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Conformal Galilei groups, Veronese curves, and Newton-Hooke spacetimes

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Abstract

Finite-dimensional nonrelativistic conformal Lie algebras spanned by polynomial vector fields of Galilei spacetime arise if the dynamical exponent is $z = 2/N$ with $N = 1, 2, \ldots$. Their underlying group structure and matrix representation are constructed (up to a covering) by means of the Veronese map of degree N. Suitable quotients of the conformal Galilei groups provide us with Newton-Hooke nonrelativistic spacetimes with a quantized reduced negative cosmological constant $\lambda = -N$.

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1 Introduction

Newton-Hooke spacetimes provide us with solutions of the nonrelativistic gravitational field equations with nonvanishing cosmological constant, and may play a role in cosmology [1, 2, 3, 4, 5, 6]. They can be viewed as deformations of their Galilean counterparts to which they reduce when the cosmological constant is turned off, and can indeed be obtained as nonrelativistic limits of the de Sitter or anti-de Sitter solutions of Einstein's equations.

Another way of constructing these nonrelativistic spacetimes is to first contract the (anti-)de Sitter group to yield the "Newton-Hooke" group(s), and then factor out the homogeneous part of the latter [3].

On the other hand, various conformal extensions of the Galilei Lie algebra have attracted much recent attention [7, 8, 9, 10, 11, 12], and one may wonder about their group structure and associated homogeneous spacetimes.

This paper is devoted to studying this question.

The most common, and historically first, of such extensions, referred to as the *Schrödinger group* [13, 14, 15], has been first discovered in classical mechanics [16], and then for the heat equation [17], before being forgotten for almost a century and then rediscovered as the maximal group of symmetries of the free Schrödinger equation [18].

The Schrödinger group admits, in addition to those of the Galilei group, two more generators given by their spacetime actions $(\boldsymbol{x}, t) \mapsto (\boldsymbol{x}^*, t^*)$, namely *dilations*

$$\boldsymbol{x}^* = a\,\boldsymbol{x}, \qquad t^* = a^2 t, \tag{1.1}$$

with $a \in \mathbb{R}^*$, and *expansions* (also called *inversions*)

$$\boldsymbol{x}^* = \Omega(t)\,\boldsymbol{x}, \qquad t^* = \Omega(t)\,t, \tag{1.2}$$

where

$$\Omega(t) = \frac{1}{ct+1}, \tag{1.3}$$

with $c \in \mathbb{R}$.

These transformations generate, along with Galilean time-translations: $\boldsymbol{x}^* = \boldsymbol{x}, t^* = t + b$, with $b \in \mathbb{R}$, the unimodular group $\mathrm{SL}(2, \mathbb{R})$. Note that the *dynamical exponent* is $z = 2$; see (1.1). Schrödinger symmetry typically arises for massive systems, as it combines with the one-parameter central extension of the Galilei group.

The *Conformal Galilei* (CG) symmetry algebra[1] [9, 11] was first found, and then discarded, by Barut in his attempt to derive the by then newly (re)discovered Schrödinger symmetry by contraction from the relativistic conformal Lie algebra [19]. At the group level, this new symmetry also features an $\mathrm{SL}(2, \mathbb{R})$ subgroup generated by time-translations augmented with modified dilations

$$\boldsymbol{x}^* = a\,\boldsymbol{x}, \qquad t^* = a\,t, \tag{1.4}$$

and expansions

$$\boldsymbol{x}^* = \Omega^2(t)\,\boldsymbol{x}, \qquad t^* = \Omega(t)\,t, \tag{1.5}$$

with the same parameters and factor $\Omega(t)$ as above.

This second type of nonrelativistic conformal symmetry has dynamical exponent is $z = 1$, and also contains *accelerations*

$$\boldsymbol{x}^* = \boldsymbol{x} + \mathbf{B}_2 t^2, \qquad t^* = t, \tag{1.6}$$

where $\mathbf{B}_2 \in \mathbb{R}^d$ (our notation will be justified below, see (4.12) and (4.14)). Moreover, this second type of conformal extension only allows for a *vanishing mass* [9]. It is rather difficult therefore to find physical systems with this kind of symmetry [20].

[1] Henkel [7] refers to it as to "Alt_1".

Both types of nonrelativistic symmetries have been related to the geometric "Newton-Cartan" structure of nonrelativistic spacetime [18, 21, 22, 23, 12].

Now, as recognized by Negro et al. [23], and by Henkel [24, 7], both infinitesimal Schrödinger and CG symmetry belong to a much larger, generally infinite dimensional, class of Lie algebras with *arbitrary*, possibly even *fractional, dynamical exponent* z; their "conformal nonrelativistic algebra" [23] is, however finite dimensional for the particular values

$$z = \frac{2}{N}\,, \qquad N = 1, 2, \ldots \tag{1.7}$$

The terminology is justified by that, for all z as in (1.7), the algebra has an $\mathfrak{sl}(2,\mathbb{R})$ Lie subalgebra, highlighted by the dilation generator

$$X = \frac{1}{z}\boldsymbol{x}\cdot\frac{\partial}{\partial\boldsymbol{x}} + t\,\frac{\partial}{\partial t}. \tag{1.8}$$

Taking into account rotations, boosts, and translations, yields, for $z = 2$, the Schrödinger algebra; the CG algebra is obtained, for $z = 1$, after incorporating also accelerations.

For both $N = 1$ (Schrödinger) and $N = 2$ (CG), the infinitesimal action integrates to a *Lie group* action, but for general z, the results known so far only concern *Lie algebras*. Our first new result is the derivation of the *global group structure* for all N as in (1.7).

A crucial observation for our purposes is the following: owing to the factor (1.3) in (1.2) and (1.5), *neither* Schrödinger, nor Conformal Galilei transformations are *globally well-defined* over ordinary Galilean spacetime. As explained in Sections 2.3 and 4, Galilei spacetime should be replaced by a "better one". Our investigations in Section 5 show indeed that the proper arena where our conformal Galilei symmetry groups act is in fact provided by Newton-Hooke spacetimes with quantized negative cosmological constant.

2 Nonrelativistic spacetimes

The standard *Galilei spacetime* is the affine space modeled on $\mathbb{R}^{d+1}$, endowed with its canonical flat affine connection Γ, and a *Galilei structure* (γ,θ) defined by a pair of (covariantly) constant tensor fields: namely by a spatial "metric" and a "clock", expressed in an affine coordinate system $(x^1,\ldots,x^d,x^{d+1})$ as

$$\gamma = \sum_{i=1}^{d}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^i}, \qquad \theta = \mathrm{d}t \tag{2.1}$$

respectively, where $t = x^{d+1}$ is an affine coordinate of the time axis, $T \cong \mathbb{R}$ [26, 27, 28, 29]. Notice that θ spans $\ker(\gamma)$.

2.1 Newton-Cartan structures

Generalized Galilei structures consist therefore of *triples* (M, γ, θ) where M is a smooth $(d+1)$-dimensional spacetime manifold, γ a twice-symmetric contravariant tensor field of M whose kernel is spanned by a nowhere vanishing closed 1-form θ. Due to the lack of a canonical affine connection on a Galilei structure, one is compelled to introduce then *Newton-Cartan* (NC) structures as *quadruples* $(M, \gamma, \theta, \Gamma)$ where (M, γ, θ) is a Galilei structure, and Γ a symmetric affine connection compatible with (γ, θ) whose curvature tensor, R, satisfies non-trivial extra symmetries which read, locally,

$$\gamma^{\mu\beta} R^{\sigma}_{\alpha\mu\rho} = \gamma^{\mu\sigma} R^{\beta}_{\rho\mu\alpha} \tag{2.2}$$

for all $\alpha, \beta, \rho, \sigma = 1, \ldots, d+1$.

Upon introducing field equations relating the Ricci tensor to the mass-density, ϱ, of the sources and the cosmological constant, Λ, viz.,

$$\mathrm{Ric} = (4\pi G\varrho - \Lambda)\theta \otimes \theta, \tag{2.3}$$

the connection Γ is interpreted as the gravitational field in a purely geometric generalization of Newtonian gravitation theory [26, 27, 28]. See [14] for a formulation of Newton-Cartan theory in a Kaluza-Klein type ("Bargmann") framework.

2.2 The Galilei group and its Lie algebra

The *Galilei group*, $\mathrm{Gal}(d)$, consists of all diffeomorphisms g of space-time which preserve all three ingredients of the Galilei structure, i.e., such that

$$g_*\gamma = \gamma, \quad g_*\theta = \theta, \quad g_*\Gamma = \Gamma. \tag{2.4}$$

This is the group of symmetries that governs nonrelativistic physics in d spatial dimensions. It clearly consists of $(d+2) \times (d+2)$ matrices of the form [25]

$$g = \begin{pmatrix} A & \mathbf{B}_1 & \mathbf{B}_0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Gal}(d), \tag{2.5}$$

where $A \in \mathrm{O}(d)$, and $\mathbf{B}_0, \mathbf{B}_1 \in \mathbb{R}^d$ stand respectively for a space translation and a boost, and $b \in \mathbb{R}$ is a time translation. The (affine) action of $\mathrm{Gal}(d)$ on space-time $\mathbb{R}^d \times \mathbb{R}$ reads

$$g_{\mathbb{R}^{d+1}} : \begin{pmatrix} \boldsymbol{x} \\ t \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} A\boldsymbol{x} + \mathbf{B}_1 t + \mathbf{B}_0 \\ t + b \\ 1 \end{pmatrix}. \tag{2.6}$$

Infinitesimal Galilei transformations form hence a Lie algebra, $\mathfrak{gal}(d)$, spanned by all vector fields X on space-time such that

$$L_X\gamma = 0, \quad L_X\theta = 0, \quad L_X\Gamma = 0 \tag{2.7}$$

(see [27, 21] for a generalization to (curved) NC structures); these vector field read

$$X = \left(\omega^i_j x^j + \beta^i_1 t + \beta^i_0\right)\frac{\partial}{\partial x^i} + \varepsilon\frac{\partial}{\partial t}, \tag{2.8}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1 \in \mathbb{R}^d$, and $\varepsilon \in \mathbb{R}$. Latin indices run in the range $1, \ldots, d$, and Einstein's summation convention is assumed throughout this article.

The Lie algebra $\mathfrak{gal}(d)$ admits the faithful $(d+2)$-dimensional anti-representation $X \mapsto Z$ where,

$$Z = \begin{pmatrix} \omega & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gal}(d) \tag{2.9}$$

with the above notation.

2.3 The Schrödinger group and its Lie algebra

Let us first discuss the *Schrödinger group*, $\mathrm{Sch}(d)$, which includes, in addition to the standard Galilei generators, those of the projective group, $\mathrm{PSL}(2,\mathbb{R})$, of the time axis. Up to a quotient that we will make more precise later on, the Schrödinger group will be defined as the matrix group whose typical element reads [30, 21]

$$g = \begin{pmatrix} A & \mathbf{B}_1 & \mathbf{B}_0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \in \mathrm{Sch}(d), \tag{2.10}$$

where $A \in \mathrm{O}(d)$, $\mathbf{B}_0, \mathbf{B}_1 \in \mathbb{R}^d$, and $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. The *projective* "action" of $g \in \mathrm{Sch}(d)$ on spacetime $\mathbb{R}^d \times \mathbb{R}$ takes the form

$$g_{\mathbb{R}^{d+1}} : \begin{pmatrix} \boldsymbol{x} \\ t \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{A\boldsymbol{x} + \mathbf{B}_1 t + \mathbf{B}_0}{ct + d} \\ \dfrac{at + b}{ct + d} \\ 1 \end{pmatrix} \tag{2.11}$$

defined on the *open subset of spacetime* where $ct + d \neq 0$.

It is an easy matter to check that the action (2.11) is consistent with the one presented in the introduction; Schrödinger dilations (1.1) correspond to $b = 0, c = 0$, and expansions (1.2) to $a = 1, b = 0, d = 1$.

The group structure is $\mathrm{Sch}(d) = (\mathrm{O}(d) \times \mathrm{SL}(2, \mathbb{R})) \ltimes (\mathbb{R}^d \times \mathbb{R}^d)$.

Now, in order to guarantee a well-behaved action of this group on spacetime, one must demand that *time be compactified*, $T \cong \mathbb{R}\mathrm{P}^1$. In fact, the Schrödinger group does *not* act on "ordinary" Galilei spacetime, but rather on the *Möbius manifold*

$$M = \big(\mathbb{R}^d \times (\mathbb{R}^2 \backslash \{0\})\big)/\mathbb{R}^* \tag{2.12}$$

fibered above the projective line, $\mathbb{R}\mathrm{P}^1$, as clear from (2.11). This point will be further developed in Section 5.1. See also [21].

Note that (2.12) can be recovered by factoring out the homogeneous subgroup generated by rotations, expansions, dilations, and boosts,

$$M = \mathrm{Sch}(d)/H \quad \text{where} \quad H = \big(\mathrm{O}(d) \times \mathrm{Aff}(1, \mathbb{R})\big) \ltimes \mathbb{R}^d, \tag{2.13}$$

where $\mathrm{Aff}(1, \mathbb{R})$ stands for the 2-dimensional group of lower-triangular matrices in $\mathrm{SL}(2, \mathbb{R})$, generated by dilations and expansions.

Note that, unlike conformally compactified Minkowski spacetime $(S^d \times S^1)/\mathbb{Z}_2$, only time, not space, is compactified here since

$$M \cong (\mathbb{R}^d \times S^1)/\mathbb{Z}_2. \tag{2.14}$$

It will be shown in Section 5 that the Möbius manifold carries a nonrelativistic Newton-Cartan structure; it is, in fact a *Newton-Hooke spacetime with cosmological constant* $\Lambda = -d$, minus the dimension of space; see (5.47).

The Schrödinger group can, indeed, be defined in a geometric way, namely in the NC framework [21, 14, 12], as the group, $\mathrm{Sch}(d)$, of all (locally defined) diffeomorphisms g such that

$$g_*(\gamma \otimes \theta) = \gamma \otimes \theta \qquad \& \qquad g \in \mathrm{Proj}(\mathbb{R}^{d+1}, \Gamma), \tag{2.15}$$

where $\mathrm{Proj}(\mathbb{R}^{d+1}, \Gamma)$ denotes the set of all projective transformations of spacetime, namely of all (local) diffeomorphisms which permute the geodesics of spacetime w.r.t. the connection Γ. Let us stress that the conditions (2.15) imply, in particular, that the diffeomorphism g projects on the time axis as an element of $\mathrm{PGL}(2, \mathbb{R})$ which must also preserve time-orientation defined by θ, namely an element of $\mathrm{PSL}(2, \mathbb{R})$. The general solution of (2.15) is therefore given by (2.10), up to a covering; see also (2.11).

The Schrödinger Lie algebra, $\mathfrak{sch}(d)$, is then the Lie algebra of those vector fields X on spacetime such that

$$L_X(\gamma \otimes \theta) = 0 \qquad \& \qquad X \in \mathfrak{proj}(\mathbb{R}^{d+1}, \Gamma). \tag{2.16}$$

In local terms, we thus require

$$L_X \gamma^{\alpha\beta} \theta_\rho + \gamma^{\alpha\beta} L_X \theta_\rho = 0 \qquad \& \qquad L_X \Gamma^\rho_{\alpha\beta} = \delta^\rho_\alpha \varphi_\beta + \delta^\rho_\beta \varphi_\alpha \tag{2.17}$$

for some 1-form φ of $\mathbb{R}^{d+1}$ depending on X, and for all $\alpha, \beta, \rho = 1, \ldots, d+1$.

We easily find that $X \in \mathfrak{sch}(d)$ iff

$$X = \left(\omega^i_j\, x^j + \kappa t x^i + \lambda x^i + \beta^i_1 t + \beta^i_0\right) \frac{\partial}{\partial x^i} + \left(\kappa t^2 + 2\lambda t + \varepsilon\right) \frac{\partial}{\partial t}\,, \tag{2.18}$$

where $\omega \in \mathfrak{so}(d)$, $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1 \in \mathbb{R}^d$, and $\kappa, \lambda, \varepsilon \in \mathbb{R}$. The Schrödinger dilation (or homothety) generator is, indeed, (1.8) with dynamical exponent $z = 2$.

The Lie algebra $\mathfrak{sch}(d)$ admits the faithful $(d+2)$-dimensional anti-representation $X \mapsto Z$, where

$$Z = \begin{pmatrix} \omega & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_0 \\ 0 & \lambda & \varepsilon \\ 0 & -\kappa & -\lambda \end{pmatrix} \in \mathfrak{sch}(d) \tag{2.19}$$

with the same notation as above.

Note that $\mathfrak{sch}(d)$ is, in fact, the (centerless) Schrödinger Lie algebra. Physical applications also involve a central extension associated with the mass; see, e.g., [21, 22, 13, 14, 15, 7, 12].

A remarkable property of the Schrödinger group, arising as a symmetry group of the classical space of motions of free spinning particle [12], and also important for studying supersymmetric extensions [31], is that it can be faithfully imbedded into the affine-symplectic Lie algebra

$$\mathrm{Sch}(d) \subset \mathrm{Sp}(d, \mathbb{R}) \ltimes \mathbb{R}^{2d}. \tag{2.20}$$

3 Conformal Newton-Cartan transformations & finite-dimensional conformal Galilei Lie algebras

In close relationship with the Lorentzian framework, we call *conformal Galilei* transformation of a general Galilei spacetime (M, γ, θ) any diffeomorphism of M that preserves the direction of γ. Owing to the fundamental constraint $\gamma(\theta) = 0$, it follows that conformal Galilei transformations automatically preserve the direction of the time 1-form θ.

In terms of infinitesimal transformations, a *conformal Galilei* vector field of (M, γ, θ) is a vector field, X, of M that Lie-transports the direction of γ; we will thus define $X \in \mathfrak{cgal}(M, \gamma, \theta)$ iff

$$L_X \gamma = f\gamma \qquad \text{hence} \qquad L_X \theta = g\,\theta \tag{3.1}$$

for some smooth functions f, g of M, depending on X. Then, $\mathfrak{cgal}(M, \gamma, \theta)$ becomes a Lie algebra whose bracket is the Lie bracket of vector fields.

The one-form θ being parallel-transported by the NC-connection, one has necessarily $d\theta = 0$; this yields $dg \wedge \theta = 0$, implying that g is (the pull-back of) a smooth function on T, i.e., that $g(t)$ depends arbitrarily on time $t = x^{d+1}$, which locally parametrizes the time axis. We thus have $dg = g'(t)\theta$.

3.1 Conformal Galilei transformations, $\mathfrak{cgal}_{2/z}(d)$, with dynamical exponent z

One can, at this stage, try and seek nonrelativistic avatars of general relativistic infinitesimal conformal transformations. Given a Lorentzian (or, more generally, a pseudo-Riemannian) manifold (M, g), the latter Lie algebra is generated by the vector fields, X, of M such that

$$L_X(g^{-1} \otimes g) = 0, \tag{3.2}$$

where g^{-1} denotes the inverse of the metric $g : TM \to T^*M$.

It has been shown [29] that one can expand a Lorentz metric in terms of the small parameter $1/c^2$, where c stands for the speed of light, as

$$g = c^2\theta \otimes \theta - {}^U\gamma + \mathcal{O}(c^{-2}), \qquad g^{-1} = -\gamma + c^{-2}U \otimes U + \mathcal{O}(c^{-4}), \tag{3.3}$$

with the previous notation. Here U is an "observer", i.e., a smooth timelike vector field of spacetime M, such that $g(U, U) = c^2$, around which the light-cone opens up in order to consistently define a procedure of nonrelativistic limit. The Galilei structure (γ, θ) is recovered via $\gamma = -\lim_{c\to\infty} g^{-1}$, and $\theta = \lim_{c\to\infty}(c^{-2}g(U))$. In (3.3) the symmetric twice-covariant tensor field ${}^U\gamma$ will define the Riemannian metric of the spacelike slices in the limiting Galilei structure.

We can thus infer that the nonrelativistic limit of Equation (3.2) would be $L_X \lim_{c\to\infty}(c^{-2}\, g^{-1} \otimes g) = 0$, viz.

$$L_X(\gamma \otimes \theta \otimes \theta) = 0. \tag{3.4}$$

More generally, we consider

$$L_X(\gamma^{\otimes m} \otimes \theta^{\otimes n}) = 0, \tag{3.5}$$

for some $m = 1, 2, 3, \ldots$, and $n = 0, 1, 2, \ldots$, to be further imposed on the vector fields $X \in \mathfrak{cgal}(M, \gamma, \theta)$. Then the quantity

$$z = \frac{2}{q} \qquad \text{where} \qquad q = \frac{n}{m} \tag{3.6}$$

matches the ordinary notion of dynamical exponent [15, 7, 12].

We will, hence, introduce the Galilean avatars, $\mathfrak{cgal}_{2/z}(M, \gamma, \theta)$, of the Lie algebra $\mathfrak{so}(d+1, 2)$ of conformal vector fields of a pseudo-Riemannian structure of signature $(d, 1)$ as the Lie algebras spanned by the vector fields X of M satisfying (3.1), and (3.5). We will call $\mathfrak{cgal}_{2/z}(M, \gamma, \theta)$ the *conformal Galilei Lie algebra with dynamical exponent* z in (3.6). This somewhat strange notation will be justified in the sequel.

The Lie algebra

$$\mathfrak{sv}(M, \gamma, \theta) = \mathfrak{cgal}_1(M, \gamma, \theta) \tag{3.7}$$

is the obvious generalization to Galilei spacetimes of the *Schrödinger-Virasoro* Lie algebra $\mathfrak{sv}(d) = \mathfrak{sv}(\mathbb{R} \times \mathbb{R}^d, \gamma, \theta)$ introduced in [15] (see also [7]) from a different viewpoint in the case of a flat NC-structure. The representations of the Schrödinger-Virasoro group and of its Lie algebra, $\mathfrak{sv}(d)$, as well as the deformations of the latter have been thoroughly studied and investigated in [32, 33].

Let us henceforth use the notation $\mathfrak{cgal}_{2/z}(d) = \mathfrak{cgal}_{2/z}(\mathbb{R}^{d+1}, \gamma, \theta)$ with γ as in (2.1) and and $\theta = dt$ respectively. Then one shows [12] that $X \in \mathfrak{cgal}_{2/z}(d)$ iff

$$X = \Big(\omega^i_j(t)x^j + \frac{1}{z}\xi'(t)x^i + \beta^i(t)\Big)\frac{\partial}{\partial x^i} + \xi(t)\frac{\partial}{\partial t}\,, \tag{3.8}$$

where $\omega(t) \in \mathfrak{so}(d)$, $\boldsymbol{\beta}(t)$, and $\xi(t)$ depend arbitrarily on time, t.

The Lie algebra $\mathfrak{cgal}_0(M, \gamma, \theta)$ corresponding to the case $z = \infty$ is also interesting; it is a Lie algebra of symplectomorphisms of the models of massless and spinning Galilean particles [21, 12].

3.2 The Lie algebra, $\mathfrak{cgal}_N(d)$, of finite-dimensional conformal Galilei transformations

Now we show that our formalism leads to a natural definition of a whole family of distinguished finite-dimensional Lie subalgebras of the conformal Galilei Lie algebra $\mathfrak{cgal}_{2/z}(d)$ with prescribed dynamical exponent z, generated by the vector fields in (3.8), where $\omega(t) \in \mathfrak{so}(d)$, $\boldsymbol{\beta}(t)$, and $\xi(t)$ depend smoothly on time, t.

Referring to [12] for details, let us restrict our attention to those vector fields $X \in \mathfrak{cgal}_{2/z}^{\mathrm{Pol}}(d)$ that are *polynomials* of fixed degree $N > 0$ in the variables $x^1, \ldots, x^d$, and $t = x^{d+1}$. We then have necessarily

$$\xi(t) = \kappa t^2 + 2\lambda t + \varepsilon, \tag{3.9}$$

with $\kappa, \lambda, \varepsilon \in \mathbb{R}$, and we find that a closed Lie algebra of polynomial vector fields of degree $N > 0$ is obtained provided

$$z = \frac{2}{N}. \tag{3.10}$$

At last, we find that $X \in \mathfrak{cgal}_N^{\mathrm{Pol}}(d)$ iff

$$X = \Big(\omega^i_j x^j + \frac{N}{2}\xi'(t)x^i + \beta^i(t)\Big)\frac{\partial}{\partial x^i} + \xi(t)\frac{\partial}{\partial t}, \tag{3.11}$$

with $\omega \in \mathfrak{so}(d)$, and $\xi(t)$ quadratic as in (3.9), together with

$$\boldsymbol{\beta}(t) = \boldsymbol{\beta}_N t^N + \cdots + \boldsymbol{\beta}_1 t + \boldsymbol{\beta}_0, \tag{3.12}$$

where $\boldsymbol{\beta}_0, \ldots \boldsymbol{\beta}_N \in \mathbb{R}^d$. The finite-dimensional Lie algebras $\mathfrak{cgal}_N^{\mathrm{Pol}}(d)$ turn out to be isomorphic to the so-called $\mathfrak{alt}_{2/N}(d)$ Lie algebras discovered by Henkel [15] in his study of scale invariance for strongly anisotropic critical systems (with $d = 1$), viz., $\mathfrak{cgal}_N^{\mathrm{Pol}}(d) \cong \mathfrak{alt}_{2/N}(d)$.

From now on we drop the superscript "Pol" as no further confusion can occur. In the case $N = 1$, we recognize the Schrödinger Lie algebra $\mathfrak{cgal}_1(d) \cong \mathfrak{sch}(d)$, see (2.18),[2] while for $N = 2$ we recover the "Conformal Galilei Algebra" (CGA) $\mathfrak{cgal}_2(d)$, called $\mathfrak{cmil}_1(d)$ in [12].

4 Conformal Galilei Groups with dynamical exponents $z = 2/N$

4.1 Veronese curves and finite-dimensional representations of $\mathrm{SL}(2, \mathbb{R})$

A *Veronese curve* is an embedding $\mathrm{Ver}_N : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^N$ defined, for $N \geq 1$ by

$$\mathrm{Ver}_N(t_1 : t_2) = (t_1^N : t_1^{N-1}t_2 : \cdots : t_2^N), \tag{4.1}$$

where $(u_1 : u_2 : \cdots : u_{N+1})$ stands for the direction of $(u_1, u_2, \ldots, u_{N+1}) \in \mathbb{R}^{N+1}\backslash\{0\}$, that is, a point in $\mathbb{R}\mathrm{P}^N$. See, e.g., [34].

[2] Strictly speaking, for the lowest level, $N = 1$, the vector fields generating $\mathfrak{cgal}_1(d)$ are polynomials of degree 2 in the *spacetime* coordinates, although $z = 2$ holds true. The higher levels $N \geq 2$ duly correspond to the actual degree of the vector fields generating $\mathfrak{cgal}_N(d)$.

With a slight abuse of notation, we will still denote by $\mathrm{Ver}_N : \mathbb{R}^2 \to \mathbb{R}^{N+1}$ the mapping defined by

$$\mathrm{Ver}_N(t_1, t_2) = (u_1, u_2, \ldots, u_{N+1}) \qquad \text{where} \qquad u_k = t_1^{N-k+1} t_2^{k-1} \tag{4.2}$$

for all $k = 1, \ldots, N+1$. Put $t = (t_1, t_2) \in \mathbb{R}^2$, and consider $t^* = Ct$ with

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{4.3}$$

The image u^* of t^* under the Veronese map is clearly a $(N+1)$-tuple of homogeneous polynomials of degree N in t; it thus depends linearly on $u = (u_1, \ldots, u_{N+1}) \in \mathbb{R}^{N+1}$, where the u_k are as in (4.2). The general formula is as follows. If $t_1^* = at_1 + bt_2$, $t_2^* = ct_1 + dt_2$, with $ad - bc = 1$, then

$$\mathrm{Ver}_N(Ct) = \mathrm{Ver}_N(C)\mathrm{Ver}_N(t), \tag{4.4}$$

where $\mathrm{Ver}_N(C)$ a nonsingular $(N+1) \times (N+1)$ matrix with entries

$$\begin{aligned} \mathrm{Ver}_N(C)^m_{m'} \quad = \quad & \sum_{k=\max(0,m'-m)}^{\min(N-m+1,m'-1)} \begin{pmatrix} N-m+1 \\ k \end{pmatrix} \begin{pmatrix} m-1 \\ m'-k-1 \end{pmatrix} \times \\ & \times a^{N-m-k+1} b^k c^{m-m'+k} d^{m'-k-1} \end{aligned} \tag{4.5}$$

for all $m, m' = 1, \ldots, N+1$. Our mapping provides us with a group homomorphism

$$\mathrm{Ver}_N : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(N+1, \mathbb{R}) \tag{4.6}$$

which constitutes (up to equivalence) the well-known $(N+1)$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{R})$; see [35]. Let us introduce the $\mathfrak{sl}(2, \mathbb{R})$ generators

$$\xi_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \xi_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.7}$$

interpreted physically as the infinitesimal generators of time translations ξ_{-1}, dilations ξ_0, and expansions ξ_1. Their images under the tangent map of of Ver_N at the identity read then

$$\mathfrak{ver}_N(\xi_{-1}) \quad = \quad \sum_{n=1}^{N+1} (N-n+1)\, u_{n+1} \frac{\partial}{\partial u_n}, \tag{4.8}$$

$$\mathfrak{ver}_N(\xi_0) \quad = \quad \sum_{n=1}^{N+1} (N-2n+2)\, u_n \frac{\partial}{\partial u_n}, \tag{4.9}$$

$$\mathfrak{ver}_N(\xi_1) \quad = \quad \sum_{n=1}^{N+1} (n-1)\, u_{n-1} \frac{\partial}{\partial u_n}. \tag{4.10}$$

One checks that $\mathfrak{ver}_N(\xi_a)$ is, indeed, divergence-free, and

$$[\mathfrak{ver}_N(\xi_a), \mathfrak{ver}_N(\xi_b)] = -\mathfrak{ver}_N([\xi_a, \xi_b]) \tag{4.11}$$

for $a, b = -1, 0, 1$, i.e., that $\mathfrak{ver}_N : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(N+1, \mathbb{R})$ is a Lie algebra anti-homomorphism.

4.2 Matrix realizations of the Conformal Galilei Groups $\mathrm{CGal}_N(d)$

Just as in the case of the Schrödinger group, see (2.10), we will strive integrating the conformal Galilei Lie algebras $\mathfrak{cgal}_N(d)$ within the matrix group $\mathrm{GL}(d+N+1, \mathbb{R})$. Let us, hence, introduce the *Conformal Galilei Group* with *dynamical exponent* $z = 2/N$ cf. (3.10), which we denote by $\mathrm{CGal}_N(d)$; it consists of those matrices of the form

$$g = \begin{pmatrix} A & \mathbf{B}_N \cdots \mathbf{B}_0 \\ 0 & \mathrm{Ver}_N(C) \end{pmatrix} \tag{4.12}$$

where $A \in \mathrm{O}(d)$, $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_N \in \mathbb{R}^d$, and $C \in \mathrm{SL}(2, \mathbb{R})$.

We now prove that the Lie algebra of $\mathrm{CGal}_N(d)$ is, indeed, $\mathfrak{cgal}_N(d)$ introduced in Section 3. In fact, putting $t = t_1/t_2$ in (4.2), wherever $t_2 \neq 0$, we easily find that the projective action $g_{\mathbb{R}^{d+1}} : (\boldsymbol{x}, t) \mapsto (\boldsymbol{x}^*, t^*)$ of $\mathrm{CGal}_N(d)$ reads, locally, as

$$\begin{pmatrix} \boldsymbol{x}^* \\ t_*^N \\ \vdots \\ t^* \\ 1 \end{pmatrix} = \mathbb{R}^* \cdot g \begin{pmatrix} \boldsymbol{x} \\ t^N \\ \vdots \\ t \\ 1 \end{pmatrix}, \tag{4.13}$$

which, with the help of (4.3) and (4.5), leaves us with

$$\boldsymbol{x}^* = \frac{A\boldsymbol{x} + \mathbf{B}_N t^N + \cdots + \mathbf{B}_1 t + \mathbf{B}_0}{(ct+d)^N}, \tag{4.14}$$

$$t^* = \frac{at+b}{ct+d}. \tag{4.15}$$

These formulæ allow for the following interpretation for the parameters in (4.12),

$$\begin{array}{rl} A: & \text{orthogonal transformation,} \\ \mathbf{B}_0: & \text{translation,} \\ \mathbf{B}_1: & \text{boost ,} \\ \mathbf{B}_2: & \text{acceleration,} \\ \vdots & \vdots \\ \mathbf{B}_N: & \text{higher-order “acceleration”,} \\ C: & \text{projective transformation of time.} \end{array} \tag{4.16}$$

Let us now write any vector in the Lie algebra of $\mathrm{CGal}_N(d)$ as

$$Z = \begin{pmatrix} \omega & \boldsymbol{\beta}_N \cdots \boldsymbol{\beta}_0 \\ 0 & \mathfrak{ver}_N(\xi) \end{pmatrix}, \tag{4.17}$$

where we have used Equations (4.8)–(4.10) with

$$\xi = \begin{pmatrix} \lambda & \varepsilon \\ -\kappa & -\lambda \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}). \tag{4.18}$$

Then the infinitesimal form of the transformation laws (4.14) and (4.15) writes as $\delta\boldsymbol{x} = \delta\boldsymbol{x}^*|_{\delta g=Z, g=\mathrm{Id}}$, together with $\delta t = \delta t^*|_{\delta g=Z, g=\mathrm{Id}}$, i.e.,

$$\delta\boldsymbol{x} = \omega\boldsymbol{x} + \boldsymbol{\beta}_N t^N + \cdots \boldsymbol{\beta}_1 t + \boldsymbol{\beta}_0 + N(\kappa t + \lambda)\boldsymbol{x}, \qquad \delta t = \kappa t^2 + 2\lambda t + \varepsilon. \tag{4.19}$$

At last, the vector field $Z_{\mathbb{R}^{d+1}} = \delta x^i\, \partial/\partial x^i + \delta t\, \partial/\partial t$ associated with Z in (4.17) is such that

$$Z_{\mathbb{R}^{d+1}} = X \in \mathfrak{cgal}_N(d), \tag{4.20}$$

where the vector field X is as in (3.11), proving our claim.

Our terminology for the dynamical exponent is justified by verifying that the dilation generator is (1.8) with $z = 2/N$.

The above definition of the conformal Galilei groups, see (4.12), yield their global structure

$$\mathrm{CGal}_N(d) \cong (\mathrm{O}(d) \times \mathrm{SL}(2,\mathbb{R})) \ltimes \mathbb{R}^{(N+1)d}, \tag{4.21}$$

and $\dim(\mathrm{CGal}_N(d)) = Nd + \frac{1}{2}d(d+1) + 3$.

• For $N = 1$, i.e., $z = 2$, we recover the Schrödinger group (2.10), and therefore

$$\mathrm{CGal}_1(d) \cong \mathrm{Sch}(d). \tag{4.22}$$

• For $N = 2$, i.e., $z = 1$, in particular, we get

$$g = \begin{pmatrix} A & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{B}_0 \\ 0 & a^2 & 2ab & b^2 \\ 0 & ac & ad+bc & bd \\ 0 & c^2 & 2cd & d^2 \end{pmatrix} \in \mathrm{CGal}_2(d), \tag{4.23}$$

with $A \in \mathrm{O}(d)$, $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{R}^d$, $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. In addition to the usual space translations $\mathbf{B}_0$, and Galilei boosts $\mathbf{B}_1$, we also have extra generators, namely *accelerations* $\mathbf{B}_2$ [9, 11]. It is an easy matter to check that the actions (1.4), (1.5), and (1.6) of dilations, expansions, and accelerations, respectively, are recovered by considering their projective action given by (4.13) with $N = 2$.

The Lie algebra $\mathfrak{cgal}_2(d)$ of the Conformal Galilei Group, $\mathrm{CGal}_2(d)$, is plainly isomorphic to the (centerless) Conformal Galilei Algebra (CGA), cf. [12, 9, 11]. It has been shown [9] that $\mathfrak{cgal}_2(d)$ admits a nontrivial 1-dimensional central extension in the planar case, $d = 2$, only.

5 Conformal Galilei spacetimes & cosmological constant

As mentioned in the Introduction, the physical spacetime can be recovered by postulating some symmetry group — defining its geometry — and then by factoring out a suitable subgroup. The simplest example is to start with the neutral component of the Galilei group (2.5), namely

$$\mathrm{Gal}_+(d) = (\mathrm{SO}(d) \times \mathbb{R}) \ltimes \mathbb{R}^{2d}, \tag{5.1}$$

and factor out rotations and boosts to yield (ordinary) Galilei spacetime,

$$\mathbb{R}^d \times \mathbb{R} = \mathrm{Gal}_+(d)/(\mathrm{SO}(d) \ltimes \mathbb{R}^d). \tag{5.2}$$

Similarly, one can start instead with a deformation of the Galilei group called Newton-Hooke group [1, 2, 3]

$$N^+(d) = \big(\mathrm{SO}(d) \times \mathrm{SO}(2)\big) \ltimes \mathbb{R}^{2d}, \tag{5.3}$$

where $\mathrm{SO}(d) \times \mathrm{SO}(2)$ is the direct product of spatial rotations and translations of (compactified) time acting on the Abelian subgroup $\mathbb{R}^{2d}$ of boosts and space-translations. Then, quotienting $N^+(d)$ by the direct product of rotations and boosts yields the *Newton-Hooke spacetime* [3]. The latter carries a non-flat nonrelativistic structure and satisfies the empty space Newton gravitational field equations with negative cosmological constant [3].[3]

Below, we extend the above-mentioned construction to our conformal Galilei groups $\mathrm{CGal}_N(d)$, at any level $N \geq 1$.

5.1 Conformal Galilei spacetimes

The conformal Galilei spacetimes M_N, associated with $z = 2/N$ where $N = 1, 2, \ldots$, are introduced by starting with the conformal Galilei groups $\mathrm{CGal}_{2/z}(d)$, viz.,

$$M_N = \mathrm{CGal}_N(d)/H_N \quad \text{where} \quad H_N = (\mathrm{O}(d) \times \mathrm{Aff}(1, \mathbb{R})) \ltimes \mathbb{R}^{Nd}. \tag{5.5}$$

[3]Starting with the group

$$N^-(d) = \big(\mathrm{SO}(d) \times \mathrm{SO}(1,1)^\uparrow\big) \ltimes \mathbb{R}^{2d} \tag{5.4}$$

we would, similarly, end up with a spacetime with positive cosmological constant. Here we will focus our attention to $N^+(d)$ in (5.3).

Explicitly, the projection $\pi_N : \mathrm{CGal}_N(d) \to M_N$ in (5.5) is defined by the direction

$$\pi_N(g) = \mathbb{R}^* \cdot g_{d+N+1} \tag{5.6}$$

of the last column-vector of the matrix, g, in (4.12). Therefore $x = \pi_N(g)$ gets locally identified with

$$\begin{pmatrix} \boldsymbol{x} \\ t^N \\ \vdots \\ t \\ 1 \end{pmatrix} = \mathbb{R}^* \cdot \begin{pmatrix} \mathbf{B}_0 \\ b^N \\ \vdots \\ bd^{N-1} \\ d^N \end{pmatrix}. \tag{5.7}$$

Intuitively, this amounts to factoring out $\mathrm{O}(d)$, dilations and expansions and all higher-than-zeroth-order accelerations, and identifying space-time with "what is left over".

Then, the action $g \mapsto g_{M_N}$ of $\mathrm{CGal}_N(d)$ on spacetime M_N is globally given by $g_{M_N}(\pi_N(h)) = \pi_N(gh)$ and, in view of (5.7), retains the local form (4.14) and (4.15). Now, by the very definition (4.12) of the conformal Galilei group at level N, we get indeed the conformal Galilei spacetime

$$M_N = (\mathbb{R}^d \times \mathrm{Ver}_N(\mathbb{R}^2 \backslash \{0\}))/\mathbb{R}^*, \tag{5.8}$$

fibered above the Veronese curve $\mathrm{Ver}_N(\mathbb{R}\mathrm{P}^1) \subset \mathbb{R}\mathrm{P}^N$, interpreted as the time axis, T.

Besides, it may be useful to view the projective line as $\mathbb{R}P^1 \cong S^1/\mathbb{Z}_2$, locally parametrized by an angle ϑ related to the above-chosen affine parameter

$$t = \tan \vartheta, \tag{5.9}$$

highlighting that $\vartheta \sim \vartheta + \pi$. This entails (see (5.8)) that

$$M_N \cong (\mathbb{R}^d \times \mathrm{Ver}_N(S^1))/\mathbb{Z}_2 \tag{5.10}$$

where $\mathrm{Ver}_N(S^1)$ stands for the image of S^1 by the mapping (4.2). To summarize, we have the following diagram

$$\begin{array}{ccc} \mathrm{CGal}_N(d) & & \\ {\scriptstyle H_N}\big\downarrow & & \\ M_N \cong (\mathbb{R}^d \times \mathrm{Ver}_N(S^1))/\mathbb{Z}_2 & \xrightarrow{\mathbb{R}^d} & T \cong \mathrm{Ver}_N(\mathbb{R}P^1) \end{array} \tag{5.11}$$

where the horizontal arrow denotes the canonical fibration of spacetime M_N onto the time axis T.

Note that for $N = 1$ the Schrödinger-homogeneous spacetime M_1, i.e., the Möbius spacetime (2.12), is obtained.

We will from now on focus our attention to the new $\mathrm{CGal}_N(d)$-homogeneous spacetimes M_N.

Let us lastly provide, for the record, explicit formulæ for the projective action $(\boldsymbol{x}, \vartheta) \mapsto (\boldsymbol{x}^*, \vartheta^*)$ of $\mathrm{CGal}_N(d)$ on our Newton-Hooke manifold M_N, in terms of the angular coordinate introduced in (5.9). We will use the local polar decomposition of any element in $\mathrm{SL}(2, \mathbb{R})$, viz.,

$$C = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \tag{5.12}$$

where $\alpha \in \mathbb{R}/(2\pi\mathbb{Z})$ is a "time-translation", $a \in \mathbb{R}^*$ a dilation, and $c \in \mathbb{R}$ an expansion. Then, Equations (4.14) and (4.15) yield

$$\begin{array}{rcll} A & : & \boldsymbol{x}^* = A\boldsymbol{x}, & \vartheta^* = \vartheta, \\ \mathbf{B}_N & : & \boldsymbol{x}^* = \boldsymbol{x} + \mathbf{B}_N \tan^N\vartheta, & \vartheta^* = \vartheta, \\ \vdots & & \vdots & \vdots \\ \mathbf{B}_1 & : & \boldsymbol{x}^* = \boldsymbol{x} + \mathbf{B}_1 \tan\vartheta, & \vartheta^* = \vartheta, \\ \mathbf{B}_0 & : & \boldsymbol{x}^* = \boldsymbol{x} + \mathbf{B}_0, & \vartheta^* = \vartheta, \\ \alpha & : & \boldsymbol{x}^* = \dfrac{\boldsymbol{x}\cos^N\vartheta}{\cos^N(\vartheta+\alpha)}, & \vartheta^* = \vartheta + \alpha, \\ a & : & \boldsymbol{x}^* = a^N\boldsymbol{x}, & \vartheta^* = \arctan(a^2\tan\vartheta), \\ c & : & \boldsymbol{x}^* = \dfrac{\boldsymbol{x}}{(c\,\tan\vartheta+1)^N}, & \vartheta^* = \arctan\left(\dfrac{\tan\vartheta}{c\,\tan\vartheta+1}\right), \end{array} \tag{5.13}$$

with the same notation as before.

These formulæ extend those derived before, at the Lie algebraic level, for $N = 1$, and $N = 2$ [5, 38]. Those in [38], for example, are obtained from (5.13) by putting $N = 2l$ and introducing, in view of (5.7) and (5.9), the new coordinates

$$\mathbf{X} = \mathbf{x}\cos^N\vartheta, \qquad t = \tan\vartheta. \tag{5.14}$$

5.2 Galilean conformal Cartan connections

Prior to introducing (flat) Cartan connections associated with our conformal Galilei groups, let us recall some basic facts about Galilei connections [28].

Given a Galilei structure (M, γ, θ) as introduced in Section 2.1, we define the bundle of Galilei frames of M as the bundle $P \to M$ of those frames $(x, e_1, \ldots, e_{d+1})$ such that

$$\sum_{K=1}^{d} e_K \otimes e_K = \gamma \qquad \text{and} \qquad \theta^{d+1} = \theta, \tag{5.15}$$

where $(\theta^1, \ldots, \theta^{d+1})$ is the coframe at $x \in M$.

The bundle of Galilei frames is a principal H-subbundle of the frame-bundle of M, with $H \subset G(= \mathrm{Gal}(d))$ the homogeneous Galilei group consisting of those matrices

$$h = \begin{pmatrix} A & \mathbf{B}_1 \\ 0 & 1 \end{pmatrix}, \tag{5.16}$$

where $A \in \mathrm{O}(d)$, and $\mathbf{B}_1 \in \mathbb{R}^d$.

A local coordinate system (x^α) on M (where $\alpha = 1, \ldots, d+1$) induces a local coordinate system $((x^\alpha), (e_a^\alpha))$ on P, where $e_a = e_a^\alpha \, \partial/\partial x^\alpha$ for all $a = 1, \ldots, d+1$; with the definition $(\theta_\alpha^a) = (e_a^\alpha)^{-1}$, the 1-forms

$$\theta^a = \theta_\alpha^a \, dx^\alpha \tag{5.17}$$

constitute the component of *soldering* 1-form of P.

Let us denote by $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of G (resp. H) so that $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^{d+1}$. A Galilei connection is a $\mathfrak{g}$-valued 1-form of P such that

$$\omega = \begin{pmatrix} (\omega_b^a) & (\theta^a) \\ 0 & 0 \end{pmatrix}, \tag{5.18}$$

where, with the notation of (2.9),

$$(\omega_b^a) = \begin{pmatrix} \omega & \boldsymbol{\beta}_1 \\ 0 & 0 \end{pmatrix} \tag{5.19}$$

is an ordinary $\mathfrak{h}$-valued connection 1-form on the H-bundle $P \to M$, and

$$(\theta^a) = \begin{pmatrix} \boldsymbol{\beta}_0 \\ \varepsilon \end{pmatrix} \tag{5.20}$$

is the $\mathbb{R}^{d+1}$-valued soldering 1-form (5.17) of P.[4]

[4]The translation components of ω are precisely chosen as those of the soldering 1-form because Galilei connection are assumed to be affine connections.

Then the structure equations provide us with the definition of the associated curvature 2-form $((\Omega^a_b), (\Omega^a))$ on P, namely

$$\Omega^a_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b, \tag{5.21}$$

$$\Omega^a = \mathrm{d}\theta^a + \omega^a_c \wedge \theta^c, \tag{5.22}$$

for all $a, b = 1, \ldots, d+1$.

Demanding now that the curvature 2-form be $\mathfrak{h}$-valued (the torsion (Ω^a) is set to zero), we end up with a symmetric connection $(\Gamma^\rho_{\alpha\beta})$, entering the following local expression

$$\omega^a_b = \theta^a_\rho (de^\rho_b + \Gamma^\rho_{\alpha\beta}\, dx^\alpha e^\beta_b), \tag{5.23}$$

such that, if ∇ stands for the associated covariant derivative of spacetime tensor fields, $\nabla_\rho \gamma^{\alpha\beta} = 0$ and $\nabla_\alpha \theta_\beta = 0$, for all $\alpha, \beta, \rho = 1, \ldots, d+1$. This finally entails that Γ (given by ω) is a *Galilei connection* on (M, γ, θ) in the sense of Section 2.1.

At this stage, it is worthwhile mentioning that Galilei connections (5.18) are special instances of "Cartan connections" on which the next developments will rely.

Let us thus recall, for completeness, the definition of a *Cartan connection* on a principal fiber bundle $P \to M$, with structural group a closed subgroup H of a Lie group G, where $\dim(M) = \dim(G/H)$. Put $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$.

Such a "connection" is given by a $\mathfrak{g}$-valued 1-form ω on the principal H-bundle P such that

1. $\omega(Z_P) = Z$ for all $Z \in \mathfrak{h}$
2. $(h_P)^*\omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$
3. $\ker \omega = \{0\}$

where the subscript P refers to the group or Lie algebra right-action on P. These connections provide a powerful means to encode the geometry of manifolds modeled on homogeneous spaces G/H, e.g., projective or conformal geometry.

We will show, in this section, that the homogeneous spaces M_N (see (5.8)) indeed admit, for all $N = 1, 2, \ldots$, a conformal Newton-Cartan structure together with a distinguished, *flat*, normal Cartan connection associated with $\mathrm{CGal}_N(d)$.[5] The general construction of the normal Cartan connection associated with a Schrödinger-conformal Newton-Cartan structure has been performed in [21].

[5]Since we are dealing here with homogeneous spaces G/H, it will naturally be given by the (left-invariant) Maurer-Cartan 1-form of the corresponding groups G.

• The Galilei group $\mathrm{Gal}(d)$ can be viewed as the bundle of Galilei frames over spacetime $\mathbb{R}^d \times \mathbb{R} = \mathrm{Gal}(d)/(\mathrm{O}(d) \ltimes \mathbb{R}^d)$, cf. (5.2). Using (2.5), we find that the (left-invariant) Maurer-Cartan 1-form $\Theta_{\mathrm{Gal}(d)} = g^{-1}\mathrm{d}g$ reads

$$\Theta_{\mathrm{Gal}(d)} = \begin{pmatrix} \omega & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.24}$$

where $\boldsymbol{\beta}_0$, and ε are interpreted as the components of the soldering 1-form (5.17) of the principal H-bundle $\mathrm{Gal}(d) \to \mathbb{R}^d \times \mathbb{R}$. Then, the Maurer-Cartan structure equations $\mathrm{d}\Theta + \Theta \wedge \Theta = 0$ read

$$0 = \mathrm{d}\omega + \omega \wedge \omega, \tag{5.25}$$

$$0 = \mathrm{d}\boldsymbol{\beta}_1 + \omega \wedge \boldsymbol{\beta}_1, \tag{5.26}$$

$$0 = \mathrm{d}\boldsymbol{\beta}_0 + \omega \wedge \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 \wedge \varepsilon, \tag{5.27}$$

$$0 = \mathrm{d}\varepsilon. \tag{5.28}$$

Clearly, the Maurer-Cartan 1-form (5.24) endows the bundle $\mathrm{Gal}(d) \to \mathbb{R}^d \times \mathbb{R}$ with a Cartan connection in view of the above defining properties of the latter. This connection is canonical and *flat*. Indeed, Equations (5.25, 5.26), specializing (5.21), entail that the connection 2-form $(\omega, \boldsymbol{\beta}_1)$ is flat, while Equations (5.27, 5.28), corresponding to $(\Omega^a) = 0$ in (5.22), guarantee having zero torsion. See, e.g., [28, 21]. We note that, with the standard notation used throughout our article, the Galilei structure is given here by $\gamma = \delta^{ij} A_i \otimes A_j$, where $A \in \mathrm{O}(d)$ represents an orthonormal frame, together with the clock 1-form $\theta = \varepsilon$; see (2.5). At last, in view of the general form (5.23) of Galilei connections, there holds

$$\Gamma^{\rho}_{\alpha\beta} = 0 \tag{5.29}$$

for all $\alpha, \beta, \rho = 1, \ldots, d+1$, in the spacetime coordinate system $(x^i = B^i_0, x^{d+1} = b)$ provided by the matrix realization (2.5) of $\mathrm{Gal}(d)$.

• Likewise, if $N = 1$, the Schrödinger group $\mathrm{Sch}(d)$ may be thought of as a subbundle of the bundle of 2-*frames* of spacetime $M_1 \cong \mathrm{Sch}(d)/H_1$, see (5.5).[6] This time, $\mathrm{Sch}(d)$ is, indeed, interpreted as the bundle of conformal Galilei 2-frames associated with the conformal class $\gamma \otimes \theta$ of a Galilei structure (γ, θ) over M_1 as given by Equation (3.5) with $m = n = 1$. The Maurer-Cartan 1-form $\Theta_{\mathrm{Sch}(d)}$ of this group actually gives rise to the canonical flat Cartan connection on the H_1-bundle $\mathrm{Sch}(d) \to M_1$. We refer to [21, 22] for a comprehensive description of Schrödinger conformal Cartan connections.

[6] The bundle of 2-frames is called upon since the vector fields (2.18) spanning $\mathfrak{sch}(d)$ are polynomials of degree 2 in the spacetime coordinates.

Using the same notation as in (2.19), we find

$$\Theta_{\mathrm{Sch}(d)} = \begin{pmatrix} \omega & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_0 \\ 0 & \lambda & \varepsilon \\ 0 & -\kappa & -\lambda \end{pmatrix} \tag{5.30}$$

and, hence, the structure equations

$$0 = \mathrm{d}\omega + \omega \wedge \omega, \tag{5.31}$$
$$0 = \mathrm{d}\boldsymbol{\beta}_1 + \omega \wedge \boldsymbol{\beta}_1 - \boldsymbol{\beta}_0 \wedge \kappa, \tag{5.32}$$
$$0 = \mathrm{d}\boldsymbol{\beta}_0 + \omega \wedge \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 \wedge \varepsilon - \boldsymbol{\beta}_0 \wedge \lambda, \tag{5.33}$$
$$0 = \mathrm{d}\lambda - \varepsilon \wedge \kappa, \tag{5.34}$$
$$0 = \mathrm{d}\varepsilon - 2\varepsilon \wedge \lambda, \tag{5.35}$$
$$0 = \mathrm{d}\kappa - 2\kappa \wedge \lambda. \tag{5.36}$$

Furthermore, a Galilei structure can be introduced on spacetime M_1 viewed as the quotient of (5.3). Our clue is that embedding SO(2) into SL(2, $\mathbb{R}$) through

$$\vartheta \mapsto \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, \tag{5.37}$$

the group

$$\mathrm{New}_1(d) = (\mathrm{O}(d) \times \mathrm{SO}(2)) \ltimes \mathbb{R}^{2d} \tag{5.38}$$

we will call the *Newton-Hooke group of level* $N = 1$ in what follows,[7] becomes a *subgroup* of the Schrödinger group,

$$\mathrm{New}_1(d) \subset \mathrm{Sch}(d). \tag{5.39}$$

Then, $\mathrm{New}_1(d)$ can readily be identified with the bundle of Galilei frames of

$$M_1 \cong \mathrm{New}_1(d)/(\mathrm{O}(d) \times \mathbb{Z}_2) \ltimes \mathbb{R}^d. \tag{5.40}$$

Introducing the pulled-back Maurer-Cartan 1-form, $\Theta_{\mathrm{New}_1(d)}$, we end up with the previous structure equations (5.31)–(5.36) specialized to the case $\lambda = 0$, and $\kappa = \varepsilon$. Comparison between the latter equations, and the (flat) Galilei structure equations shows that both sets coincide except for the equations characterizing $\mathrm{d}\boldsymbol{\beta}_1$. This entails that M_1 acquires curvature through the 2-form [21]

$$\boldsymbol{\Omega}_1 = \boldsymbol{\beta}_0 \wedge \varepsilon. \tag{5.41}$$

[7] Its neutral component is $N^+(d)$ in (5.3).

Let us present, for completeness, a local expression of the NC-Cartan connection on M_1 generated by $\Theta_{\mathrm{New}_1(d)}$. Putting $\boldsymbol{x} = \mathbf{B}_0$ defines, along with $\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})$ introduced in (5.37), a coordinate system on spacetime M_1. We readily find the components of the soldering 1-form to be

$$\boldsymbol{\beta}_0 = A^{-1}(\mathrm{d}\boldsymbol{x} - \mathbf{B}_1 \mathrm{d}\vartheta) \qquad \text{and} \qquad \varepsilon = \mathrm{d}\vartheta, \tag{5.42}$$

see (5.20), while those of the $\mathfrak{h}$-connection write

$$\omega = A^{-1}\mathrm{d}A \qquad \text{and} \qquad \boldsymbol{\beta}_1 = A^{-1}(\mathrm{d}\mathbf{B}_1 + \boldsymbol{x}\,\mathrm{d}\vartheta), \tag{5.43}$$

see (5.19). From the last expression we infer (exploiting the general form (5.23) of Galilei connections) that the only nonzero components of the Christoffel symbols of the connection are

$$\Gamma^i_{\vartheta\vartheta} = x^i \tag{5.44}$$

for all $i = 1, \ldots, d$. This entails that the nonzero components of curvature tensor R_1, associated with $\boldsymbol{\Omega}_1$, are $(R_1)^i_{j\vartheta\vartheta} = \delta^i_j$ for all $i, j = 1, \ldots, d$. This Galilei connection is clearly a NC-connection since (2.2) holds true.

The only nonvanishing component of the Ricci tensor is therefore

$$(\mathrm{Ric}_1)_{\vartheta\vartheta} = d. \tag{5.45}$$

Hence, our NC-connection $(\omega, \boldsymbol{\beta}_1)$ provides us with a solution of the NC-field equations (2.3),

$$\mathrm{Ric}_1 + \widetilde{\Lambda}_1 \theta \otimes \theta = 0, \tag{5.46}$$

where $\theta = \mathrm{d}\vartheta$; the "cosmological constant"

$$\widetilde{\Lambda}_1 = -d \tag{5.47}$$

is therefore given by the *dimension of space*.

The angular parameter ϑ being dimensionless, it might be worth introducing, at this stage, a (circular) *time parameter*

$$\tau = H_0^{-1}\vartheta, \tag{5.48}$$

where H_0 is a "Hubble constant" whose inverse would serve as a time unit.[8] This entails that $(\mathrm{Ric}_1)_{\tau\tau} = H_0^2 d$ (compare (5.45)), and, hence, provides us with the cosmological constant

$$\Lambda_1 = -H_0^2 d. \tag{5.49}$$

[8]Unlike in general relativity, no specific parameter such as the de Sitter spacetime radius is available in our nonrelativistic framework; whence this somewhat arbitrary choice of a time unit.

Now, cosmologists usually introduce the *reduced cosmological constant*, λ, in terms of the cosmological constant, Λ, the Hubble constant, H_0, and the dimension of space, d, via

$$\Lambda = \lambda H_0^2 d. \tag{5.50}$$

In view of (5.49), our model therefore yields a reduced cosmological constant

$$\lambda_1 = -1. \tag{5.51}$$

- For the case $N > 1$, the Maurer-Cartan 1-form of $\mathrm{CGal}_N(d)$ reads

$$\Theta_{\mathrm{CGal}_N(d)} = \begin{pmatrix} \omega & \boldsymbol{\beta}_N & \boldsymbol{\beta}_{N-1} & \cdots & \boldsymbol{\beta}_1 & \boldsymbol{\beta}_0 \\ 0 & N\lambda & N\varepsilon & \cdots & 0 & 0 \\ 0 & -\kappa & (N-1)\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 2\varepsilon & 0 \\ 0 & 0 & \cdots & \cdots & (1-N)\lambda & \varepsilon \\ 0 & 0 & 0 & 0 \cdots 0 & -N\kappa & -N\lambda \end{pmatrix} \tag{5.52}$$

with the same notation as before.

Then, the preceding computations can be reproduced, *mutatis mutandis*, for the group $\mathrm{CGal}_N(d)$ which serves as the bundle of conformal Galilei N-frames of M_N. At that point, as a straightforward generalization of (5.38), we can define the *Newton-Hooke group at level N* as

$$\mathrm{New}_N(d) = (\mathrm{O}(d) \times \mathrm{SO}(2)) \ltimes \mathbb{R}^{d(N+1)} \subset \mathrm{CGal}_N(d). \tag{5.53}$$

Much in the same manner as in the case $N = 1$, the NC-connection obtained from the pull-back $\Theta_{\mathrm{New}_N(d)}$ of the Maurer-Cartan 1-form (5.52) can be computed. It is easily shown that the curvature of the homogeneous spacetimes M_N is now given by

$$\boldsymbol{\Omega}_N = N\boldsymbol{\Omega}_1. \tag{5.54}$$

This connection turns out to produce an exact solution of the vacuum NC-field equations (2.3), the reduced cosmological constant at level N being now given by

$$\lambda_N = -N \tag{5.55}$$

for all $N = 1, 2, \ldots$.

6 Conclusion and outlook

Our main results are two-fold: firstly, we have found that the infinite-dimensional Lie algebra of infinitesimal conformal Galilei transformations with rational dynamical exponent admits, in fact, finite dimensional Lie subalgebras provided the dynamical exponent is $z = 2/N$, where $N = 1, 2, \ldots$. Then, we have proposed a natural construction devised to integrate, for each N, these Lie algebras into Lie groups, named *Conformal Galilei groups at level* N, by means of the classic Veronese embeddings [34]. The values $N = 1$ and $N = 2$ correspond to the Schrödinger Lie algebra, and to the CGA, respectively. Our results are somewhat unexpected in that, starting with the symmetry problem in the Galilean context, we end up with Newton-Hooke space-times and their symmetries.

Let us shortly list some of the applications beyond the by-now standard $N = 1$, i.e., $z = 2$ Schrödinger symmetry.

The $N = 2$, i.e., $z = 1$ conformal extension of the [exotic] Galilean algebra was studied in [8], and later extended so as to include also accelerations [9].

Newton-Hooke symmetry and spacetime have been considered in nonrelativistic cosmology [1, 2, 3, 4, 5, 6]. It has been proved [36] that the $N = 1$ Galilean Conformal algebra is isomorphic to the Newton-Hooke string algebra studied in string theory [37].

The values $N = 4$, and $N = 6$, i.e., the dynamical exponents $z = 1/2$ and $z = 1/3$ arise in statistical mechanics, namely for the spin-spin correlation function in the axial next-nearest-neighbor spherical model at its Lifschitz points of first and second order [24, 7].

Our Conformal Galilei groups, $\mathrm{CGal}_N(d)$, which generalize the Schrödinger ($N = 1$) and the Conformal Galilei ($N = 2$) cases to any integer N, do *not* act regularly on ordinary flat Galilean spacetime, however; they act rather on manifolds constructed from them, which are analogous to the conformal compactification of Minkowski spacetime. Moreover, their group structure allows us to recover these spacetimes as homogeneous spaces of the Newton-Hooke groups $\mathrm{New}_N(d)$ as defined in (5.53), and generalizing to level N the Newton-Hooke group (5.3). These associated spacetimes M_N are endowed with a (conformal) Newton-Cartan structure by construction. Remarkably, they are identified as *Newton-Hooke spacetimes with quantized negative reduced cosmological constant*, λ_N in (5.55).[9]

An intuitive way of understanding our strategy is to consider, say, Schrödinger expansions in (1.2), or in (2.11), and observe that it is the *denominator* which makes the group action singular. Our way of removing this "hole" singularity is factorize

[9]Spacetimes with positive cosmological constant would be obtained by replacing SO(2) by $\mathrm{SO}(1,1)^{\uparrow}$ as in (5.4); see also [3].

the denominator in these expressions, as dictated by the projective action in (4.13).

Having constructed our finite-dimensional conformal extensions for each N, there remains the task to find *physical realizations*.

Firstly, in the Schrödinger case $N = 1$, one can verify directly that the geodesic equations, i.e., the *free Newton equations*,

$$\frac{d^2\mathbf{x}}{dt^2} = 0, \tag{6.1}$$

are, indeed, preserved due to the following transformation of the acceleration

$$\frac{d^2\mathbf{x}^*}{dt^{*2}} = A\frac{d^2\mathbf{x}}{dt^2}\frac{dt}{dt^*}, \tag{6.2}$$

where $A \in \mathrm{O}(d)$. Similarly, the equations of *uniform acceleration*,

$$\frac{d^3\mathbf{x}}{dt^3} = 0, \tag{6.3}$$

are preserved by the conformal Galilei group, $\mathrm{CGal}_1(d)$, in view of the following transformation law, namely,

$$\frac{d^3\mathbf{x}^*}{dt^{*3}} = A\frac{d^3\mathbf{x}}{dt^3}\left(\frac{dt}{dt^*}\right)^2. \tag{6.4}$$

These examples make it plausible that the N-conformal symmetry, $\mathrm{CGal}_N(d)$, is realized by the higher-order geodesic equations,

$$\frac{d^{N+1}\mathbf{x}}{dt^{N+1}} = 0. \tag{6.5}$$

Such a statement is suggested by the quantum formulas in [24, 7] and is consistent, for $N = 2$, with Ref. [10]. Another promising approach is [39].

We would finally like to mention that a complete, general, construction of Cartan connections for conformal Galilei structures, modeled on the $\mathrm{CGal}_N(d)$-homogeneous manifolds M_N, still remains to be undertaken in order to extend that of normal Schrödinger-Cartan connections carried out in [22]. This would lead to a satisfactory, brand-new, geometric definition of the group of N-conformal Galilei transformations of a NC-structure as the group of automorphisms of such an associated normal Cartan connection.

Let us now discuss the relationship of our procedures and technique to some other work on the same subject, namely to conformal Galilean symmetries.

Gibbons and Patricot [3] derive the Newton-Cartan structure of Newton-Hooke spacetime in the "Bargmann" framework of Ref. [14]. This null "Kaluza-Klein-type" approach provides us indeed with a preferred way of defining a nonrelativistic structure "downstairs". A similar explanation holds in a uniform magnetic field [40, 41, 42, 43, 44, 45].

No central terms are considered in this paper: our Lie algebras are represented by vector fields on (flat) Newton-Cartan spacetime. Central terms, and the mass in particular, are important, though, and are indeed necessary for physical applications. Henkel [24, 7] does actually consider mass terms: he works with *operators* with such terms involving c^{-2}, where c is the speed of light. In his approach, those appear in the coefficients of *powers* of $\partial/\partial t$. Considering higher-order terms in powers of $\partial/\partial t$ goes beyond our framework, though. For finite c, Henkel's boosts are Lorentzian, not Galilean, however; our center-free Lie algebras with genuine Galilei boosts are recovered as $c \to \infty$, yielding also $(N-1)$ accelerations in addition. At last, the mass terms disappear, as they should: mass and accelerations are indeed inconsistent [9].

Central extensions of the conformal Galilei algebra have been considered in [9, 44] in the planar case, $d = 2$, and in [23] for $d = 3$.

Let us finally mention a natural further program, in the wake of the present study, namely the determination of the group-cohomologies of our Conformal Galilei groups. Also would it be worthwhile to classify all symplectic homogeneous spaces [25] of the latter, likely to unveil new, physically interesting, systems.

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