

## Equivalence of Linear Time-Delay Systems

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**Abstract**—The question of characterizing those time-delay systems which are equivalent (up to bicausal transformations) to a delay-free system is a natural and fundamental problem which has been raised since the early 1980s. Surprisingly, this problem remains largely open despite that some partial answers have been provided. This technical note tackles this problem for linear systems with constant time delays, and gives necessary and sufficient conditions to eliminate, whenever is possible, or to reduce the number of time delays and a constructive way to get the corresponding bicausal change of coordinates.

**Index Terms**—Delay systems, linear systems, system analysis and design.

### I. INTRODUCTION

The aftereffect phenomenon appears naturally in many control processes in areas as different as chemistry, biology, communications and robotics, (see [2], [23], [24] and references therein). Practically, every transport of material, data or energy involves time delays. New mathematical tools and a theory adapted to this class of systems is developed in the current literature [19] since the standard control methodologies yield to anticipating (noncausal) solutions.

A natural and fundamental problem, which has been raised since the early 1980s, is the characterization of those time-delay systems (TDS) which are equivalent (up to bicausal transformations) to a delay-free system or to a system with a reduced number of delays. This problem can be trivially solved for discrete time-delay systems by introducing new independent variables [8]. However, for continuous time-delay systems the situation is completely different. The paper of Kwon and Pearson [12] suggested the possibility to transform a linear system with delayed control action to a delay-free system. This idea inspired Artstein, who proposed an interesting transformation to solve this problem in 1982 [1]. However, although such transformation is a useful tool for this kind of systems, it is not a bijective mapping. At the same time, Lee *et al.* [14] developed a sufficient condition to reduce the number of delay terms in the state via unimodular transformations, unfortunately this condition does not include a large class of linear time-delay systems (LTDS).

This research is motivated by several reasons: a TDS equivalent to a delay-free system can be dealt with classical control techniques; moreover the delay reduction expand the possibilities to handle TDS, there are some stability criteria developed for systems with a few quantity of time delays (see for example [9], [19] and references inside). Surprisingly, this problem remains open and only sufficient conditions to

reduce delays have been given despite that it is still an active research topic, as shown in [3], [4], [6], [21].

A complete solution to this open problem is derived in this Note for continuous time-delay systems. Necessary and sufficient conditions to reduce the number of delay terms in a LTDS are given. These conditions are constructive and give an easy way to test if an equivalent delay-free system exists. The involved state transformation is unimodular, and it preserves the relevant system properties of the original system.

Basic assumptions on the time-delay systems under interest, the problem statement and some preliminaries are given in Section II. Necessary and sufficient conditions to reduce or eliminate the time delays are presented in Section III. Finally, some concluding remarks are stated in Section IV.

### II. PRELIMINARIES

This section states definitions and the mathematical background, as well as the class of systems considered throughout the paper.

#### A. Considered Systems

In this work, we consider linear systems with constant, commensurable delays described by

$$\dot{x}(t) = \sum_{i=0}^{\tau_a} A_i x(t - ih) + \sum_{j=0}^{\tau_b} B_j u(t - jh)$$

where the state  $x(t) \in \mathbb{R}^n$ , the input  $u(t) \in \mathbb{R}^m$ ,  $A_i \in \mathbb{R}^{n \times n}$ ,  $B_j \in \mathbb{R}^{n \times m}$ ,  $\tau_a, \tau_b \in \mathbb{N}$ .  $h \in \mathbb{R}^+$  is the base delay. For notation simplicity, and without loss of generality, we will assume that the time  $t$  has been rescaled to have  $h = 1$ , so previous equation becomes

$$\dot{x}(t) = \sum_{i=0}^{\tau_a} A_i x(t - i) + \sum_{j=0}^{\tau_b} B_j u(t - j). \quad (1)$$

Define  $\nabla$  as the delay operator, (i.e.,  $\nabla^k x(t) = x(t - k)$ , with  $k \in \mathbb{N}$ ). Let  $\mathbb{R}[\nabla]$  be the ring of polynomials in  $\nabla$  with coefficients in  $\mathbb{R}$ . Every element of this ring may be written as

$$a[\nabla] = a_0 + a_1 \nabla + \dots + a_{r_a} \nabla^{r_a}, \quad a_i \in \mathbb{R}$$

where  $r_a$  is the polynomial degree (deg) of  $a[\nabla]$ . The addition and multiplication are defined as usual

$$a[\nabla] + b[\nabla] = \sum_{i=0}^{\sup\{r_a, r_b\}} (a_i + b_i) \nabla^i$$

$$a[\nabla]b[\nabla] = b[\nabla]a[\nabla] = \sum_{i=0}^{r_a} \sum_{j=0}^{r_b} a_i b_j \nabla^{i+j}.$$

Delays are computed with finite precision and a degree of uncertainty in practical applications. From this point of view to consider the case of commensurate delays is not very restrictive. One way to get rid of the time variations of transmission delays in some applications is to use a delay maximizing strategy: considering a virtual delay, buffer or waiting strategy to make a bounded delay to become constant and known [5], [16], [20]. Other kind of delay is due to the signal reconstruction (like the use of ZOH, as discussed in [7]). Although it cannot be made constant, it is assumed that it is small enough to be neglected with the appropriate sampling rate.

Examples of systems with commensurate delays include conveyor belts, rolling steel mills and some population models (see [11] and [17]).

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The system (1) can always be represented as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{l=1}^{\tau_a} A_l \xi_l(t) + B_0 u(t) + \sum_{m=1}^{\tau_b} B_m \psi_m(t) \\ \xi_l(t) &= x(t-l), \quad l > 0 \\ \psi_m(t) &= u(t-m), \quad m > 0 \end{aligned} \quad (2)$$

where  $\xi_l(t) \in \mathbb{R}^n$  and  $\psi_m(t) \in \mathbb{R}^m$ . The rest of the variables are defined as in system (1).

### B. Mathematical Setting

Some notions, fundamental for the rest of this work, are now recalled. A nice introduction to unimodular matrices can be found in [10] and [22].

**Definition 1. (Unimodular Matrix):** A polynomial matrix  $A \in \mathbb{R}[\nabla]^{n \times n}$  is said to be unimodular if it has a polynomial inverse on the same ring.

**Definition 2. (Smith Invariant Polynomials) [22]:** Every  $m \times n$  matrix polynomial  $P(\lambda)$  of rank  $r$  is equivalent to the matrix  $S(\lambda)$ :

$$P(\lambda) = U_1(\lambda)S(\lambda)U_2(\lambda)$$

with  $S(\lambda) = \begin{bmatrix} \Delta(\lambda) & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\Delta(\lambda) = \text{diag}[d_1(\lambda), \dots, d_r(\lambda), 0, \dots, 0]$  such that  $d_i(\lambda)$  is divisible by  $d_{i-1}(\lambda)$  for  $i = 2, \dots, r$  for some unimodular matrices  $U_1(\lambda)$  and  $U_2(\lambda)$ . The matrix polynomial  $S(\lambda)$  is called the Smith form of  $P(\lambda)$  and the diagonal elements  $d_i(\lambda)$  are the invariant polynomials.

The standard notion of change of coordinates requires to associate the state coordinates  $x(t)$  to the new coordinates  $z(t)$  by a linear invertible mapping. This is well-known for systems without delays; however, the invertibility requirement yields the following definition, specific for LTDS.

**Definition 3. (Change of Coordinates) [18]:** Considering the time-delay system (1), with state coordinates  $x(t)$ , then

$$z(t) = T[\nabla]x(t), \quad \text{with } T[\nabla] \in \mathbb{R}[\nabla]^{n \times n} \quad (3)$$

is a causal change of coordinates if the Smith invariant polynomials of  $T[\nabla]$  are of the form  $\nabla^{\tau_i}$  for some  $\tau_i \in \mathbb{N}$ . If  $\tau_i = 0 \forall i = 1 \dots n$ , then the change of coordinates is said to be bicausal.

**Remark 1:**  $z(t) = x(t-\tau)$  is a causal change of coordinates but its inverse is not.  $z(t) = x(t) - x(t-\tau)$  is not a change of coordinates because  $z(t) \equiv 0 \not\Rightarrow x(t) \equiv 0$ . There are several functions  $x(t) = \phi(t)$  of period  $\tau$  which map to  $z(t) = 0$ , therefore the only way to differentiate them from an initial time  $t_0$  is to know the initial condition in a period of time  $[t_0 - \tau, t_0]$ . However for causal and bicausal change of coordinates  $z(t) \equiv 0 \Rightarrow x(t) \equiv 0$  after a period of time. The Cauchy initial value problem is not important for the purpose of this paper. The interested reader can see [13, Example 2] for a detailed discussion.

The transformation (3) is a *bicausal change of coordinates* if and only if  $T[\nabla]$  is unimodular.

**Example 1:** Consider the change of coordinates

$$z(t) = T[\nabla]x(t) = \begin{bmatrix} \nabla & 0 \\ \nabla & 1 \end{bmatrix} x(t).$$

We have that  $T[\nabla]^{-1} \in \mathbb{R}[\nabla]^{n \times n}$ . Note that

$$\begin{aligned} x_1(t) &= z_1(t+1) \\ x_2(t) &= -z_1(t) + z_2(t) \end{aligned} \quad (4)$$

$z(t)$  is a causal transformation since the Smith invariant polynomials are  $\{1, \nabla\}$ . Applying (4) to a given system (e.g.  $\dot{z}_1(t) = z_2(t)$ ;  $\dot{z}_2(t) = u(t)$ ) may yield a non-causal one. Thus, only *bicausal transformations* will be considered in the rest of the paper.

The general notion of system equivalence is introduced next.

**Definition 4. (Equivalence of Systems):** Two linear systems

$$\dot{x}(t) = \sum_{i=0}^{\tau_a} A_i x(t-i) + \sum_{j=0}^{\tau_b} B_j u(t-j) \quad (5)$$

and

$$\dot{z}(t) = \sum_{j=0}^{\tau_c} \bar{A}_j z(t-j) + \sum_{k=0}^{\tau_d} \bar{B}_k u(t-k) \quad (6)$$

are equivalent if there exists a unimodular matrix  $T[\nabla]$  such that  $z(t) = T[\nabla]x(t)$ .

### C. Problem Statement

Considering the time-delay system described by (1) and a bicausal change of coordinates

$$z(t) = T[\nabla]x(t) \quad (7)$$

two problems are considered in the rest of this paper.

First, to characterize those LTDS which are equivalent to a delay-free system via (7).

Second, if such a transformation does not exist, then we want to get (7) such that the new system representation

$$\dot{z}(t) = \bar{A}[\nabla]z(t) + \bar{B}[\nabla]u(t) \quad (8)$$

has the minimal polynomial degree on  $\nabla$ .

### D. The Cyclic Vector Approach

The definition of cyclic vector will be recalled for the sake of completeness.

**Definition 5. (Cyclic Vector) [15]:** The module  $\mathbb{R}[\nabla]$  will be called cyclic (relative to  $A[\nabla]$ ) and  $A[\nabla]$  will be called cyclic if there exists a single vector  $b[\nabla] \in \mathbb{R}[\nabla]^n$  called a generator, which generates  $\mathbb{R}[\nabla]^n$ . In other words,  $A[\nabla]$  is cyclic with generator  $b[\nabla]$  if and only if the vectors  $[b[\nabla] \ A[\nabla]b[\nabla] \ \dots \ A[\nabla]^{n-1}b[\nabla]]$  are a basis for  $\mathbb{R}[\nabla]^n$ . The generator  $b[\nabla]$  is called a cyclic vector.

In [13], a sufficient condition to reduce the number of delay terms for unforced LTDS was given. This result is based on the existence of a cyclic vector  $b[\nabla]$  and whenever it exists, the form with a reduced number of time delays can be found with the transformation

$$z(t) = [b[\nabla] \ A[\nabla]b[\nabla] \ \dots \ A[\nabla]^{n-1}b[\nabla]]^{-1} x(t). \quad (9)$$

Moreover, (9) yields a delay-free system in the case of single input strongly controllable systems with  $b[\nabla] = B$ .

These results are a partial solution to the problem stated in Section II-C, although in [13] the considered delay reduction was quantified by the addition of each column degree on  $\nabla$  of  $A[\nabla]$  and not just for the greatest polynomial degree as in this note.

However, it is necessary to point out that the cyclic vector approach did not provide a full characterization of those unforced, weakly controllable or uncontrollable LTDS equivalents to a delay-free system, and the existence of such cyclic vector is not necessary to reduce the system's polynomial degree on  $\nabla$ , as is shown by the following example.

### E. Example 2

Consider the following system:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - x_2(t-1) + u(t) \\ \dot{x}_2(t) &= -2x_2(t) \\ \dot{x}_3(t) &= -x_2(t-1) - x_3(t).\end{aligned}\quad (10)$$

From Theorem 3 of [15] it can be shown that (10) does not have any cyclic vector. Later on, a full answer to this problem will be developed and we will see that all the delay terms of (10) can be eliminated even though no cyclic vector does exist.

## III. MAIN RESULTS

The principal contributions of this note are divided in three parts: necessary and sufficient conditions to assure the equivalence of LTDS are presented in Theorem 1, followed by the conditions to eliminate time delays. In the last part it is shown how to apply both results to the delay reduction problem.

### A. Equivalence of LTDS

Although the equivalence problem may have a limited practical interest, its solution, provided in the following Theorem, is instrumental to find a complete solution to the delay elimination problem which is given by Theorem 2.

**Theorem 1:** Consider two linear systems, described by (5) and (6). Let  $\tau_x = \sup(\tau_a, \tau_b)$ ,  $\tau_z = \sup(\tau_c, \tau_d)$ . Assume, without loss of generality, that  $\tau_x \geq \tau_z$ . Then, both systems are equivalent if and only if there exist  $T_i \in \mathbb{R}^{n \times n}$ , for  $i = 0, 1, \dots, \tau_m$ , with  $\tau_m = \sup(\tau_a, (n-1)\tau_b)$ , such that:

- I)  $\sum_{i=0}^k T_i A_{k-i} = \sum_{j=0}^k \bar{A}_j T_{k-j}$  for  $k = 0, 1, \dots, \gamma$ , where  $\gamma = \tau_a + \tau_m$  if  $\tau_a > \tau_c$ ; otherwise  $\gamma = \tau_c + \tau_m$ .  $\bar{A}_i = 0_{n \times n}$  for  $i > \tau_c$ ;
- II)  $\sum_{i=0}^k T_i B_{k-i} = \bar{B}_k$   $k = 0, 1, \dots, \vartheta$ , where  $\vartheta = \tau_b + \tau_m$  if  $\tau_a > \tau_c$ ; otherwise  $\vartheta = \tau_d + \tau_m$ .  $\bar{B}_i = 0_{n \times 1}$  for  $i > \tau_d$ ;
- III)  $\det(\sum_{i=0}^{\tau_m} T_i \nabla^i) \in \mathbb{R} \setminus \{0\}$ .

*Proof:* Let  $T[\nabla] = \sum_{i=0}^{\tau_m} T_i \nabla^i$ . Without loss of generality, assume that  $\tau_x \geq \tau_z$ .

*Necessity:* Assume that systems (1) and (8) are equivalent. Then there exists a unimodular transformation  $T[\nabla]$  satisfying (7). Unimodularity implies iii).

From (2) and (7) one has

$$\dot{z}(t) = T[\nabla] \left[ A_0 x(t) + \sum_{l=1}^{\tau_a} A_l \xi_l(t) + B_0 u(t) + \sum_{m=1}^{\tau_b} B_m \psi_m(t) \right]. \quad (11)$$

Equations (6) and (11) imply

$$\begin{aligned}\dot{z}(t) &= \bar{A}[\nabla] z(t) + \bar{B}[\nabla] u(t) \\ &= \bar{A}[\nabla] T[\nabla] x(t) + \bar{B}[\nabla] u(t).\end{aligned}\quad (12)$$

Considering  $T[\nabla] = \sum_{i=0}^{\tau_m} T_i \nabla^i$  with  $T_i \in \mathbb{R}^{n \times n}$ , after applying the delay operator, we can express (11) and (12) in matrix form as

$$\begin{aligned}\dot{z}(t) &= [T_0 \ T_1 \ \dots \ T_{\tau_m}] \\ &\times \begin{bmatrix} A_0 x(t) + A_1 \xi_1(t) + \dots + A_{\tau_a} \xi_{\tau_a}(t) \\ A_0 \xi_1(t) + A_1 \xi_2(t) + \dots + A_{\tau_a} \xi_{\tau_a+1}(t) \\ \vdots \\ A_0 \xi_{\tau_m}(t) + A_1 \xi_{\tau_m+1}(t) + \dots + A_{\tau_a} \xi_{\tau_m+\tau_a}(t) \end{bmatrix} \\ &+ [T_0 \ T_1 \ \dots \ T_{\tau_m}]\end{aligned}$$

$$\begin{aligned}&\times \begin{bmatrix} B_0 u(t) + B_1 \psi_1(t) + \dots + B_{\tau_b} \psi_{\tau_b}(t) \\ B_0 \psi_1(t) + B_1 \psi_2(t) + \dots + B_{\tau_b} \psi_{\tau_b+1}(t) \\ \vdots \\ B_0 \psi_{\tau_m}(t) + B_1 \psi_{\tau_m+1}(t) + \dots + B_{\tau_b} \psi_{\tau_m+\tau_b}(t) \end{bmatrix} \\ &= \bar{A}[\nabla] [T_0 \ T_1 \ \dots \ T_{\tau_m}] \begin{bmatrix} x(t) \\ \xi_1(t) \\ \vdots \\ \xi_{\tau_m}(t) \end{bmatrix} + \bar{B}[\nabla] u(t). \\ &= \bar{A}_0 [T_0 \ T_1 \ \dots \ T_{\tau_m}] \begin{bmatrix} x(t) \\ \xi_1(t) \\ \vdots \\ \xi_{\tau_m}(t) \end{bmatrix} \\ &\quad + \bar{A}_1 [T_0 \ T_1 \ \dots \ T_{\tau_m}] \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{\tau_m+1}(t) \end{bmatrix} \\ &\quad + \dots + \bar{A}_{\tau_c} [T_0 \ T_1 \ \dots \ T_{\tau_m}] \begin{bmatrix} \xi_{\tau_c}(t) \\ \xi_{\tau_c+1}(t) \\ \vdots \\ \xi_{\tau_m+\tau_c}(t) \end{bmatrix} \\ &\quad + \bar{B}_0 u(t) + \bar{B}_1 \psi_1(t) + \dots + \bar{B}_{\tau_d} \psi_d(t).\end{aligned}\quad (13)$$

The coefficients of  $x(t)$ ,  $\xi_i(t)$ ,  $u(t)$  and  $\psi_i(t)$  have to be equal, which implies i), ii).

Let  $\tau_m = \deg(T[\nabla])$ . Since the determinant of a unimodular matrix is a scalar quantity, then it can be easily proved, using the adjoint matrix, that  $\deg(T[\nabla]^{-1}) \leq \tau_m(n-1)$ .

From (5), (6) and (7)

$$A[\nabla] = T[\nabla]^{-1} \bar{A}[\nabla] T[\nabla] \quad (14)$$

and

$$B[\nabla] = T[\nabla]^{-1} \bar{B}[\nabla]. \quad (15)$$

To have the same polynomial matrix degree in both sides of (14) the worst case is to have  $\tau_m = \tau_a$ , and for (15) the worst case is to need  $\tau_m = (n-1)\tau_b$ . Then  $\tau_m = \sup(\tau_a, (n-1)\tau_b)$ .

*Sufficiency:* From iii)  $T[\nabla] = \sum_{i=0}^{\tau_m} T_i \nabla^i$  is a bicausal transformation. From i) and ii), it can be directly checked that (7) implies (13).  $\square$

From the previous Theorem, it is possible to give conditions to eliminate the time delays in system (1).

### B. Elimination of Time Delays

The necessary and sufficient conditions for the existence of a change of coordinates such that the delay terms can be eliminated are presented in this section. It is shown, as well, that it is not restrictive to set  $T_0$  as the identity matrix to compute the transformation  $T[\nabla]$ . This Theorem is a special case of Theorem 1.

**Theorem 2:** The system (1) is equivalent to a delay-free system if and only if there exist  $T_i \in \mathbb{R}^{n \times n}$  for  $i = 0, 1, \dots, \tau_m$ , with  $\tau_m = \sup(\tau_a, (n-1)\tau_b)$ , such that:

- i)  $\sum_{i=0}^k T_i A_{k-i} = \bar{A}_0 T_k$  for  $k = 0, 1, \dots, \tau_m + \tau_a$ , with  $\bar{A}_0 = A_0$ ;
- ii)  $\sum_{i=0}^k T_i B_{k-i} = \bar{B}_k$  for  $k = 0, 1, \dots, \tau_m + \tau_b$ , with  $\bar{B}_0 = B_0$  and  $\bar{B}_k = 0_{n \times n} \ \forall k > 0$ ;
- iii)  $\det(\sum_{i=0}^{\tau_m} T_i \nabla^i) \in \mathbb{R} \setminus \{0\}$ .

*Proof:* From Theorem 1, and considering  $\bar{A}_k = 0 \ \forall k \geq 1$  and  $\bar{B}_k = 0 \ \forall k \geq 1$ .  $\square$

Note that the matrix  $\bar{A}_0$  of the delay-free system can always be set equal to  $A_0$ , since there is no loss of generality if  $T_0$  is set to the identity matrix. To see this, consider first the general change of coordinates  $\tilde{z}(t)$

$$\begin{aligned}\tilde{z}(t) &= \tilde{T}[\nabla]x(t) \\ \tilde{z}(t) &= [\tilde{T}_0 + \tilde{T}_1\nabla + \cdots + \tilde{T}_r\nabla^r]x(t).\end{aligned}$$

Let

$$\begin{aligned}z(t) &= \tilde{T}_0^{-1}\tilde{z}(t) \\ z(t) &= [\mathbb{I} + \tilde{T}_0^{-1}\tilde{T}_1\nabla + \cdots + \tilde{T}_0^{-1}\tilde{T}_r\nabla^r]x(t) \\ z(t) &= [\mathbb{I} + T_1\nabla + \cdots + T_r\nabla^r]x(t)\end{aligned}$$

which implies  $\bar{A}_0 = A_0$  and  $\bar{B}_0 = B_0$  and the conditions i) and ii).

*Remark 2:* The transformation matrix  $T[\nabla]$  can be found by solving a system of linear equations, which results in a checkable and fully constructive method.

*Example 2. Continued:*

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - x_2(t-1) + u(t) \\ \dot{x}_2(t) &= -2x_2(t) \\ \dot{x}_3(t) &= -x_2(t-1) - x_3(t)\end{aligned}\quad (16)$$

which can be expressed as

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1\xi_1(t) + B_0u(t) \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}x(t) + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\xi_1(t) \\ &\quad + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\psi_1(t) \\ \xi_1(t) &= x(t-1) \\ \psi_1(t) &= u(t-1).\end{aligned}$$

Conditions i) and ii) of Theorem 2 gives the following system of equations:

$$\begin{aligned}T_0A_0 &= \bar{A}_0T_0 \\ T_0A_1 + T_1A_0 &= \bar{A}_0T_1 \\ T_0A_2 + T_1A_1 + T_2A_0 &= \bar{A}_0T_2 \\ T_0B_0 &= \bar{B}_0 \\ T_0B_1 + T_1B_0 &= 0_{n \times n}.\end{aligned}\quad (17)$$

Let  $T_0 = \mathbb{I}$ ,  $A_0 = \bar{A}_0$ ,  $B_0 = \bar{B}_0$ , and  $T_2 = 0_{n \times n}$  since  $\tau_a = 1$  and  $\tau_b = 0$ . Solving (17), we have that

$$T_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & k & 0 \\ 0 & -1 & 0 \end{bmatrix}\quad (18)$$

where  $k \in \mathbb{R}$ .

Condition iii) is fulfilled if and only if we have  $k = 0$ . Then

$$T[\nabla] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\nabla = \begin{bmatrix} 1 & -\nabla & 0 \\ 0 & 1 & 0 \\ 0 & -\nabla & 1 \end{bmatrix}.$$

By computing the Smith's normal form for the transformation matrix, we find that the invariant polynomials are  $\{1, 1, 1\}$ , the condition of Definition 3 is fulfilled and  $T[\nabla]$  is a bicausal transformation.

The delay-free system equivalent to (16) is

$$\begin{aligned}\dot{z}_1(t) &= -z_1(t) + u(t) \\ \dot{z}_2(t) &= -2z_2(t) \\ \dot{z}_3(t) &= -z_3(t).\end{aligned}$$

For unforced systems, that is, systems described by

$$\dot{x}(t) = A[\nabla]x(t)$$

the conditions given by Theorem 2 reduce to i) and iii). For a driftless system, described by

$$\dot{x}(t) = B[\nabla]u(t)$$

they reduce to ii) and iii). This can be obtained directly from the proof of Theorem 2 setting  $B_j = 0_{n \times m}$  and  $A_j = 0_{n \times n}\forall j$ , respectively.

### C. Reduction of Time Delays

For those systems on which the delays cannot be eliminated, a minimal-delay representation can be found by the iterative application of conditions (i)–(iii) of Theorem 1.

$T_0$  can be set to the identity matrix, which is not restrictive, as it was shown for Theorem 2. System (1) is equivalent to a system with  $\tau_z = l$  as maximum delay if there is a solution for  $\bar{A}_j$ ,  $\bar{B}_j$  and  $T_i$ , with  $j \leq l$  and  $i = \{1, \dots, \tau_m\}$ . See the following example to illustrate the procedure.

*Example 3:*

$$\begin{aligned}\dot{x}_1(t) &= x_1(t-1) + 2x_1(t) + x_2(t-2) - x_2(t-1) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t-1) \\ \dot{x}(t) &= A_0x(t) + A_1\xi_1(t) + A_2\xi_2(t) \\ &= \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}x(t) + \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\xi_1(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi_2(t) \\ \xi_1(t) &= x(t-1) \\ \xi_2(t) &= x(t-2).\end{aligned}\quad (19)$$

The system of equations, given by condition i) of Theorem 2, are inconsistent. Thus, delays cannot be eliminated. However, solving i) of Theorem 1 in  $\bar{A}_1, T_1, T_2$ , with  $T_0 = \mathbb{I}$ ,  $\bar{A}_0 = A_0$ , we have the following solution:

$$\begin{aligned}T_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ T_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{A}_1 &= \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}\end{aligned}\quad (21)$$

and

$$\begin{aligned}T[\nabla] &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\nabla + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\nabla^2 \\ &= \begin{bmatrix} 1 & \nabla \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

$\det(T[\nabla]) = 1$  and the Smith invariant polynomials are  $\{1, 1\}$ , then the transformation is bicausal

$$\begin{aligned}\dot{z}(t) &= \begin{bmatrix} 2 & -3\nabla \\ -1 & 0 \end{bmatrix}z(t) \\ \dot{z}_1(t) &= 2z_1(t) - 3z_2(t-1) \\ \dot{z}_2(t) &= -z_1(t).\end{aligned}$$

Thus, system (19) is equivalent to a system which has  $\tau_a = 1$  as the maximum delay.

Conditions to reduce time delays in a driftless system remain described by a system of linear equations. They are reduced to conditions ii) and iii) of the Theorem 1, which are directly obtained setting  $A_j = 0_{n \times n} \forall j$ .

For those systems (1) which admit a so-called cyclic vector  $b[\nabla]$  the transformation

$$z(t) = [b[\nabla] \ A[\nabla]b[\nabla] \ \dots \ A[\nabla]^{n-1}b[\nabla]]^{-1} x(t)$$

is a solution in the sense of Theorem 1, which yields a canonical Brunovsky form for  $\bar{A}[\nabla]$ .

#### IV. CONCLUSION

A full characterization of the equivalence of linear time-delay systems has been given. We claim that the necessary and sufficient conditions proposed in this paper are easy to test for the case of equivalence to delay-free systems. The quantity and complexity of equations to solve is increased for the case of delay-reduction. However, the approach is useful and gives a constructive way to find the transformation. Theorem 2 is a full answer to a longstanding open problem. Further research is required to extend this result to a broad class of nonlinear systems.

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