbb{R}^{d+1}, \Gamma)$.

Let us stress that the conditions (2.15) imply, in particular, that the diffeomorphism g projects on the time axis as an element of $\text{PGL}(2, \mathbb{R})$ which must also preserve time-orientation defined by θ , namely an element of $\text{PSL}(2, \mathbb{R})$. The general solution of (2.15) is therefore given by (2.10), up to a covering; see also (2.11).

²Here, $\text{Aff}(1, \mathbb{R})$ stands for the 2-dimensional group of lower-triangular matrices in $\text{SL}(2, \mathbb{R})$, generated by dilations and expansions.

The Schrödinger Lie algebra, $\mathfrak{sch}(d)$, is then the Lie algebra of those vector fields $X \in \text{Vect}(\mathbb{R}^{d+1})$ such that³

$$L_X(\gamma \otimes \theta) = 0 \quad \& \quad X \in \mathfrak{proj}(\mathbb{R}^{d+1}, \Gamma). \quad (2.16)$$

We easily find that $X \in \mathfrak{sch}(d)$ iff

$$X = (\omega_j^i x^j + \kappa t x^i + \lambda x^i + \beta_1^i t + \beta_0^i) \frac{\partial}{\partial x^i} + (\kappa t^2 + 2\lambda t + \varepsilon) \frac{\partial}{\partial t}, \quad (2.17)$$

where $\omega \in \mathfrak{so}(d)$, $\beta_0, \beta_1 \in \mathbb{R}^d$, and $\kappa, \lambda, \varepsilon \in \mathbb{R}$. The Schrödinger dilation (or homothety) generator is, indeed, (1.7) with the dynamical exponent $z = 2$.

Note that $\mathfrak{sch}(d)$ is, in fact, the (centerless) Schrödinger Lie algebra. Physical applications also involve a central extension associated with the mass; see, e.g., [17, 18, 9, 10, 11, 5, 8].

The Lie algebra $\mathfrak{sch}(d)$ admits the faithful $(d+2)$ -dimensional (anti-)representation $X \mapsto Z$ where

$$Z = \begin{pmatrix} \omega & \beta_1 & \beta_0 \\ 0 & \lambda & \varepsilon \\ 0 & -\kappa & -\lambda \end{pmatrix} \in \mathfrak{sch}(d) \quad (2.18)$$

with the same notation as above.

2.3 Schrödinger symplectomorphisms

Let us recall here how Schrödinger symmetry arises in mechanics [8]. The space of free motions of particle of mass m moving in Euclidean space \mathbb{R}^d admits, as we shall see, the group $\text{Sch}(d)$ as a group of symplectomorphisms.

The $(2d+1)$ -dimensional evolutions space $V = T\mathbb{R}^d \times \mathbb{R}$ described by the triples $(\mathbf{x}, \mathbf{v}, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ is endowed with a presymplectic 2-form

$$\sigma = m \delta_{ij} dv^i \wedge (dx^j - v^j dt). \quad (2.19)$$

The leaves of the null foliation $\ker(\sigma)$ define the motions (or classical states) of the particle, and organize themselves in a symplectic manifold (U, ω) ; here $U \cong T^*\mathbb{R}^d$, and $\omega = dp_i \wedge dq^i$ with $\mathbf{p} = m\mathbf{v}$, and $\mathbf{q} = \mathbf{x} - \mathbf{v}t$.

³In other words, $L_X \Gamma_{\mu\nu}^\lambda = \delta_\mu^\lambda \varphi_\nu + \delta_\nu^\lambda \varphi_\mu$ for some 1-form φ of \mathbb{R}^{d+1} depending on X , and for all $\lambda, \mu, \nu = 1, \dots, d+1$.

The Schrödinger “action” (2.11) can be canonically lifted to V according to $g_V : (\mathbf{x}, \mathbf{v}, t) \mapsto (\mathbf{x}^*, \mathbf{v}^*, t^*)$ where

$$\mathbf{x}^* = \frac{A\mathbf{x} + \mathbf{B}_1 t + \mathbf{B}_0}{ct + d}, \quad (2.20)$$

$$\mathbf{v}^* = (ct + d)(A\mathbf{v} + \mathbf{B}_1 - c\mathbf{x}^*), \quad (2.21)$$

$$t^* = \frac{at + b}{ct + d}, \quad (2.22)$$

and a short calculation shows that $g_V^* \sigma = \sigma$ for all $g \in \text{Sch}(d)$.

This group preserves the presymplectic form (2.19), hence its kernel, and therefore passes to the quotient $U = V/\ker(\sigma)$ as a group of symplectomorphisms of (U, ω) . The neat outcome is that the Schrödinger group appears as a subgroup of the affine-symplectic group of $U \cong \mathbb{R}^{2d}$, namely⁴

$$\text{Sch}(d) \subset \text{Sp}(d, \mathbb{R}) \times \mathbb{R}^{2d}; \quad (2.23)$$

we indeed find that

$$\begin{pmatrix} \mathbf{p}^* \\ \mathbf{q}^* \\ 1 \end{pmatrix} = \begin{pmatrix} dA & -mcA & m(d\mathbf{B}_1 - c\mathbf{B}_0) \\ -\frac{b}{m}A & aA & a\mathbf{B}_0 - b\mathbf{B}_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ 1 \end{pmatrix} \quad (2.24)$$

and, moreover, $dp_i^* \wedge dq_*^i = dp_i \wedge dq^i$, proving the assertion [17]. Although the Schrödinger group does not actually “act” on the affine Galilei spacetime where the massive particle dwells, it nevertheless acts in a well-behaved fashion on the symplectic space of motions.

3 Conformal Newton-Cartan transformations & finite-dimensional conformal Galilei Lie algebras

In close relationship with the Lorentzian framework, we call *conformal Galilei* transformation of a general Galilei spacetime (M, γ, θ) any diffeomorphism of M that preserves the direction of γ . Owing to the fundamental constraint $\gamma^{ij}\theta_i = 0$, it follows that conformal Galilei transformations automatically preserve the direction of the time 1-form θ .

⁴Imbedding the Schrödinger algebra into the affine-symplectic Lie algebra is the key for studying supersymmetric extensions [27].

In terms of infinitesimal transformations, a *conformal Galilei* vector field of (M, γ, θ) is a vector field, X , of M that Lie-transport the direction of γ ; we will thus define $X \in \mathbf{cgal}(M, \gamma, \theta)$ iff

$$L_X \gamma = f \gamma \quad \text{hence} \quad L_X \theta = g \theta \quad (3.1)$$

for some smooth functions f, g of M , depending on X . Then, $\mathbf{cgal}(M, \gamma, \theta)$ becomes a Lie algebra whose bracket is the Lie bracket of vector fields.

The one-form θ being parallel-transported by the NC-connection, one has necessarily $d\theta = 0$; this yields $dg \wedge \theta = 0$, implying that g is (the pull-back of) a smooth function on T , i.e., that $g(t)$ depends arbitrarily on time $t = x^{d+1}$, which locally parametrizes the time axis. We thus have $dg = g'(t)\theta$.

3.1 Conformal Galilei transformations, $\mathbf{cgal}_{2/z}(d)$, with dynamical exponent z

One can, at this stage, try and seek nonrelativistic avatars of general relativistic infinitesimal conformal transformations. Given a Lorentzian (or, more generally, a pseudo-Riemannian) manifold (M, g) , the latter Lie algebra is generated by the vector fields, X , of M such that

$$L_X(g^{-1} \otimes g) = 0 \quad (3.2)$$

where g^{-1} denotes the inverse of the metric $g : TM \rightarrow T^*M$.

It has been shown [25] that one can expand a Lorentz metric in terms of the small parameter $1/c^2$, where c stands for the speed of light, as

$$g = c^2 \theta \otimes \theta - {}^U \gamma + \mathcal{O}(c^{-2}), \quad g^{-1} = -\gamma + c^{-2} U \otimes U + \mathcal{O}(c^{-4}), \quad (3.3)$$

with the previous notation. Here U is an ‘‘observer’’, i.e., a smooth timelike vector field of spacetime M , such that $g(U, U) = c^2$, around which the light-cone opens up in order to consistently define a procedure of nonrelativistic limit. The Galilei structure (γ, θ) is recovered via $\gamma = -\lim_{c \rightarrow \infty} g^{-1}$, and $\theta = \lim_{c \rightarrow \infty} (c^{-2} g(U))$. In (3.3) the symmetric twice-covariant tensor field ${}^U \gamma$ will define the Riemannian metric of the spacelike slices in the limiting Galilei structure.

We can thus infer that the nonrelativistic limit of Equation (3.2) would be $L_X \lim_{c \rightarrow \infty} (c^{-2} g^{-1} \otimes g) = 0$, viz.

$$L_X(\gamma \otimes \theta \otimes \theta) = 0. \quad (3.4)$$

More generally, we will consider

$$L_X(\gamma^{\otimes m} \otimes \theta^{\otimes n}) = 0 \quad (3.5)$$

for some $m = 1, 2, 3, \dots$, and $n = 0, 1, 2, \dots$, to be further imposed on the vector fields $X \in \mathbf{cgal}(M, \gamma, \theta)$. This is equivalent to Equation (3.1) together with the extra condition

$$f + qg = 0 \quad \text{where} \quad q = \frac{n}{m}. \quad (3.6)$$

Indeed, $L_X(\gamma^{\otimes m} \otimes \theta^{\otimes n}) = 0$ implies $L_X\gamma = f\gamma$ and $L_X\theta = g\theta$ for some functions f and g of M such that $mf + ng = 0$. Equation (3.4) plainly corresponds to the special case $m = 1, n = 2$. From now on, we will call *dynamical exponent* the quantity

$$z = \frac{2}{q} \quad (3.7)$$

where q is as in (3.6). This quantity will be shown to match the ordinary notion of dynamical exponent; see, e.g., [11, 5] and [8].

We will, hence, introduce the Galilean avatars, $\mathbf{cgal}_{2/z}(M, \gamma, \theta)$, of the Lie algebra $\mathfrak{so}(d+1, 2)$ of conformal vector fields of a pseudo-Riemannian structure of signature $(d, 1)$ as the Lie algebras spanned by the vector fields X of M satisfying (3.1), and (3.5) — or (3.6) for some rational number z . We will call $\mathbf{cgal}_{2/z}(M, \gamma, \theta)$ the *conformal Galilei Lie algebra with dynamical exponent z* (see (3.7)). This somewhat strange notation will be justified in the sequel.

The Lie algebra

$$\mathfrak{sv}(M, \gamma, \theta) = \mathbf{cgal}_2(M, \gamma, \theta) \quad (3.8)$$

is the obvious generalization to Galilei spacetimes of the *Schrödinger-Virasoro* Lie algebra $\mathfrak{sv}(d) = \mathfrak{sv}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta)$ introduced in [11] (see also [5]) from a different viewpoint in the case of a flat NC-structure. The representations of the Schrödinger-Virasoro group and of its Lie algebra, $\mathfrak{sv}(d)$, as well as the deformations of the latter have been thoroughly studied and investigated in [28, 29].

Let us henceforth use the notation $\mathbf{cgal}_{2/z}(d) = \mathbf{cgal}_{2/z}(\mathbb{R}^{d+1}, \gamma, \theta)$ with γ and θ as in (2.1) and (2.2) respectively. An easy calculation using the new constraints (3.6), and (3.7) shows that $X \in \mathbf{cgal}_{2/z}(d)$ iff

$$X = \left(\omega_j^i(t)x^j + \frac{1}{z}\xi'(t)x^i + \beta^i(t) \right) \frac{\partial}{\partial x^i} + \xi(t) \frac{\partial}{\partial t}, \quad (3.9)$$

where $\omega(t) \in \mathfrak{so}(d)$, $\beta(t)$, and $\xi(t)$ depend arbitrarily on time, t .

The Lie algebra $\mathbf{cgal}_0(M, \gamma, \theta)$ corresponding to the case $z = \infty$ is also interesting; it is a Lie algebra of symplectomorphisms of the models of massless and spinning Galilean particles [17, 8].

3.2 The Lie algebra, $\mathfrak{cgal}_N(d)$, of finite-dimensional conformal Galilei transformations

Our endeavor here is to show that our formalism leads to a natural definition of a whole family of distinguished finite-dimensional Lie subalgebras of the conformal Galilei Lie algebra $\mathfrak{cgal}_{2/z}(d)$ with prescribed dynamical exponent z , generated by those vector fields in (3.9), where $\omega(t) \in \mathfrak{so}(d)$, $\beta(t)$, and $\xi(t)$ depend smoothly on time, t .

To that end, let us restrict our attention to vector fields $X \in \mathfrak{cgal}_{2/z}(d)$ that are *polynomials* of fixed degree $N > 0$ in the variables $x^1, \dots, x^d, t = x^{d+1}$. This entails the following decompositions

$$\omega(t) = \sum_{n=0}^{N-1} \omega_n t^n, \quad \beta(t) = \sum_{n=0}^N \beta_n t^n \quad \text{and} \quad \xi(t) = \sum_{n=0}^N \xi_n t^n, \quad (3.10)$$

since the spatial components of X are already of first-order in x^1, \dots, x^d . Bearing in mind that $X \mapsto \xi$ is a Lie algebra homomorphism, we claim that the $\xi = \xi(t)\partial/\partial t$ do span a polynomial Lie subalgebra of $\text{Vect}(\mathbb{R})$, hence a Lie subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ since, as a classic result, the latter (spanned by polynomial vector fields of degree 2) is *maximal* in the Lie algebra, $\text{Vect}^{\text{Pol}}(\mathbb{R})$, of polynomial vector fields of \mathbb{R} . Indeed, the Lie bracket of two polynomial vector fields of degree k has degree $2k - 2$, and so $2k - 2 \leq k$ requires $k \leq 2$. Hence

$$\xi(t) = \kappa t^2 + 2\lambda t + \varepsilon \quad (3.11)$$

with $\kappa, \lambda, \varepsilon \in \mathbb{R}$.

Condition (3.11) being granted, let us seek the conditions under which the Lie bracket $X_{12} = [X_1, X_2]$ of two such polynomial vector fields X_1 and X_2 is, itself, polynomial of degree N . Straightforward calculation yields

$$\xi_{12} = \xi_1 \xi_2' - \xi_2 \xi_1' \quad (3.12)$$

$$\omega_{12} = [\omega_2, \omega_1] + \xi_1 \omega_2' - \xi_1' \omega_2 \quad (3.13)$$

$$\beta_{12} = \omega_2 \beta_1 - \omega_1 \beta_2 + \xi_1 \beta_2' - \xi_2 \beta_1' - \frac{1}{z} (\xi_1' \beta_2 - \xi_2' \beta_1). \quad (3.14)$$

Condition (3.12) brings no further restriction in view of (3.11). From (3.13), we discover that, necessarily, $\omega_1' = \omega_2' = 0$; this entails that $\omega \in \mathfrak{so}(d)$ is a constant infinitesimal rotation in (3.9). At last, we readily find that the right-hand side of Equation (3.14) turns out to be a polynomial of degree $N + 1$ in t , namely $\beta_{12} = \sum_{n=0}^{N+1} (\beta_{12})_n t^n$. In order to acquire a closed Lie algebra of polynomial vector fields of degree $N > 0$, we must have $(\beta_{12})_{N+1} = (N - 2z)(\kappa_1(\beta_2)_N - \kappa_2(\beta_1)_N) = 0$, enforcing the condition

$$z = \frac{2}{N}, \quad (3.15)$$

consistently with [19]. At last, we have shown that, in Equation (3.9), there holds

$$\boldsymbol{\beta}(t) = \boldsymbol{\beta}_N t^N + \cdots + \boldsymbol{\beta}_1 t + \boldsymbol{\beta}_0 \quad (3.16)$$

with $\boldsymbol{\beta}_n \in \mathbb{R}^d$ for all $n = 0, 1, \dots, N = 2/z$. We finally get

$$X = \left(\omega_j^i x^j + \frac{N}{2} \xi'(t) x^i + \beta^i(t) \right) \frac{\partial}{\partial x^i} + \xi(t) \frac{\partial}{\partial t} \in \mathbf{cgal}_N^{\text{Pol}}(d) \quad (3.17)$$

with $\omega \in \mathfrak{so}(d)$, $\xi(t)$ and $\boldsymbol{\beta}(t)$ as in (3.11) and (3.16) respectively.

The finite-dimensional Lie algebras $\mathbf{cgal}_N^{\text{Pol}}(d)$ are isomorphic to the so-called $\mathfrak{alt}_{2/N}(d)$ Lie algebras discovered by Henkel [11] in his study of scale invariance for strongly anisotropic critical systems (with $d = 1$),

$$\mathbf{cgal}_N^{\text{Pol}}(d) \cong \mathfrak{alt}_{2/N}(d). \quad (3.18)$$

From now on we will drop the superscript ‘‘Pol’’ as no further confusion can occur. In the case $N = 1$, we recognize the Schrödinger Lie algebra $\mathbf{cgal}_1(d) \cong \mathfrak{sch}(d)$, see (2.17), while for $N = 2$ we recover the ‘‘Conformal Galilei Algebra’’ (CGA) $\mathbf{cgal}_2(d)$, called $\mathbf{cmit}_1(d)$ in [8].

It would be highly desirable to find a geometric definition of such Lie subalgebras of the Lie algebra of conformal Galilei Lie algebras, $\mathbf{cgal}(M, \gamma, \theta)$, in the most general case of an arbitrary NC structure.

4 Conformal Galilei Groups with dynamical exponents $z = 2/N$

4.1 Veronese curves and finite-dimensional representations of $\text{SL}(2, \mathbb{R})$

A *Veronese curve* is an embedding $\text{Ver}_N : \mathbb{RP}^1 \rightarrow \mathbb{RP}^N$ defined, for $N \geq 1$ by

$$\text{Ver}_N(t_1 : t_2) = (t_1^N : t_1^{N-1} t_2 : \cdots : t_2^N) \quad (4.1)$$

where $(u_1 : u_2 : \cdots : u_{N+1})$ stands for the direction of $(u_1, u_2, \dots, u_{N+1}) \in \mathbb{R}^{N+1} \setminus \{0\}$, that is a point in \mathbb{RP}^N ; see, e.g. [30]. With a slight abuse of notation, we will still denote by $\text{Ver}_N : \mathbb{R}^2 \rightarrow \mathbb{R}^{N+1}$ the mapping $(t_1, t_2) \mapsto (u_1, u_2, \dots, u_{N+1})$, where

$$u_k = t_1^{N-k+1} t_2^{k-1} \quad (4.2)$$

for all $k = 1, \dots, N + 1$.

Put $t = (t_1, t_2) \in \mathbb{R}^2$, and consider $t^* = Ct$ with

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \quad (4.3)$$

The image u^* of t^* under the Veronese map is clearly a $(N+1)$ -tuple of homogeneous polynomials of degree N in t ; it thus depends linearly on $u = (u_1, \dots, u_{N+1}) \in \mathbb{R}^{N+1}$, where the u_k are as in (4.2). The general formula is as follows. If $t_1^* = at_1 + bt_2$, $t_2^* = ct_1 + dt_2$, with $ad - bc = 1$, then $\mathrm{Ver}_N(Ct) = \mathrm{Ver}_N(C)\mathrm{Ver}_N(t)$ where $\mathrm{Ver}_N(C)$ a nonsingular $(N+1) \times (N+1)$ matrix with entries

$$\begin{aligned} \mathrm{Ver}_N(C)_{m'}^m &= \sum_{k=\max(0, m'-m)}^{\min(N-m+1, m'-1)} \binom{N-m+1}{k} \binom{m-1}{m'-k-1} \times \\ &\times a^{N-m-k+1} b^k c^{m-m'+k} d^{m'-k-1} \end{aligned} \quad (4.4)$$

for all $m, m' = 1, \dots, N+1$. Our mapping provides us with a group homomorphism

$$\mathrm{Ver}_N : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(N+1, \mathbb{R}) \quad (4.5)$$

which constitutes (up to equivalence) the well-known $(N+1)$ -dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{R})$; see [31].

Let us introduce the $\mathfrak{sl}(2, \mathbb{R})$ generators

$$\xi_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (4.6)$$

interpreted physically as the infinitesimal generators of time translations ξ_{-1} , dilations ξ_0 , and expansions ξ_1 . Their images under the tangent map of Ver_N at the identity read then

$$\mathbf{ver}_N(\xi_{-1}) = \sum_{n=1}^{N+1} (N-n+1) u_{n+1} \frac{\partial}{\partial u_n} \quad (4.7)$$

$$\mathbf{ver}_N(\xi_0) = \sum_{n=1}^{N+1} (N-2n+2) u_n \frac{\partial}{\partial u_n}, \quad (4.8)$$

$$\mathbf{ver}_N(\xi_1) = \sum_{n=1}^{N+1} (n-1) u_{n-1} \frac{\partial}{\partial u_n}. \quad (4.9)$$

One checks that $\mathbf{ver}_N(\xi_a)$ is, indeed, divergence-free, and

$$[\mathbf{ver}_N(\xi_a), \mathbf{ver}_N(\xi_b)] = -\mathbf{ver}_N([\xi_a, \xi_b]) \quad (4.10)$$

for $a, b = -1, 0, 1$, i.e., that $\mathbf{ver}_N : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(N+1, \mathbb{R})$ is a Lie algebra anti-homomorphism.

4.2 Matrix realizations of the Conformal Galilei Groups $\text{CGal}_N(d)$

Just as in the case of the Schrödinger group, see (2.10), we will strive integrating the conformal Galilei Lie algebras $\mathfrak{cgal}_N(d)$ within the matrix group $\text{GL}(d + N + 1, \mathbb{R})$. Let us, hence, introduce the *Conformal Galilei Group* with *dynamical exponent* $z = 2/N$ cf. (3.15), which we denote by $\text{CGal}_N(d)$; it consists of those matrices of the form

$$g = \begin{pmatrix} A & \mathbf{B}_N \cdots \mathbf{B}_0 \\ 0 & \text{Ver}_N(C) \end{pmatrix} \quad (4.11)$$

where $A \in \text{O}(d)$, $\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_N \in \mathbb{R}^d$, and $C \in \text{SL}(2, \mathbb{R})$.

We now prove that the Lie algebra of $\text{CGal}_N(d)$ is, indeed, $\mathfrak{cgal}_N(d)$ introduced in Section 3. In fact, putting $t = t_1/t_2$ in (4.2), wherever $t_2 \neq 0$, we easily find that the projective action $\mathfrak{g}_{\mathbb{R}^{d+1}} : (\mathbf{x}, t) \mapsto (\mathbf{x}^*, t^*)$ of $\text{CGal}_N(d)$ reads, locally, as

$$\begin{pmatrix} \mathbf{x}^* \\ t_*^N \\ \vdots \\ t^* \\ 1 \end{pmatrix} = \mathbb{R}^* \cdot g \begin{pmatrix} \mathbf{x} \\ t^N \\ \vdots \\ t \\ 1 \end{pmatrix}, \quad (4.12)$$

which, with the help of (4.3) and (4.4), leaves us with

$$\mathbf{x}^* = \frac{A\mathbf{x} + \mathbf{B}_N t^N + \cdots + \mathbf{B}_1 t + \mathbf{B}_0}{(ct + d)^N}, \quad (4.13)$$

$$t^* = \frac{at + b}{ct + d}. \quad (4.14)$$

These formulæ authorize the following interpretation for the parameters in (4.11), viz.,

$$\begin{aligned} A &: \text{orthogonal transformation,} \\ \mathbf{B}_0 &: \text{translation,} \\ \mathbf{B}_1 &: \text{boost,} \\ \mathbf{B}_2 &: \text{acceleration,} \\ &\vdots \\ \mathbf{B}_N &: \text{higher-order "acceleration",} \\ C &: \text{projective transformation of time.} \end{aligned} \quad (4.15)$$

Let us now write any vector in $\text{Lie}(\text{CGal}_N(d))$ as

$$Z = \begin{pmatrix} \omega & \boldsymbol{\beta}_N \cdots \boldsymbol{\beta}_0 \\ 0 & \mathbf{ver}_N(\xi) \end{pmatrix} \quad (4.16)$$

where we have used Equations (4.7)–(4.9) with

$$\xi = \begin{pmatrix} \lambda & \varepsilon \\ -\kappa & -\lambda \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}). \quad (4.17)$$

Then, the infinitesimal form of the transformation laws (4.13) and (4.14) writes as $\delta \mathbf{x} = \delta \mathbf{x}^*|_{\delta \mathbf{g}=Z, \mathbf{g}=\text{Id}}$, together with $\delta t = \delta t^*|_{\delta \mathbf{g}=Z, \mathbf{g}=\text{Id}}$, i.e.,

$$\delta \mathbf{x} = \omega \mathbf{x} + \boldsymbol{\beta}_N t^N + \cdots \boldsymbol{\beta}_1 t + \boldsymbol{\beta}_0 + N(\kappa t + \lambda) \mathbf{x}, \quad (4.18)$$

and

$$\delta t = \kappa t^2 + 2\lambda t + \varepsilon. \quad (4.19)$$

At last, the vector field $Z_{\mathbb{R}^{d+1}} = \delta x^i \partial / \partial x^i + \delta t \partial / \partial t$ associated with Z in (4.16) is such that

$$Z_{\mathbb{R}^{d+1}} = X \in \mathfrak{cgal}_N(d), \quad (4.20)$$

where the vector field X is as in (3.17), proving our claim.

Let us notice that our terminology for the dynamical exponent is justified by verifying that the dilation generator is (1.7) with $z = 2/N$.

The above definition of the conformal Galilei groups, see (4.11), yield their global structure

$$\text{CGal}_N(d) \cong (\text{O}(d) \times \text{SL}(2, \mathbb{R})) \ltimes \mathbb{R}^{(N+1)d}, \quad (4.21)$$

and $\dim(\text{CGal}_N(d)) = Nd + \frac{1}{2}d(d+1) + 3$.

- For $N = 1$, i.e., $z = 2$, we recover the Schrödinger group (2.10), and therefore

$$\text{CGal}_1(d) \cong \text{Sch}(d). \quad (4.22)$$

- For $N = 2$, i.e., $z = 1$, in particular, we get

$$\mathfrak{g} = \begin{pmatrix} A & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_0 \\ 0 & a^2 & 2ab & b^2 \\ 0 & ac & ad+bc & bd \\ 0 & c^2 & 2cd & d^2 \end{pmatrix} \in \text{CGal}_2(d) \quad (4.23)$$

with $A \in \text{O}(d)$, $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{R}^d$, $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. In addition to the usual space translations \mathbf{B}_0 , and Galilei boosts \mathbf{B}_1 , we also have extra generators, namely *accelerations* \mathbf{B}_2 [6, 7]. It is an easy matter to check that the actions (1.3), (1.4), and (1.5) of dilations, expansions, and accelerations, respectively, are recovered by considering their projective action given by (4.12) with $N = 2$.

The Lie algebra $\mathfrak{cgal}_2(d)$ of the Conformal Galilei Group, $\text{CGal}_2(d)$, is plainly isomorphic to the (centerless) Conformal Galilei Algebra (CGA), cf. [8, 6, 7]. It has been shown [6] that $\mathfrak{cgal}_2(d)$ admits a nontrivial 1-dimensional central extension in the planar case, $d = 2$, only.

5 Conformal Galilei spacetimes & cosmological constant

As mentioned in the Introduction, the physical spacetime can be recovered by positing some symmetry group — defining its geometry — and then by factoring out a suitable subgroup. The simplest example is to start with the neutral component of the Galilei group (2.5), namely

$$\text{Gal}_+(d) = (\text{SO}(d) \times \mathbb{R}) \ltimes \mathbb{R}^{2d}, \quad (5.1)$$

and factor out rotations and boosts to yield (ordinary) Galilei spacetime,

$$\mathbb{R}^d \times \mathbb{R} = \text{Gal}_+(d)/(\text{SO}(d) \times \mathbb{R}^d). \quad (5.2)$$

Similarly, one can start instead with a deformation of the Galilei group called Newton-Hooke group [1, 2, 3]

$$N^+(d) = (\text{SO}(d) \times \text{SO}(2)) \ltimes \mathbb{R}^{2d}, \quad (5.3)$$

$$N^-(d) = (\text{SO}(d) \times \text{SO}(1, 1)^\uparrow) \ltimes \mathbb{R}^{2d}, \quad (5.4)$$

where $\text{SO}(d) \times \text{SO}(2)$ and $\text{SO}(d) \times \text{SO}(1, 1)^\uparrow$ are, respectively, the direct products of spatial rotations, and translations of (compactified) time acting on the Abelian subgroup \mathbb{R}^{2d} of boosts and space-translations. Then, quotienting $N^\pm(d)$ by the direct product of rotations and boosts yields the *Newton-Hooke spacetimes* [3]. The latter carry a non-flat nonrelativistic structure and satisfies the Newton gravitational field equations with cosmological constant [3].

Below, we extend the above-mentioned construction to our conformal Galilei groups $\text{CGal}_N(d)$, at any level $N \geq 1$.

5.1 Conformal Galilei spacetimes

The conformal Galilei spacetimes M_N , associated with $z = 2/N$ where $N = 1, 2, \dots$, are introduced by starting with the conformal Galilei groups $\text{CGal}_{2/z}(d)$, viz.,

$$M_N = \text{CGal}_N(d)/H_N \quad \text{where} \quad H_N = (\text{O}(d) \times \text{Aff}(1, \mathbb{R})) \ltimes \mathbb{R}^{Nd}. \quad (5.5)$$

Explicitly, the projection $\pi : \text{CGal}_N(d) \rightarrow M_N$ in (5.5) is defined by the direction $\pi(\mathfrak{g}) = \mathbb{R}^* \cdot \mathfrak{g}_{d+N+1}$ of the last column-vector of the matrix, \mathfrak{g} , in (4.11). More intuitively, this amounts to factoring out $O(d)$, dilations and expansions and all higher than zeroth-order accelerations. By the very definition (4.11) of the conformal Galilei group at level N , we get indeed the conformal Galilei spacetime

$$M_N = (\mathbb{R}^d \times \text{Ver}_N(\mathbb{R}^2 \setminus \{0\})) / \mathbb{R}^*, \quad (5.6)$$

fibered above the Veronese curve $\text{Ver}_N(\mathbb{R}P^1) \subset \mathbb{R}P^N$ interpreted as the time axis, T .

To summarize, we have the following diagram

$$\begin{array}{ccc} \text{CGal}_N(d) & & \\ H_N \downarrow & & (5.7) \\ M_N \cong (\mathbb{R}^d \times \text{Ver}_N(S^1)) / \mathbb{Z}_2 & \xrightarrow{\mathbb{R}^d} & T \cong \text{Ver}_N(\mathbb{R}P^1). \end{array}$$

Note that for $N = 1$ the Schrödinger-homogeneous spacetime M_1 , i.e., the Möbius spacetime (2.12), is obtained.

We will from now on focus our attention to the new $\text{CGal}_N(d)$ -homogeneous spacetimes M_N .

5.2 Galilean conformal Cartan connections

Let us recall, for completeness, the definition of a *Cartan connection* on a H -principal fiber bundle $P \rightarrow M$ where H is a closed subgroup of a Lie group G , and where $\dim(M) = \dim(G/H)$. Put $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{h} = \text{Lie}(H)$.

Such a “connection” is given by a \mathfrak{g} -valued 1-form ω of P such that

1. $\omega(Z_P) = Z$ for all $Z \in \mathfrak{h}$
2. $(h_P)^*\omega = \text{Ad}(h^{-1})\omega$ for all $h \in H$
3. $\ker \omega = \{0\}$

where the subscript P refers to the group or Lie algebra (anti-)action on P . These “connections” provide a means to encode the geometry of manifolds modeled on homogeneous spaces, e.g., projective or conformal geometry.

We will show, in this section, that the homogeneous spaces M_N (see (5.6)) indeed admit, for all $N = 1, 2, \dots$, a conformal Newton-Cartan structure together with a distinguished, *flat*, normal Cartan connection associated with $\text{CGal}_N(d)$.⁵ The

⁵Since we are dealing here with homogeneous spaces G/H , it will naturally be given by the (left-invariant) Maurer-Cartan 1-form of the corresponding groups G .

general construction of the normal Cartan connection associated with a Schrödinger-conformal Newton-Cartan structure has been performed in [17].

- The Galilei group $\text{Gal}(d)$ can be viewed as the bundle of Galilei frames over spacetime $\mathbb{R}^d \times \mathbb{R} = \text{Gal}(d)/(O(d) \times \mathbb{R}^d)$, cf. (5.2). Using (2.5), we find that the (left-invariant) Maurer-Cartan 1-form $\Theta_{\text{Gal}(d)} = g^{-1}dg$ reads

$$\Theta_{\text{Gal}(d)} = \begin{pmatrix} \omega & \beta_1 & \beta_0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.8)$$

where the 1-forms β_0 , and ε are interpreted as the components of the soldering 1-form of the principal bundle $\text{Gal}(d) \rightarrow \mathbb{R}^d \times \mathbb{R}$. Then the Maurer-Cartan structure equations $d\Theta + \Theta \wedge \Theta = 0$ read

$$0 = d\omega + \omega \wedge \omega, \quad (5.9)$$

$$0 = d\beta_1 + \omega \wedge \beta_1, \quad (5.10)$$

$$0 = d\beta_0 + \omega \wedge \beta_0 + \beta_1 \wedge \varepsilon, \quad (5.11)$$

$$0 = d\varepsilon. \quad (5.12)$$

Equations (5.9, 5.10) entails that the NC-connection 2-form (ω, β_1) is flat, while (5.11, 5.12) insure zero torsion.⁶ See, e.g., [24, 17].

- Likewise, if $N = 1$, the Schrödinger group $\text{Sch}(d)$ may be thought of as a subbundle of the bundle of 2-frames of spacetime $M_1 \cong \text{Sch}(d)/H_1$, see (5.5).⁷ This time, $\text{Sch}(d)$ is, indeed, interpreted as the bundle of conformal Galilei 2-frames associated with the conformal class $\gamma \otimes \theta$ of a Galilei structure (γ, θ) over M_1 as given by Equation (3.5) with $m = n = 1$. The Maurer-Cartan 1-form $\Theta_{\text{Sch}(d)}$ of this group actually gives rise to the canonical flat Cartan connection on the H_1 -bundle $\text{Sch}(d) \rightarrow M_1$. We refer to [17, 18] for a comprehensive description of Schrödinger conformal Cartan connections.

Using the same notation as in (2.18), we find

$$\Theta_{\text{Sch}(d)} = \begin{pmatrix} \omega & \beta_1 & \beta_0 \\ 0 & \lambda & \varepsilon \\ 0 & -\kappa & -\lambda \end{pmatrix} \quad (5.13)$$

⁶We note that the Galilei structure is given by $\gamma = \delta^{ij} A_i \otimes A_j$, where $A \in O(d)$ represents an orthonormal frame (see (2.5)), together with the clock 1-form $\theta = \varepsilon$.

⁷The bundle of 2-frames is called upon since the vector fields (2.17) spanning $\mathfrak{sch}(d)$ are polynomials of degree 2 in the spacetime coordinates.

and, hence, the structure equations

$$0 = d\omega + \omega \wedge \omega, \quad (5.14)$$

$$0 = d\beta_1 + \omega \wedge \beta_1 - \beta_0 \wedge \kappa, \quad (5.15)$$

$$0 = d\beta_0 + \omega \wedge \beta_0 + \beta_1 \wedge \varepsilon - \beta_0 \wedge \lambda, \quad (5.16)$$

$$0 = d\lambda - \varepsilon \wedge \kappa, \quad (5.17)$$

$$0 = d\varepsilon - 2\varepsilon \wedge \lambda, \quad (5.18)$$

$$0 = d\kappa - 2\kappa \wedge \lambda. \quad (5.19)$$

Furthermore, a Galilei structure can be introduced on spacetime M_1 viewed as the quotient of the Newton-Hooke group (5.3). Our clue is that embedding $\text{SO}(2)$ into $\text{SL}(2, \mathbb{R})$ through

$$\vartheta \mapsto \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}, \quad (5.20)$$

the Newton-Hooke group⁸

$$\text{New}_1(d) = (\text{O}(d) \times \text{SO}(2)) \ltimes (\mathbb{R}^d \times \mathbb{R}^d) \quad (5.21)$$

becomes a *subgroup* of the Schrödinger group,

$$\text{New}_1(d) \subset \text{Sch}(d). \quad (5.22)$$

Then, $\text{New}_1(d)$ can readily be identified with the bundle of Galilei frames of

$$M_1 \cong \text{New}_1(d) / (\text{O}(d) \times \mathbb{Z}_2) \ltimes \mathbb{R}^d. \quad (5.23)$$

Introducing the pulled-back Maurer-Cartan 1-form, $\Theta_{\text{New}_1(d)}$, we end up with the previous structure equations (5.14)–(5.19) specialized to the case $\lambda = 0$, and $\kappa = \varepsilon$. Comparison between the latter equations, and the (flat) Galilei structure equations shows that both sets coincide except for the equations characterizing $d\beta_1$. This entails that M_1 acquires curvature through the 2-form [17]

$$\Omega_1 = \beta_0 \wedge \varepsilon. \quad (5.24)$$

Let us present, for completeness, a local expression of the NC-Cartan connection on M_1 generated by $\Theta_{\text{New}_1(d)}$. Putting $\mathbf{x} = \mathbf{B}_0$ defines, along with $\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})$ introduced in (5.20), a coordinate system on spacetime M_1 . We readily find the components of the soldering 1-form to be $(\beta_0 = A^{-1}(d\mathbf{x} - \mathbf{B}_1 d\vartheta), \varepsilon = d\vartheta)$, while those of the NC-connection write $(\omega = A^{-1}dA, \beta_1 = A^{-1}(d\mathbf{B}_1 + \mathbf{x} d\vartheta))$. From the

⁸Its neutral component is $N^+(d)$, as given in (5.3).

last expression we infer that the only nonzero components of the Christoffel symbols of the NC-connection are

$$\Gamma_{\vartheta\vartheta}^i = x^i, \quad i = 1, \dots, d. \quad (5.25)$$

This entails that the nonzero components of curvature tensor R_1 , associated with $\mathbf{\Omega}_1$, are $(R_1)_{j\vartheta\vartheta}^i = \delta_j^i$ for all $i, j = 1, \dots, d$. The only nonvanishing component of the Ricci tensor is therefore

$$(\text{Ric}_1)_{\vartheta\vartheta} = d. \quad (5.26)$$

Hence, our NC-connection (ω, β_1) provides us with a solution of the NC-field equations (2.3),

$$\text{Ric}_1 + \Lambda_1 \theta \otimes \theta = 0, \quad (5.27)$$

where the clock is $\theta = d\vartheta$, and the cosmological constant is given by the dimension of space,

$$\Lambda_1 = -d. \quad (5.28)$$

- For the case $N > 1$, the Maurer-Cartan 1-form of $\text{CGal}_N(d)$ reads

$$\Theta_{\text{CGal}_N(d)} = \begin{pmatrix} \omega & \beta_N & \beta_{N-1} & \cdots & \beta_1 & \beta_0 \\ 0 & N\lambda & N\varepsilon & \cdots & 0 & 0 \\ 0 & -\kappa & (N-1)\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 2\varepsilon & 0 \\ 0 & 0 & \cdots & \cdots & (1-N)\lambda & \varepsilon \\ 0 & 0 & 0 & 0 \cdots 0 & -N\kappa & -N\lambda \end{pmatrix} \quad (5.29)$$

with the same notation as before.

Then, the preceding computations can be reproduced, *mutatis mutandis*, for the group $\text{CGal}_N(d)$ which serves as the bundle of conformal Galilei N -frames of M_N . At that point, as a straightforward generalization of (5.21), we can define the *Newton-Hooke group at level N* as

$$\text{New}_N(d) = (\text{O}(d) \times \text{SO}(2)) \ltimes \mathbb{R}^{d(N+1)} \subset \text{CGal}_N(d). \quad (5.30)$$

Much in the same manner as in the case $N = 1$, the NC-connection obtained from the pull-back $\Theta_{\text{New}_N(d)}$ of the Maurer-Cartan 1-form (5.29) can be computed. It is easily shown that the curvature of the homogeneous spacetimes M_N is now given by

$$\mathbf{\Omega}_N = N\mathbf{\Omega}_1. \quad (5.31)$$

This connection turns out to produce an exact solution of the vacuum NC-field equations (2.3), the cosmological constant at level N being now given by

$$\Lambda_N = -Nd \tag{5.32}$$

for all $N = 1, 2, \dots$

6 Conclusion and outlook

Our main results are two-fold: firstly, we have found that the infinite-dimensional Lie algebra of infinitesimal conformal Galilei transformations with rational dynamical exponent admits, in fact, finite dimensional Lie subalgebras provided the dynamical exponent is $z = 2/N$, where $N = 1, 2, \dots$. Then, we have proposed a natural construction devised to integrate, for each N , these Lie algebras into Lie groups, named *Conformal Galilei groups at level N* , by means of the classic Veronese embeddings [30]. The values $N = 1$ and $N = 2$ correspond to the Schrödinger Lie algebra, and to the CGA, respectively. It is worth noting that the values $N = 4$ and $N = 6$, i.e., the dynamical exponents $z = 1/2$ and $z = 1/3$ arise in statistical mechanics, namely for the spin-spin correlation function in the axial next-nearest-neighbor spherical model at its Lifschitz points of first and second order [20, 5].

Our Conformal Galilei groups, $\text{CGal}_N(d)$, which generalize the Schrödinger ($N = 1$) and the Conformal Galilei ($N = 2$) cases to any integer N , do *not* act regularly on ordinary flat Galilean spacetime, however; they act rather on manifolds constructed from them, which are analogous to the conformal compactification of Minkowski spacetime. Moreover, their group structure allows us to recover these spacetimes as homogeneous spaces which turn out to be endowed with a (conformal) Newton-Cartan structure by construction. Remarkably, they are identified as *Newton-Hooke spacetimes with quantized negative cosmological constant*, Λ_N in (5.32).⁹

We would like to mention that a complete, general, construction of Cartan connections for conformal Galilei structures, modeled on the $\text{CGal}_N(d)$ -homogeneous manifolds M_N , still remains to be undertaken in order to extend that of normal Schrödinger-Cartan connections carried out in [18].

Let us now discuss the relationship of our procedures and technique to some other work on the same subject, namely to conformal Galilean symmetries.

Gibbons and Patricot [3] derive the Newton-Cartan structure of Newton-Hooke spacetime in the ‘‘Bargmann’’ framework of Ref. [10]. This null ‘‘Kaluza-Klein-type’’ approach provides us indeed with a preferred way of defining a nonrelativistic

⁹Spacetimes with positive cosmological constant would be obtained by replacing $\text{SO}(2)$ by $\text{SO}(1, 1)^\uparrow$ as in (5.4); see also [3].

structure on spacetime, “downstairs”. Then the (mass-centrally extended) Galilei and Newton-Hooke groups both appear as isometries, whereas their conformal extensions appear as conformal transformations of the respective Bargmann spaces. The fact that both the “empty” and Newton-Hooke spaces carry a Schrödinger symmetry [32] is then explained by the fact that their Bargmann manifolds are (Bargmann-)conformally related [33].

A similar explanation holds for the results in [34, 35], where motions in a homogeneous magnetic field are considered. This problem fits also into the Bargmann framework, and provides us in fact with yet another example of a conformally flat Bargmann spacetime [36, 32].

No central terms are considered in this paper: our Lie algebras are represented by vector fields on (flat) Newton-Cartan spacetime. Central terms, and the mass in particular, are important, though, and are indeed necessary for physical applications. Henkel [20, 5] does actually consider mass terms: he works with *operators* with such terms involving c^{-2} , where c is the speed of light. In his approach, those appear in the coefficients of *powers* of $\partial/\partial t$. Considering higher-order terms in powers of $\partial/\partial t$ goes beyond our framework, though. For finite c , Henkel’s boosts are Lorentzian, not Galilean, however; our center-free Lie algebras with genuine Galilei boosts are recovered as $c \rightarrow \infty$, yielding also $(N-1)$ accelerations in addition. At last, the mass terms disappear, as they should: mass and accelerations are indeed inconsistent [6].

Central extensions of the conformal Galilei algebra have been considered in [6, 35] in the planar case, $d = 2$, and in [19] for $d = 3$.

Let us finally mention a natural further program, in the wake of the present study, namely the determination of the group-cohomologies of our Conformal Galilei groups. Also would it be worthwhile to classify all symplectic homogeneous spaces [21] of the latter, likely to unveil new, physically interesting, systems.

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