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# REALIZATION OF MULTIVARIABLE NONLINEAR SYSTEMS VIA THE APPROACHES OF DIFFERENTIAL FORMS AND DIFFERENTIAL ALGEBRA

JIANGFENG ZHANG, CLAUDE H. MOOG AND XIAOHUA XIA

In this paper differential forms and differential algebra are applied to give a new definition of realization for multivariable nonlinear systems consistent with the linear realization theory. Criteria for the existence of realization and the definition of minimal realization are presented. The relations of minimal realization and accessibility and finally the computation of realizations are also discussed in this paper.

*Keywords:* realization, nonlinear system, differential ideal, differential form

*Classification:* 93B15, 93C10

## 1. INTRODUCTION

Various approaches have been developed to find a suitable definition of realization for nonlinear systems since the late 1970's ([9, 12, 16, 22, 46, 50, 51]). These definitions are not equivalent and some are not consistent with the linear theory as this review will show. The purpose of this paper is to study this problem and present a new definition of realization for nonlinear systems which is consistent with the linear realization theory.

In general a single input single output (SISO) linear time-invariant system can be written in its *state space* form as follows

$$\Sigma : \begin{cases} \dot{x} &= Ax + bu, \\ y &= cx, \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^{n \times 1}$  and  $c \in \mathbb{R}^{1 \times n}$ . By using Laplace transform, this SISO system can also be rewritten in the *transfer function* form

$$H(s) = \frac{b_{k-1}s^{k-1} + \cdots + b_1s + b_0}{s^k + a_{k-1}s^{k-1} + \cdots + a_1s + a_0}. \quad (2)$$

The relation between the state space form and transfer function form is clear: every system in state space form  $\Sigma$  admits a transfer function  $c(sI - A)^{-1}b$ , and every transfer function  $H(s)$  is realized by a system in a state space form  $\Sigma$  [3]. In the

later case,  $\Sigma$  is called a *realization* of  $H(s)$  if

$$H(s) = c(sI - A)^{-1}b.$$

The above realization theory between state space form and transfer function form for SISO systems is easily generalized to multi-input multi-output (MIMO) linear time-invariant systems [3]; however, the corresponding generalization to nonlinear systems has been a longstanding problem. For nonlinear systems, the realization problem is finding a suitable state equation for a given system of input-output equations. Recall that the success of the concept of transfer function in solving the linear realization problem is because those transfer functions could be computed either from the input-output equations or from the state space equations. However, for nonlinear systems it is impossible to define transfer functions by using the Laplace transform which results in great difficulty in the study of nonlinear realization theory. Furthermore, the problem of minimal realization for nonlinear systems is not adequately understood.

Since transfer function is powerful for linear realization theory, it is natural to consider its nonlinear generalizations. By using the differential 1-form method introduced in [13] and the noncommutative ring theory ([42, 43]), references [56] and [23] define transfer functions/matrices for nonlinear systems, with a focus on the SISO case. Reference [36] applies the same techniques to discuss the irreducibility of nonlinear systems; however, its further application in nonlinear realization theory is not discussed. [24] presents a general framework for the MIMO case, while a comprehensive study on the existence and computation of realization is lacking. These approaches based on transfer functions generally depend on state variables and are not helpful for our purpose.

An early attempt at nonlinear realization theory investigated whether a nonlinear state space system admits an input-output equation. Under some *regularity* conditions, the state variable could be expressed as a function of the input, the output and their derivatives. Though such state elimination procedures can be found here and there in the early literature on nonlinear observability and observers ([26, 38, 39, 46]), and later in books by Isidori [27] and by Nijmeijer and van der Schaft [41], some detailed descriptions, with a view of realization, were first given in [14, 22] and [49]. Different state elimination processes for the same set of state space equations sometimes end up with different sets of input-output equations, and the trajectories of the obtained input-output equations may also be different.

The *trajectories* (or *behavior* as referred to in [47])  $\Sigma_d$  of a state space nonlinear system and the *trajectories* (or *behavior*)  $\Sigma_e$  of a system of input-output equations are defined and [54] calls the state space system *admits an input-output equation* if  $\Sigma_d \subset \Sigma_e$ . This is the case when the state space variables in the state space equations can be eliminated (via observability) to yield (or to generate as in [40]) the input-output equations. Besides this inclusion definition of realization, there are two other definitions: [47] defines realization through the equality condition  $\Sigma_e = \Sigma_d$ , while [54] defines realization by the inclusion  $\Sigma_e \subset \Sigma_d$ . From a similar point of view as in [54], bilinear realizability was investigated in [21] and [50], polynomial realizability in [2], rational realizability in [52], and the more general form of realizability in [9, 10]

and [11]. Some necessary or sufficient conditions and constructive procedures were given for realizing an input-output equation in these approaches. These trajectory based definitions are, however, conceptually inconsistent with the linear realization as the subsequent example will show. In fact, it follows from the linear realization theory that any of the following three state space systems

$$\Sigma_1 : \begin{cases} \dot{x} &= u, \\ y &= x, \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_1 &= u, \\ \dot{x}_2 &= 0, \\ y &= x_1, \end{cases} \quad \Sigma_3 : \begin{cases} \dot{x}_1 &= x_2 + u, \\ \dot{x}_2 &= 0, \\ y &= x_1, \end{cases} \quad (3)$$

is a realization of both the transfer functions  $H_1(s) = \frac{1}{s}$  and  $H_2(s) = \frac{s}{s^2}$ . These transfer functions are equivalent to the following input-output equations (4) and (5) respectively:

$$\dot{y} - u = 0, \tag{4}$$

$$\ddot{y} - \dot{u} = 0. \tag{5}$$

It is easy to check that  $\Sigma_d \subset \Sigma_e$  is not satisfied for systems  $\Sigma_3$  and (4),  $\Sigma_d = \Sigma_e$  is violated for  $\Sigma_2$  and (4), and  $\Sigma_e \subset \Sigma_d$  does not hold for  $\Sigma_1$  and (5).

There is also the point of view to understand the realization theory as the relation between the input-output map and its state-space representation. By this understanding, a lot of work is done for nonlinear theory (see, for example, [4, 27, 28, 30, 31, 32, 53]). This paper focuses on the relations between the input-output equation and the corresponding state-space realization, therefore the input-output map approach is not adopted here. In the literature reviewed above, the minimal realization for nonlinear MIMO systems is still far from being adequately understood. It is therefore necessary to study the realization of nonlinear MIMO systems including the notion of minimality.

In this paper a new definition of realization is given for nonlinear MIMO systems in accordance with the linear theory. Differential algebra (see [7, 15, 17, 18, 19, 20, 33, 34, 44]) and the method of differential 1-form (see [6, 13]) are the main tools for this new approach. In Section 2 differential ideals, notations and conventions are introduced. In Section 3 the definition of realization is given. Criteria to check whether a system is realizable is presented in Section 4. In the subsequent section the definition of minimal realization is provided and the relation with accessibility is found. In Section 6 the general scheme for the computation of realization and minimal realization is presented. Conclusions are presented in Section 7. The definition of a differential ring and the proof that the new definition of nonlinear realization is consistent with the linear theory are given in the Appendix.

## 2. PRELIMINARIES

Terminologies used throughout this paper emanates from several sources. Definitions of observability, accessibility and integrability are described by [6]. *Differential 1-form* can be found in [35], the definitions of *ring* and *ideal* are referred to in [29]. The notions of *derivation operator*, *differential ring* and *differential ideal* are taken from [34]. For any given set  $\mathcal{S}$  in a differential ring, the symbol  $\langle \mathcal{S} \rangle$  denotes the differential ideal generated by  $\mathcal{S}$  in this differential ring.

### 2.1. Notations and Hypotheses on General Input-Output Equations

For any given nonnegative integers  $k_0$  and  $s_0$ , and a contractible open subset  $U_0$  in  $\mathbb{R}^{p(k_0+1)+m(s_0+1)}$ , consider the system of input-output equations

$$\Phi_i(y, \dot{y}, \dots, y^{(k_0)}, u, \dot{u}, \dots, u^{(s_0)}) = 0, i = 1, 2, \dots, n_0, \tag{6}$$

where  $y = (y_1, \dots, y_p)^T \in \mathbb{R}^p, u = (u_1, \dots, u_m)^T \in \mathbb{R}^m$ , and each  $\Phi_i$  is real analytic in its arguments on  $U_0$ . Here a real analytic function on  $U_0$  means that it can be expanded as a convergent power series with real coefficients at any point inside  $U_0$ . For any  $k'_0 \geq k_0, s'_0 \geq s_0$  define the projection

$$(y, \dot{y}, \dots, y^{(k'_0)}, u, \dot{u}, \dots, u^{(s'_0)}) \mapsto (y, \dot{y}, \dots, y^{(k_0)}, u, \dot{u}, \dots, u^{(s_0)}),$$

which induces a natural projection from some contractible open set  $U'_0 \subseteq \mathbb{R}^{p(k'_0+1)+m(s'_0+1)}$  to  $U_0 \subseteq \mathbb{R}^{p(k_0+1)+m(s_0+1)}$ .

Let  $R_1(U_0)$ (or  $R_1$  for simplicity) be the set of all the real analytic functions in some finite variables in  $\{y_i^{(j)}, u_r^{(l)} : j, l \geq 0; i = 1, \dots, p; r = 1, \dots, m\}$  such that each element is analytic over a contractible open set  $U'_0$  for which  $U_0$  is the projection of  $U'_0$ . That is, each element of  $R_1$  is real analytic on a contractible open set  $U'_0$ , and for different elements in  $R_1$  the corresponding open sets  $U'_0$  may be different as they are possibly lying in different spaces. In Subsection A of the Appendix, operations are introduced in  $R_1(U_0)$  so that it becomes an integral domain and a differential ring. Denote the fraction field of  $R_1(U_0)$  by  $\mathcal{K}(U_0)$  (or  $\mathcal{K}$  when the corresponding open set is clear from the context).

For any linear space  $M$  over the field  $\mathcal{K}(U_0)$  with generators  $\{\beta_\mu : \mu \in \Lambda'\}$  for an index set  $\Lambda'$ , the following expression is used to denote  $M$ :

$$M = \text{span}_{\mathcal{K}(U_0)}\{\beta_\mu : \mu \in \Lambda'\}.$$

Let  $I_\Phi$  be the differential ideal (see [33] or [34]) of  $R_1$  which is generated by  $\Phi_i, i = 1, \dots, n_0$ , and denote it by  $I_\Phi = \langle \Phi_1, \dots, \Phi_{n_0} \rangle$ . For any element  $\phi(y, \dot{y}, \dots, y^{(k)}, u, \dot{u}, \dots, u^{(r)})$  in  $\mathcal{K}$  define formally the derivative and the differential

$$\begin{aligned} \dot{\phi} &= \sum_{i=0}^k \frac{\partial \phi}{\partial y^{(i)}} y^{(i+1)} + \sum_{j=0}^r \frac{\partial \phi}{\partial u^{(j)}} u^{(j+1)}, \\ d\phi &= \sum_{i=0}^k \frac{\partial \phi}{\partial y^{(i)}} dy^{(i)} + \sum_{j=0}^r \frac{\partial \phi}{\partial u^{(j)}} du^{(j)}. \end{aligned}$$

In this paper the analytic version of Poincaré Lemma and Frobenius Theorem, which means that all the functions and differential forms in these results are analytic, will be used. The proofs are almost the same as the classical proofs of the smooth version, and therefore omitted and referred to results in [5].

Let  $R_2 = R_1[D]$  be the set of polynomials in the variable  $D$  with coefficients in  $R_1$ , where  $D$  is the operator that computes the derivative of a function. Define the

multiplication in  $R_2$  as the composition of operators, where each element of  $R_2$  is viewed as an operator, then  $R_2$  is noncommutative [55]. Let

$$E = d\mathcal{K} = \text{span}_{\mathcal{K}}\{dg : g \in \mathcal{K}\}.$$

Note that the above  $R_2$  and  $E$  are actually defined over the same contractible open set  $U_0$  as  $R_1$ . For any differential ideal  $L$ , define

$$dL = \text{span}_{\mathcal{K}}\{dg : g \in L\}.$$

It is clear that  $dL$  is a linear subspace of  $E$ , and one can define the quotient space  $E/dL$ . Note that each element in the quotient space  $E/dL$  can be written as  $[w]$  or  $w + dL$ , where  $w$  is an element of  $E$ . This element  $[w]$  or  $w + dL$  is also called the equivalent class of  $w$ . Although both notations  $[w]$  and  $w + dL$  are adopted in literature, we often use the latter since it obviously states that the equivalent relation is defined by  $dL$ . The following lemma is required to define the degree of a nonzero element in  $R_2$ , and its proof can easily be calculated and is thus omitted.

**Lemma 2.1.** For any  $\lambda \in R_1$ , the equality  $D^k\lambda = \sum_{i=0}^k \binom{k}{i}\lambda^{(i)}D^{k-i}$  holds in  $R_2$ , where  $k = 1, 2, 3, \dots$ , and  $\lambda^{(i)}$  denotes the derivative of  $\lambda$  up to order  $i$ .

A general element of  $R_2$  can be a finite sum of the terms like  $a_1D^{i_1}a_2D^{i_2} \dots a_kD^{i_k}a_{k+1}$ , where  $a_1, a_2, \dots, a_{k+1} \in \mathcal{K}$ , and  $i_1, i_2, \dots, i_k$  are nonnegative integers. By the above Lemma 2.1, any nonzero element  $\rho \in R_2$  can be written uniquely in the form

$$\rho = \sum_{i=0}^k \lambda_i D^i, \lambda_i \in R_1, i = 0, \dots, k; \lambda_k \neq 0.$$

Define the *degree* of the above  $\rho$  to be  $\text{deg } \rho = k$ .

Without loss of generality, suppose that system (6) is right-invertible, i.e. that the functions  $\{y_i^{(j)} : j \geq 0, i = 1, \dots, p\}$  are *differentially analytically independent* on  $U_0$  in the sense that  $\{dy_i^{(j)} + dI_{\Phi} : j \geq 0, i = 1, \dots, p\}$  is an independent set in the quotient space  $E/dI_{\Phi}$  on  $U_0$ . This assumption is also called the *differential analytical independence* of  $\{y_1, \dots, y_p\}$ . It implies that we do not consider the realization problem for any equation which has only variables in  $\{y_l^{(i)} : i \geq 0; l = 1, \dots, p\}$  but has no variable in  $\{u_r^{(j)} : j \geq 0; r = 1, \dots, m\}$ . Assume also that  $\{u_1, \dots, u_m\}$  are differentially analytically independent which in turn implies that  $\{du_r^{(j)} + dI_{\Phi} : j \geq 0; r = 1, \dots, m\}$  are independent in  $E/dI_{\Phi}$ . The two assumptions are fixed throughout the paper. The assumption about the differential analytic independence of  $\{y_1, \dots, y_p\}$  or  $\{u_1, \dots, u_m\}$  does not make the approach of the paper less general. For example, if  $\{y_1, \dots, y_p\}$  is differentially analytically dependent, then with some suitable initial values, some of  $\{y_1, \dots, y_p\}$  are functions of the other independent variables. Now these variables can be eliminated from the input-output equation, and one can define the original input-output equation to have the same realization as the reduced input-output equation.

**Remark 2.2.** In the above notion of differential analytical independence, the symbols  $y_i^{(j)}$  and  $u_r^{(l)}$  are considered as variables and not functions of  $t$ . Therefore,  $y_i^{(j)}$  and  $u_r^{(l)}$  are treated as variables instead of functions of  $t$  when we refer to input-output equations, differential analytical independence, and differential ideals. They are treated as functions of time  $t$  only when state equations are considered.

**2.2. From General Input-Output Equations to Standard Form**

A standard form hypothesis on (6) is now introduced. An elimination process which triangularizes a system of analytic functions is needed to obtain the standard form of input-output equations. This is a generalization of Gaussian elimination for linear system of equations and Ritt’s elimination theory for differential algebraic equations [44]. Define the following local operations for the functions in  $R_1$ .

- (i) Use the implicit function theorem for the equation  $\phi(y_1^{(i_1)}, y_1^{(i_2)}, \dots) = 0$ ,  $i_2 < i_1$ , to obtain locally that  $y_1^{(i_1)} - g_0(y_1^{(i_2)}, \dots) = 0$  when  $\frac{\partial \phi}{\partial y_1^{(i_1)}}$  is nonzero on certain open set.
- (ii) For  $\phi'(y_1^{(i_1+k)}, y_1^{(i_1+k-1)}, \dots, y_1^{(i_1)}, y_1^{(i_2)}, \dots) = 0$ ,  $i_2 < i_1$ , and the  $\phi = 0$  in the above (i), substitute  $y_1^{(i_1)} = g_0, y_1^{(i_1+1)} = g_1, \dots, y_1^{(i_1+k)} = g_k$  into  $\phi'$  and obtain  $\phi''(g_k, g_{k-1}, \dots, g_0, y_1^{(i_2)}, \dots) = 0$ , where  $g_i = g_0^{(i)}$  for  $i = 1, \dots, k$ .
- (iii) Permute two equations.

The above elementary operations differ from the transformation in [48]. In fact, [48] aims to eliminate latent variables so as to obtain a system which consists of input and output variables only. Such an algorithm will not work for (6) since there are no latent variables in (6). Furthermore, the aim of the elementary operations in this paper is to transform (6) into (7) where the function has an explicit leading term  $y_i^{(m_i)}$ , while [48] will not transform a function into such explicit form as  $y_i^{(m_i)} + \phi_i$  with  $LT(\phi_i) < y_i^{(m_i)}$  (see the text below for the notations about ordering).

For the set of functions  $\mathcal{T} = \{y_i^{(j)}, u_r^{(l)} : i = 1, \dots, p; r = 1, \dots, m; j, l \geq 0\}$ , define the following ordering:

$$\begin{aligned}
 &y_i^{(j)} > u_r^{(l)} && \text{for all } i, j, l, r; \\
 &y_i^{(j)} > y_{i'}^{(j')} && \text{if and only if } i < i' \text{ or } i = i', j > j'; \\
 &u_r^{(l)} > u_{r'}^{(l')} && \text{if and only if } r < r' \text{ or } r = r', l > l'.
 \end{aligned}$$

Then the above ordering is well-defined (see [8] for more information about ordering). For any meromorphic function  $\phi(y_i^{(j)}, u_r^{(l)} : i = 1, \dots, p; r = 1, \dots, m; j, l \geq 0)$ , define its leading term  $LT(\phi)$  to be the greatest variable in  $\mathcal{T}$  such that the partial derivative of  $\phi$  with respect to this variable is nonzero.

Now consider operations on the contractible open set  $U_0$ . If, in system (6),  $LT(\Phi_i) = y_1^{(j_i)}$  for all  $i = 1, \dots, n_0$ , and  $j_1 < j_2 < \dots < j_{n_0}$ , then the first elementary operation is used to solve  $y_1^{(j_1)}$  from  $\Phi_1 = 0$  and substitute it into the

other equations. Then the derivatives of  $y_1$  with order higher than or equal to  $j_1$  are eliminated. Since the first operation is only a local operation and may not hold on the whole  $U_0$ , we shrink the open set  $U_0$  into some smaller, contractible open set  $\overline{U_0}$  so that the resulting equation after the first operation is analytic on some  $\overline{U_0}'$  whose projection equals  $\overline{U_0}$ . Repeating the process for other variables and shrinking the underlying open set when necessary, system (6) is transformed into the so-called *echelon form*. This is that, for any  $y_i$ , there is at most one nonzero function, whose leading term is some derivative of  $y_i$ , in the resulting set of functions obtained by the above operations. After using the first elementary operation, the nonzero element of the echelon form with leading term  $y_i^{(j)}$  can be written as

$$y_i^{(m_i)} + \phi_i \left( y_i^{(j_0)}, y_{j_1}^{(j_2)}, u_{j_3}^{(j_4)} : j_0 = 0, \dots, m_i - 1; \right. \\ \left. j_1 = i + 1, \dots, p; j_3 = 1, \dots, m; j_2 \leq m_i, 0 \leq j_4 \right). \tag{7}$$

Since  $\{u_1, \dots, u_m\}$  are differentially analytically independent, any nonzero element in the echelon form can not have a leading term  $u_r^{(l)}$ , that is, its leading form must be some  $y_i^{(j)}$ . Therefore, all the nonzero elements in the echelon form can be written as (7) and the number of the nonzero elements, denoted by  $p'$ , may also be smaller than  $p$ . After back substitution, the following hypothesis is made on (6) such that  $p' = p$ . Note that the functions in the echelon form are analytic only on a contractible open subset  $V_0$  which is derived from  $U_0$ . For simplicity, the following hypothesis is made.

**Standard Form Hypothesis 1.** Assume the functions in system (6) are real analytic on  $U_0$  and (6) can be transformed on  $U_0$  by the above three kinds of elementary operations and the rearrangement of  $\{y_1, \dots, y_p\}$  into the form

$$y_i^{(m_i)} + \zeta_i \left( y_i^{(j_0)}, y_{j_1}^{(j_2)}, u_{j_3}^{(j_4)} : 0 \leq j_0 \leq m_i - 1; 0 \leq j_2 < m_{j_1}; 0 \leq j_4 < s_1; \right. \\ \left. i + 1 \leq j_1 \leq p; 1 \leq j_3 \leq m \right) = 0, i = 1, \dots, p, \tag{8}$$

where  $s_1$  is a positive integer. Furthermore, assume that there exists a point  $P_0$ , which is called an *operating point* of the above system of equations (8) or of the differential ideal  $\langle y_i^{(m_i)} + \zeta_i : i = 1, \dots, p \rangle$ , such that  $P_0$  belongs to  $U_0$  and  $P_0$  is a zero of all the equations in (8).

Note that the above hypothesis assumes that one can obtain the standard form from (6) on the set  $U_0$ . This hypothesis is quite general and does not lose any generality. In fact, an open set  $V_0$  is obtained after performing the three kinds of elementary operations. This  $V_0$  may lie in an Euclidean space whose dimension is higher than  $p(k_0 + 1) + m(s_0 + 1)$ ; however, its projection to  $\mathbb{R}^{p(k_0+1)+m(s_0+1)}$  is a subset of  $U_0$ . We can work on this  $V_0$ , perform the three kind of elementary operations, and obtain the standard form on  $V_0$ .

This hypothesis means that any nonzero function  $\phi$  in the echelon form must contain a variable in the set  $\{y_i^{(j)} : i = 1, \dots, p; j \geq 0\}$  and there are exactly  $p$  nonzero elements in the echelon form. Let  $\mathcal{I}$  be the differential ideal generated by



the functions  $\{y_1^{(m_1)} + \zeta_1, \dots, y_p^{(m_p)} + \zeta_p\}$ . The equations  $y_i^{(m_i)} + \zeta_i = 0, i = 1, \dots, p$ , defined on  $U_0$  with the operation point  $P_0$ , will be the starting point of the realization problem. That is, consider the equations  $y_i^{(m_i)} + \zeta_i = 0, i = 1, \dots, p$ , over the contractible open set  $U_0$  and an operation point  $P_0 \in U_0$ , instead of the equations in (6). The reason why the integers  $m_i$  and  $s_1$  are introduced in equation (8) is for the convenience of the computation of  $V_{\max}$  which is defined at the beginning of Section 4.

The above hypothesis can also be expressed in another form which is easier to check. The following three types of invertible operations are needed for any  $k$ -tuple  $(w_1, w_2, \dots, w_k)$ , where  $w_i \in E, i = 1, \dots, k$ .

(i') Substitute  $f(D)w_1 + w_2$  to  $w_2$ , where  $f(D) \in R_2$ ;

(ii') Substitute  $\lambda w_1$  to  $w_1$ , where  $\lambda \in \mathcal{K}$  and  $\lambda \neq 0$ ;

(iii') Permute two differential 1-forms.

The above three operations are obviously invertible. Assume that the functions  $\Phi_i, i = 1, \dots, n_0$ , in (6) satisfy the following condition.

**Standard Form Hypothesis 1'.** Suppose  $(d\Phi_1, \dots, d\Phi_{n_0})$  can be transformed on  $U_0$  by the above three kind of elementary operations and the rearrangement of  $(y_1, \dots, y_p)$  into  $(\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_p, 0, \dots, 0)$ , where

$$\begin{aligned} \tilde{w}_i &= d(y_i^{(m_i)} + \zeta_i) \\ &= dy_i^{(m_i)} + \sum_{j_1=0}^{m_i-1} a_{j_1}^i dy_i^{(j_1)} + \sum_{j_1=i+1}^p \sum_{0 \leq j_2 < m_{j_1}} b_{j_1, j_2} dy_{j_1}^{(j_2)} \\ &\quad + \sum_{j_1=1}^m \sum_{0 \leq j_3 < s_1} c_{j_1, j_3} du_{j_1}^{(j_3)}, i = 1, \dots, p, \end{aligned} \tag{9}$$

$\zeta_i$  is a real analytic function on  $U_0$  and  $LT(\zeta_i) < y_i^{(m_i)}, i = 1, \dots, p$ , and  $s_1$  is a positive integer. Furthermore, assume that there exists a point  $P_0$  such that  $P_0$  belongs to  $U_0$  and is a zero of  $y_i^{(m_i)} + \zeta_i = 0$  for all  $i = 1, \dots, p$ .

Obviously the above two forms of hypotheses are equivalent. The first form will be used in the definition of realization, and the second form is used to check if the standard form hypothesis holds.

**Definition 2.3.** A differential ideal  $L$  of  $R_1(U_0)$  is called standard with basis  $\{\phi_1, \dots, \phi_{n_1}\}$  and indices  $(m_1, \dots, m_p, s_1)$  on a contractible open set  $U_3 \subseteq U_0$  if it has a finite set of generators  $\{\phi_1, \dots, \phi_{n_1}\}$  which satisfies the Standard Form Hypothesis 1 on the set  $U_3$ , where  $U_3$  contains also an operating point  $P_0$ . The set of nonzero elements in the corresponding echelon form (8) or (9) is called the standard form of  $L$  with basis  $\{\phi_1, \dots, \phi_{n_1}\}$  and indices  $(m_1, \dots, m_p, s_1)$  on  $U_3$  or simply standard form. A real analytic function  $g \in R_1(U_0)$  is called in standard form if there exists a function  $\zeta$  and integers  $(i, j)$  or  $(r, l)$  such that  $g = y_i^{(j)} + \zeta, LT(g) = y_i^{(j)} > LT(\zeta)$ ; or  $g = u_r^{(l)} + \tilde{\zeta}, LT(g) = u_r^{(l)} > LT(\tilde{\zeta})$ , where  $1 \leq i \leq p, j \geq 0, 1 \leq r \leq m, l \geq 0$ . For simplicity, denote  $g = s(\xi)$ .

Given any standard differential ideal  $L$  and the corresponding finite basis, it follows directly from the elimination process of the Standard Form Hypothesis 1 that the resulting standard form of  $L$  must be unique. That is, the above indices  $(m_1, \dots, m_p, s_1)$  are well defined for any given basis  $\mathcal{B} := \{\phi_1, \dots, \phi_{n_1}\}$  of  $L$ . Define  $s(L, \mathcal{B})$  as the differential ideal which is generated by the standard form. Note that  $s(L, \mathcal{B})$  depends on the basis  $\mathcal{B}$ . For simplicity the following convention is made:

*When a system of input-output equations  $\xi_1 = \dots = \xi_{n_1} = 0$  is considered or when a differential ideal is defined by  $\langle \xi_1, \dots, \xi_{n_1} \rangle$ , then the basis of the corresponding ideal will be taken as  $\{\xi_1, \dots, \xi_{n_1}\}$ . This basis will be fixed to compute the standard form. The resulting  $s(L, \mathcal{B})$  will be denoted by  $s(L)$  for simplicity.*

Therefore,  $s(I_\Phi) = \mathcal{I}$ . Note that  $s(L) \neq L$  in general. For example, when  $L$  is generated by the function  $\sin(y) - u$  on the open set  $\{(y, u) : -\frac{\pi}{2} < y < \frac{\pi}{2}, -1 < u < 1\}$ , then  $s(L) = \langle y - \arcsin(u) \rangle \neq L$ .

For the operating point  $P_0$  of (6) and the contractible and open set  $U_0$ , consider any differential ideal  $L(U_0)$  with a basis  $\mathcal{B}$ , and define its differential closure  $\overline{L}(U_0, P_0, \mathcal{B})$  (sometimes write  $\overline{L}(U_0, P_0)$ ,  $\overline{L}(U_0)$ , or  $\overline{L}$ , for simplicity), with respect to (6), as the differential ideal generated by the set

$$\left\{ g \in R_1(U_0) : g \text{ is in standard form, } g(P_0) = 0, \text{ and there exist } k_1 \geq 0, \right. \\ \left. \alpha_i \in R_1(U_0), \text{ and } \beta_i \in R_1(U_0), \text{ such that } \alpha_i \notin s(L(U_0)), \right. \\ \left. \beta_i \notin s(L(U_0)), \text{ and } \sum_{j=0}^{k_1} \beta_j D^j (\alpha_j g) \in s(L(U_0)), \text{ where } i = 0, 1, \dots, k_1 \right\}. \tag{10}$$

Now introduce another stronger hypothesis.

**Standard Form Hypothesis 2.** Assume that the ideal  $\overline{s(I_\Phi)} = \overline{\mathcal{I}}$  is standard with respect to the basis  $\{y_i^{(m_i)} + \zeta_i : i = 1, \dots, p\}$  on some contractible open set  $U_0''$  whose projection equals  $U_0$ .

The following convention is made throughout this paper on the differential analytical independence of the  $y_i$ 's and  $u_r$ 's since the standard form hypotheses are made:

Suppose that both  $\{dy_i^{(j)} + d\overline{\mathcal{I}} : i = 1, \dots, p; j \geq 0\}$  and  $\{du_r^{(l)} + d\overline{\mathcal{I}} : r = 1, \dots, m; l \geq 0\}$  are linearly independent sets in  $E/d\overline{\mathcal{I}}$ .

Define  $\mathcal{U} := \text{span}_{\mathcal{K}}\{du_r^{(l)} : r = 1, \dots, m; l \geq 0\}$  which is a subspace of  $E$ . For any standard ideal  $L$ , it is clear that the standard form of  $L$  in the Standard Form Hypothesis 1' contains no nonzero element in  $\mathcal{U}$ . For this standard ideal  $L$ , define the quotient space  $\overline{\mathcal{U}} := \mathcal{U}/(dL \cap \mathcal{U}) \cong (\mathcal{U} + dL)/dL$ .

**Proposition 2.4.** Fix the notation  $\mathcal{U}$  as above, and suppose  $L$  is a standard ideal on a contractible open set  $U_2$ , then the equivalent classes of 1-forms in the set  $\{du_r^{(l)} : r = 1, \dots, m; l \geq 0\}$  are linearly independent in the quotient space  $\overline{\mathcal{U}} = \mathcal{U}/(dL \cap \mathcal{U})$ .

*Proof.* Suppose  $\{[du_r^{(l)}] : r = 1, \dots, m; l \geq 0\}$  is dependent, then there exist  $\{\alpha_{rl} \in R_1(U_2) : r = 1, \dots, m; l = 0, \dots, n_1\}$ , which are not all zeros, such that

$\sum_{r=1}^m \sum_{l=0}^{n_1} \alpha_{rl} du_r^{(l)} \in dL$ . Since  $L$  is standard, one can assume that there exists  $g_1, g_2, \dots, g_{n_2} \in L$  and nonzero  $\beta_1, \beta_2, \dots, \beta_{n_2} \in R_1(U_2)$  such that  $LT(g_1) = y_{i_1}^{(j_1)} > LT(g_k) > LT(g_{n_2})$  for all  $k = 2, 3, \dots, n_2$ , and

$$\sum_{r=1}^m \sum_{l=0}^{n_1} \alpha_{rl} du_r^{(l)} = \sum_{i=1}^{n_2} \beta_i dg_i \tag{11}$$

holds in  $E$ . Then the above equality can be rewritten locally as  $dy_{i_1}^{(j_1)} + \sum_{i,j} \gamma_{ij} dy_i^{(j)} + \sum_{r,l} \delta_{rl} du_r^{(l)} = 0$  holds in  $E$ , where  $LT(y_{i_1}^{(j_1)}) > LT(y_i^{(j)}) > LT(u_r^{(l)})$  for all possible indices  $i, j, r, l$  in (11). However, this is impossible since all the 1-forms  $\{dy_{i_2}^{(j_2)}, du_{r_2}^{(l_2)} : 1 \leq i_2 \leq p, j_2 \geq 0, 1 \leq r_2 \leq m, l_2 \geq 0\}$  are independent in  $E$  over any open set.  $\square$

It follows from the above proposition that the equivalent classes of the 1-forms in  $\{du_r^{(l)} : r = 1, \dots, m; l \geq 0\}$  are independent on  $U_0$  when they are considered in the quotient space  $\mathcal{U}/(dI_{\mathbb{F}} \cap \mathcal{U})$ .

For a standard ideal  $L$  with indices  $(m_1, \dots, m_p, s_1)$  on any contractible open set  $U_3 \subseteq U_0$  which contains also the operating point  $P_0$ , define the linear space  $H_0(L)$  (or  $H_0$ ) on  $U_3$  by

$$H_0(L) = \text{span}_{\mathcal{K}} \left\{ dy_i^{(j)} + dL, du_r^{(l)} + dL : j = 0, 1, \dots, m_i - 1; l = 0, 1, \dots, s_1 - 1; i = 1, \dots, p; r = 1, \dots, m \right\}. \tag{12}$$

Obviously  $H_0(L)$  is a subspace of  $E/dL$  on  $U_3$ .

### 2.3. Notations and Conventions on State Equations

Consider the following system of state equations

$$\dot{x} = f(x, u), \tag{13}$$

$$y = h(x), \tag{14}$$

where  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ,  $y = (y_1, \dots, y_p)^T \in \mathbb{R}^p$ ,  $u = (u_1, \dots, u_m)^T \in \mathbb{R}^m$ ,  $f$  and  $h$  are real analytic functions of their arguments on a contractible open set  $U'_0$  whose projection equals  $U_0$ .

If the system (13–14) is observable [6] with observability indices  $(k_1, \dots, k_p)$  and  $k_1 \geq \dots \geq k_p$  (see [49]), then  $x$  can be solved as

$$x = \xi(y, u) = \xi(y_i^{(j)}, u_r^{(l)} : 0 \leq j \leq k_i - 1; 0 \leq l \leq k_1 - 1; 1 \leq i \leq p; 1 \leq r \leq m).$$

Denote by  $J$  the differential ideal of  $R_1$  on  $U_0$  which is generated by the standard forms of the components of the vector functions  $\dot{x} - f(x, u)$  and  $y - h(x)$  and their derivatives, where  $x = \xi(y, u)$ . For simplicity write  $J = \langle s(\dot{\xi} - f(\xi, u)), s(y - h(\xi)) \rangle$  or  $J = \langle \dot{\xi} - f(\xi, u), y - h(\xi) \rangle$ , where  $s(\dot{\xi} - f(\xi, u))$  denotes the vector whose components

are the standard forms of the components of  $\dot{\xi} - f(\xi, u)$ . Similar convention is made on  $s(y - h(\xi))$  as well.

If the system (13)–(14) is not observable, then let  $x = ((x')^T, (x'')^T)^T$ ,  $f = ((f')^T, (f'')^T)^T$ , such that  $x'$  is the maximal observable part,  $\dot{x}' = f'(x', u)$ ,  $y = h(x')$ , and assume that the rank of  $\frac{\partial(y_i^{(j)} : j=0,1,\dots,k_i-1; i=1,\dots,p)}{\partial x}$  is constant on some contractible open subset  $U'_0 \subseteq U_0$ . Then there exists  $\xi'$  such that  $x' = \xi'(y_i^{(j)}, u_r^{(l)} : j = 0, 1, \dots, k_i - 1; l = 0, 1, \dots, k_1 - 1; i = 1, \dots, p; r = 1, \dots, m)$  on  $U'_0$  for some integers  $k_i, i = 1, \dots, p$ , with  $k_1 \geq k_2 \geq \dots \geq k_p$ . Now consider the realization problem on  $U'_0$  instead of  $U_0$ , and define similarly the ideal  $J$  which is generated by the components of the vector functions  $\dot{x}' - f'(x', u)$  and  $y - h(x')$  and their derivatives, where  $x' = \xi'$ .

Assume that the functions  $\{y_1, \dots, y_p\}$  in (13)–(14) are differentially analytically independent on  $U_0$  with respect to the differential ideal  $J$ . Then it is clear that  $\text{rank} \frac{\partial h}{\partial x} = p$  on  $U_0$ .

**Remark 2.5.** In the above definition of  $J$ , the direct substitution of  $\xi$  into  $\dot{x} - f(x, u)$  and  $y - h(x)$  will sometimes give zeros; however, the substitution of  $\xi$  into the derivatives of  $\dot{x} - f(x, u)$  and  $y - h(x)$  may not result in zeros (see the computation of  $J$  in Example 3.3 in Section 3). Therefore, the definition of  $J$  assumes that we compute the derivatives before the substitution of  $\xi$ ; and the above  $J = \langle \dot{\xi} - f(\xi, u), y - h(\xi) \rangle$  is simply a notation. However, in the linear case one can substitute  $\xi$  in  $\dot{\xi} - f(\xi, u)$ ,  $y - h(\xi)$  first and then compute the derivatives, the resulting  $J$ 's are the same. This is due to the fact that  $(\dot{\xi} - A\xi - Bu)^{(i)} = \xi^{(i+1)} - A\xi^{(i)} - Bu^{(i)}$ ,  $(y - C\xi)^{(i)} = y^{(i)} - C\xi^{(i)}$ ,  $i = 0, 1, 2, \dots$ . Thus, it makes no difference whether or not  $\xi$  is substituted before or after the computation of the derivatives. In the proof of Lemma B.2 this observation is used without any comment.

The following lemma is clear since the ideal  $J$  is generated by  $\{s(\dot{\xi} - f(\xi, u)), s(y - h(\xi))\}$ .

**Lemma 2.6.** For any system of equations (13–14) and the ideal  $J$  defined above, suppose the functions  $y, u$  also satisfy system (6) on  $U_0$ , then there exists a finite set of generators of  $J$  such that  $J$  is standard on  $U_0$  with respect to this basis.

Therefore,  $s(J)$  and  $H_0(J)$  can be defined for the basis  $\{y_i^{(l)} - g_i^l : x = \xi, l \geq k_i, i = 1, \dots, p\}$  of  $J$ , and this basis will be fixed for the definitions of  $s(J)$  and  $H_0(J)$ .

All the assumptions and notations in this section are fixed throughout the paper. For the convenience of the reader the main symbols are listed.

$n, p, m$	the dimensions of $x, y$ and $u$ respectively ;
$n_0$	the number of equations in (6);
$k_0, s_0$	the highest orders of $y$ and $u$ , respectively, in equation (6);
$k_1, \dots, k_p$	the observability indices of (13–14);
$m_1, \dots, m_p, s_1$	$m_i$ is the highest orders of $y$ in the $i$ th equation of the standard form, while $s_1$ is the highest order of $u$ in the $p$ equations of the standard form, see Standard Form Hypothesis 1 and Definition 2.3;

$I_\Phi$	the differential ideal generated by $\{\Phi_1, \dots, \Phi_{n_0}\}$ , see the fourth paragraph of Section 2.1;
$\mathcal{I}$	the differential ideal generated by the functions in the standard form of system (6), see the second paragraph after Standard Form Hypothesis 1 in Section 2.2;
$\overline{\mathcal{I}}$	the differential closure of $\mathcal{I}$ , see (10);
$J$	the differential ideal generated by the components of $\dot{x} - f(x, u)$ and $y - h(x)$ and their derivatives, where $x = \xi(y, u)$ , see the second paragraph of Section 2.3;
$s(J), \overline{J}$	the standard form and closure of $J$ , respectively;
$H_0(L), H_0$	the linear space defined in (12);
$R_1$	the ring of real analytic functions, see the second paragraph of Section 2.1;
$R_2$	the noncommutative ring of operators $R_1(D)$ ;
$R_0$	the linear space defined in (32);
$V_{\max}(H_0, L)$	the linear space defined in the beginning of Section 4;
$\mathcal{U}^{(i)}$	the linear space spanned by $\{du_r^{(l)} : r = 1, \dots, m, 0 \leq l \leq i\}$ , see the beginning of Section 4;
$\mathcal{U}$	the linear space spanned by $\{du_r^{(l)} : r = 1, \dots, m, l \geq 0\}$ , see the paragraph before Proposition 2.4 in Section 2.2.

### 3. DEFINITION OF REALIZATION

**Definition 3.1.** Given system (6), the system (13–14) is called a realization of (6) on  $(U_0, P_0)$  (or  $U_0$  for short) if  $s(J) = \overline{\mathcal{I}}$  holds on  $U_0$ .

In Subsection B of the Appendix, it is proved that in the special case of linear systems the above definition reduces to equality of transfer function matrices computed either from the input-output equations or from the state equations.

Now consider the three systems in (3). Suppose the corresponding ideals  $J$  for the three systems are  $J_1, J_2, J_3$  respectively. Take  $U_0$  to be the whole Euclidean space, and its origin as the operating point  $P_0$ . Let  $I_{\Phi_1} = \langle \dot{y} - u \rangle$ ,  $I_{\Phi_2} = \langle \dot{y} - \dot{u} \rangle$ , then the two ideals are standard and  $\mathcal{I}_1 := s(I_{\Phi_1}) = I_{\Phi_1}$ ,  $\mathcal{I}_2 := s(I_{\Phi_2}) = I_{\Phi_2}$ . Now the two input-output equations have the same  $\overline{\mathcal{I}}$  which equals  $\mathcal{I}_1 = \overline{\mathcal{I}_1} = \overline{\mathcal{I}_2} = \langle \dot{y} - u \rangle$ . It is easy to compute that  $\overline{J_1} = s(J_1) = \langle \dot{y} - u \rangle = \overline{\mathcal{I}_1}$ ,  $J_2 = s(J_2) = \langle \dot{y} - u \rangle = \overline{\mathcal{I}_1}$ ,  $J_3 = s(J_3) = \langle \dot{y} - \dot{u} \rangle \subset \overline{\mathcal{I}_1}$ ,  $s(J_1) = s(J_2) = s(J_3) = \overline{\mathcal{I}_1}$ . Thus, all three systems are realizations of both input-output equations.

The following example shows how the realization varies with respect to different open sets.

**Example 3.2.** Consider the input-output equation  $\Phi = (\dot{y} - u)(\dot{y} - 2u) = 0$ , then  $I_\Phi = \langle (\dot{y} - u)(\dot{y} - 2u) \rangle$ . Define four contractible open sets  $U_1 = \{(\dot{y}, y, u) : \dot{y} - 2u > 0\}$ ,  $U_2 = \{(\dot{y}, y, u) : \dot{y} - u > 0\}$ ,  $U_3 = \{(\dot{y}, y, u) : \dot{y} - 2u < 0\}$ ,  $U_4 = \{(\dot{y}, y, u) : \dot{y} - u < 0\}$ . It is clear that the standard form of  $I_\Phi$  can not be defined on the whole space, therefore consider on the open sets  $U_i$  separately,  $i = 1, 2, 3, 4$ . Take the operating points  $P_1 = (-1, 0, -1)$ ,  $P_2 = (2, 0, 1)$ ,  $P_3 = (1, 0, 1)$ ,  $P_4 = (-2, 0, -1)$ , then  $P_i \in U_i$ ,

where  $i = 1, 2, 3, 4$ . Define the following two systems

$$\Sigma : \begin{cases} \dot{x} = u, \\ y = x, \end{cases} \quad \Sigma' : \begin{cases} \dot{x} = 2u, \\ y = x. \end{cases}$$

Then the corresponding ideals  $J$  for the two systems are, respectively,  $\overline{J} = \langle \dot{y} - u \rangle = \overline{s(J)}$  and  $J' = \langle \dot{y} - 2u \rangle = \overline{s(J')}$ . On the set  $U_1$ ,  $I_\Phi = \langle \dot{y} - u \rangle$ ,  $\overline{\mathcal{I}} = \overline{s(J)}$ . Thus,  $\Sigma$  is a realization of  $\Phi = 0$  on  $(U_1, P_1)$ . Similarly,  $\Sigma$  is also a realization on  $(U_3, P_3)$ ; and  $\Sigma'$  is a realization of  $\Phi = 0$  on  $(U_2, P_2)$ , and on  $(U_4, P_4)$  as well. Therefore, the system has no realization on the whole space but it has realizations on smaller open sets.

**Example 3.3.** Consider the equation  $\Phi = \ddot{y}(2\dot{y} - 3u) - \dot{u}(3\dot{y} - 4u) = 0$ , then  $\Phi = \frac{d((\dot{y}-u)(\dot{y}-2u))}{dt}$ . Let  $U_0 = \{(\dot{y}, \dot{y}, y, \dot{u}, u) : 2\dot{y} - 3u > 0\}$ , and take an operating point  $P_0 = (0, 1, 0, 0, 0)$  in  $U_0$ . Then  $\mathcal{I}(U_0, P_0) = \langle \dot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u) \rangle$ ,  $\overline{\mathcal{I}}(U_0, P_0) = \langle \dot{y} - u, \dot{y} - 2u \rangle = \langle y, u \rangle$ . Define  $U_1 = \{(y, u) : 2\dot{y} - 3u > 0, \dot{y} - u > 0\}$ ,  $U_2 = \{(y, u) : 2\dot{y} - 3u > 0, \dot{y} - 2u > 0\}$ . Then  $\overline{\mathcal{I}}(U_1, P_0) = \langle \dot{y} - u \rangle$ ,  $\overline{\mathcal{I}}(U_2, P_0) = \langle \dot{y} - 2u \rangle$ , and system  $\Sigma$  (respectively,  $\Sigma'$ ) is a realization on  $(U_1, P_0)$  (respectively,  $(U_2, P_0)$ ). The following system is a realization of  $\Phi$  on  $(U_0, P_0)$ .

$$\begin{cases} \dot{x}_1 = f_1(x, u) = (3u + \sqrt{u^2 + 4x_2})/2, \\ \dot{x}_2 = f_2(x, u) = 0, \\ y = h(x) = x_1. \end{cases} \tag{15}$$

This system is obtained following the detail in Section 6. It can be shown that (15) is a realization on  $U_0$ . In fact,  $x_1 = \xi_1(y, u) = y, x_2 = \xi_2(y, u) = \dot{y}^2 - 3u\dot{y} + 2u^2$ . Then  $\dot{\xi} - f(\xi, u) = 0, y - h(\xi) = 0, \frac{d(\xi_1 - f_1(\xi, u))}{dt} = \dot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u)$ . Now  $J = \mathcal{I}(U_0), \overline{s(J)} = \overline{\mathcal{I}}(U_0)$ , and one concludes that (15) is a realization on  $U_0$ .

#### 4. CRITERIA OF REALIZABILITY

In the following, criteria are sought for the existence of realization. For any standard differential ideal  $L$  of  $R_1$  on a contractible open set  $U_3 \subseteq U_0$  which contains also the operating point  $P_0$ , define the linear space  $V_{\max}(H_0, L)$  on  $U_3$  with respect to  $H_0 = H_0(L)$  on  $U_3$  (see (12) for definition) by

$$V_{\max}(H_0, L) = \max\{V : \text{The set } V \text{ is a } \mathcal{K}\text{-subspace of } H_0 \text{ and } \dot{V} \subseteq V + \overline{\mathcal{U}^0}\}.$$

The maximum means the maximal subspace with respect to inclusion; the symbol  $\dot{V}$  denotes the linear space which is generated by the equivalent classes of the time derivatives of all the elements of  $V$ ;  $\mathcal{U}^k := \text{span}_{\mathcal{K}}\{du_r^{(l)} : 0 \leq l \leq k, r = 1, \dots, m\}$ ; and  $\overline{\mathcal{U}^{(k)}} = \mathcal{U}^{(k)}/(\mathcal{U}^{(k)} \cap dL) \cong (\mathcal{U}^{(k)} + dL)/dL, k = 0, 1, \dots$

Now it is shown that the above  $V_{\max}(H_0, L)$  remains unchanged if  $H_0$  is replaced by any bigger  $H'_0 \supseteq H_0$ .

**Lemma 4.1.** Suppose  $L \subseteq \overline{\mathcal{I}}$  is a standard differential ideal of  $R_1$  on some contractible open set  $U_2 \subseteq U_0$  with indices  $(m_1, \dots, m_p, s_1)$  for some fixed finite set of

generators, where  $U_2$  contains also the operating point  $P_0$ . Define  $H_0^1$  to be the right hand side of the equality (12). Also define

$$H_0^2 = \text{span}_{\mathcal{K}}\{dy_i^{(j)} + dL, du_r^{(l)} + dL : 0 \leq j \leq k'_0; 0 \leq l \leq s'_0; 1 \leq i \leq p; 1 \leq r \leq m\},$$

where  $s'_0$  and  $k'_0$  are arbitrary integers such that  $s'_0 \geq s_1 - 1, k'_0 \geq \max\{m_i - 1 : i = 1, \dots, p\}$ , and  $H_0^1 \neq H_0^2$ . Let  $V_1 = V_{\max}(H_0^1, L), V_2 = V_{\max}(H_0^2, L)$ , then  $V_1 = V_2$  on  $U_2$ .

*Proof.* For any  $dy_i^{(j')} + dL \in H_0^2 \setminus H_0^1$ , the standard form in Standard Form Hypothesis 1' can be used to represent  $dy_i^{(j')}$  as a linear combination of  $\mathcal{T}_0 := \{dy_i^{(j)}, du_r^{(l)} : 0 \leq j \leq m_i - 1; 0 \leq l \leq s''_0; 1 \leq i \leq p; 1 \leq r \leq m\}$  on  $U_2$ , where  $s''_0 \geq s'_0$ . Thus, without loss of generality suppose that  $H_0^2$  is the span of the equivalent classes of the functions in  $\mathcal{T}_0$  on  $U_2$ . Let  $Y_1 = \text{span}_{\mathcal{K}}\{dy_i^{(j)} : 0 \leq j \leq m_i - 1; 1 \leq i \leq p\}$ . The inclusion  $V_1 \subseteq V_2$  is clear. If  $V_1 \neq V_2$  then  $V_2 = V_1 \oplus V_0$  where  $V_1$  has a basis  $\{w_1 + dL, \dots, w_a + dL\}$ ,  $V_0$  has a basis  $\{\tau_1 + dL, \dots, \tau_b + dL\}$ ,  $w = (w_1, \dots, w_a)^T, \tau = (\tau_1, \dots, \tau_b)^T, w_1, \dots, w_a \in E$ , and  $\tau_1, \dots, \tau_b \in (\mathcal{U}^{s''_0} + Y_1) \setminus (\mathcal{U}^{s_1-1} + Y_1)$ . Then there exist matrices  $A, B, C, S, F$ , with elements in  $\mathcal{K}$ , such that  $\dot{w} \equiv Aw + Bdu \pmod{dL}$ ,  $\dot{\tau} \equiv Cw + S\tau + Fdu \pmod{dL}$ , where  $du = (du_1, \dots, du_m)^T$ , and the notation  $X \equiv Y \pmod{dL}$  means each component of the vector  $(X - Y)$  belongs to  $dL$ . Thus,

$$\dot{\tau} - Fdu \equiv Cw + S\tau \pmod{dL}. \tag{16}$$

Suppose  $i_0 = \max\{i : \text{there exists some } r, 1 \leq r \leq m, \text{ and some function } \alpha \in \mathcal{K} \text{ such that } \alpha du_r^{(i)} \text{ is a nonzero term of } \tau\}$ , then  $i_0 > s_1 - 1$ . It follows from (16) that  $v_{i_0+1} := \dot{\tau} - Fdu \in (\mathcal{U}^{i_0+1} + Y_1) \setminus (\mathcal{U}^{i_0} + Y_1), v_{i_0+1} + dL \in V_2$ , where the notation ' $v_{i_0+1} \in$ ' means that each component of  $v_{i_0+1}$  belongs to the set in the right hand side. By computing the derivative of (16) one has  $v_{i_0+2} := \dot{v}_{i_0+1} - (CB + SF)du = (\dot{C} + CA + SC)w + (\dot{S} + S^2)\tau$ . There may be some term  $dy_i^{(m_i)}$  in the above  $v_{i_0+2}$ . Use the standard form again and note that  $i_0 > s_1 - 1$ ; one obtains  $v_{i_0+2} \in ((\mathcal{U}^{i_0+2} + Y_1) \setminus (\mathcal{U}^{i_0+1} + Y_1))$ , and  $v_{i_0+2} + dL \in V_2$ . A similar process results in the existence of  $v_k \in (\mathcal{U}^k + Y_1) \setminus (\mathcal{U}^{k-1} + Y_1)$  which satisfies also  $v_k + dL \in V_2$ , where  $k$  is any integer greater than  $i_0$ . By Proposition 2.4, the 1-forms  $\{du_r^{(l)} + dL : l \geq 0, r = 1, \dots, m\}$  are independent. Then it follows that  $\dim_{\mathcal{K}} \text{span}_{\mathcal{K}}\{v_k + dL : k \geq i_0 + 1\} = \infty$ . This contradicts the fact that  $V_2$  is finite dimensional.  $\square$

By Lemma 4.1 one can use  $V_{\max}(L)$  (or  $V_{\max}$ ) to denote  $V_{\max}(H_0^2, L)$  for any  $H_0^2$  in which the order of  $dy$  (respectively,  $du$ ) is not less than  $\max\{m_1 - 1, \dots, m_p - 1\}$  (respectively,  $s_1 - 1$ ).

**Lemma 4.2.** For any observable system (13–14) which is a realization of (6) on some contractible open set  $U_2 \subseteq U_0$  with  $P_0 \in U_2$ , define the observability indices  $k_1, \dots, k_p$  and the functions  $\xi_1, \dots, \xi_n$  as that in Section 2.3. Let

$$H'_0 = H_0(J) + \text{span}_{\mathcal{K}}\{dy_i^{(j)} + dJ, du_r^{(l)} + dJ : 0 \leq j \leq k'_0; 0 \leq l \leq s'_0; 1 \leq i \leq p; 1 \leq r \leq m\},$$

where  $k'_0 > k_1, s'_0 > k'_0$ . Then  $V_{\max}(H'_0, J) = \text{span}_{\mathcal{K}}\{d\xi_1 + dJ, \dots, d\xi_n + dJ\}$  on  $U_2$ .

**Proof.** Let  $W_0$  be the space  $\text{span}_{\mathcal{K}}\{d\xi_1 + dJ, \dots, d\xi_n + dJ\}$ , then it is a subspace of  $H'_0$  and  $\dot{W}_0 \subseteq W_0 + \overline{\mathcal{U}^0}$ . The maximality of  $V_{\max}(H'_0, J)$  implies  $V_{\max}(H'_0, J) \supseteq W_0$ .

If  $V_{\max}(H'_0, J) \neq W_0$  then  $V_{\max}(H'_0, J)$  has a basis  $\{d\xi_1 + dJ, \dots, d\xi_n + dJ, v_1 + dJ, \dots, v_r + dJ\}$ . It follows from  $y = h(x)$  that  $dy_i + dJ \in W_0, i = 1, \dots, p$ , and  $dy_i^{(j)} + dJ \in W_0 + \overline{\mathcal{U}^{j-1}}, i = 1, \dots, p; j = 1, 2, \dots$ . Thus, one can suppose that each  $v_i$  has the form  $v_i = \sum_{i_1, i_2} c_{i_1, i_2}^i du_{i_1}^{(i_2)}, i = 1, \dots, r$ . Define  $j_0 = \max\{i_2: \text{there exists a nonzero term } c_{i_1, i_2}^i du_{i_1}^{(i_2)} \text{ in some } v_i\}$ , then it corresponds to some nonzero term  $c_{i_0, j_0}^i du_{i_0}^{(j_0)}$  in some  $v_i$ , say  $v_1$ . Hence  $\dot{v}_1$  contains the term  $c_{i_0, j_0}^i du_{i_0}^{(j_0+1)}$ , and  $\dot{v}_1 + dJ \in V_{\max}(H'_0, J) + \overline{\mathcal{U}^0}$ . Thus, there exists a nonzero element  $\tau_1 \in V_{\max}(H'_0, J) \cap (\overline{\mathcal{U}^{j_0+1}} \setminus \overline{\mathcal{U}^{j_0}})$ . It follows from Proposition 2.4 that  $\{du_r^{(l)} + dJ : 1 \leq r \leq m; l \geq 0\}$  are independent on  $U_2$ , therefore a similar process as that of the proof of Lemma 4.1 completes the proof.  $\square$

**Lemma 4.3.** Suppose  $L$  is a standard differential ideal of  $R_1$  on some contractible open set  $U_2 \subseteq U_0$  with indices  $(m_1, \dots, m_p, s_1)$ ,  $P_0$  belongs to  $U_2$ , and  $s(L) = L$  is generated by the standard basis  $y_1^{(m_1)} + \zeta_1, \dots, y_p^{(m_p)} + \zeta_p$ . Assume that  $\phi$  is an analytic function in the variables in  $\{y_i^{(j)}, u_r^{(l)} : 0 \leq j \leq m_i - 1, 1 \leq i \leq p, 1 \leq r \leq m, l \geq 0\}$  on some contractible open set  $U'_2$  whose projection equals  $U_2$ ,  $\phi(P_0) = 0$ , and  $d\phi \in dL$ . Then  $\phi \equiv 0$  on  $U'_2$ .

**Proof.** Suppose  $\phi \neq 0$ , then  $\phi$  does not equal any constant. If  $LT(\phi) = u_{r_0}^{(l_0)}$  for some  $(r_0, l_0)$ , then it contradicts the result of Proposition 2.4, therefore  $LT(\phi) = y_{i_0}^{(j_0)}$  for some  $(i_0, j_0)$ . It follows that  $j_0 \leq \min\{m_1 - 1, \dots, m_p - 1\}$ . Consider the system of equations determined by the standard basis of  $s(L)$ , then  $\phi = 0$  on the trajectories of this system of equations. Assume that  $u$  is given, then by the Implicit Function Theorem there exists a function  $\rho$  such that  $y_{i_0}^{(j_0)} = \rho(y_i^{(j)} : y_i^{(j)} < y_{i_0}^{(j_0)})$  on some open subset  $U'_3 \subseteq U'_2$ .

Note that  $s(L)$  determines a system of ordinary differential equations  $y_i^{(m_i)} + \zeta_i = 0, i = 1, \dots, p$ . The system has a unique local solution for any initial condition  $\{y_i^{(j_i)}(t_0) : 0 \leq j_i \leq m_i - 1; 1 \leq i \leq p\}$  in  $U'_3$ . However, the relation  $y_{i_0}^{(j_0)} = \rho(y_i^{(j)} : y_i^{(j)} < y_{i_0}^{(j_0)})$  yields that  $\{y_i^{(j_i)}(t_0) : 0 \leq j_i \leq m_i - 1 \text{ for } i = 1, \dots, i_0 - 1, i_0 + 1, \dots, p, \text{ and } j_{i_0} = 0, 1, \dots, j_0 - 1\}$  is enough to determine  $y$  uniquely, this is a contradiction.  $\square$

**Theorem 4.4.** Suppose Standard Form Hypothesis 1 and 2 hold, then (6) has a realization on  $U_0$  if and only if there exists a differential ideal  $L$  in  $R_1$ , which is standard with respect to some finite basis  $\mathcal{S}$  of  $L$  on  $U_0$  and  $P_0$  is a zero of all the functions of  $\mathcal{S}$ , such that the following three conditions

- (i)  $\overline{s(L)} = \overline{\mathcal{I}}$ ;
- (ii)  $V_{\max}(H''_0, L)$  is integrable;
- (iii)  $dy_1 + dL, \dots, dy_p + dL \in V_{\max}(H''_0, L)$



hold on  $U_0$ , where  $H_0''$  is any linear space such that  $H_0'' \supseteq H_0(L)$  and  $H_0(L)$  is defined by the indices corresponding to the basis  $\mathcal{S}$  as (12), and the integrability of  $V_{\max}(H_0'', L)$  means that there exist functions  $\zeta_1, \dots, \zeta_{n_1} \in \mathcal{K}$  such that  $V_{\max}(H_0'', L) = \text{span}_{\mathcal{K}}\{d\zeta_1 + dL, \dots, d\zeta_{n_1} + dL\}$ .

**Remark 4.5.** In the linear case, condition (ii) is trivially satisfied and condition (iii) implies the strict properness condition (see Example 6.3 in Section 6). Condition (i) reduces to equality of transfer matrices computed either from the input-output equations or from the state equations. An integrability condition similar to condition (ii) was established in [37] for discrete-time systems.

**Proof of Theorem 4.4** Suppose (13–14) is an observable realization of (6) then  $J$  is standard and  $\overline{s(J)} = \overline{\mathcal{I}}$  on  $U_0$ . By Lemma 4.2 one has that  $V_{\max}(H_0'', J) = \text{span}_{\mathcal{K}}\{d\zeta_1 + dJ, \dots, d\zeta_n + dJ\}$  which is integrable on  $U_0$ . The equation  $y = h(\xi)$  implies that  $dy_1 + dJ, \dots, dy_p + dJ \in V_{\max}(H_0'', J)$ . Thus, the necessity follows.

Now prove the sufficiency. Assume that  $s(L)$  is generated by the standard basis  $\{y_1^{(m_1)} + \zeta_1, \dots, y_p^{(m_p)} + \zeta_p\}$ . Let  $V_{\max}(H_0'', L) = \text{span}_{\mathcal{K}}\{d\xi_1 + dL, \dots, d\xi_n + dL\}$  for some functions  $\xi_i$  on  $U_0$ ,  $\xi = (\xi_1, \dots, \xi_n)^T$ , and  $x_i = \xi_i$ ,  $i = 1, \dots, n$ . By  $V_{\max}(H_0'', L) \subseteq V_{\max}(H_0'', L) + \overline{\mathcal{U}^0}$ ,  $dy_1 + dL, \dots, dy_p + dL \in V_{\max}(H_0'', L)$ , and Poincaré Lemma [5], there exist functions  $f(\cdot, u), h(\cdot)$  on  $U_0$  such that

$$d(\dot{\xi} - f(\xi, u)) \in dL, d(y - h(\xi)) \in dL, (\dot{\xi} - f(\xi, u))|_{P_0} = 0, (y - h(\xi))|_{P_0} = 0. \tag{17}$$

Consider the following state equation

$$\dot{\xi} - f(\xi, u) = 0, y - h(\xi) = 0. \tag{18}$$

Let  $J_1 = \langle \dot{\xi} - f(\xi, u), y - h(\xi) \rangle$  be the differential ideal in  $R_1$ , one only needs to prove that  $s(J_1) = \overline{\mathcal{I}}$ . It suffices to prove the stronger result  $s(J_1) = s(L)$ . In the following, the fact in (17) is used when applying Lemma 4.3.

By Lemma 2.6 there exist functions  $g_i^l(\xi, u, \dot{u}, \dots, u^{(l-1)})$  such that  $J_1 = \langle y_i^{(l)} - g_i^l : i = 1, \dots, p; l \geq 0 \rangle$ . For  $l < m_i, y_i^{(l)} - g_i^l \equiv 0$  by Lemma 4.3. For  $l = m_i, y_i^{(l)} - g_i^l = (y_i^{(m_i)} + \zeta_i) - (\zeta_i + g_i^l)$ . By Lemma 4.3 one has  $\zeta_i + g_i^l \equiv 0$ , hence  $y_i^{(l)} - g_i^l \in s(L)$  for  $l = m_i$ . For  $l = m_i + 1, y_i^{(l)} - g_i^l = (y_i^{(m_i)} + \zeta_i)^{(l-m_i)} - \dot{\zeta}_i - g_i^l = (y_i^{(m_i)} + \zeta_i)^{(l-m_i)} - \sum_{j=1}^p \frac{\partial \zeta_i}{\partial y_j^{(m_j-1)}} (y_j^{(m_j)} + \zeta_j) + \sum_{j=1}^p \frac{\partial \zeta_i}{\partial y_j^{(m_j-1)}} \zeta_j - \sum_{j=1}^p \sum_{k=0}^{m_j-2} \frac{\partial \zeta_i}{\partial y_j^{(k)}} y_j^{(k+1)} - \sum_{l \geq 0} \frac{\partial \zeta_i}{\partial u^{(l)}} u^{(l+1)} - g_i^l$ . It follows from Lemma 4.3 that  $\sum_{j=1}^p \frac{\partial \zeta_i}{\partial y_j^{(m_j-1)}} \zeta_j - \sum_{j=1}^p \sum_{k=0}^{m_j-2} \frac{\partial \zeta_i}{\partial y_j^{(k)}} y_j^{(k+1)} - \sum_{l \geq 0} \frac{\partial \zeta_i}{\partial u^{(l)}} u^{(l+1)} - g_i^l \equiv 0$ , hence  $y_i^{(l)} - g_i^l \in s(L)$  for  $l = m_i + 1$ . Similarly one proves that  $y_i^{(l)} - g_i^l \in s(L)$  for all  $l \geq 0$  and  $i = 1, \dots, p$ . Therefore,  $J_1 \subseteq s(L)$ .

From the above  $\zeta_i + g_i^l \equiv 0$  for  $l = m_i$ . Therefore,  $y_i^{(m_i)} + \zeta_i \equiv y_i^{(m_i)} - g_i^l \in J_1$ . Thus,  $s(L) \subseteq J_1$ , and  $s(L) = J_1$ . Then  $s(J_1) = J_1 = s(L)$ .  $\square$

The system of state equations (18) constructed in the proof of Theorem 4.4 is called *the system determined by  $V_{\max}(L)$  or  $L$* .

**Remark 4.6.** It is worth noting that for any differential ideal  $L$ , which satisfies the conditions in Theorem 4.4, the dimension of  $V_{\max}(L)$  on  $U_0$  must be greater than or equal to  $p$ . In fact, suppose  $\dim V_{\max}(L) < p$ , then it follows from  $dy_i + dL \in \dim V_{\max}(L)$ ,  $i = 1, \dots, p$ , that there exist analytic functions  $\lambda_1, \dots, \lambda_p$ , which are not all zero, such that  $\sum_{i=1}^p \lambda_i dy_i \in dL$ . This contradicts the hypothesis in Section 2.1 that  $\{y_1, \dots, y_p\}$  are differentially analytically independent. Therefore, one must have  $\dim \text{span}_{\mathcal{K}}\{dy_1, \dots, dy_p\} = p$  and  $\dim V_{\max}(L) \geq p$ . The dimension of the state space of the corresponding realization is greater than or equal to  $p$ .

**Lemma 4.7.** Suppose that, on a contractible open set  $U_2 \subseteq U_0$  with  $P_0 \in U_2$ ,  $\overline{L_1}$  and  $\overline{L_2}$  are two standard differential ideals of  $R_1$ ,  $L_1 = s(L_1) \subset L_2 = s(L_2)$ ,  $\overline{s(L_1)} = \overline{s(L_2)} = \overline{\mathcal{I}}$ ,  $H_0 = H_0(L_1) + H_0(L_2)$ ,  $V_{\max}(H_0, L_1)$  is integrable,  $dy_i + dL_1 \in V_{\max}(H_0, L_1)$ ,  $i = 1, \dots, p$ . Assume also  $V_{\max}(H_0, L_1) = \text{span}_{\mathcal{K}}\{d\xi_j + dL_1 : j = 1, \dots, r\}$ ,  $\xi_j \in R_1, j = 1, \dots, r$ . Denote  $V_{\max}(L_1) = V_{\max}(H_0, L_1)$ ,  $V_{\max}(L_2) = V_{\max}(H_0, L_2)$ ,  $W_0 = \text{span}_{\mathcal{K}}\{d\xi_j + dL_2 : j = 1, \dots, r\}$ , then  $V_{\max}(L_2) = W_0$ .

*Proof.* By the definition of  $V_{\max}(H_0, L_1)$  and the condition  $dL_1 \subseteq dL_2$ , it is easy to obtain  $\dot{W}_0 \subseteq W_0 + (\mathcal{U}^0 + dL_2)/dL_2$ . Thus,  $W_0 \subseteq V_{\max}(L_2)$ . If the two are not equal, then take any nonzero  $w + dL_2 = \sum_{ij} a_{ij} dy_i^{(j)} + \sum_{r1} b_{r1} du_r^{(1)} + dL_2 \in V_{\max}(L_2) \setminus W_0$ . Note that  $V_{\max}(L_1) = \text{span}\{d\xi_1 + dL_1, \dots, d\xi_r + dL_1\}$  determines a system of state equations, say (13–14), then one has  $dy_i^{(j)} + dL_1 \in V_{\max}(L_1) + (\mathcal{U} + dL_1)/dL_1$ . Thus,  $w + dL_2 \in W_0 + (\mathcal{U} + dL_2)/dL_2$  and  $W_0 \subset V_{\max}(L_2) \subseteq W_0 + (\mathcal{U} + dL_2)/dL_2$ .

Suppose  $V_{\max}(L_2) = \text{span}\{w_1 + dL_2, \dots, w_s + dL_2\}$ ,  $w = (w_1, \dots, w_s)^T$ ,  $\xi = (\xi_1, \dots, \xi_r)^T$ . Then there exists a matrix  $A$  such that  $w \equiv \text{Ad}\xi + \bar{u} \pmod{dL_2}$ , where  $\bar{u}$  is a nonzero vector whose components belong to  $\mathcal{U}$ . Assume that  $\dot{w} \equiv Bw + Cdu \pmod{dL_2}$ ,  $d\xi \equiv Sd\xi + Fdu \pmod{dL_2}$  for some matrices  $B, C, S, F$ . Then  $\dot{w} \equiv \text{Ad}\xi + ASd\xi + AFdu + \dot{\bar{u}} \pmod{dL_2}$ ,  $\dot{w} \equiv B\text{Ad}\xi + B\bar{u} + Cdu \pmod{dL_2}$ . Therefore,

$$(BA - \dot{A} - AS)d\xi \equiv \dot{\bar{u}} - B\bar{u} + (AF - C)du \pmod{dL_2}. \tag{19}$$

The highest order of  $du$  in the above equality is in  $\dot{\bar{u}}$ , and the above equality means that the subspace  $W_0 \cap ((\mathcal{U} + dL_2)/dL_2)$  is nonzero. Let the highest order of  $du$  in  $\dot{\bar{u}}$  be  $i_0$ , then  $W_0 \cap ((\mathcal{U}^{i_0} + dL_2)/dL_2) \setminus ((\mathcal{U}^{i_0-1} + dL_2)/dL_2) \neq 0$ . Compute the derivatives of (19) and use again  $d\xi \equiv Sd\xi + Fdu \pmod{dL_2}$  one obtains similarly  $W_0 \cap ((\mathcal{U}^{i_0+1} + dL_2)/dL_2) \setminus ((\mathcal{U}^{i_0} + dL_2)/dL_2) \neq 0$ . A similar process as the one in the proof of Lemma 4.1 ends the proof.  $\square$

The following theorem follows directly from Theorem 4.4 and Lemma 4.7.

**Theorem 4.8.** Suppose Standard Form Hypothesis 1 and 2 hold, then (6) admits a realization on  $U_0$  if and only if  $V_{\max}(L_0)$  is integrable and  $dy_i + dL_0 \in V_{\max}(L_0)$ ,  $i = 1, \dots, p$ , on  $U_0$ , where  $L_0$  is a maximal differential ideal in the set  $\{L : L \text{ is a standard differential ideal on } U_0, L = s(L) \text{ and } \overline{s(L)} = \overline{\mathcal{I}}\}$ , and the maximum is in the sense of inclusion.

When  $s(\overline{\mathcal{I}}) = \overline{\mathcal{I}}$  it is clear that the above  $L_0$  is unique and  $L_0 = \overline{\mathcal{I}}$ .

5. MINIMAL REALIZATION

**Definition 5.1.** The system (13–14) is called a minimal realization of (6) on a contractible open subset  $U_1 \subseteq U_0$  with respect to the operating point  $P_0$  if it is an observable realization of (6) on the set  $U_1$ ,  $P_0 \in U_1$ , and the ideal  $J$  generated by (13–14) satisfies

$$\begin{aligned} & \dim V_{\max}(H_0(J), J) \\ &= \min \left\{ \dim V_{\max}(H_0(L), L) : \text{On the open set } U_1, L \text{ is standard, } s(J) \subseteq L = s(L), \right. \\ & \left. \overline{s(J)} = \overline{s(L)} = \overline{\mathcal{I}}, V_{\max}(H_0(L), L) \text{ is integrable, } dy_i \in V_{\max}(H_0(L), L), i = 1, \dots, p \right\}. \end{aligned}$$

By the above definition, a minimal realization on some contractible open set  $U_1$  is just the one which has the minimal dimension of states in all the possible realizations on  $U_1$ . It is worth noting that a possible realization should satisfy the hypothesis in Section 2.3, that is, the functions  $\{y_1, \dots, y_p\}$  must be differentially analytically independent. In Example 3.2,  $\Sigma$  is a minimal realization on  $U_1$ ,  $\Sigma'$  is a minimal realization on  $U_2$ . In Example 3.3,  $\Sigma$  and  $\Sigma'$  in Example 3.2 are minimal realizations respectively on  $U_1$  and  $U_2$ . It is easy to prove that the system (15) is a minimal realization on  $U_0$ . In fact, if it is not a minimal realization, then there exists an ideal  $L = s(L) \supset s(J)$  (i. e.  $L \supseteq s(J)$  and  $L \neq s(J)$ ), such that  $\overline{s(L)} = \overline{\mathcal{I}}$ ,  $V_{\max}(L)$  is integrable,  $dy + dL \in V_{\max}(L)$ ,  $\dim V_{\max}(L) < 2$ . Thus,  $V_{\max}(L) = \text{span}_{\mathcal{K}}\{dy + dL\}$ , and it follows that the indices of  $L$  are  $(m_1, s_1) = (1, 1)$ , the standard form of  $L$  must be  $\dot{y} - g(y, u) = 0$  for some analytic function  $g$ . Since  $\overline{\mathcal{I}} = \overline{\mathcal{I}} = \langle y, u \rangle$  on  $U_0$  (see Example 3.3), one must have  $g(y, u) = 0$ , and the standard form of  $L$  is  $\dot{y} = 0$ , which contradicts the differential algebraic independence hypothesis on  $y$ . Therefore, (15) is a minimal realization on  $U_0$ . This example shows that a system of input-output equations may have different minimal realizations on different open sets, and the dimensions of the minimal realizations can be different as well.

The following proposition follows directly from Definition 5.1 and Theorem 4.8.

**Proposition 5.2.** Suppose (6) has a realization on the contractible open set  $U_0$  and  $L_0$  is the differential ideal defined in Theorem 4.8. Then the system of state equations determined by  $V_{\max}(L_0)$  is a minimal realization of (6) on  $U_0$ .

For any standard differential ideal  $L$  which satisfies the conditions of Theorem 4.4, it determines a realization of (6). Suppose (13–14) is the corresponding system of state equations and it is observable. For the state equations, [1] defines  $\mathcal{K}_1$  to be the field of meromorphic functions of  $\{x_1, \dots, x_n, u_r^{(l)} : 1 \leq r \leq m; l \geq 0\}$ , and

$$\begin{aligned} \tilde{H}_0 &= \text{span}_{\mathcal{K}_1} \{dx_1 + dJ, \dots, dx_n + dJ, du_1 + dJ, \dots, du_m + dJ\}, \\ \tilde{H}_k &= \{w + dJ \in \tilde{H}_{k-1} : \dot{w} + dJ \in \tilde{H}_{k-1}\}, \quad k \geq 1. \end{aligned}$$

Then there exists a  $k^*$  such that  $\tilde{H}_{k^*+1} = \tilde{H}_{k^*+2} = \dots$ . Define  $\tilde{H}_\infty = \tilde{H}_{k^*+1}$ .  $\tilde{H}_\infty$  is always integrable [1] and (13–14) is accessible if and only if  $\tilde{H}_\infty = 0$ . This result is now used to find the relation between minimal realization and accessibility.

**Theorem 5.3.** Suppose  $p \geq m$ , and Standard Form Hypothesis 1 and 2 hold. Then the system (13–14) is a minimal realization of (6) on a contractible open set  $U_1 \subseteq U_0$  with respect to the operating point  $P_0 \in U_1$  if it is both observable and accessible on  $U_1$ .

*Proof.* Suppose the system (13–14) is both observable and accessible but not a minimal realization of (6) on  $U_1$ . Then one has  $x = \xi(y, u)$ ,  $V_{\max}(J) = \text{span}\{d\xi_1 + dJ, \dots, d\xi_n + dJ\}$ ,  $\dim V_{\max}(J) = n$ , and a standard differential ideal  $L = s(L) \supseteq s(J)$  such that  $\bar{L} = \overline{s(L)} = \overline{s(J)}$  and  $L$  corresponds to a minimal realization. By Lemma 4.7,  $V_{\max}(L)$  can be written as  $\text{span}_{\mathcal{K}}\{d\xi_1 + dL, \dots, d\xi_n + dL\}$ ,  $\dim V_{\max}(L) < \dim V_{\max}(J)$ . Thus,  $\{d\xi_1 + dL, \dots, d\xi_n + dL\}$  are linearly dependent. Without loss of generality suppose  $\dim V_{\max}(L) = n - r$ ,  $w = (w_1, \dots, w_r)^T$ ,  $\xi' = (\xi'_1, \dots, \xi'_{n-r})^T$  such that  $\text{span}\{w_1 + dJ, \dots, w_r + dJ, d\xi'_1 + dJ, \dots, d\xi'_{n-r} + dJ\} = V_{\max}(J)$ ,  $\text{span}\{d\xi'_1 + dL, \dots, d\xi'_{n-r} + dL\} = V_{\max}(L)$  and  $w_i \in dL, i = 1, \dots, r$ . It is clear that there exist matrices  $A_{r \times n}, B_{(n-r) \times n}, A_1, B_1, C, S_{n \times m}$  such that  $w \equiv Ad\xi(\text{mod } dJ)$ ,  $d\xi' \equiv Bd\xi(\text{mod } dJ)$ ,  $d\xi \equiv A_1w + B_1d\xi'(\text{mod } dJ)$ ,  $d\dot{\xi} \equiv Cd\xi + Sdu(\text{mod } dJ)$ , and  $(A_1, B_1)$  is the inverse of the matrix  $(A^T, B^T)^T$ , where  $A$  and  $B$  are both of full row rank. Note that the matrices  $C$  and  $S$  are known while  $A$  and  $B$  are to be determined. It clearly follows that  $\dot{w} \equiv \dot{A}d\xi + Ad\dot{\xi} \equiv (\dot{A} + AC)A_1w + (\dot{A} + AC)B_1d\xi' + ASdu(\text{mod } dJ)$ .

Since  $dy_1 + dL, \dots, dy_p + dL \in V_{\max}(L)$  one has  $n - r \geq p$ . Assume that  $\text{rank}S = s$  and denote by  $\mathcal{A}$  the subspace spanned by the rows of  $A$ . Let  $\mathcal{D}$  be the subspace spanned by the columns of  $S$ , and  $\mathcal{D}^\perp$  the subspace orthogonal to  $\mathcal{D}$ . Then  $\dim \mathcal{D}^\perp = n - s$ . Since  $p \geq m$  one has  $n - r \geq p \geq m \geq s$  which implies immediately that  $n - s \geq r$  or equivalently  $\dim \mathcal{D}^\perp \geq \dim \mathcal{A}$ . Suppose the  $n - s$  rows of the matrix  $G_{(n-s) \times n}$  consists of a basis of  $\mathcal{D}^\perp$ , and let  $A = VG$ , where  $V$  is  $r \times (n - s)$  and of full row rank. It follows that  $AS = VGS = 0$  and  $(\dot{A} + AC)B_1 = (\dot{V}G + V\dot{G} + VGC)B_1$ .

The ordinary differential system  $(\dot{V}G + V\dot{G} + VGC)B_1 = 0$  with respect to  $V$  has  $r$  equations and  $r(n - s)$  unknowns. Note that  $G$  is a known matrix which can be determined by  $S$ , and  $B_1$  is a function of  $A = VG$  and  $B$ . Thus, for any given initial condition  $V(t_0) = V_0$  with  $\text{rank}V_0 = r$  the system has always a solution locally such that  $V(t)$  is of full row rank. Hence for the corresponding  $A(t)$ , one has  $\dot{w} \equiv (\dot{A} + AC)A_1w(\text{mod } dJ)$ , and  $w_i^{(j)} + dJ \in \text{span}\{w_1 + dJ, \dots, w_r + dJ\}$  for all  $j \geq 0$  and  $i = 1, \dots, r$ . Therefore,  $\text{span}\{w_1 + dJ, \dots, w_r + dJ\} \subseteq \tilde{H}_\infty$ , this contradicts the accessibility of the system.  $\square$

The system (15) in Example 3.3 shows that a minimal realization may not be accessible. The following is another example to illustrate this point again, and furthermore it shows that the ideal generated by the minimal realization may be strictly smaller than the differential closure  $\bar{L}$ .

**Example 5.4.** Consider an input-output equation  $u\ddot{y} - \dot{u}\dot{y} = 0$  on the open set  $U_0 = \{(\ddot{y}, \dot{y}, y, \dot{u}, u) : u > 0\}$  with an operating point  $P_0 = (0, 0, 0, 0, 1)$ . Then

$$\begin{cases} \dot{x}_1 &= u, & \dot{x}_2 &= 0, \\ y &= x_1x_2, \end{cases} \tag{20}$$

is a realization since  $J = \langle u\ddot{y} - \dot{u}\dot{y} \rangle = \langle \ddot{y} - \dot{u}\dot{y}/u \rangle = s(J) = \mathcal{I}$  on  $U_0$ . Suppose it is not a minimal realization on  $U_0$ , then there exists a standard differential ideal  $L = s(L) \supseteq s(J)$ , such that  $V_{\max}(L)$  is integrable,  $dy + dL \in V_{\max}(L)$ ,  $\dot{V}_{\max}(L) \subseteq V_{\max}(L) + \text{span}\{du + dL\}$ , and  $\dim V_{\max}(L) < 2$ . By Remark 4.6, one has  $dy \notin dL$  and  $\dim V_{\max}(L) \neq 0$ . Thus,  $\dim V_{\max}(L) = 1$  and  $V_{\max}(L) = \text{span}\{dy + dL\}$ . It is easy to compute  $\overline{\mathcal{I}} = \overline{L} = \overline{\langle \ddot{y} - \dot{u}\dot{y}/u \rangle} = \overline{\langle \frac{d}{dt}(\dot{y}/u) \rangle} = \overline{\langle \dot{y} \rangle} = \langle y \rangle$  on  $U_0$ . Note that in the above computation,  $\overline{\langle \frac{d}{dt}(\dot{y}/u) \rangle} \neq \overline{\langle \dot{y}/u + c \rangle}$  for any nonzero constant  $c$ , because  $P_0$  is not a zero of  $\dot{y}/u + c$  when  $c \neq 0$ . If  $L = \langle y \rangle$  or  $L = \langle \dot{y} \rangle$ , then it contradicts the hypothesis that  $\{y\}$  is differentially analytically independent. Similarly  $L$  can not be generated by elements which are analytic functions of  $\{y^{(k)} : k \geq 0\}$  only. That is,  $u$  or its derivatives has to appear in the generators of  $L$ . It follows from the condition  $\dot{V}_{\max}(L) \subseteq V_{\max}(L) + \text{span}\{du + dL\}$  that there exists a nonzero analytic function  $\theta$  such that  $\dot{y} - \theta(y, u)$  is the standard form of  $L$ . By the condition  $\overline{L} = \langle y \rangle$ , there exists a differential operator  $g(D) \in R_2$  such that  $g(D)(y) = \dot{y} - \theta(y, u)$ . Then  $\theta(y, u) = y\mu(y, u)$  for a nonzero analytic function  $\mu$ . By  $L \supseteq s(J) = \mathcal{I}$ , there exist analytic functions  $\varsigma_1$  and  $\varsigma_2$  such that  $\varsigma := \varsigma_1 D + \varsigma_2 \in R_2$ ,  $\varsigma(\dot{y} - y\mu(y, u)) = \ddot{y} - \dot{u}\dot{y}/u$ . Then  $\varsigma_1(\ddot{y} - \dot{y}\mu - y\dot{\mu}) + \varsigma_2(\dot{y} - y\mu) = \ddot{y} - \dot{u}\dot{y}/u$ , or equivalently  $\ddot{y}\varsigma_1 + \dot{y}(-\varsigma_1\mu + \varsigma_2) - y(\varsigma_1\dot{\mu} + \varsigma_2\mu) = \ddot{y} - \dot{u}\dot{y}/u$ . Since the right-hand side of the above equality contains no  $y$ , one must have  $\varsigma_1\dot{\mu} + \varsigma_2\mu \equiv 0$ , or equivalently  $\varsigma_2 = -\varsigma_1\dot{\mu}/\mu$ . Then  $\ddot{y}\varsigma_1 + \dot{y}(-\varsigma_1\mu + \varsigma_2) = \varsigma_1(\ddot{y} - \dot{y}(\mu + \dot{\mu}/\mu)) = \ddot{y} - \dot{u}\dot{y}/u$ . By expanding  $\varsigma_1$  into the power series of  $y$  and its derivatives, and identifying the coefficients of  $\ddot{y}$  in the above equalities, one has  $\varsigma_1 = 1$ . Thus,  $\mu + \dot{\mu}/\mu = \dot{u}/u$ . It is clear that  $\mu$  must be a function of  $u$ , that is,  $\mu = \mu(u)$ . Now  $\mu(u) + \frac{\mu^{(1)}(u)\dot{u}}{\mu(u)} = \frac{\dot{u}}{u}$ . Then  $\mu(u) = 0$ ,  $\frac{\mu^{(1)}(u)}{\mu(u)} = \frac{1}{u}$ , which is impossible. The contradiction shows that  $\dim V_{\max}(L) = 2$  and the system (20) is a minimal realization on  $U_0$ . It is clearly not accessible.

6. COMPUTATION OF  $V_{\max}$

In this section the computation of  $V_{\max}(L)$  is discussed for any given standard differential ideal  $L$  on  $U_0$  with indices  $(m_1, \dots, m_p, s_1)$ . Let  $H_0 = \text{span}_{\mathcal{K}}\{dy_i^{(j)} + dL, du_r^{(l)} + dL : 0 \leq j \leq m_i - 1; i = 1, \dots, p; 0 \leq l \leq s_1 - 1; r = 1, \dots, m\}$ ,  $\bar{y} = (dy_1, d\dot{y}_1, \dots, dy_1^{(m_1-1)}, dy_2, d\dot{y}_2, \dots, dy_2^{(m_2-1)}, \dots, dy_p, d\dot{y}_p, \dots, dy_p^{(m_p-1)})^T$ ,  $\bar{u} = (du_1, d\dot{u}_1, \dots, du_1^{(s_1-1)}, du_2, d\dot{u}_2, \dots, du_2^{(s_2-1)}, \dots, du_m, \dots, du_m^{(s_1-1)})^T$ . By the standard form one can compute the matrices  $A$  and  $B$  such that  $\dot{\bar{y}} \equiv A\bar{y} + B\bar{u} \pmod{dL}$ . Denote  $du = (du_1, \dots, du_m)^T$ . Let  $V$  be the subspace of  $H_0$  which is generated by the equivalent classes of the components of the vector  $\bar{y} + C\bar{u}$ , where  $C$  is a matrix such that the following equality holds for some matrices  $S$  and  $F$

$$\overbrace{\bar{y} + C\bar{u}} \equiv S(\bar{y} + C\bar{u}) + Fdu \pmod{dL}. \tag{21}$$

One can find some  $(C, S, F)$  which satisfies (21) and may not be unique. In fact, substituting  $\dot{\bar{y}} \equiv A\bar{y} + B\bar{u} \pmod{dL}$  into the above equality, one has

$$\dot{\bar{y}} + \dot{C}\bar{u} + C\dot{\bar{u}} \equiv A\bar{y} + (B + \dot{C})\bar{u} + C\dot{\bar{u}} \equiv S\bar{y} + SC\bar{u} + Fdu \pmod{dL}.$$

Denote  $N_1 = m_1 + \dots + m_p, N_2 = s_1 m$ , then  $\bar{y}$  is an  $N_1$ -dimensional vector,  $A$  is of the size  $N_1 \times N_1$ ,  $B$  and  $C$  are of the size  $N_1 \times N_2$ . Let  $C = (C_1, \dots, C_{N_2}), B = (B_1, \dots, B_{N_2}), S = A, F = (F_1, \dots, F_m)$ . Then  $(B + \dot{C})\bar{u} + C\dot{\bar{u}} \equiv AC\bar{u} + Fdu \pmod{dL}$ , or equivalently

$$\sum_{i=1}^m \sum_{j=1}^{s_1} (B_{(i-1)s_1+j} + \dot{C}_{(i-1)s_1+j} - AC_{(i-1)s_1+j}) du_i^{(j-1)} + \sum_{i=1}^m \sum_{j=1}^{s_1} C_{(i-1)s_1+j} du_i^{(j)} - \sum_{i=1}^m F_i du_i \equiv 0 \pmod{dL}.$$

The above equality holds if

$$(B_{(i-1)s_1+1} + \dot{C}_{(i-1)s_1+1} - AC_{(i-1)s_1+1} - F_i) du_i + \sum_{j=2}^{s_1} (B_{(i-1)s_1+j} + \dot{C}_{(i-1)s_1+j} - AC_{(i-1)s_1+j} + C_{(i-1)s_1+j-1}) du_i^{(j-1)} + C_{(i-1)s_1+s_1} du_i^{(s_1)} \equiv 0 \pmod{dL}$$

holds for each  $i = 1, \dots, m$ . The new equalities hold if all the coefficients vanish, that is, the following equalities hold.

$$C_{(i-1)s_1+s_1} = 0; \tag{22}$$

$$C_{(i-1)s_1+j-1} = -B_{(i-1)s_1+j} - \dot{C}_{(i-1)s_1+j} + AC_{(i-1)s_1+j}, j = 2, \dots, s_1; \tag{23}$$

$$F_i = B_{(i-1)s_1+1} + \dot{C}_{(i-1)s_1+1} - AC_{(i-1)s_1+1}, i = 1, \dots, m. \tag{24}$$

The equalities in (22–24) give the procedure for determining the matrices  $C$  and  $F$ . Therefore, one can use (22–24) to construct the subspace  $V$  such that (21) holds, that is,  $\dot{V} \subseteq V + \overline{\mathcal{U}}^0$ . It is now shown that this subspace is just  $V_{\max}(L)$ .

**Proposition 6.1.** The above  $V$  equals  $V_{\max}(H_0, L)$ , and  $\dim V = m_1 + m_2 + \dots + m_p$ .

*Proof.* For convenience denote  $V_{\max}(H_0, L)$  by  $V'$ , then  $V' \cap \mathcal{U} = 0$  must hold. In fact, if there exists some nonzero  $w_1 + dL \in V' \cap \mathcal{U}$ , then let the highest order of  $du$  in  $w_1$  be  $k$ . By  $\dot{V}' \subseteq V' + \overline{\mathcal{U}}^0$  and computing the derivative of  $w_1$  one obtains a nonzero element  $w_2 + dL \in V' \cap (\overline{\mathcal{U}}^{k+1} \setminus \overline{\mathcal{U}}^k)$ . By induction one has  $w_i + dL \in V' \cap (\overline{\mathcal{U}}^{k+i-1} \setminus \overline{\mathcal{U}}^{k+i-2}), i \geq 2$ . A similar proof as Lemma 4.1 leads to a contradiction. Therefore,  $V' \cap \overline{\mathcal{U}} = 0$ .

The maximality of  $V'$  implies that  $V \subseteq V'$ . If  $V \neq V'$ , then  $\dim V < \dim V'$ . Note that for  $V'$  one can use elementary operations to transform a basis of  $V'$  into echelon form. Any nonzero element of the echelon form does not belong to  $\overline{\mathcal{U}}$  since  $V' \cap \overline{\mathcal{U}} = 0$ , therefore it contains some term  $dy_i^{(j)} + dL$ . However,  $V$  has a basis which consists of the equivalent classes of the elements in the vector  $\bar{y} + C\bar{u}$ , and it is the greatest subspace in  $H_0$  such that each nonzero element in the echelon form contains some term  $dy_i^{(j)} + dL$ . Thus,  $\dim V' \leq \dim V$ , which contradicts  $\dim V < \dim V'$ . This contradiction shows that  $V = V'$ .

The result  $\dim V = m_1 + m_2 + \dots + m_p$  comes from the clear fact that the equivalent classes of the components of  $\bar{y} + C\bar{u}$  are linearly independent.  $\square$

For any given (6), one can use the following steps to check the realizability and compute the realization and minimal realization when it is realizable.

1. Compute the ideals  $\mathcal{I}, \bar{\mathcal{I}}$  and their standard forms on some contractible open set  $U_1$  which contains the operating point  $P_0$ . If they are not standard, then there is no realization on  $U_1$ , and one can consider a smaller open set. Otherwise, go to the next step.
2. Compute the ideal  $L$ , which satisfies the conditions in Theorem 4.4 or equals  $L_0$  and satisfies the conditions in Theorem 4.8, and its standard form on  $U_1$ . Then compute  $V_{\max}(L)$  by (22–24).
3. Check by the Frobenius Theorem [5] if  $V_{\max}(L)$  is integrable. Check also if  $dy_1 + dL, \dots, dy_p + dL \in V_{\max}(L)$ . If both are true, then one obtains a minimal realization on  $U_1$  by letting  $x_1 = \xi_1, \dots, x_n = \xi_n$  and computing  $f(\cdot, u)$  and  $h(\cdot)$ , where  $V_{\max}(L) = \text{span}_{\mathcal{K}}\{d\xi_1 + dL, \dots, d\xi_n + dL\}$  and the functions  $f, h$  are obtained by Poincaré Lemma.

**Example 3.3.** (continued) Now compute the system (15) from the input-output equation  $\ddot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u) = 0$  in Example 3.3 on the open set  $U_0$ . Let  $L$  be the differential ideal generated by  $\dot{y} - \dot{u}(3\dot{y} - 4u)/(2\dot{y} - 3u)$ . It is clear that  $m = 1, p = 1, m_1 = 2, s_1 = 2, N_1 = 2, N_2 = 2, \bar{y} = (dy, d\dot{y})^T, \bar{u} = (du, d\dot{u})^T$ . Computing differentials one has

$$\begin{aligned} \dot{y} &\equiv \begin{pmatrix} d\dot{y} \\ \alpha_1 d\dot{y} + \alpha_2 du + \alpha_3 d\dot{u} \end{pmatrix} \\ &\equiv A\bar{y} + B\bar{u} \pmod{dL}, \end{aligned}$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0 & \alpha_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix}, \\ \alpha_1 &= \frac{-\dot{u}u}{(2\dot{y} - 3u)^2}, \alpha_2 = \frac{\dot{u}\dot{y}}{(2\dot{y} - 3u)^2}, \alpha_3 = \frac{3\dot{y} - 4u}{2\dot{y} - 3u}. \end{aligned}$$

By (22–24) one has

$$\begin{aligned} C_2 &= 0, C_1 = AC_2 - B_2 - \dot{C}_2 = -B_2 = \begin{pmatrix} 0 \\ -\alpha_3 \end{pmatrix}, \\ \bar{y} + C\bar{u} &\equiv \begin{pmatrix} dy \\ d\dot{y} - \alpha_3 du \end{pmatrix} \pmod{dL}. \end{aligned}$$

Therefore,  $V_{\max} = \text{span}_{\mathcal{K}}\{dy + dL, d\dot{y} - \alpha_3 du + dL\}$ . Let  $w_1 = dy, w_2 = d\dot{y} - \alpha_3 du$ , then

$$d\alpha_3 = \frac{-u}{(2\dot{y} - 3u)^2} d\dot{y} + \frac{\dot{y}}{(2\dot{y} - 3u)^2} du, dw_2 = -d\alpha_3 \wedge du = \frac{-u}{(2\dot{y} - 3u)^2} d\dot{y} \wedge du.$$

Thus,  $dw_2 \wedge w_2 = 0$ ,  $w_2$  is closed by Poincaré Lemma. In fact,  $(2\dot{y} - 3u)w_2 = d(\dot{y}^2 - 3u\dot{y} + 2u^2)$ . Hence  $V_{\max} = \text{span}_{\mathcal{K}}\{dy + dL, d(\dot{y}^2 - 3u\dot{y} + 2u^2) + dL\}$  and one can let  $x_1 = y, x_2 = \dot{y}^2 - 3u\dot{y} + 2u^2$ . Now  $y = x_1, \dot{x}_2 = 0, \dot{x}_1 = \dot{y} = (3u \pm \sqrt{u^2 + 4x_2})/2$ . Suppose  $\dot{x}_1 = \dot{y} = (3u - \sqrt{u^2 + 4x_2})/2$ , then  $\xi_1 - f_1(\xi, u) = (2\dot{y} - 3u) > 0$  on  $U_0$ . Note that  $2\dot{y} - 3u$  is an invertible element in  $R_1$  and it belongs to  $J$ , hence the corresponding  $J = R_1$  and can not be contained in  $\bar{\mathcal{I}} = \langle y, u \rangle$ . In this case it is not a realization. Thus,  $\dot{x}_1 = \dot{y} = (3u + \sqrt{u^2 + 4x_2})/2$ , and one obtains finally the system (15).

**Example 6.2.** (Moog et al. [40]) Let  $\Phi = \ddot{y} - \dot{y}u - y\dot{u} = 0$ ,  $U_0$  be the whole Euclidian space, and the origin be the operating point. Then  $\mathcal{I} = \langle \ddot{y} - \dot{y}u - y\dot{u} \rangle$ ,  $\bar{\mathcal{I}} = \langle \dot{y} - yu \rangle$ . For  $\mathcal{I}$  one has  $H_0(\mathcal{I}) = \text{span}_{\mathcal{K}}\{dy + d\mathcal{I}, d\dot{y} + d\mathcal{I}, du + d\mathcal{I}, d\dot{u} + d\mathcal{I}\}$ ,  $\bar{y} = (dy, d\dot{y})^T, \bar{u} = (du, d\dot{u})^T$ . It is easy to compute

$$\dot{\bar{y}} \equiv \begin{pmatrix} d\dot{y} \\ u d\dot{y} + \dot{u} dy + \dot{y} du + y d\dot{u} \end{pmatrix} \equiv A\bar{y} + B\bar{u} \pmod{d\bar{\mathcal{I}}},$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \dot{u} & u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \dot{y} & y \end{pmatrix}.$$

It follows from (22–24) and the Frobenius Theorem that  $V_{\max} = \text{span}_{\mathcal{K}}\{dy + d\bar{\mathcal{I}}, d\dot{y} - ydu + d\bar{\mathcal{I}}\} = \text{span}_{\mathcal{K}}\{dy + d\bar{\mathcal{I}}, d(\dot{y} - yu) + d\bar{\mathcal{I}}\}$  which is integrable. Let  $x_1 = y, x_2 = \dot{y} - yu$ , then one has a system

$$\begin{cases} \dot{x}_1 &= x_2 + ux_1, \\ \dot{x}_2 &= 0, \\ y &= x_1. \end{cases}$$

The corresponding ideal  $J = s(J) = \mathcal{I}$ , therefore it is a realization of  $\Phi = 0$ . However, it is not a minimal realization. Note that the ideal  $\bar{\mathcal{I}}$  has a generator  $\dot{y} - yu$ , and let  $H_0(\bar{\mathcal{I}}) = \text{span}_{\mathcal{K}}\{d\dot{y} + d\bar{\mathcal{I}}, du + d\bar{\mathcal{I}}\}$ , then  $C$  is a  $1 \times 1$  matrix. It is easy to show that  $C = 0$  and  $V_{\max} = \text{span}_{\mathcal{K}}\{dy + d\bar{\mathcal{I}}\}$ . Now the following minimal realization is obtained

$$\begin{cases} \dot{x} &= ux, \\ y &= x. \end{cases}$$

The following linear system is not realizable since the transfer function matrix is not strictly proper. The corresponding  $V_{\max}$  is computed to check how the conditions of Theorem 4.8 are violated.

**Example 6.3.** Consider the system

$$\begin{cases} \Phi_1 &= \dot{y}_1 - u_1 - \dot{y}_2 = 0, \\ \Phi_2 &= \dot{y}_2 - u_2 = 0, \end{cases}$$

with  $U_0$  the whole Euclidean space, and the origin the operating point, then  $I_{\Phi} = \langle \dot{y}_1 - u_1 - \dot{y}_2, \dot{y}_2 - u_2 \rangle$ . Now compute the standard form of  $I_{\Phi}$ . The standard form of  $I_{\Phi}$  is

$$\begin{cases} \dot{y}_1 - u_1 - \dot{u}_2, \\ \dot{y}_2 - u_2. \end{cases}$$



Then it is straightforward to check  $\mathcal{I} = \bar{\mathcal{I}}$ . By the standard form one has

$$H_0(\bar{\mathcal{I}}) = \text{span}_{\mathcal{K}}\{dy_1 + d\bar{\mathcal{I}}, dy_2 + d\bar{\mathcal{I}}, du_1 + d\bar{\mathcal{I}}, du_2 + d\bar{\mathcal{I}}, d\dot{u}_1 + d\bar{\mathcal{I}}, d\dot{u}_2 + d\bar{\mathcal{I}}\}.$$

One can compute that  $V_{\max} = \text{span}_{\mathcal{K}}\{dy_1 - du_2 + d\bar{\mathcal{I}}, dy_2 + d\bar{\mathcal{I}}\}$ . Clearly  $dy_1 + d\bar{\mathcal{I}} \notin V_{\max}$ , and the system is not realizable by Theorem 4.8.

The following MIMO system is realizable and it shows the general procedure to obtain a realization.

**Example 6.4.** Consider the system

$$\begin{cases} \Phi_1 & := \dot{y}_1 - u_1 = 0, \\ \Phi_2 & := u_1\ddot{y}_2 - \dot{u}_1\dot{y}_2 - u_2u_1^2 = 0, \end{cases} \tag{25}$$

on the open set  $U_0 := \{(y_1, \dot{y}_1, y_2, \dot{y}_2, \ddot{y}_2, u_1, \dot{u}_1, u_2) : u_1 > 0\}$  with an operating point  $P_0 = (0, 0, 0, 0, 0, 1, 0, 0)$ . System (25) is easily transformed into the standard form:

$$\begin{cases} \Phi'_1 & := \dot{y}_1 - u_1 = 0, \\ \Phi'_2 & := \dot{y}_2 - \frac{\dot{u}_1}{u_1}\dot{y}_2 - u_2u_1 = 0. \end{cases}$$

Let  $L$  be the differential ideal generated by  $\Phi'_1$  and  $\Phi'_2$ , then  $\bar{y} = (dy_1, dy_2, dj_2)^T$ ,  $\bar{u} = (du_1, d\dot{u}_1, du_2, d\dot{u}_2)^T$ ,  $\bar{y} \equiv A\bar{y} + B\bar{u} \pmod{dL}$ ,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{\dot{u}_1}{u_1} \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u_2 - \frac{\dot{y}_2\dot{u}_1}{u_1^2} & \frac{\dot{y}_2}{u_1} & u_1 & 0 \end{pmatrix},$$

$$\bar{y} + C\bar{u} \equiv \bar{y} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\dot{y}_2}{u_1} & 0 & 0 & 0 \end{pmatrix} \bar{u} \equiv \begin{pmatrix} dy_1 \\ dy_2 \\ dj_2 - \frac{\dot{y}_2}{u_1} du_1 \end{pmatrix} \pmod{dL},$$

and  $V_{\max} = \text{span}_{\mathcal{K}}\{dy_1 + dL, dy_2 + dL, dj_2 - \frac{\dot{y}_2}{u_1}du_1 + dL\} = \text{span}_{\mathcal{K}}\{dy_1 + dL, dy_2 + dL, u_1d(\frac{\dot{y}_2}{u_1}) + dL\}$  which is obviously integrable. Let  $x_1 = y_1, x_2 = y_2, x_3 = \frac{\dot{y}_2}{u_1}$ , then the following system of state equations is formulated

$$\begin{cases} \dot{x}_1 & = u_1, & \dot{x}_2 & = x_3u_1, & \dot{x}_3 & = u_2, \\ y_1 & = x_1, & y_2 & = x_2. \end{cases} \tag{26}$$

By Definition 3.1 system (26) is a realization of (25) on  $U_0$ . Furthermore, it follows from Theorem 5.3 and the observability and accessibility of (26) that it is also a minimal realization of (25) on  $U_0$ .

### 7. CONCLUSIONS

In this paper, the realization problem has been studied for multi-input multi-output nonlinear systems. A standard form differential ideal is defined for a given set of input-output differential equations. The corresponding notion of differential closure

yields a natural definition of realization as stated in Definition 3.1. The Appendix proves that this new definition coincides completely with the linear theory. Definition 3.1 also generalizes the idea of transfer equivalence in [45] and [40] to multi-input multi-output systems. That is, if a system of state equations is a realization of a single input-output equation in the sense of [40], then it is also a realization in the sense of Definition 3.1. Criteria for the realizability have been developed in the main theorems. The definition of minimal realization is also presented which is consistent with the linear theory in the sense that it is minimal with respect to the state space dimension. By this definition, an observable and accessible realization is minimal but the converse is not true. An algorithm for the computation of realization and minimal realization is provided, and the examples have shown its effectiveness. The problem whether the minimal realization at a given operating point is unique, up to a diffeomorphism, remains open for further research.

APPENDIX

**A. Operations in  $R_1(U_0)$**

For any two elements  $g_1, g_2 \in R_1(U_0)$ , one can assume that  $g_1$  is real analytic on a contractible open  $U'_0 \subseteq \mathbb{R}^{n'}$  and  $g_2$  is real analytic on a contractible open set  $U''_0 \subseteq \mathbb{R}^{n''}$ , and there are two projection mappings  $P_1$  and  $P_2$  which map  $U'_0$  and  $U''_0$  onto  $U_0$  respectively. Then there exists a contractible open set  $U'''_0 \subseteq \mathbb{R}^{n'''}$  and two projection mappings  $P'_3 : \mathbb{R}^{n'''} \rightarrow \mathbb{R}^{n'}$ ,  $P''_3 : \mathbb{R}^{n'''} \rightarrow \mathbb{R}^{n''}$ , such that

$$P_1(P'_3(U'''_0)) = P_2(P''_3(U'''_0)) = U_0, P_1(P'_3(x)) = P_2(P''_3(x)) \text{ for any } x \in U'''_0. \quad (27)$$

The triple  $(U'''_0, P'_3, P''_3)$  with the property (27) may not be unique; however, there does exist such a unique triple  $(\overline{U'''_0}, \overline{P'_3}, \overline{P''_3})$  satisfying the following:

- (i)  $\overline{U'''_0}$  is open and contractible in  $\mathbb{R}^{n'''}$ ,  $\overline{P'_3}$  is a mapping from  $\mathbb{R}^{n'''}$  to  $\mathbb{R}^{n'}$ ,  $\overline{P''_3}$  is a mapping from  $\mathbb{R}^{n'''}$  to  $\mathbb{R}^{n''}$ ,  $P_1(P'_3(U'''_0)) = P_2(P''_3(U'''_0)) = U_0$ , and  $P_1 \circ \overline{P'_3} = P_2 \circ \overline{P''_3}$ ;
- (ii) For any  $(U'''_0, P'_3, P''_3)$  satisfying (27), there exists a mapping  $\theta$  from  $\mathbb{R}^{n'''}$  to  $\mathbb{R}^{n'''}$  such that

$$P'_3 = \overline{P'_3} \circ \theta, P''_3 = \overline{P''_3} \circ \theta.$$

Proving the existence of  $(\overline{U'''_0}, \overline{P'_3}, \overline{P''_3})$  is simple; actually one can define  $\overline{U'''_0} = (U'_0 \times U''_0) / \sim$  and  $\overline{P'_3}([x_1, x_2]) = x_1, \overline{P''_3}([x_1, x_2]) = x_2$ , where  $x_1$  and  $x_2$  are arbitrary points in  $U'_0$  and  $U''_0$  respectively,  $\sim$  is the equivalent relation defined by  $\{(x_1, x_2) : P_1(x_1) - P_2(x_2) = 0, x_1 \in U'_0, x_2 \in U''_0\}$ , and  $[x_1, x_2]$  denotes the equivalent class of  $(x_1, x_2) \in U'_0 \times U''_0$  with respect to the relation  $\sim$ . Note that  $\overline{U'''_0}$  can be embedded into some Euclidean space  $\mathbb{R}^{n'''}$ . By the universal property of Cartesian product, properties in the above (i) and (ii) hold. The uniqueness of  $(\overline{U'''_0}, \overline{P'_3}, \overline{P''_3})$  follows directly from the property in (ii). This construction is also similar to the construction of fibred product of schemes in algebraic geometry (see Section 2, Chapter 2 of [25]).

Now one can define addition and multiplication of  $g_1$  and  $g_2$  on  $\overline{U_0''}$  formally as the usual operations of real functions on  $\mathbb{R}^{n''}$ , that is, for any  $x_1 \in U_0', x_2 \in U_0''$ , define

$$(g_1 + g_2)([x_1, x_2]) = g_1(x_1) + g_2(x_2), (g_1 g_2)([x_1, x_2]) = g_1(x_1)g_2(x_2).$$

By the properties in (i) and (ii), the above addition and multiplication are well defined. Then it is simple to check that  $R_1(U_0)$  is an integral domain.

Note that any function in  $R_1(U_0)$  has only a finite number of variables from the infinite set  $\{y_i^{(j)}, u_r^{(l)} : j, l \geq 0; i = 1, \dots, p; r = 1, \dots, m\}$ , therefore derivatives of any function in  $R_1(U_0)$  still belongs to  $R_1(U_0)$ . Then by the usual differentiation operation,  $R_1(U_0)$  becomes a differential ring.

### B. Realization of Linear Systems

In the following it is proved that Definition 3.1 is consistent with the definition of realization of linear systems. Consider the following linear system

$$\dot{x} = Ax + Bu, \tag{28}$$

$$y = Cx. \tag{29}$$

in which  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$ . Let  $U_0$  be the whole Euclidean space, and the operating point  $P_0$  be the origin of  $U_0$ . Without loss of generality, assume that  $(C, A)$  is observable. Then the matrix  $Q = (C^T, (CA)^T, \dots, (CA^{n-1})^T)^T$  has rank  $n$ . Let  $P$  be the left inverse of  $Q$ , and define  $\bar{y} = (y^T, \dot{y}^T, \dots, (y^{(n-1)})^T)^T$ ,  $\bar{u} = (u^T, \dot{u}^T, \dots, (u^{(n-2)})^T)^T$ , and

$$W = \begin{pmatrix} 0 & 0 & \dots & 0 \\ CB & 0 & \dots & 0 \\ CAB & CB & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \dots & CB \end{pmatrix}.$$

Then  $x$  can be solved as

$$x = \xi(y, u) = P\bar{y} - PW\bar{u}. \tag{30}$$

The corresponding ideal  $J$  defined in Section 2.3 is  $J = \langle \dot{\xi} - A\xi - Bu, y - C\xi \rangle$ .

For any system of linear equations of the form

$$\Phi_i := \sum_{k=0}^{k_0} F_k^i y^{(k)} - \sum_{j=0}^{s_0} H_j^i u^{(j)} = 0, i = 1, \dots, p, \tag{31}$$

suppose the Laplace transform of the system is  $M(s)Y(s) - N(s)U(s) = 0$ , where  $Y(s)$  (respectively,  $U(s)$ ) is the Laplace transform of  $y$  (respectively,  $u$ ), and the elements of the matrices  $M(s), N(s)$  are polynomials in the variable  $s$ . Let  $\mathbb{R}[s]$  be the ring of polynomials of  $s$  and  $\mathbb{R}(s)$  its fraction field. Suppose (31) satisfies the Standard Form Hypothesis, then the matrix  $M(s)$  is invertible over  $\mathbb{R}(s)$  and

the matrix  $M(s)^{-1}N(s)$  is the so-called transfer matrix. The system (28–29) is an observable realization if  $M(s)^{-1}N(s) = C(sI - A)^{-1}B$ . Note also that the ideal  $I_{\Phi}$  defined in Section 2.3 is  $I_{\Phi} = \langle \Phi_1, \dots, \Phi_p \rangle$ .

It is clear that for linear systems the set of all the possible input-output equations are linear equations in the derivatives of  $y$  and  $u$ . Thus, consider the following set

$$R_0 = \text{span}_{\mathbb{R}} \{y_i^{(j)}, u_r^{(l)} : j \geq 0, l \geq 0, i = 1, \dots, p; r = 1, \dots, m\} \tag{32}$$

instead of  $R_1$ . The following results are needed to show that Definition 3.1 is consistent with the linear theory.

**Lemma B.1** For any functions  $\phi_1, \dots, \phi_r \in R_0$ , let  $L$  be the differential ideal of  $R_1$  which is generated by  $\phi_1, \dots, \phi_r$ . Then

$$\begin{aligned} \overline{s(L)} \cap R_0 &= \overline{L} \cap R_0 = \text{span}_{\mathbb{R}} \{ \phi_i^{(j)} : j \geq 0, i = 1, \dots, r \}; \\ \overline{s(L)} \cap R_0 &= \overline{L} \cap R_0 = \{ g \in R_0 : \text{there exist some real numbers } \lambda_0, \lambda_1, \\ &\quad \dots, \lambda_l \text{ such that } \sum_{j=0}^l \lambda_j g^{(j)} \in L \}. \end{aligned}$$

**Proof.** The equalities follow directly from the definitions of  $L$ ,  $s(L)$ ,  $\overline{L}$ ,  $\overline{s(L)}$ , and the fact that  $R_0$  contains no product of any  $\phi_{i_1}^{(j_1)}$  and  $\phi_{i_2}^{(j_2)}$ . □

Denote by  $\mathcal{L}(\cdot)$  the Laplace transform,  $\mathcal{L}(L)$  the set of Laplace transforms of all the elements of  $L$ . For simplicity, it is assumed that the initial value is zero whenever a Laplace transform is applied to a function.

**Lemma B.2** With the notations above one has

$$\begin{aligned} &\mathcal{L}(R_0 \cap \overline{I_{\Phi}}) \\ &= \{ \mathcal{L}(g) : g \in R_0, \text{ and there exist polynomials } \lambda(s), a_1(s), \dots, a_p(s) \in \mathbb{R}[s] \text{ such} \\ &\quad \text{that } \lambda(s)\mathcal{L}(g) = (a_1(s), \dots, a_p(s))(M(s)Y(s), -N(s)U(s)) \} \\ &\cong \{ \beta(s) : \beta(s) \text{ is a row vector in } \mathbb{R}[s]^{p+m} \text{ such that there exist a polynomial} \\ &\quad \lambda(s) \in \mathbb{R}[s] \text{ and a row vector } \alpha(s) \text{ in } \mathbb{R}[s]^p \text{ which satisfy } \lambda(s)\beta(s) \\ &\quad = \alpha(s)(M(s), -N(s)) \} \text{(isomorphic as } \mathbb{R}\text{-vector spaces), } \mathcal{L}(R_0 \cap \overline{J}) \\ &= \{ \mathcal{L}(g) : g \in R_0, \text{ and there exist a row vector } \alpha(s) \in \mathbb{R}[s]^p \text{ and a polynomial} \\ &\quad \lambda(s) \in \mathbb{R}[s] \text{ such that } \lambda(s)\mathcal{L}(g) = \alpha(s)G(s)(Y(s)^T, U(s)^T)^T \} \\ &\cong \{ \beta(s) : \beta(s) \text{ is a row vector in } \mathbb{R}[s]^{p+m} \text{ such that there exist a polynomial} \\ &\quad \lambda(s) \in \mathbb{R}[s] \text{ and a row vector } \alpha(s) \text{ in } \mathbb{R}[s]^p \text{ which satisfy } \lambda(s)\beta(s) = \alpha(s)G(s) \} \\ &\text{(isomorphic as } \mathbb{R}\text{-vector spaces),} \end{aligned}$$

where  $G(s)$  is defined as

$$G(s) = \begin{pmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{pmatrix},$$

and  $G_1(s) = I_p - CP(I_p, sI_p, \dots, s^{n-1}I_p)^T$ ,  $G_2(s) = CPW(I_m, sI_m, \dots, s^{n-2}I_m)^T$ ,  $G_3(s) = P(sI_p, s^2I_p, \dots, s^nI_p)^T - AP(I_p, sI_p, \dots, s^{n-1}I_p)^T$ ,  $G_4(s) = -PW(sI_m, s^2I_m, \dots, s^{n-1}I_m)^T + APW(I_m, sI_m, \dots, s^{n-2}I_m)^T - B$ . Here  $I_p$  is the  $p \times p$  identity matrix.

**Proof.** The equality about  $\mathcal{L}(R_0 \cap \overline{I_\Phi})$  follows from Lemma B.1, while the equality for  $\mathcal{L}(R_0 \cap \overline{J})$  is due to Lemma B.1 and the following computation  $\mathcal{L}(y - C\xi) = \mathcal{L}(y - CP\bar{y} + CPW\bar{u}) = G_1(s)Y(s) + G_2(s)U(s)$ ,  $\mathcal{L}(\dot{\xi} - A\xi - Bu) = G_3(s)Y(s) + G_4(s)U(s)$ . The two isomorphisms follow from the two equalities respectively.  $\square$

**Lemma B.3** The equality  $\mathcal{L}(R_0 \cap \overline{I_\Phi}) = \mathcal{L}(R_0 \cap \overline{J})$  holds if and only if  $C(sI - A)^{-1}B = M(s)^{-1}N(s)$ .

**Proof.** By Lemma B.2 one needs only to prove that  $C(sI - A)^{-1}B = M(s)^{-1}N(s)$  holds if and only if the two row spaces spanned by the rows of  $(M(s), -N(s))$  and  $G(s)$  over  $\mathbb{R}(s)$  respectively are equal. Let

$$P_1 = \begin{pmatrix} I_p, & C(sI_n - A)^{-1} \\ 0, & I_n \end{pmatrix}, P_2 = \begin{pmatrix} I_p, & 0 \\ -(sI_n - A)P(I_p, sI_p, \dots, s^{n-1}I_p)^T, & I_n \end{pmatrix},$$

then  $P_1$  and  $P_2$  are elementary matrices over  $\mathbb{R}(s)$  and

$$P_2P_1G(s) = \begin{pmatrix} I_p, & -C(sI_n - A)^{-1}B \\ 0, & P_3 \end{pmatrix},$$

where  $P_3 = (sI_n - A)P(I_p, sI_p, \dots, s^{n-1}I_p)^T C(sI_n - A)^{-1}B - (sI_n - A)PW(I_m, sI_m, \dots, s^{n-2}I_m)^T - B$ .

In the following we show that  $P_3 = 0$ . Suppose the matrix  $P$  is partitioned into the block form  $P = (Z_1, Z_2, \dots, Z_n)$  such that the condition  $PQ = I_n$  can be expressed as  $I_n = \sum_{i=1}^n Z_iCA^{i-1}$ . Let  $W_1 = \left(0, C^T, (CA + sC)^T, \dots, (CA^{n-2} + sCA^{n-3} + \dots + s^{n-2}C)^T\right)^T$ , then  $W(I_m, sI_m, \dots, s^{n-2}I_m)^T = W_1(sI_n - A)(sI_n - A)^{-1}B$  and therefore

$$P_3 = (sI - A) \left( P(I_p, sI_p, \dots, s^{n-1}I_p)^T C - PW_1(sI_n - A) \right) (sI_n - A)^{-1}B - B.$$

To show that  $P_3 = 0$  one needs only to prove  $P_4 := P(I_p, sI_p, \dots, s^{n-1}I_p)^T C - PW_1(sI_n - A) = I_n$ . In fact,

$$\begin{aligned} P_4 &= \left( \sum_{i=1}^n s^{i-1}Z_i \right) C - \left( (Z_2C + Z_3CA + Z_4CA^2 + \dots + Z_nCA^{n-2}) \right. \\ &\quad \left. + s(Z_3C + Z_4CA + \dots + Z_nCA^{n-3}) + \dots + s^{n-2}Z_nC \right) (sI_n - A) \\ &= \left( \sum_{i=1}^n s^{i-1}Z_i \right) C - \left( - \sum_{i=2}^n Z_iCA^{i-1} + s \left( \sum_{i=2}^n Z_iCA^{i-2} - \sum_{i=3}^n Z_iCA^{i-3} + 1 \right) \right. \\ &\quad \left. + \dots + s^{n-2} \left( \sum_{i=n-1}^n Z_iCA^{i-(n-1)} - \sum_{i=n} Z_iCA^{i-n+1} \right) + s^{n-1} \sum_{i=n} Z_iCA^{i-n} \right) \end{aligned}$$

$$\begin{aligned}
 &= \left( \sum_{i=1}^n s^{i-1} Z_i \right) C - (Z_1 C - I_n + s Z_2 C + s^2 Z_3 C + \dots + s^{n-2} Z_{n-1} C + s^{n-1} Z_n C) \\
 &= I_n.
 \end{aligned}$$

Therefore,  $P_3 = 0$  and the row space of  $G(s)$  over  $\mathbb{R}(s)$ , denoted by  $S_1$ , is spanned by the rows of  $(I_p, -C(sI_n - A)^{-1}B)$ . Note that  $(M(s), -N(s)) = M(s)(I_p, -M(s)^{-1}N(s))$ , hence the row space of  $(M(s), -N(s))$ , denoted by  $S_2$ , is spanned by the rows of  $(I_p, M(s)^{-1}N(s))$ . It follows that  $S_1 = S_2$  if and only if  $M(s)^{-1}N(s) = C(sI_n - A)^{-1}B$ .  $\square$

**Theorem B.4** With the notations above,  $R_0 \cap \overline{\mathcal{I}} = R_0 \cap \overline{s(\mathcal{J})}$  if and only if  $\overline{\mathcal{I}} = \overline{s(\mathcal{J})}$ . The equation  $C(sI - A)^{-1}B = M(s)^{-1}N(s)$  holds if and only if  $R_0 \cap \overline{\mathcal{I}} = R_0 \cap \overline{s(\mathcal{J})}$ .

In plain words, Definition 3.1 reduces to equality of transfer function matrices computed either from the input-output equations or from the state equations. Therefore, this theorem shows that Definition 3.1 is consistent with the linear theory (see the definition of realization for linear systems in [3]).

*Proof of Theorem B.4* Since  $I_\Phi$  and  $J$  are generated by the elements of  $R_0$ , it is clear that  $R_0 \cap \overline{\mathcal{I}} = R_0 \cap \overline{s(\mathcal{J})}$  if and only if  $\overline{\mathcal{I}} = \overline{s(\mathcal{J})}$ . Note that  $R_0 \cap \overline{I_\Phi} = R_0 \cap \overline{\mathcal{J}}$  if and only if  $\mathcal{L}(R_0 \cap \overline{I_\Phi}) = \mathcal{L}(R_0 \cap \overline{\mathcal{J}})$ , then the second result follows from Lemma B.1 and Lemma B.3.  $\square$

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