



HAL
open science

Approximations of degree zero in the Poisson problem

Cesare Davini, Franck Jourdan

► **To cite this version:**

Cesare Davini, Franck Jourdan. Approximations of degree zero in the Poisson problem. Communications on Pure and Applied Analysis, 2005, 4 (2), pp.269-283. 10.3934/cpaa.2005.4.267 . hal-00583225

HAL Id: hal-00583225

<https://hal.science/hal-00583225v1>

Submitted on 5 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

APPROXIMATIONS OF DEGREE ZERO IN THE POISSON PROBLEM

C. DAVINI

Dipartimento di Georisorse e Territorio
University of Udine, 33100 Udine, Italy

F. JOURDAN

LMGC, Université de Montpellier II
Montpellier, France

ABSTRACT. We discuss a technique for the approximation of the Poisson problem under mixed boundary conditions in spaces of piece-wise constant functions. The method adopts ideas from the theory of Γ -convergence as a guideline. Some applications are considered and numerical evaluation of the convergence rate is discussed.

1. Introduction. We consider the mixed Poisson problem in variational form: Find $u \in H_{\partial_u}^1(\Omega)$ such that

$$\mathcal{F}(u) = \inf_{v \in H_{\partial_u}^1(\Omega)} \int_{\Omega} |\nabla v|^2 dx - 2 \langle f, v \rangle, \quad (1.1)$$

where $\mathcal{F}(\cdot)$ is the functional on the right hand side. We stipulate that: Ω is an open bounded set of \mathbb{R}^2 with smooth enough boundary; $H_{\partial_u}^1(\Omega)$ is the space of the H^1 -functions that vanish at some given connected and relatively open portion $\partial_u\Omega$ of the boundary $\partial\Omega$; $f \in (H_{\partial_u}^1(\Omega))'$, with $\langle \cdot, \cdot \rangle$ denoting the pairing between $H_{\partial_u}^1(\Omega)$ and its dual. Our aim is to construct approximations by introducing a suitable sequence of unconstrained minimum problems for discrete functionals defined on piece-wise constant functions over a regular family of triangulations of Ω .

Equation (1.1) describes the simplest problem one can think of in the context of Finite Element theory. However, here the attention is focused on the approximation of problem (1.1) by means of a non conforming technique that does not require giving up its main feature of being an unconstrained minimum problem. Thus, the approach distinguishes from the most classical ones that introduce additional variables and related constraint equations, or that resort to the introduction of suitable penalizing terms in order to account for non conformity, see *e.g.* [1] [14]. Namely, the method is based on the substitution of the functional \mathcal{F} with a sequence of discrete functionals defined in spaces of poor regularity whose minima and minimizers are shown to converge to those of the original problem. Both the construction of

2000 *Mathematics Subject Classification.* 65N12, 65N30, 46N40, 74K15.

Key words and phrases. Numerical methods, non-conforming approximations, Γ -convergence, Poisson problem.

these functionals and the proof of convergence draw inspiration from the theory of Γ -convergence, which is here used in a rather unconventional way. Analogous ideas for second order problems were discussed by Davini [5] [6], Davini and Pitacco [10] [11], Davini and Paroni [9].

In the present paper we introduce the averages around the mesh nodes of the distributional gradient of what in other contexts would be called the primal variable v . This enables us to define a corresponding generalized gradient operator that applies to the piece-wise constant functions and can be used in the original functional, thus broadening the space where the minimum is searched for. Things are not that straightforward because a correction term in the discrete functionals is needed in order to take non-conformity into account. Yet, this term turns out to converge to zero in the limit, so that the gradient of the solution u is approximated by these averages eventually. It's worth saying that the correction term and its role in the theory emerge quite naturally in the convergence proof.

The passage through an averaging operation is a characteristic feature of the method. In fact, the computation of the discretized gradient doesn't require derivation and this reflects into a degree of accuracy that is definitely better than what one would expect. This occurs in most examples we have treated and is common to other applications of a method based on similar ideas to second order problems, as is given account elsewhere.

The present paper relies upon [7], but generalizes it in two ways. First, the case of mixed boundary conditions is considered. In particular, we show that the Dirichlet boundary conditions, as well as the natural ones, are accounted for directly in the functional, rather than by making an appropriate selection of the function space. Incidentally, we note that this way of taking account of the Dirichlet boundary conditions is different from that followed in [7]. Secondly, we give some numerical applications that illustrate the convergence property and enable an evaluation of the convergence rate. It's worth noticing that the method can be extended to the non homogeneous and non quadratic, although convex, first order functionals and the reader is addressed to [9] for a guideline on how to do it. Applications of the method to non-homogeneous and non quadratic functionals and the deduction of error estimates are postponed to future work.

2. Preliminaries and Notations. Following [7], let $\mathcal{T}_h := \{T_j\}_{j=1, \dots, P_h}$ be a sequence of regular triangulations of Ω in the sense of Ciarlet [2], with h taking values in some countable set \mathcal{H} of real numbers and $h \rightarrow 0$. The quantity h is defined as $h := \sup_j \{\text{diam } T_j\}$ and describes the fineness of the mesh. We shall

call \mathcal{T}_h the *primal mesh* and require that the domains $\Omega_h := \overline{\bigcup_{T_j \in \mathcal{T}_h} T_j}$ are such that

$\text{dist}(x, \partial\Omega) < ch^2 \forall x \in \partial\Omega_h$ for some c .¹ We denote by x_i the nodes of the mesh and indicate by $\mathcal{P}_h := \{1, 2, \dots, P_h\}$ and $\mathcal{N}_h := \{1, 2, \dots, N_h\}$ the sets where the indexes of the triangles and of the mesh nodes, respectively, take value. We will indicate by \mathcal{I}_h and \mathcal{B}_h the sets of the index values corresponding to the nodes in $\overset{\circ}{\Omega}_h$ and $\partial\overset{\circ}{\Omega}_h$, respectively. In the latter set we denote by $\mathcal{B}_{h, \partial_t} = \{i \in \mathcal{B}_h : \text{dist}(x_i, \partial_t\Omega) < ch^2\}$ the subset corresponding to the nodes that are close to $\partial_t\Omega = \partial\Omega \setminus \partial_u\Omega$ and by

¹In fact we never use this requirement explicitly. Its role is to guarantee that $\partial\Omega_h$ provide a C^1 -approximation of $\partial\Omega$, thus avoiding oscillations.

$\mathcal{B}_{h\partial_u} = \mathcal{B}_h \setminus \mathcal{B}_{h\partial_t}$ its complement. Let $\partial_t\Omega_h$ be the polygonal line with vertices x_i , $i \in \mathcal{B}_{h\partial_t}$, and $\partial_u\Omega_h := \partial\Omega_h \setminus \partial_t\Omega_h$.

For convenience we also introduce a *dual mesh* $\hat{\mathcal{T}}_h := \{\hat{T}_i\}_{i=1,\dots,N_h}$ consisting of disjoint open polygonal domains each of which contains just one primal node, as shown in Figure 1, where the dual elements are drawn with dashed lines. We assume that the sequence of the dual meshes is also regular and that $\Omega_h = \overline{\bigcup_{\hat{T}_i \in \hat{\mathcal{T}}_h} \hat{T}_i}$.

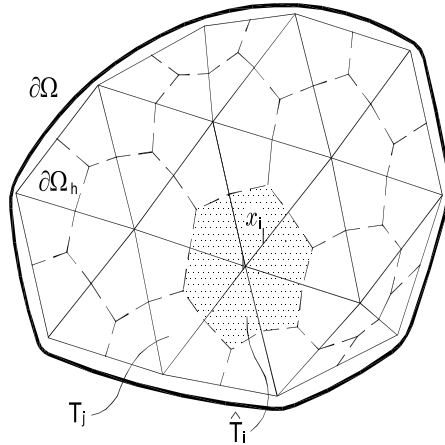


FIGURE 1. Ω, Ω_h , the primal and the dual mesh.

X_h shall be the space of functions which are affine on T_j and continuous on Ω_h (briefly, the *polyhedral functions* over \mathcal{T}_h), defined by

$$X_h := \left\{ q = \sum_{\mathcal{N}_h} q(x_i) \hat{\varphi}_i(x) \right\} \quad (2.1)$$

with $\hat{\varphi}_i$ the polyhedral functions characterized by the condition: $\hat{\varphi}_i(x_j) = \delta_{ij}$ for $i, j = 1, \dots, N_h$. We introduce also the subspaces

$$X_{h\partial_t} := \left\{ q = \sum_{\mathcal{I}_h \cup \mathcal{B}_{h\partial_u}} q(x_i) \hat{\varphi}_i(x) \right\} \quad (2.2)$$

and

$$X_{h\partial_u} := \left\{ q = \sum_{\mathcal{I}_h \cup \mathcal{B}_{h\partial_t}} q(x_i) \hat{\varphi}_i(x) \right\}, \quad (2.3)$$

corresponding to the polyhedral functions that vanish at the nodes in $\mathcal{B}_{h\partial_t}$ and $\mathcal{B}_{h\partial_u}$, respectively.

We assume that the area of the dual elements \hat{T}_i satisfies the condition

$$|\hat{T}_i| = \int_{\Omega_h} \hat{\varphi}_i dx \left(= \frac{1}{3} |\text{supp}(\hat{\varphi}_i)| \right). \quad (2.4)$$

As noted in [7], it is always possible to satisfy this condition, e.g., by taking a dual mesh where the boundary of the elements \hat{T}_i passes through the centroid and the middle points of the sides of the primal triangles. But this is not really important, because the method does not require the construction of the dual mesh in practice.

Define

$$Y_h := \{v \in L^2(\Omega) : v = \text{const on } T_j \in \mathcal{T}_h, \text{ with } v = 0 \text{ in } \Omega \setminus \Omega_h\}. \quad (2.5)$$

We denote by \bar{v}_h the functions of Y_h and imagine that they are extended to zero in the whole \mathbb{R}^2 .

Finally, let $H(\text{div}, \Omega)$ be the Hilbert space

$$H(\text{div}, \Omega) := \{\mathbf{q} \in (L^2(\Omega))^2 : \text{div} \mathbf{q} \in L^2(\Omega)\}$$

endowed with norm

$$\|\mathbf{q}\|_{H(\text{div}, \Omega)} = \left\{ \|\mathbf{q}\|_{L^2(\Omega)}^2 + \|\text{div} \mathbf{q}\|_{L^2(\Omega)}^2 \right\}^{1/2}, \quad (2.6)$$

see Girault and Raviart [12]. We introduce the subspace

$$H_{\partial_t}(\text{div}, \Omega) := \{\mathbf{q} \in H(\text{div}, \Omega) : \langle \gamma_n \mathbf{q}, \phi \rangle_{\partial\Omega} = 0 \quad \forall \phi \in \tilde{H}^{1/2}(\partial_t\Omega)\}, \quad (2.7)$$

where, with Grisvard [13], we have indicated by $\tilde{H}^{1/2}(\partial_t\Omega)$ the subspace of functions in $H^{1/2}(\partial\Omega)$ that vanish on $\partial_u\Omega$. The symbol γ_n stands for the linear continuous mapping from $H(\text{div}, \Omega)$ into $H^{-1/2}(\partial\Omega)$ that extends by continuity the mapping $\gamma_n : \mathbf{q} \mapsto \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega}$ defined on $C^\infty(\bar{\Omega})^2$.

In fact, see [8], it turns out that

$$H_{\partial_t}(\text{div}, \Omega) = \overline{C_{\partial_t}^\infty(\bar{\Omega})^2}, \quad (2.8)$$

where $C_{\partial_t}^\infty(\bar{\Omega})^2$ is the subset of functions in $C^\infty(\bar{\Omega})^2$ that vanish in a neighborhood of $\partial_t\Omega$ and the closure is made with respect to the norm defined in (2.6). In particular, a density argument and the fact that the duality $\langle \gamma_n \mathbf{q}, \phi \rangle_{\partial\Omega}$ reduces to an integral when $\gamma_n \mathbf{q}$ and ϕ are L^2 -functions yield that

$$\phi \in H^{1/2}(\partial\Omega), \quad \langle \gamma_n \mathbf{q}, \phi \rangle_{\partial\Omega} = 0 \quad \forall \mathbf{q} \in H_{\partial_t}(\text{div}, \Omega) \quad \Rightarrow \quad \phi = 0 \text{ in } \partial_u\Omega. \quad (2.9)$$

3. Generalized Gradient and a Sequence of Approximating Functionals.

Let $D\bar{v}_h$ be the distributional gradient of \bar{v}_h . By extending it to a linear operator on $H^1(\Omega_h)$ by density, we can write

$$\langle D\bar{v}_h, \hat{q} \rangle = - \int_{\Omega_h} \bar{v}_h \nabla \hat{q} dx \quad \forall \hat{q} \in X_h. \quad (3.1)$$

Thence, from $\hat{q}(x) = \sum_{i \in \mathcal{N}_h} \hat{q}(x_i) \hat{\varphi}_i(x)$ it follows that

$$\langle D\bar{v}_h, \hat{q} \rangle = - \sum_{i \in \mathcal{N}_h} \left(\int_{\Omega_h} \bar{v}_h \nabla \hat{\varphi}_i dx \right) \hat{q}(x_i) = \sum_{i \in \mathcal{N}_h} D_h \bar{v}_h(x_i) \hat{q}(x_i) |\hat{T}_i| \quad (3.2)$$

for every $\bar{v}_h \in Y_h$ and every $\hat{q} \in X_h$, where we have set

$$D_h \bar{v}_h(x_i) := - \frac{1}{|\hat{T}_i|} \int_{\Omega_h} \bar{v}_h \nabla \hat{\varphi}_i dx. \quad (3.3)$$

We call the simple function

$$D_h \bar{v}_h := \sum_{i \in \mathcal{N}_h} D_h \bar{v}_h(x_i) \chi_{\hat{T}_i}, \quad (3.4)$$

with $\chi_{\hat{T}_i}$ the characteristic function of \hat{T}_i , the *generalized gradient* of \bar{v}_h . Notice that with this definition a discrete form of the Green's formula

$$\int_{\Omega_h} \bar{v}_h \nabla \hat{q} \, dx = - \sum_{i \in \mathcal{N}_h} D_h \bar{v}_h(x_i) \hat{q}(x_i) |\hat{T}_i| \quad \forall \bar{v}_h \in Y_h, \forall \hat{q} \in X_h \quad (3.5)$$

holds true, by definition.

The interpretation of quantities $D_h \bar{v}_h(x_i)$ is suggestive and we refer the reader to [7] for an illustration. There one can also find suitable formulas useful to calculate these quantities. It's only worth recalling that, if the distributional gradient is regarded as a Radon measure (supported at the discontinuity set of \bar{v}_h) and the dual elements \hat{T}_i meet the sides of the primal mesh at the midpoints, we have

$$D_h \bar{v}_h(x_i) = \frac{D \bar{v}_h(\hat{T}_i)}{|\hat{T}_i|}.$$

Thus, $D_h \bar{v}_h(x_i)$ is the *mean value* of the gradient of \bar{v}_h over \hat{T}_i , according to the idea originally followed by Davini and Pitacco [10] in working out their external approximation for the biharmonic problem.

Let us define the functionals

$$J_h(\bar{v}_h) := \sum_{\mathcal{I}_h \cup \mathcal{B}_h \partial_u} |D_h \bar{v}_h(x_i)|^2 |\hat{T}_i| + \eta h^2 \sum_{i \in \mathcal{N}_h} \sum_{t=1}^{t_i} |[[\bar{v}_h]](t)|^2, \quad (3.6)$$

where η is an arbitrarily chosen and strictly positive number. In this formula we have adopted a local (counterclockwise) labeling of the nodes $x_t^{(i)}$, $t = 1, \dots, t_i$, around the node x^i and indicated by $[[\bar{v}_h]](t)$ the discontinuity of \bar{v}_h across the side $(x^i, x_t^{(i)})$. Note that only the nodes sitting on $\partial_u \Omega_h$ are taken into account in the first sum in (3.6). We shall see below that the sequence of these functionals provides the basis for the approximation scheme discussed in the paper.

4. Some Properties of a Sequential Topology in $\bigcup_{h \in \mathcal{H}} Y_h$.

The functionals $J_h(\cdot)$ endow $\bigcup_{h \in \mathcal{H}} Y_h$ with a sequential topology described below. Our exposition follows the line of [7], where the case of Dirichlet data was considered. While the reader is addressed to that paper for the details of the proofs, we confine here to the changes that are needed when more general boundary conditions are taken into account.

The following theorem establishes a compactness property.

THEOREM 4.1 (see [7, Theorem 1]). *If $\{\bar{v}_h\} \subset \bigcup_{h \in \mathcal{H}} Y_h$ and $v \in L^2(\Omega)$ are such that*

$$\bar{v}_h \xrightarrow{L^2(\Omega)} v \quad \text{and} \quad \liminf_h J_h(\bar{v}_h) < +\infty,$$

then,

$$v \in H_{\partial_u}^1(\Omega).$$

Moreover, by possibly passing to a subsequence,

$$D_h \bar{v}_h \xrightarrow{L^2(\Omega)} \nabla v.$$

Proof. By making use of Green's formula (3.5) and straightforward inequalities we first prove that

$$\begin{aligned} |\langle Dv, q \rangle| &\leq \liminf_h \left(\sum_{\mathcal{I}_h} |D_h \bar{v}_h(x_i)|^2 |\hat{T}_i| \right)^{1/2} \|q\|_{L^2(\Omega)} \\ &\leq \liminf_h J_h(\bar{v}_h)^{1/2} \|q\|_{L^2(\Omega)} \quad \forall q \in C_0^\infty(\Omega). \end{aligned} \quad (4.1)$$

It follows that $\langle Dv, \cdot \rangle$ can be uniquely extended to a linear vector valued functional on $L^2(\Omega)$. So, the distributional gradient Dv can be identified with the vector valued function $\mathbf{g} \in L^2(\Omega)^2$ that, according to Riesz's theorem, represents this extension. This implies that $v \in H^1(\Omega)$.

Secondly, consider any subsequence, still denoted as $\{\bar{v}_h\}$, such that $\sup_h J_h(\bar{v}_h) < +\infty$ and let $\mathbf{q} \in C_0^\infty(\Omega)^2$. Then,

$$\int_{\Omega} (D_h \bar{v}_h - \nabla v) \cdot \mathbf{q} dx = \int_{\Omega} (D_h \bar{v}_h - \nabla v) \cdot (\mathbf{q} - r_h \mathbf{q}) dx + \int_{\Omega} (D_h \bar{v}_h - \nabla v) \cdot r_h \mathbf{q} dx,$$

where $r_h \mathbf{q} \in X_h^2$ is the interpolant of \mathbf{q} . Here the integrals are in fact limited to the support of \mathbf{q} . Therefore, observing that $\text{supp } \mathbf{q} \subset \bigcup_{i \in \mathcal{I}_h} \hat{T}_i$ for h small enough, the first integral tends to zero with h , because $\int_{\text{supp } \mathbf{q}} |D_h \bar{v}_h - \nabla v|^2 dx \leq c(\sup_h J_h(\bar{v}_h) + \|v\|_{H^1(\Omega)}^2)$ and $r_h \mathbf{q} \rightarrow_{H^1(\Omega)} \mathbf{q}$. In particular, $\|D_h \bar{v}_h\|_{L^2(\Omega)}$ are equibounded when $\sup_h J_h(\bar{v}_h) < +\infty$. By the same considerations, the second one reads

$$\int_{\Omega} (D_h \bar{v}_h - \nabla v) \cdot r_h \mathbf{q} dx = \int_{\Omega} (v - \bar{v}_h) \text{div} r_h \mathbf{q} dx + \sum_{i \in \mathcal{I}_h} \int_{\hat{T}_i} D_h \bar{v}_h \cdot \nabla r_h \mathbf{q}(x) (x - x_i) dx,$$

having used Eq. (3.2) and the fact that in each \hat{T}_i we can write $r_h \mathbf{q} = r_h \mathbf{q}(x_i) + \nabla r_h \mathbf{q}(x) (x - x_i)$. Thus, the second integral also tends to zero and it follows that

$$\lim_h \int_{\Omega} (D_h \bar{v}_h - \nabla v) \cdot \mathbf{q} dx = 0 \quad \forall \mathbf{q} \in C_0^\infty(\Omega)^2,$$

i. e.,

$$D_h \bar{v}_h \xrightarrow{L^2(\Omega)} \nabla v.$$

Finally, let us show that $v \in H_{\partial_u}^1(\Omega)$. Recalling property (2.8), let $\mathbf{q} \in C_{\partial_t}^\infty(\bar{\Omega})^2$ and apply Green's formula

$$\int_{\Omega} \nabla v \cdot \mathbf{q} dx = \int_{\partial\Omega} v \gamma_n \mathbf{q} dt - \int_{\Omega} v \text{div} \mathbf{q} dx. \quad (4.2)$$

If we again indicate by $r_h \mathbf{q}$ the interpolant of \mathbf{q} in X_h^2 , by manipulations similar to those used above:

$$\int_{\partial\Omega} v \gamma_n \mathbf{q} dt = \int_{\Omega} \nabla v \cdot \mathbf{q} dx + \int_{\Omega} v \text{div}(\mathbf{q} - r_h \mathbf{q}) dx + \int_{\Omega} (v - \bar{v}_h) \text{div} r_h \mathbf{q} dx + \int_{\Omega} \bar{v}_h \text{div} r_h \mathbf{q} dx,$$

we finally get

$$\left| \int_{\partial\Omega} v \gamma_n \mathbf{q} dt \right| \leq [\liminf_h J_h(\bar{v}_h)^{1/2} + |v|_{H^1(\Omega)}] \|q\|_{L^2(\Omega)}. \quad (4.3)$$

By continuity it follows that

$$|\langle \gamma_n \mathbf{q}, v \rangle_{\partial\Omega}| \leq \left[\liminf_h J_h(\bar{v}_h)^{1/2} + |v|_{H^1(\Omega)} \right] \|q\|_{L^2(\Omega)} \quad \forall \mathbf{q} \in H_{\partial_t}(\text{div}, \Omega)$$

which implies

$$\langle \gamma_n \mathbf{q}, v \rangle_{\partial\Omega} = 0 \quad \forall \mathbf{q} \in H_{\partial_t}(\text{div}, \Omega),$$

because for every $\mathbf{q} \in H_{\partial_t}(\text{Div}, \Omega)$ it is always possible to find a sequence of $\mathbf{q}_k \in H_{\partial_t}(\text{div}, \Omega)$ such that $\gamma_n \mathbf{q}_k = \gamma_n \mathbf{q}$, with $\mathbf{q}_k \rightarrow 0$ in the L^2 -norm. By (2.9) we conclude that $v = 0$ in $\partial_u \Omega$. \square

The next theorem establishes a density result.

THEOREM 4.2 (see [7, Theorem 2]). *For every $v \in H_{\partial_u}^1(\Omega)$ there is a sequence $\{\bar{v}_h\} \subset \bigcup_{h \in \mathcal{H}} Y_h$ such that*

$$\bar{v}_h \xrightarrow{L^2(\Omega)} v, \quad J_h(\bar{v}_h) \rightarrow \|\nabla v\|_{L^2(\Omega)}^2. \quad (4.4)$$

This result is first proved for $v \in C_{\partial_u \Omega}^\infty(\bar{\Omega})$ and then it follows by density. The proof strictly mimics that given in [7], with the proviso that the projection defined in formula (3.11), then, is here substituted by

$$R_h(\psi) := \sum_{i \in \mathcal{I}_h \cup \mathcal{B}_{h\partial_u}} \frac{\int_{\Omega_h} \psi \hat{\varphi}_i dx}{\int_{\Omega_h} \hat{\varphi}_i dx} \chi_{\hat{T}_i},$$

that also satisfies the property

$$R_h(\psi) \xrightarrow{L^2(\Omega)} \psi \quad \forall \psi \in H^1(\Omega)$$

as an immediate consequence of a classical result in the theory of approximation, cf. [3], see also [11, Proposition 6]. Observed that $v = 0$ on $\partial_u \Omega_h$ for sufficiently small h , formula (3.13) of [7] keeps on holding true for $i \in \mathcal{I}_h \cup \mathcal{B}_{h\partial_u}$. Thence, the rest of the proof remains unchanged. We refer the reader to that paper for the details.

Also unchanged is the proof of the following theorem, that states a further compactness property of the set $\bigcup_{h \in \mathcal{H}} Y_h$ under the sequential topology defined by means of the functionals J_h .

THEOREM 4.3 (see [7, Theorem 3]). *If the sequence $\{\bar{v}_h\} \subset Y_h$ and $v \in H_{\partial_u}^1(\Omega)$ are such that*

$$\bar{v}_h \xrightarrow{L^2(\Omega)} v \quad \text{and} \quad J_h(\bar{v}_h) \rightarrow \|\nabla v\|_{L^2(\Omega)}^2, \quad (4.5)$$

then

$$\bar{v}_h \xrightarrow{L^2(\Omega)} v. \quad (4.6)$$

5. An External Approximation Scheme for the Poisson Problem. Let us extend J_h to $L^2(\Omega)$ as follows

$$J_h(v) := \begin{cases} \sum_{\mathcal{I}_h \cup \mathcal{B}_{h\partial_u}} |D_h \bar{v}_h(x_i)|^2 |\hat{T}_i| + \eta h^2 \sum_{i \in \mathcal{N}_h} \sum_{t=1}^{t_i} \|[\bar{v}_h](t)\|^2 & v \in Y_h, \\ +\infty, & v \in L^2(\Omega) \setminus Y_h. \end{cases} \quad (5.1)$$

The properties proved in Section 4, namely that established by Theorem 4.3, yield

THEOREM 5.1 (see [7, Theorem 4]). *The functionals J_h are equicoercive in $L^2(\Omega)$, that is:*

$$\exists \gamma > 0, \text{ independent of } h, \text{ such that: } J_h(v) \geq \gamma \|v\|_{L^2(\Omega)}^2 \quad \forall v \in L^2(\Omega). \quad (5.2)$$

It follows from Theorem 5.1 that a sequential characterization of Γ -convergence holds true, see [4] for the details, and this eventually allows us to prove that

THEOREM 5.2 (see [7, Theorem 5]). *The sequence $\{J_h(\cdot)\}$ Γ -converges to $J(\cdot)$ in $L^2(\Omega)$ endowed with the weak topology:*

$$\Gamma\text{-}\lim_h J_h = J, \quad (5.3)$$

where the functional $J(\cdot)$ is defined by

$$J(v) := \begin{cases} \int_{\Omega} |\nabla v|^2 dx & \text{for } v \in H_{\partial_u}^1(\Omega) \\ +\infty & \text{for } v \in L^2(\Omega) \setminus H_{\partial_u}^1(\Omega) \end{cases} \quad (5.4)$$

Theorem 5.2 leads to a convergent non-conforming approximation scheme for the mixed Poisson problem (1.1) as follows. Let J and J_h be given by (5.4) and (5.1), respectively, and extend the functional $\mathcal{F}(\cdot)$ in (1.1) by setting

$$\mathcal{F}(v) := J(v) - 2\langle f, v \rangle, \quad (5.5)$$

with $\langle f, v \rangle$ set equal to 0 if $v \in L^2(\Omega) \setminus H_{\partial_u}^1(\Omega)$. Introduce also the discrete functionals

$$\mathcal{F}_h(v) := J_h(v) - 2\langle\langle f, v \rangle\rangle_h, \quad (5.6)$$

with $\langle\langle \cdot, \cdot \rangle\rangle_h$ defined by

$$\langle\langle f, v \rangle\rangle_h := \begin{cases} 0 & \text{if } v \in L^2(\Omega) \setminus Y_h \\ \int_{\Omega} f^0 v + \mathbf{f}^1 \cdot D_h v dx & \text{if } v \in Y_h, \end{cases} \quad (5.7)$$

where, according to the representation theorem of $(H_{\partial_u}^1)'$ we have denoted by (f^0, \mathbf{f}^1) the pair of L^2 -functions such that: $f = f^0 - \operatorname{div} \mathbf{f}^1$.

The terms $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle_h$ are continuous perturbations of J and J_h , so that Γ -convergence of the functionals is maintained. Moreover, \mathcal{F} has a unique minimizer and the \mathcal{F}_h are equicoercive. Therefore, theory's general format assures that

$$\min \mathcal{F}(v) = \lim_h \min \mathcal{F}_h(v) \quad (5.8)$$

and

$$u_h \xrightarrow[L^2(\Omega)]{} u, \quad (5.9)$$

u and u_h being the minimizers of \mathcal{F} and \mathcal{F}_h , respectively. In (5.9) we have specialized the type of convergence taking Theorem 4.3 into account.

It follows from (5.9) that we can use the discrete minima u_h in order to approximate u . In order to compute them, we have to solve a sequence of unconstrained minimum problems that can be treated by standard algorithms for symmetric quadratic forms. In the following we discuss some numerical implementations of the method.

6. Numerical Applications. Let us illustrate the above method by discussing some examples. The aim is to give a numerical estimate of the convergence rate and to compare it with more traditional finite element techniques.

The method leads to the following optimization problem. Let $\bar{v}_h(j)$ be the displacement in T_j and vector $\bar{\mathbf{v}}_h = \{\bar{v}_h(1), \dots, \bar{v}_h(P_h)\}$ be the collection of all the displacements. We recall that the value of \bar{v}_h in the boundary triangles is unrestricted, no matter what the boundary conditions are, whereas the natural boundary conditions are taken into account by simply ignoring the contribution of the

corresponding boundary nodes in the discrete functional J_h , see definition (3.6). We also recall that the jumps $[[\bar{v}_h]](t)$ on $\partial\Omega_h$ in the second sum are computed by taking into account that \bar{v}_h are extended to zero outside Ω_h .

The optimization problem is to find $\bar{\mathbf{u}}_h \in \mathbb{R}^{P_h}$ such that it minimizes the function

$$\mathcal{F}_h(\bar{\mathbf{v}}_h) = \bar{\mathbf{v}}_h \cdot \mathbf{A}_h \bar{\mathbf{v}}_h - 2 \mathbf{f}_h \cdot \bar{\mathbf{v}}_h, \quad (6.1)$$

where \mathbf{A}_h is a symmetric positive definite matrix and \mathbf{f}_h is the vector defined by (5.7). As recalled above, the Dirichlet boundary conditions directly affect the definition of the stiffness matrix \mathbf{A}_h .

This minimization problem has been numerically treated by a software named CAMP (Constant Approximations in Membrane Problems). We consider the four examples described in Figure 2. The first consists of a circular membrane supported along the boundary and subject to a uniform pressure; the second and third correspond to a rectangular membrane supported along two opposite sides and free on the remaining ones, respectively subject to a uniform pressure and a distributed load acting on the symmetry axis; the last example is a circular membrane supported at the boundary under a distributed load acting along a diameter.

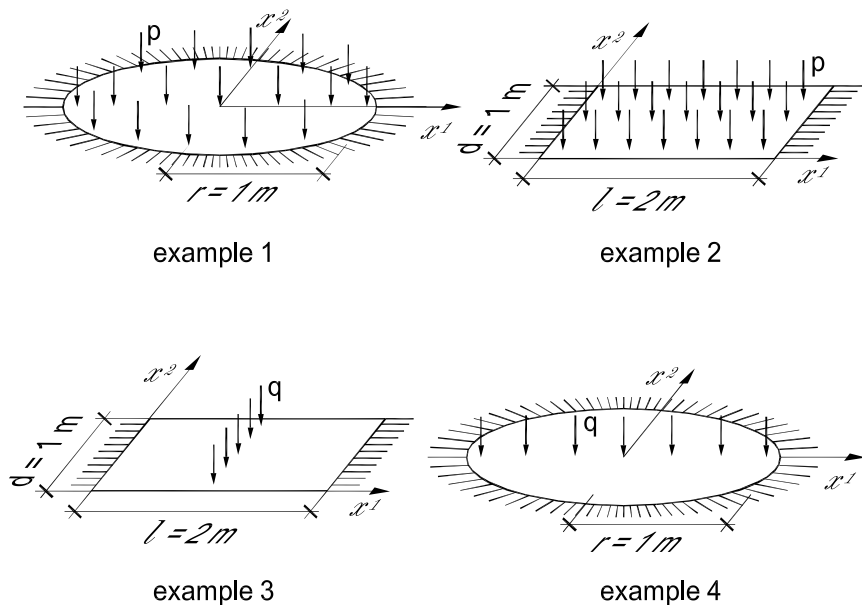


FIGURE 2. Examples

We assume that in Examples 1 and 2 the pressure takes the value $p = -1$ MPa, and that the applied line load in Examples 3 and 4 is $q = -1$ N/m. For the latter cases the loading is described according to definition (5.7). For Example 4, for instance, the load is given by the vector valued function

$$\mathbf{f}^1(x) = \left[\frac{1}{2} \chi_{D_1}(x) - \frac{1}{2} \chi_{D_2}(x) \right] \mathbf{e}_2, \quad (6.2)$$

where $D_1 = \{x \in \Omega : x^2 > 0\}$ and $D_2 = \{x \in \Omega : x^2 < 0\}$ are the domains above and below the loaded diameter, respectively, and $\chi_{D_\alpha}(\cdot)$ is the characteristic function of D_α . Thence, the external work takes the form

$$\langle \langle f, v \rangle \rangle_h = \frac{1}{2} \int_{D_1} \mathbf{e}_2 \cdot D_h v \, dx - \frac{1}{2} \int_{D_2} \mathbf{e}_2 \cdot D_h v \, dx \quad (6.3)$$

which yields

$$\langle \langle f, v \rangle \rangle_h = \frac{1}{2} \sum_{i \in \mathcal{N}_h} \left[|D_1 \cap \hat{T}_i| \mathbf{e}_2 \cdot D_h \bar{v}_h(x_i) - |D_2 \cap \hat{T}_i| \mathbf{e}_2 \cdot D_h \bar{v}_h(x_i) \right]. \quad (6.4)$$

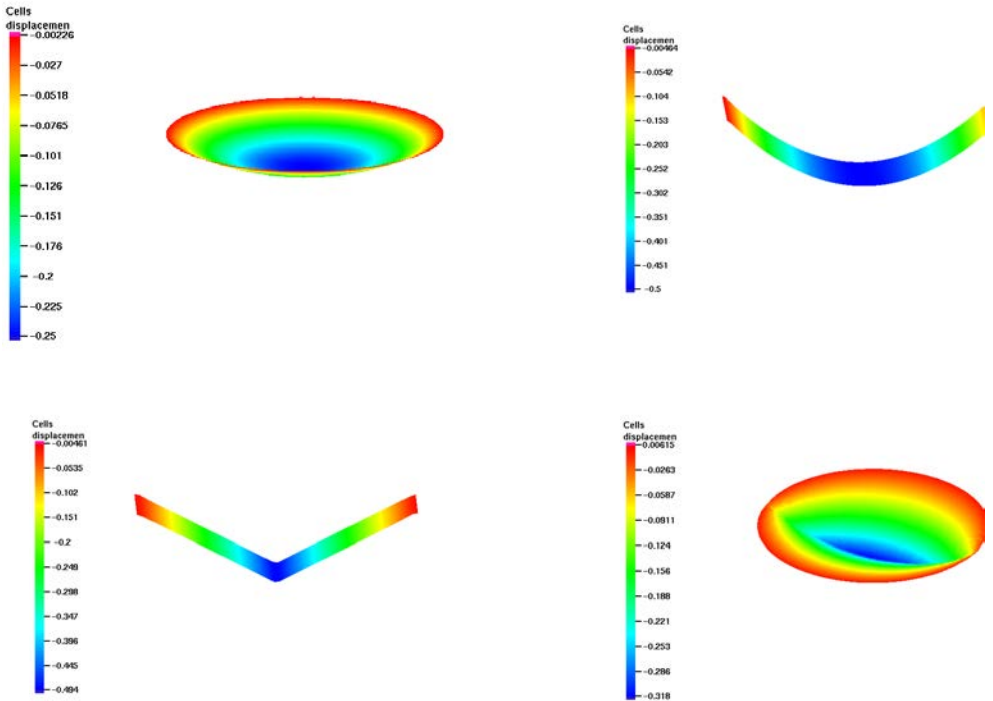


FIGURE 3. Numerical results.

Numerical results are presented in Figure 3 for each example. The computations refer to a parameter $\eta = 10^{-4}$, a value that optimizes the results in the range we have explored. As is made clear from the proof, the value of this parameter has no influence on convergence. So, it is irrelevant when the mesh is fine enough, although it may lead to bad results for large h .

Errors versus the mesh size are plotted in Figures 4-10.

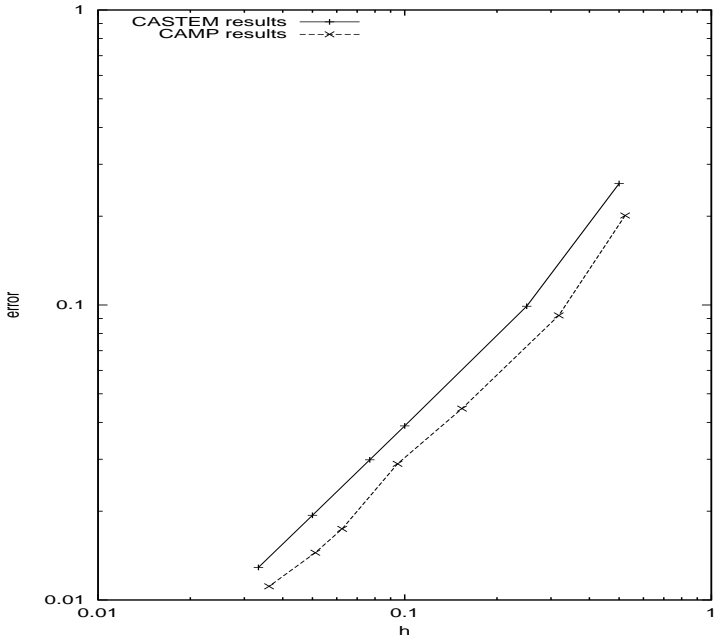


Figure 4. Error $\frac{\|u_h - u\|_{L^2}}{\|u\|_{L^2}}$ versus the mesh size in Example 1.

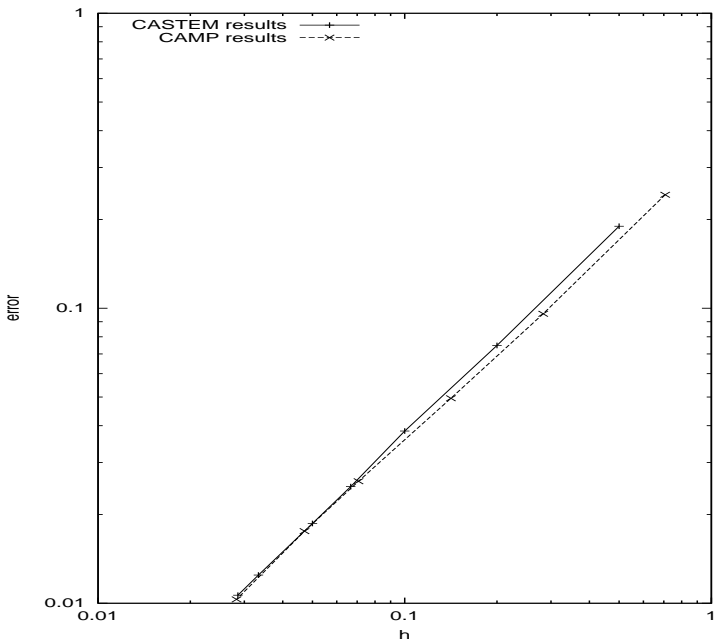


Figure 5. Error $\frac{\|u_h - u\|_{L^2}}{\|u\|_{L^2}}$ versus the mesh size in Example 2.

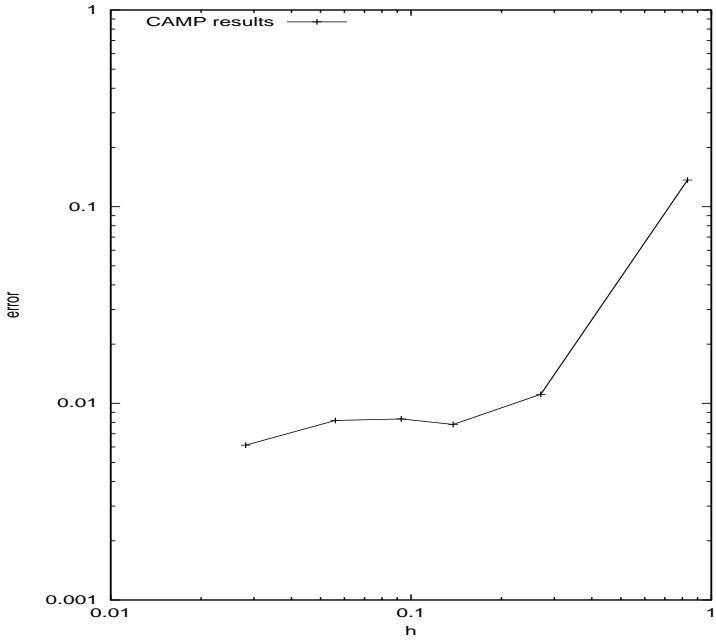


Figure 6. Error $\frac{\|u_h - u\|_{L^2}}{\|u\|_{L^2}}$ versus the mesh size in Example 3.

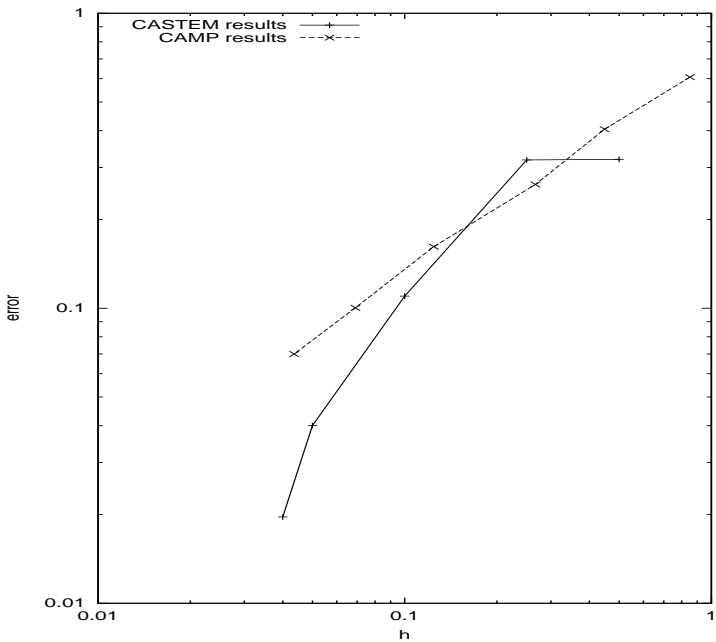


Figure 7. Error $\frac{\|u_h - u\|_{L^2}}{\|u\|_{L^2}}$ versus the mesh size in Example 4.

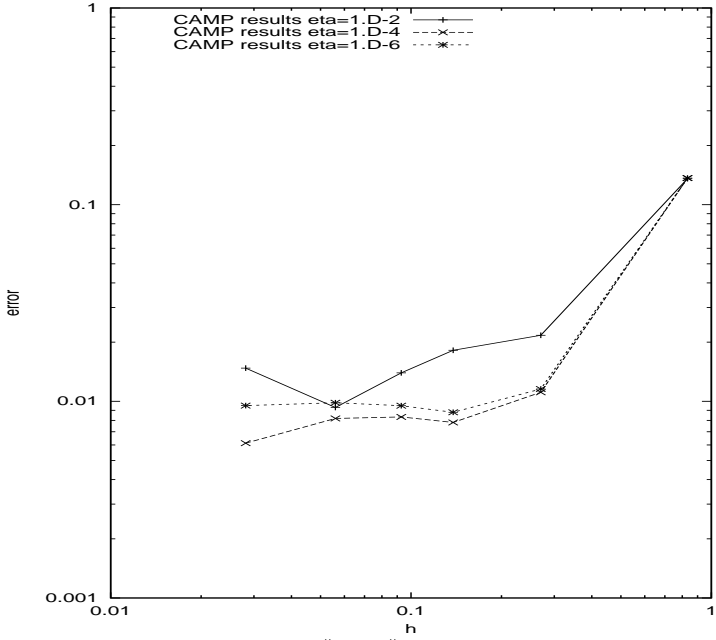


Figure 8. Parametric study of error $\frac{\|u_h - u\|_{L^2}}{\|u\|_{L^2}}$ versus the mesh size in Example 3.

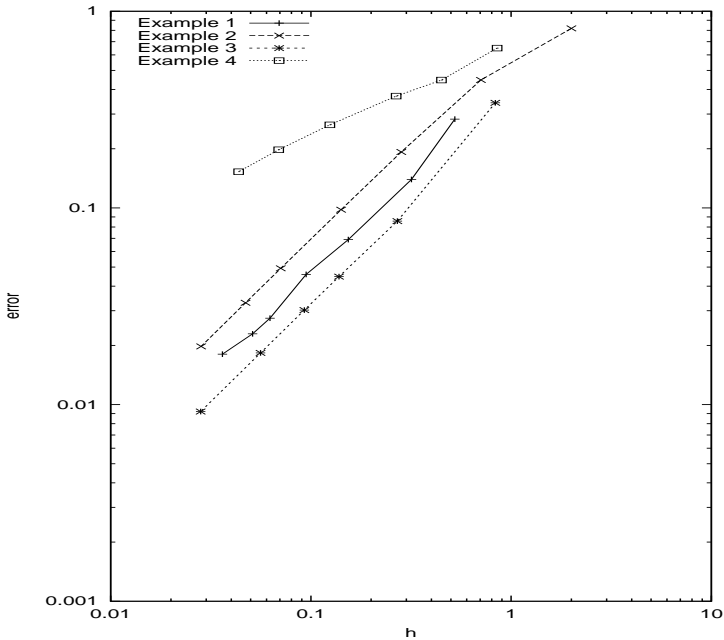


Figure 9. Error $\frac{\|D_h u_h - \nabla u\|_{L^2}}{\|\nabla u\|_{L^2}}$ versus the mesh size (CAMP).

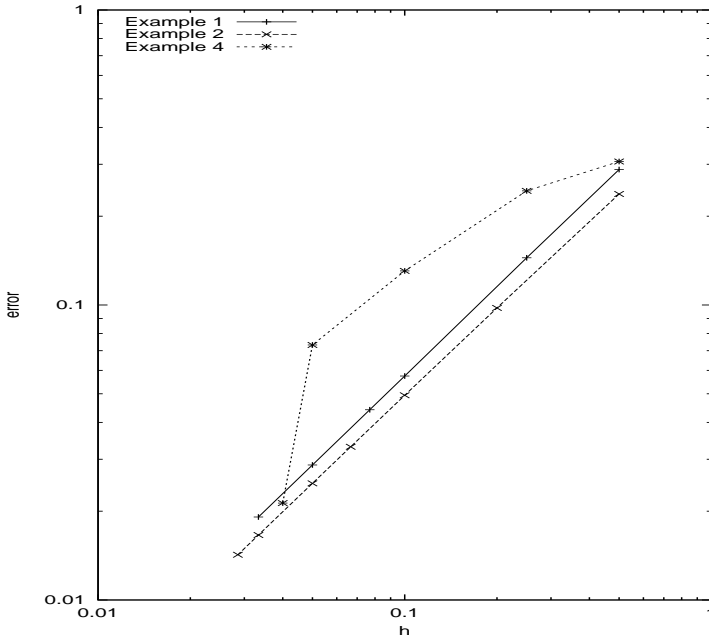


Figure 10. Error $\frac{\|D_h u_h - \nabla u\|_{L^2}}{\|\nabla u\|_{L^2}}$ versus the mesh size (CASTEM).

The quantity

$$\frac{\|u_h - u\|_{L^2}}{\|u\|_{L^2}}$$

is plotted in Figures 4-7. In cases 1-3 we used for u the exact solution, whereas for Example 4 we used a classical finite element approximation computed by a software named CASTEM developed at the Commissariat à l’Energie Atomique (CEA), France, with a mesh size $h = 2.5 \times 10^{-2}$. With the exception of case 3, for which CASTEM provides the exact solution because it is linear, the errors got by CASTEM are also plotted in the figures. It is seen that the error rate is rather uniform and close to 1, for cases 1 and 2, and to $2/3$ for case 4. We observe that in cases 3 and 4, where the solution is less regular, the way the mesh covers the singularity domain may play a role in the numerical performance for coarse discretizations. In particular, the results for case 3 are sensitive to the value of parameter η , as is shown in Figure 8.

In Figure 9 we plotted the L^2 -error on the gradient for all the examples

$$\frac{\|D_h u_h - \nabla u\|_{L^2}}{\|\nabla u\|_{L^2}}.$$

It turns out that the convergence rate is now close to 1 for the first three cases, and to $1/2$ for case 4. Finally, Figure 10 shows the error when the problems are treated by CASTEM package.

REFERENCES

- [1] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Spriger-Verlag, New York, 1991.
- [2] P.G. Ciarlet, “The finite element method for elliptic problems”, North Holland, Amsterdam, 1978.

- [3] P.G. Ciarlet, *Basic error estimates for elliptic problems*, in Ciarlet P. G. and Lions J.L. ed., “Handbook of numerical analysis”, North Holland, Amsterdam.
- [4] G. Dal Maso, “An introduction to Γ -convergence”, Birkäuser, Boston, 1991.
- [5] C. Davini, *Note on a parameter lumping in the vibrations of elastic beams*, Rend. Ist. Matematica Università di Trieste, 28 (1996), 83–99.
- [6] C. Davini, *Γ -convergence of external approximations in boundary value problems involving the bi-laplacian*, J. Comp. Appl. Math., 140 (2002), 185–208.
- [7] C. Davini, *Piece-wise constant approximations in the membrane problem*, Meccanica, 38 (2003), 555–569.
- [8] C. Davini, *On space $H_{\partial_t}(div, \Omega)$* , private communication, 2003.
- [9] C. Davini and R. Paroni, *Generalized Hessian and external approximations in variational problems of second order*, J. Elasticity, 70 (2003), 149–174.
- [10] C. Davini and I. Pitacco, *Relaxed notions of curvature and a lumped strain method for elastic plates*, SIAM J. Numer. Anal., 35 (1998), 677–691.
- [11] C. Davini and I. Pitacco, *An unconstrained mixed method for the biharmonic problem*, SIAM J. Numer. Anal., 38 (2000), 820–836.
- [12] V. Girault and P.A. Raviart, “Finite element methods for Navier-Stokes equations”, Springer-Verlag, Berlin Heidelberg, 1986.
- [13] P. Grisvard, “Singularities in boundary value problems”, Springer-Verlag, Berlin, 1992.
- [14] J.E. Roberts and J.M. Thomas, *Mixed and Hybrid methods*, in Ciarlet P. G. and Lions J.L. ed., “Handbook of numerical analysis”, North Holland, Amsterdam, 1991.