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# HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, II: Verification and Optimal Feedbacks \*

Salvatore Federico<sup>†</sup>      Ben Goldys<sup>‡</sup>      Fausto Gozzi<sup>§</sup>

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## Abstract

This paper, which is the natural continuation of [14], studies a class of optimal control problems with state constraints where the state equation is a differential equation with delays. In [14] the problem is embedded in a suitable Hilbert space  $H$  and the regularity of the associated Hamilton-Jacobi-Bellman (HJB) equation is studied. The goal of the present paper is to exploit the regularity result of [14] to prove a Verification Theorem and find optimal feedback controls for the problem. While it is easy to define a feedback control formally following the classical case, the proof of its existence and optimality is hard due to lack of full regularity of  $V$  and to the infinite dimensionality of the problem. The theory developed is applied to study economic problems of optimal growth for nonlinear time-to-build models. In particular, we show the existence and uniqueness of optimal controls and their characterization as feedbacks.

**Keywords:** Hamilton-Jacobi-Bellman equation, optimal control, delay equations, verification theorem.

**A.M.S. Subject Classification:** 34K35, 49L25, 49K25.

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## 1 Introduction

The aim of this paper is to prove a Verification Theorem and the existence of optimal feedback controls for a class of optimal control problems of delay differential equations (DDE's) arising in economic models.

This work is a natural continuation of [14]. Let us recall that in [14] we study optimal control of differential equations with delays and state constraints. This class includes some problems arising in economics, in particular growth models with time-to-build (see [2, 3, 4, 17]). In [14] the problem is embedded in a suitable Hilbert space  $H$  and the associated Hamilton-Jacobi-Bellman (HJB) equation is studied<sup>1</sup>. Therein the main result is the regularity of solutions to such a HJB equation. More precisely it is shown that the value function has continuous classical derivative in the direction of the “present”. This allows to define a feedback control in (an almost) classical sense. In the present paper we use this regularity result to prove a Verification Theorem and the existence of optimal feedback controls.

The class of optimal control problems we are going to study can be described as follows. Given a control  $c(\cdot) \geq 0$ , the state  $x(\cdot)$  satisfies a DDE

$$\begin{cases} x'(t) = rx(t) + f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0), \end{cases} \quad (1)$$

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<sup>1</sup>A similar problem, but without state constraints and with no regularity result is studied in [10].

with state constraint  $x(\cdot) > 0$  or  $x(\cdot) \geq 0$ . The objective is to maximize the functional

$$J(\eta; c(\cdot)) := \int_0^{+\infty} e^{-\rho t} (U_1(c(t)) + U_2(x(t))) dt, \quad \rho > 0, \quad (2)$$

over all admissible controls  $c(\cdot)$  that are locally integrable and such that the associated state trajectory satisfies the state constraint above. The structure of the problem (state equation, objective functional and state constraint) is determined by the usual intertemporal discounted utility from consumption used in economic problems (economic growth with time to build and vintage capital, see e.g. [2, 6, 7], and, in the stochastic case, optimal portfolio problems with consumption). The treatment of such problems is indeed the main goal of our paper, as can be seen from the examples in Section 5. Our techniques can be adapted to deal with other problems, such as the minimization of quadratic (or, more generally, coercive) functionals representing the energy or the distance to a target. This would require however nontrivial technical modifications and is beyond the scope of this paper. It will be the subject of further research.

The results of [14] allow us to state the existence of the directional derivative  $V_{\eta_0}$  of the value function

$$V(\eta) = \sup_{c(\cdot)} J(\eta, c(\cdot)), \quad \eta = (\eta_0, \eta_1),$$

with respect to  $\eta_0$ . Since the feedback map can be written in terms of  $V_{\eta_0}$ , we can address the problem of proving a Verification Theorem stating the optimality of a control satisfying a closed loop property (see (24)).

This is a non-trivial result since the classical gradient of the value function does not exist in this case and we need to prove first a verification theorem for viscosity solutions which is new in this context and of independent interest. Let us recall that a verification theorem in the framework of viscosity solutions is proved in the finite dimensional case in [21]. Adapting the technique of that proof to our case is difficult due to the infinite dimensional nature of our problem. Moreover, there is a mistake in the key Lemma 5.2, Chapter 5 of [21], see Remark 3.4 for more details. We provide a correct version of this lemma (Lemma 3.3) and then exploit it to prove the Verification Theorem 3.2<sup>2</sup>.

The next step is to study the closed loop equation associated to the feedback map. In order to prove that the feedback control satisfies the hypotheses of the Verification Theorem, we prove that the closed loop equation admits local solutions (Proposition 4.4) and that these solutions are global under further hypotheses (Proposition 4.9). Then Verification Theorem 3.2 yields the existence of a unique *locally* optimal feedback control in the general case (Theorem 4.5) and of a unique optimal feedback control (Theorem 4.10) under the hypotheses of Proposition 4.9. Unfortunately, the assumptions of Proposition 4.9 are not satisfied in the case of economic application we are interested in. In order to cover this case, we use an approximation procedure that allows us to get rid of the aforementioned additional hypotheses and obtain the existence of a unique optimal control and its characterization as feedback in the interior region (Theorem 4.15). We do not address in this paper the issue of computing the feedback control law. We believe that our results provide a basis for such computations. Indeed, if one was able to approximate numerically the value function and its derivative with respect to the “present”, then our result would allow us to approximate the optimal feedback control

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<sup>2</sup>This result has been exploited in the paper [12] to prove a weaker verification theorem in a different context.

law. In Section 5 we apply this result to the growth models with distributed time-to-build and nonlinear production.

## 2 The optimal control problem and the value function

In this section we formulate the optimal control problem studied in this paper and recall, for the reader's convenience, the main results of [14]. We use the notations

$$L_T^2 := L^2([-T, 0]; \mathbb{R}), \quad W_T^{1,2} := W^{1,2}([-T, 0]; \mathbb{R}).$$

We denote by  $H$  the Hilbert space

$$H := \mathbb{R} \times L_T^2,$$

endowed with the inner product  $\langle \cdot, \cdot \rangle$  defined by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}} + \langle \cdot, \cdot \rangle_{L_T^2},$$

and the norm  $\| \cdot \|$  defined by

$$\| \cdot \|^2 = | \cdot |_{\mathbb{R}}^2 + \| \cdot \|_{L_T^2}^2.$$

We denote by  $\eta = (\eta_0, \eta_1(\cdot))$  a generic element of this space. For convenience we also set

$$\begin{aligned} H_+ &:= (0, +\infty) \times L_T^2, & H_{++} &:= (0, +\infty) \times \{\eta_1(\cdot) \in L_T^2 \mid \eta_1(\cdot) \geq 0 \text{ a.e.}\}, \\ \bar{H}_+ &:= [0, +\infty) \times L_T^2, & \bar{H}_{++} &:= [0, +\infty) \times \{\eta_1(\cdot) \in L_T^2 \mid \eta_1(\cdot) \geq 0 \text{ a.e.}\}. \end{aligned}$$

**Remark 2.1.** *Economic motivations we are mainly interested in (see [2, 3, 4] and Section 5) require to study the optimal control problem with the initial condition in  $H_{++}$  in the case of state constraint  $x(\cdot) > 0$  or in  $\bar{H}_+$  in the case of state constraint  $x(\cdot) \geq 0$ . However the sets  $H_{++}$  and  $\bar{H}_{++}$  are not convenient to work with in infinite dimension, since their topological interior with respect to the  $\| \cdot \|$ -norm is empty. This is why we take initial states belonging to  $H_+$  or  $\bar{H}_+$  (respectively in the case of state constraint  $x(\cdot) > 0$  or  $x(\cdot) \geq 0$ ).*

For  $\eta \in H_+$  (respectively,  $\eta \in \bar{H}_+$ ), we consider the controlled delay differential equation (1) with the state constraint  $x(\cdot) > 0$  (respectively,  $x(\cdot) \geq 0$ ) and control constraint  $c(\cdot) \geq 0$ .

The following will be standing assumptions on the functions  $a$  and  $f_0$ . They will hold throughout the whole paper and will not be repeated.

### Hypothesis 2.2.

- (i)  $a(\cdot) \in W_T^{1,2}$  is such that  $a(\cdot) \geq 0$  and  $a(-T) = 0$ ;
- (ii)  $f_0 : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is jointly concave, nondecreasing with respect to the second variable, Lipschitz continuous with Lipschitz constant  $C_{f_0}$ , and

$$f_0(0, 0) \geq 0. \tag{3}$$

**Remark 2.3.** *In this remark we discuss Hypothesis 2.2-(ii).*

- (a) *Hypothesis 2.2-(ii) is assumed in [14] to prove the regularity of the value function (see [14, Theorem 4.6], which corresponds to the second part of Theorem 2.13 in the present paper). We explain in detail how it is used:*

- The concavity of  $f_0$  is necessary to prove the concavity of the value function, which is essential to prove the regularity result.
  - The monotonicity is essential to address the problem of the state constraint. This assumption could be avoided if one wanted to consider the problem without state constraint but with more general state equation.
  - The global Lipschitz continuity of  $f_0$  with respect to both variables is assumed is needed to obtain “good” results on the existence, uniqueness and continuous dependence on data of the solutions to the state equation (1). In fact the local Lipschitz continuity is required in the variable  $y$  only, since concavity and monotonicity automatically guarantee the global Lipschitz continuity in any half-line of kind  $[a, +\infty)$ , which is the part of our interest due to the state constraint. In economic applications it often happens that  $f_0$  is a production function defined only for positive  $y$ . A typical example is  $f_0(x, y) = y^\alpha$ ,  $\alpha \in (0, 1)$ , which is not Lipschitz continuous at  $0^+$ . This case cannot be covered directly by our approach as we cannot extend such a function to a locally Lipschitz continuous function on  $\mathbb{R}$ . An attempt to study such a case would lead to a non trivial technical problem in the proof of the continuity of the value function. Indeed, in this case we cannot enlarge the natural set of initial data, which is the positive cone  $\bar{H}_{++} \subset H$ , to an open set without destroying the concavity of the value function. This difficulty arises since the cone  $\bar{H}_{++}$  has empty interior part in  $H$  and is due to the infinite dimensional nature of our problem.
- (b) Given the considerations of (a), we observe that Hypotesis 2.2-(ii) is clearly satisfied by linear functions  $f_0$ , but also by some nonlinear functions that are, e.g., meaningful examples of production functions. Indeed we can have  $f_0(x, y) = g(y)$ , where  $g$  is as follows on  $[0, +\infty)$  and is extended to a concave Lipschitz function on  $(-\infty, 0)$  (the possibility of this extension is exactly what we need to enlarge the set of initial data and preserve the concavity):
- $g(y) = a_0 - b_0 e^{-\gamma y}$ ,  $a_0 \in \mathbb{R}$ ,  $b_0 > 0$ ,  $\gamma > 0$ ,  $a_0 - b_0 \geq 0$ ;
  - $g(y) = a_0(y + y_0)^\gamma - b_0$ ,  $\gamma \in (0, 1)$ ,  $a_0 > 0$ ,  $b_0 \in \mathbb{R}$ ,  $y_0 > 0$ ,  $a_0 y_0^\gamma - b_0 \geq 0$ ;
  - $g(y) = -a_0(y + y_0)^\gamma + b_0$ ,  $\gamma \in (-\infty, 0)$ ,  $a_0 > 0$ ,  $b_0 \in \mathbb{R}$ ,  $y_0 > 0$ ,  $-a_0 y_0^\gamma + b_0 \geq 0$ ;
  - $g(y) = a_0 \log(y + y_0) + b_0$ ,  $a_0 > 0$ ,  $b_0 \in \mathbb{R}$ ,  $y_0 > 0$ ,  $a_0 \log y_0 + b_0 \geq 0$ .
- (c) However the requirement of global Lipschitz continuity of  $f_0$  is not needed to get the main results of the present paper, if we know a priori that the results of [14] hold true. We refer to Remark 4.16 for comments on that.

From now on we will assume that  $f_0$  is extended to a Lipschitz continuous map on  $\mathbb{R}^2$  setting

$$f_0(x, y) := f_0(0, y), \quad \text{for } x < 0.$$

Without loss of generality (see [14]) we assume that  $r > 0$ .

We say that a function  $x : [-T, +\infty) \rightarrow \mathbb{R}$  is a solution to equation (1) if  $x(t) = \eta_1(t)$  for  $t \in [-T, 0)$  and

$$x(t) = \eta_0 + \int_0^t r x(s) ds + \int_0^t f_0 \left( x(s), \int_{-T}^0 a(\xi) x(s + \xi) d\xi \right) ds - \int_0^t c(s) ds, \quad t \geq 0. \quad (4)$$

From [14] we have that, for each  $\eta \in H$  and  $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+)$ , equation (1) admits a unique solution that is locally<sup>3</sup> absolutely continuous on  $[0, +\infty)$ . We denote by  $x(\cdot; \eta, c(\cdot))$  the unique solution of (1) with initial value  $\eta \in H_+$  and control  $c$ . We emphasize that this is a solution to the integral equation (4), hence it satisfies differential equation (1) for only almost every  $t \in [0, \infty)$ .

For an initial condition  $\eta \in H_+$  we define a class of admissible controls for the problem with state constraint  $x(\cdot) > 0$  as

$$\mathcal{C}(\eta) := \{c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+) \mid x(\cdot; \eta, c(\cdot)) > 0\}. \quad (5)$$

In analogous way, for an initial condition  $\eta \in \bar{H}_+$ , we define a class of admissible controls for the problem with state constraint  $x(\cdot) \geq 0$  as

$$\bar{\mathcal{C}}(\eta) := \{c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+) \mid x(\cdot; \eta, c(\cdot)) \geq 0\}. \quad (6)$$

In both cases, setting  $x(\cdot) := x(\cdot; \eta, c(\cdot))$ , the problem consists in maximizing the functional (2), where  $U_1, U_2$  satisfy Hypothesis 2.4 below, over the set of corresponding admissible controls.

Differently from [14], for some reasons that will be clearer later, here we will deal explicitly with the case of large state constraint  $x(\cdot) \geq 0$ . Subsection A.3 will be devoted to study the relationship of the two problems starting from the interior region. From now on we will consider the problem with state constraint  $x(\cdot) \geq 0$ , for which the set of admissible controls is given in (6).

The following will be standing assumptions on the utility functions  $U_1, U_2$  and on the discounting rate  $\rho$ . They will hold throughout the whole paper and will not be repeated.

**Hypothesis 2.4.**

(i)  $U_1 \in C([0, +\infty); \mathbb{R}) \cap C^2((0, +\infty); \mathbb{R})$  and

$$U'_1 > 0, U''_1 < 0; \quad U'_1(0^+) = +\infty; \quad (7)$$

$$\exists \beta_1 \in [0, 1), C_1 > 0 \text{ such that } U_1(c) \leq C_1(1 + c^{\beta_1}). \quad (8)$$

Without loss of generality we will assume  $U_1(0) = 0$ . We note that (7) and (8) imply  $\lim_{c \rightarrow +\infty} U'_1(c) = 0$ .

(ii)  $U_2 : [0, +\infty) \rightarrow [-\infty, +\infty)$ , and  $U_2 \in C((0, +\infty); \mathbb{R})$  is increasing and concave. Moreover

$$\int_0^{+\infty} e^{-\rho t} U_2(e^{-C_{f_0} t}) dt > -\infty, \quad (9)$$

and

$$\exists \beta_2 \in [0, 1), C_2 > 0 \text{ such that } U_2(x) \leq C_2(1 + x^{\beta_2}). \quad (10)$$

(iii) The discounting rate  $\rho$  is such that

$$\rho > r + C_{f_0} \left( 1 + T \cdot \sup_{\xi \in [-T, 0]} a(\xi) \right), \quad (11)$$

where  $C_{f_0}$  is the Lipschitz constant of  $f_0$ .

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<sup>3</sup>In [14] the fact that the absolute continuity is local is not remarked.

**Remark 2.5.** We refer to [14] for comments on the assumptions above. Comparing to [14], we observe that for simplicity of presentation only we have replaced the assumption

$$\rho > (\beta_1 \vee \beta_2) \left( r + C_{f_0} \left( 1 + T \cdot \sup_{\xi \in [-T, 0]} a(\xi) \right) \right),$$

with stronger assumption (11). We note also that this would allow us to weaken assumptions (8), (10) on  $U_1, U_2$ . However, we do not go deeper into these technical problems in order to focus on other topics such as the existence of optimal controls and their characterization as feedback controls.

For  $\eta \in \bar{H}_+$  the value function of our problem is defined by

$$V(\eta) := \sup_{c(\cdot) \in \bar{\mathcal{C}}(\eta)} J(\eta, c(\cdot)), \quad (12)$$

with  $\sup \emptyset = -\infty$ . The domain of the value function is the set

$$\mathcal{D}(V) := \{\eta \in \bar{H}_+ \mid V(\eta) > -\infty\}.$$

What is fundamental throughout the paper is the concavity of  $V$  (see [14, Prop. 2.10-(2) and Prop. 2.14]).

**Proposition 2.6.** *The set  $\mathcal{D}(V)$  is convex and the value function  $V$  is a concave proper function on  $\mathcal{D}(V)$ .  $\square$*

Since  $V$  is concave and proper, there exist  $b_0, b_1 \in \mathbb{R}$  such that

$$V(\eta) \leq b_0 + b_1 \|\eta\|, \quad \eta \in \mathcal{D}(V). \quad (13)$$

Also we have the strict monotonicity of  $V$  with respect to the first component (see [14, Prop. 2.16]).

**Proposition 2.7.** *The function  $\eta_0 \mapsto V(\eta_0, \eta_1(\cdot))$  is strictly increasing for every  $\eta_1 \in L_T^2$  over the set  $\{\eta_0 \geq 0 \mid (\eta_0, \eta_1(\cdot)) \in \mathcal{D}(V)\}$ .  $\square$*

## 2.1 The delay problem rephrased in infinite dimension

Here we recall how to rewrite the original delay state equation (1) as an ordinary differential equation in the space  $H$ . Formally, the unknown is

$$X(t) = (x(t), x(t + \xi)_{\xi \in [-T, 0]})$$

and the equation for it is

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n}, \\ X(0) = \eta. \end{cases} \quad (14)$$

where

- $A : \mathcal{D}(A) \subset H \rightarrow H$  is an unbounded operator defined by  $A(\eta_0, \eta_1(\cdot)) := (r\eta_0, \eta_1'(\cdot))$  with the domain

$$\mathcal{D}(A) := \{\eta \in H \mid \eta_1(\cdot) \in W_T^{1,2}, \eta_1(0) = \eta_0\};$$



- $F : H \longrightarrow H$  is a Lipschitz continuous map defined by

$$F(\eta_0, \eta_1(\cdot)) := (f(\eta_0, \eta_1(\cdot)), 0),$$

where  $f(\eta_0, \eta_1(\cdot)) := f_0\left(\eta_0, \int_{-T}^0 a(\xi)\eta_1(\xi)d\xi\right)$ ;

- $\hat{n} = (1, 0) \in H_+$ ,  $\eta \in H$ ,  $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+)$ .

It is well known that  $A$  is the infinitesimal generator of a strongly continuous semigroup  $(S(t))_{t \geq 0}$  on  $H$ . A mild solution of (14) is a function  $X \in C([0, +\infty); H)$  which satisfies the integral equation

$$X(t) = S(t)\eta + \int_0^t S(t-\tau)F(X(\tau))d\tau + \int_0^t c(\tau)S(t-\tau)\hat{n}d\tau, \quad \forall t \geq 0. \quad (15)$$

We refer to [14] for the proofs of the following results.

**Theorem 2.8.** *For each  $\eta \in H$ ,  $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+)$ , there exists a unique mild solution to (14).*  $\square$

We denote by  $X(\cdot; \eta, c(\cdot)) = (X_0(\cdot; \eta, c(\cdot)), X_1(\cdot; \eta, c(\cdot)))$  the unique solution to (14) for the initial state  $\eta \in H$  and under the control  $c(\cdot) \in L^1([0, +\infty); \mathbb{R}^+)$ .

**Proposition 2.9.** *Let  $\eta \in H$ ,  $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+)$ . Set  $x(\cdot) := x(\cdot; \eta, c(\cdot))$  and  $X(\cdot) := X(\cdot; \eta, c(\cdot))$ . Then we have the equality in  $H$*

$$X(t) = (x(t), x(t + \xi)_{\xi \in [-T, 0]}), \quad \forall t \geq 0. \quad \square$$

The previous result justifies the infinite-dimensional approach. Indeed, due to Proposition 2.9, the original optimization problem can be rewritten as

$$\text{Maximize } \int_0^{+\infty} e^{-\rho t} (U_1(c(t)) + U_2(X_0(t; \eta, c(\cdot)))) dt, \quad \text{over } c(\cdot) \in \bar{\mathcal{C}}(\eta).$$

## 2.2 Continuity of the value function

Here we state a continuity property of the value function. Note that the generator  $A$  of the semigroup  $(S(t))_{t \geq 0}$  in  $H$  has bounded inverse

$$A^{-1}(\eta_0, \eta_1)(s) = \left( \frac{\eta_0}{r}, \frac{\eta_0}{r} - \int_s^0 \eta_1(\xi)d\xi \right), \quad s \in [-T, 0].$$

We define the  $\|\cdot\|_{-1}$ -norm on  $H$  by

$$\|\eta\|_{-1} := \|A^{-1}\eta\|,$$

and the set

$$\mathcal{D}^\circ := \text{Int}_{(H, \|\cdot\|_{-1})}(\mathcal{D}(V)).$$

Note that  $\mathcal{D}^\circ$  is open in  $(H, \|\cdot\|)$ . A priori this set might be empty, but this is not the case due to the following result (whose proof can be found in [14]).

**Proposition 2.10.**

1. We have  $H_{++} \subset \mathcal{D}^\circ$ . In particular, the set  $\mathcal{D}(V)$  has not empty interior part  $\mathcal{D}^\circ$  in the space  $(H, \|\cdot\|_{-1})$ .
2. The value function  $V$  is continuous with respect to  $\|\cdot\|_{-1}$  on  $\mathcal{D}^\circ$ . □

We denote the boundary of  $\mathcal{D}(V)$  in the space  $(H, \|\cdot\|_{-1})$  by  $\mathcal{B}$ , i.e.

$$\mathcal{B} := \text{Fr}_{(H, \|\cdot\|_{-1})}(\mathcal{D}(V)).$$

Clearly we have

$$\mathcal{D}^\circ \subset \mathcal{D}(V) \subset \text{Clos}_{(H, \|\cdot\|_{-1})}(\mathcal{D}(V)) = \mathcal{D}^\circ \cup \mathcal{B}.$$

In general the inclusions above are proper. We note that we have some information on  $\mathcal{D}^\circ$  and  $\mathcal{B}$ . For example, we know that

$$H_{++} \subset \mathcal{D}^\circ, \quad \{\eta \in \bar{H}_{++} \mid \eta_0 = 0\} \subset \mathcal{B}.$$

However,  $\mathcal{D}^\circ$  and  $\mathcal{B}$  may contain other points. In particular,  $\mathcal{D}^\circ$  may contain points from  $H_+$  and  $\mathcal{B}$  may contain points from  $\bar{H}_+$ . We emphasize that  $\mathcal{B}$  may also contain points from  $H_+$ . Thus, points  $\eta \in \mathcal{D}(V)$  such that  $\eta_0 > 0$  can be also boundary points. We refer to Subsection A.2 for a characterization of these sets.

### 2.3 The operator $A^*$ and the superdifferential of $V$

Due to [14, Prop. 3.4], we have a characterization of the operator  $A^*$ , adjoint of  $A$ , and of its domain  $\mathcal{D}(A^*)$ . Indeed

$$\begin{aligned} \mathcal{D}(A^*) &= \{\eta \in H \mid \eta_1 \in W_T^{1,2}, \eta_1(-T) = 0\}, \\ A^*\eta &= (r\eta_0 + \eta_1(0), -\eta_1'(\cdot)). \end{aligned}$$

Note that  $V : \mathcal{D}^\circ \rightarrow \mathbb{R}$  is a continuous concave function, so that the superdifferential  $D^+V(\eta)$

$$D^+V(\eta) := \left\{ \alpha \in H \mid V(\zeta) - V(\eta) \leq \langle \zeta - \eta, \alpha \rangle, \forall \zeta \in H \right\}$$

is not empty at each  $\eta \in \mathcal{D}^\circ$ . Due to [14, Prop. 3.12], we have

$$D^+V(\eta) \subset \mathcal{D}(A^*), \quad \forall \eta \in \mathcal{D}^\circ. \tag{16}$$

Moreover, defining the directional superdifferential of  $V$  along  $\hat{n} = (1, 0) \in H$  at  $\eta \in \mathcal{D}^\circ$  as

$$D_{\hat{n}}^+V(\eta) = \{\alpha_0 \in \mathbb{R} \mid V(\zeta_0, \eta_1) - V(\eta_0, \eta_1) \leq \alpha_0(\zeta_0 - \eta_0), \forall \zeta_0 \in \mathbb{R}\},$$

we have the representation (see [14, Sect. 3.3])

$$D_{\hat{n}}^+V(\eta) = \{\alpha_0 \mid \alpha \in D^+V(\eta)\}. \tag{17}$$

**Remark 2.11** (Errata). *Here we take the opportunity to correct a slight mistake contained in our previous paper [14] (we use the notation used in the present paper). There, in Section 3.3, we introduced the set*

$$D^*V(\eta) := \left\{ \alpha \in H \mid \exists \eta_m \rightarrow \eta, \eta_m \in \mathcal{D}^\circ, \text{ such that } \exists \nabla V(\eta_m) \text{ and } \nabla V(\eta_m) \rightarrow \alpha \right\}, \tag{18}$$

claiming that

$$D^+V(\eta) = \overline{c\circ}(D^*V(\eta)), \quad \eta \in \mathcal{D}^\circ. \quad (19)$$

Actually this is not true if  $D^*V(\eta)$  defined as in (18) with strong convergence of the gradients; it is true (see [9, Cor. 4.7]) if weak convergence of the gradients is taken in the definition of  $D^*V(\eta)$ :

$$D^*V(\eta) := \left\{ \alpha \in H \mid \exists \eta_n \rightarrow \eta, \eta_n \in \mathcal{D}^\circ, \text{ such that } \exists \nabla V(\eta_n) \text{ and } \nabla V(\eta_n) \rightharpoonup \alpha \right\}.$$

The claim 4 of [14, Prop. 3.12] has to be correspondingly corrected taking the weak convergence for the gradients. However, this weaker version of the result is still sufficient to prove the main result [14, Th. 4.6], as in the proof of such result the weak convergence of the gradients is projected in the direction  $\hat{n} = (1, 0)$  yielding the convergence of the respective terms.

## 2.4 The HJB equation in the viscosity sense and the directional regularity

The infinite-dimensional HJB equation associated to our optimization problem in the space  $H$  is

$$\rho v(\eta) = \langle \eta, A^* \nabla v(\eta) \rangle + f(\eta) v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta)), \quad \eta \in \mathcal{D}^\circ, \quad (20)$$

where  $\mathcal{H}$  is the Legendre transform of  $U_1$ , i.e.

$$\mathcal{H}(\zeta_0) := \sup_{c \geq 0} (U_1(c) - \zeta_0 c), \quad \zeta_0 > 0.$$

In [14] we have studied this equation by means of the viscosity approach as follows. Define the set of regular test functions

$$\tau := \left\{ \varphi \in C^1(H) \mid \nabla \varphi(\cdot) \in \mathcal{D}(A^*), \eta \mapsto A^* \nabla \varphi(\eta) \text{ is continuous} \right\}. \quad (21)$$

**Definition 2.12.** (i) A continuous function  $v : \mathcal{D}^\circ \rightarrow \mathbb{R}$  is called a viscosity subsolution of (20) on  $\mathcal{D}^\circ$  if for any  $\varphi \in \tau$  and any  $\eta_M \in \mathcal{D}^\circ$  such that  $v - \varphi$  has a  $\|\cdot\|$ -local maximum at  $\eta_M$  we have

$$\rho v(\eta_M) \leq \langle \eta_M, A^* \nabla \varphi(\eta_M) \rangle + f(\eta_M) \varphi_{\eta_0}(\eta_M) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_M)).$$

(ii) A continuous function  $v : \mathcal{D}^\circ \rightarrow \mathbb{R}$  is called a viscosity supersolution of (20) on  $\mathcal{D}^\circ$  if for any  $\varphi \in \tau$  and any  $\eta_m \in \mathcal{D}^\circ$  such that  $v - \varphi$  has a  $\|\cdot\|$ -local minimum at  $\eta_m$  we have

$$\rho v(\eta_m) \geq \langle \eta_m, A^* \nabla \varphi(\eta_m) \rangle + f(\eta_m) \varphi_{\eta_0}(\eta_m) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_m)).$$

(iii) A continuous function  $v : \mathcal{D}^\circ \rightarrow \mathbb{R}$  is called a viscosity solution of (20) on  $\mathcal{D}^\circ$  if it is both a viscosity sub and supersolution.

Theorems 4.4 and 4.6 in [14] are summarized in the following

**Theorem 2.13.** The value function  $V$  is a viscosity solution to (20) on  $\mathcal{D}^\circ$ . Moreover  $V$  is differentiable along the direction  $\hat{n} = (1, 0)$  continuously in  $\mathcal{D}^\circ$ .  $\square$

### 3 Verification Theorem

Before to proceed with our first main result (Verification Theorem 3.2) we provide the definition of optimal controls and we briefly discuss the main technical issues that arise in proving it.

**Definition 3.1.** *Let  $\eta \in \mathcal{D}(V)$ . A control  $c^*(\cdot) \in \bar{\mathcal{C}}(\eta)$  is said to be optimal for the initial state  $\eta$  if  $J(\eta; c^*(\cdot)) = V(\eta)$ . If  $X^*(\cdot)$  is the associated infinite dimensional state trajectory, then  $X^*(\cdot)$  is said an optimal state and the pair  $(X^*(\cdot), c^*(\cdot))$  is said to be an optimal pair.  $\square$*

Due to the regularity provided by Theorem 2.13, we can define a feedback map on  $\mathcal{D}^\circ$ , which is expected to yield an optimal feedback control. Given  $p_0 \in (0, +\infty)$  we denote by  $\operatorname{argmax}_{c \geq 0}(U_1(c) - cp_0)$  the unique maximizer of  $[0, +\infty) \rightarrow \mathbb{R}$ ,  $c \mapsto U_1(c) - cp_0$  (existence and uniqueness of follow from the assumptions on  $U_1$ ). Then we may define the feedback map for our problem as

$$C(\eta) := \operatorname{argmax}_{c \geq 0}(U_1(c) - cV_{\eta_0}(\eta)), \quad \eta \in \mathcal{D}^\circ. \quad (22)$$

We note that  $C$  is well-defined (and nonnegative) on  $\mathcal{D}^\circ$ . Indeed, by Proposition 2.7,  $V$  is strictly increasing with respect to  $\eta_0$  on this set, so we have  $V_{\eta_0}(\eta) \in (0, +\infty)$  for all  $\eta \in \mathcal{D}^\circ$ . Moreover, since  $V_{\eta_0}$  is continuous on  $\mathcal{D}^\circ$ , also  $C$  is continuous on  $\mathcal{D}^\circ$ . Note also that in general *we cannot extend this map by continuity up to the boundary  $\mathcal{B}$* . In particular, we cannot extend this map by continuity at the points  $\eta \in \bar{H}_{++}$  such that  $\eta_0 = 0$ .

Define for  $c \geq 0$  the operator  $\mathcal{L}^c$  acting on  $\tau$  by

$$[\mathcal{L}^c \varphi](\eta) := -\rho \varphi(\eta) + \langle \eta, A^* \nabla \varphi(\eta) \rangle + f(\eta) \varphi_{\eta_0}(\eta) - c \varphi_{\eta_0}(\eta).$$

By Lemma 4.2 in [14], for every  $\varphi \in \tau$ ,  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ , setting  $X(\cdot) := X(\cdot; \eta, c(\cdot))$  we have

$$\frac{d}{dt} [e^{-\rho t} \varphi(X(t))] = [\mathcal{L}^{c(t)} \varphi](X(t)), \quad \text{for a.e. } t \geq 0.$$

If  $c(\cdot)$  is continuous, then

$$\frac{d}{dt} [e^{-\rho t} \varphi(X(t))] = [\mathcal{L}^{c(t)} \varphi](X(t)), \quad \forall t \geq 0. \quad (23)$$

We are ready to present our Verification Theorem.

**Theorem 3.2** (Verification). *Let  $\eta \in \mathcal{D}^\circ$ . Let  $c^*(\cdot) \in \bar{\mathcal{C}}(\eta)$  and set  $X^*(\cdot) := X(\cdot; \eta, c^*(\cdot))$ . Assume that  $X^*(t) \in \mathcal{D}^\circ$  for every  $t \geq 0$ , so that  $t \mapsto C(X^*(t))$  is well-defined and continuous in  $[0, +\infty)$ , and that*

$$c^*(t) = C(X^*(t)), \quad \text{for a.e. } t \geq 0. \quad (24)$$

*Then  $c^*(\cdot)$  is optimal starting from  $\eta$ .*

Note that the formulation of the Verification Theorem is the same as it would be in the classical case, i.e. for  $V \in C^1$  (see Theorem 3.7, Chapter 5, of [21]). The proof here is much more difficult since we can not assume the existence of the derivative of  $V$  with respect to  $\eta_1$ . In the classical case the main step of the proof is the computation of the derivative

$$t \mapsto \frac{d}{dt} [e^{-\rho t} V(X^*(t))]. \quad (25)$$

and then using the HJB equation and integrating the resulting equality.

We want to get exactly the classical statement but we cannot proceed with the classical proof since we cannot compute the derivative (25). So we proceed using the fact that  $V$  is a viscosity solution (as e.g. in Theorem 3.9, Chapter 5, of [21] and in Theorems 5.4, 5.5, Chapter 6, of [18]). But two main difficulties arise (strongly connected with each other):

- We cannot say ex ante that the function

$$t \mapsto e^{-\rho t} V(X^*(t)) \quad (26)$$

is absolutely continuous. This would be true if  $X^*(t) \in \mathcal{D}(A)$  for almost every  $t \geq 0$  and  $X^* \in L^1_{loc}([0, +\infty); \mathcal{D}(A))$ , as required for example in Theorems 5.4, 5.5, Chapter 6, of [18]. But we do not have these conditions, since we do not require that the initial datum  $\eta$  belongs to  $\mathcal{D}(A)$  and since the semigroup generated by the operator  $A$  acts as a left shift operator on the infinite-dimensional component. Without this regularity we cannot apply the Fundamental Theorem of Calculus.

- Consequently we have to deal with the concept of Dini derivatives of the function (26) and, since we want to integrate them, we need a version of the Fundamental Theorem of Calculus in inequality form relating the function and the integral of its Dini derivative. Such a result in the context of stochastic verification theorems for viscosity solutions is given in [21], Lemma 5.2, Chapter 5. Unfortunately, this result is not true in the version given in the paper (we give a counterexample in Remark 3.4), so we have to use a more refined result, the Saks Theorem, that needs stronger assumptions and that is based on the theory of Dini derivatives.

To proceed we recall first that, if  $g$  is a continuous function on some interval  $[\alpha, \beta] \subset \mathbb{R}$ , the right Dini derivatives of  $g$  are defined by

$$D^+g(t) = \limsup_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}, \quad D_+g(t) = \liminf_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}, \quad t \in [\alpha, \beta),$$

and the left Dini derivatives by

$$D^-g(t) = \limsup_{h \uparrow 0} \frac{g(t+h) - g(t)}{h}, \quad D_-g(t) = \liminf_{h \uparrow 0} \frac{g(t+h) - g(t)}{h}, \quad t \in (\alpha, \beta].$$

The following Lemma is a special case of the Saks Theorem (see [20, Ch.VI, p.204, Theorem 7.3])

**Lemma 3.3.** *Let  $g \in C([0, +\infty); \mathbb{R})$  and assume that there exists  $\mu \in L^1_{loc}((0, +\infty); \mathbb{R})$  such that*

$$D_-g(t) \geq \mu(t), \quad \text{for a.e. } t \in (0, +\infty). \quad (27)$$

and that

$$D_-g(t) > -\infty \quad \forall t \in (0, +\infty) \quad (28)$$

except at most for those of a countable set. Then, for every  $0 \leq \alpha \leq \beta < +\infty$ ,

$$g(\beta) - g(\alpha) \geq \int_{\alpha}^{\beta} \mu(t) dt. \quad (29)$$

□

**Remark 3.4.** We give some remarks on Lemma 3.3.

(a) If  $\mu$  is continuous and the condition (27) holds for all  $t > 0$  then (28) holds and so the claim of Lemma 3.3 holds without assuming (28).

(b) We cannot avoid condition (28). If it does not hold then (29) is no longer true. For example, if  $g = -f$  on  $[0, 1]$ , where  $f$  is the Cantor function and  $\mu \equiv 0$ , we have

$$\mu(t) = 0 = g'(t) = D_-g(t) \quad \text{for a.e. } t \in (0, 1].$$

Therefore, taking  $\alpha = 0$ ,  $\beta = 1$ , the left handside of (29) is  $-1$ , while the right handside is 0. Indeed, in this case  $D_-g = -\infty$  on the Cantor set. So Lemma 5.2, Chapter 5, of [21] is not correct. Indeed the condition required therein is not sufficient to apply Fatou's Lemma: it is assumed that only the limsup of difference quotients is estimated from above by an integrable function while all difference quotients (for  $h$  sufficiently small) should be also estimated from above by the same integrable function (and this is not true in the case of our counterexample).

(c) Following item (b) above, one could substitute the assumption (28) with the following: there exists  $\rho \in L^1(0, +\infty; \mathbb{R})$  such that, for some  $h_0 > 0$ , we have  $\frac{g(t+h)-g(t)}{h} \geq \rho(t)$ , for  $-h_0 < h \leq 0$ , for a.e.  $t > 0$  (see e.g. Lemma 2.3 of [19]). However this assumption is more difficult to check in our case than the one of our Lemma 3.3.

**Proof of Theorem 3.2.** Set  $X^*(s) := X(s; \eta, c^*(\cdot))$  for  $s \geq 0$  and notice that  $X^*$  is continuous as solution of (15). Since the feedback map  $C$  is continuous on  $\mathcal{D}^\circ$ , we see from (24) that the control  $c^*(\cdot)$  admits a continuous version. We will refer to this continuous version.

Since  $V$  is concave, (17) holds. Therefore, for every  $s > 0$  there exists  $p_1(s) \in L_T^2$  such that

$$(V_{\eta_0}(X^*(s)), p_1(s)) \in D^+V(X^*(s)).$$

Let

$$\varphi(\zeta) := V(X^*(s)) + \langle (V_{\eta_0}(X^*(s)), p_1(s)), \zeta - X^*(s) \rangle, \quad \zeta \in H,$$

so that

$$\varphi(X^*(s)) = V(X^*(s)), \quad \varphi(\zeta) \geq V(\zeta), \quad \zeta \in H.$$

By (16) we have  $\varphi \in \tau$ . As we have noticed, the control  $c^*(\cdot)$  is continuous. Therefore, by (23)

$$\begin{aligned} \liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)}V(X^*(s+h)) - e^{-\rho s}V(X^*(s))}{h} &\geq \liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)}\varphi(X^*(s+h)) - e^{-\rho s}\varphi(X^*(s))}{h} \\ &= e^{-\rho s} \left[ \mathcal{L}^{c^*(s)}\varphi \right] (X^*(s)) = e^{-\rho s} \left[ -\rho V(X^*(s)) + \langle X(s), A^*(V_{\eta_0}(X^*(s)), p_1(s)) \rangle \right. \\ &\quad \left. + f(X^*(s))V_{\eta_0}(X^*(s)) - c^*(s)V_{\eta_0}(X^*(s)) \right], \quad \forall s > 0. \end{aligned}$$

Due to the definition of  $c^*(\cdot)$  we get

$$\begin{aligned} \liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)}V(X^*(s+h)) - e^{-\rho s}V(X^*(s))}{h} &+ e^{-\rho s}[U_1(c^*(s)) + U_2(X_0^*(s))] \\ &\geq e^{-\rho s} \left[ -\rho V(X^*(s)) + \langle X^*(s), A^*(V_{\eta_0}(X^*(s)), p_1(s)) \rangle \right. \\ &\quad \left. + f(X^*(s))V_{\eta_0}(X^*(s)) + \mathcal{H}(X^*(s)) + U_2(X_0^*(s)) \right], \quad \forall s > 0. \end{aligned}$$

Due to the subsolution property of  $V$  we get

$$\liminf_{h \uparrow 0} \frac{e^{-\rho(s+h)}V(X^*(s+h)) - e^{-\rho s}V(X^*(s))}{h} + e^{-\rho s}(U_1(c^*(s)) + U_2(X_0^*(s))) \geq 0, \quad \forall s > 0.$$

The function  $s \mapsto e^{-\rho s}V(X^*(s))$  and the function  $s \mapsto e^{-\rho s}(U_1(c^*(s)) + U_2(X_0^*(s)))$  are continuous. Therefore we can apply Lemma 3.3 (in the version of Remark 3.4-(a)) on  $[0, M]$ ,  $M > 0$ , getting

$$e^{-\rho M}V(X^*(M)) + \int_0^M e^{-\rho s}(U_1(c^*(s)) + U_2(X_0^*(s)))ds \geq V(\eta). \quad (30)$$

Now we want to take the limsup for  $M \rightarrow +\infty$  in (30). Since  $U_1$  is nonnegative we have by monotone convergence

$$\limsup_{M \rightarrow +\infty} \int_0^M e^{-\rho s}U_1(c^*(s))ds = \lim_{M \rightarrow +\infty} \int_0^M e^{-\rho s}U_1(c^*(s))ds = \int_0^{+\infty} e^{-\rho s}U_1(c^*(s))ds. \quad (31)$$

Moreover, the functions  $(f_M)_{M>0}$  defined as

$$f_M : [0, +\infty) \rightarrow \mathbb{R}, \quad s \mapsto \mathbf{1}_{[0, M]}e^{-\rho s}U_2(X_0^*(s)),$$

are dominated from above by the function  $[0, +\infty) \rightarrow \mathbb{R}, \quad s \mapsto e^{-\rho s}U_2^+(X_0^*(s))$ . The latter function is integrable due to (10), (70), Lemma A.3 and Proposition 2.9. Therefore Fatou's Lemma yields

$$\limsup_{M \rightarrow +\infty} \int_0^M e^{-\rho s}U_2(X_0^*(s))ds \leq \int_0^{+\infty} e^{-\rho s}U_2(X_0^*(s))ds. \quad (32)$$

Furthermore, again from estimate (70), Lemma A.3 and Proposition 2.9, we get

$$\|X^*(t)\|^2 \leq \|(x(t; \eta, 0), x(t + \cdot; \eta, 0))|_{[-T, 0]}\|^2 \leq (1 + T)(a_0 + a_1\|\eta\|)^2 e^{2K_0 t}, \quad \forall t \geq 0. \quad (33)$$

Therefore (13) and (33) yield

$$V(X^*(t)) \leq b_0 + b_1(1 + T)^{1/2}(a_0 + a_1\|\eta\|)e^{K_0 t}.$$

Since (11) holds, we have

$$\limsup_{M \rightarrow +\infty} [e^{-\rho M}V(X^*(M))] = 0. \quad (34)$$

So, taking the limsup for  $M \rightarrow +\infty$  in (30) and considering (31), (32) and (34), we get

$$J(\eta; c^*(\cdot)) = \int_0^{+\infty} e^{-\rho s}(U_1(c^*(s)) + U_2(X_0^*(s)))ds \geq V(\eta), \quad (35)$$

which gives the claim. □

**Remark 3.5.**

- (i) Observe that no continuity or measurability property of  $p_1(s)$  with respect to  $s$  is needed in the proof of the theorem above.

- (ii) *The Verification Theorem 3.2 gives as consequence a “half” of the Comparison Theorem for viscosity solutions of the HJB equation (20). Indeed, suppose that in the definition of the feedback map (22) and in the proof of Theorem 3.2 we replace the value function  $V$  with another viscosity solution  $v$  of the HJB equation (20). Assume that such  $v$  is concave, strictly increasing with respect to the first variable and with at most linear growth from above at infinity. Then the feedback map is well defined due to Theorem 2.13. Therefore, arguing as in the proof of Theorem 3.2, we would obtain (35) with  $v$  in place of  $V$  and the inequality  $V \geq v$  follows immediately. So, every viscosity solution  $v$  to the HJB equation (20) verifying the properties above (concavity, strict monotonicity with respect to the first variable and linear growth from above) is such that  $v \leq V$ .*
- (iii) *In Theorem 3.2 we provide a sufficient condition for optimality. Indeed, we have proved that if the feedback map defines an admissible control then such a control is optimal. A natural question arises whether, at least with a special choice of data, such a condition is also necessary for the optimality, i.e. if, given any optimal control, it is always possible to write it in feedback form. At this stage, i.e. if we do not know whether the closed loop equation admits a strictly positive solution or not, from the viscosity point of view the answer to this question relies in requiring that the value function is a bilateral viscosity subsolution of (20) along the optimal state trajectory, i.e. requiring that the value function satisfies the property of Definition 2.12-(i) also with the reversed inequality along this trajectory.*

*Such a property of the value function is related to the so-called backward dynamic programming principle which is, in turn, related to the backward study of the state equation (see [5], Chapter III, Section 2.3). Differently from the finite-dimensional case, this topic is not standard in infinite-dimension unless the operator  $A$  is the generator of a strongly continuous group, which is not our case.*

*However, in our case we could use the original setting of the state equation with delays to approach this problem. Then the problem reduces to finding, at least for sufficiently regular data, a backward continuation of the solution. This problem is studied, e.g., in [15], Chapter 2, Section 5. Unfortunately, our equation does not satisfy the main assumption required in [15], which in our setting basically corresponds to the requirement that the function  $a(\cdot)$ , when considered as a measure, has an atom at  $-T$ . Investigation of this issue is left for future research.*

## 4 Optimal feedbacks

In this section we use Theorem 3.2 to study the existence (and uniqueness) of optimal feedbacks for our problem. The key point is to study the existence and uniqueness of the *closed-loop delay state equation* associated to the map (22) in order to provide a control satisfying the assumptions of Theorem 3.2. Given  $\eta \in \mathcal{D}^\circ$ , the closed loop delay state equation takes the form

$$\begin{cases} x'(t) = rx(t) + f_0 \left( x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi \right) - C \left( (x(t), x(t+\cdot))|_{[-T,0]} \right), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0). \end{cases} \quad (36)$$



#### 4.1 Maximal solutions of the closed loop equations and locally optimal feedbacks

We recall (see [14]) that the Dynamic Programming Principle for our problem states that, for every  $\eta \in \mathcal{D}(V)$ ,

$$V(\eta) = \sup_{c(\cdot) \in \bar{\mathcal{C}}(\eta)} \left[ \int_0^\tau e^{-\rho t} (U_1(c(t)) + U_2(X_0(t; \eta, c(\cdot)))) dt + e^{-\rho \tau} V(X(\tau; \eta, c(\cdot))) \right], \quad \forall \tau > 0. \quad (37)$$

In particular, for every  $\eta \in \mathcal{D}(V)$ ,  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ ,  $\tau > 0$

$$V(\eta) \geq \int_0^\tau e^{-\rho t} (U_1(c(t)) + U_2(X_0(t; \eta, (\cdot)))) dt + e^{-\rho \tau} V(X(\tau; \eta, c(\cdot))). \quad (38)$$

Given  $\tau > 0$ ,  $\eta \in \mathcal{D}(V)$ , we define the convex set  $\bar{\mathcal{C}}_\tau(\eta)$  as the set of restrictions of the elements of  $\bar{\mathcal{C}}(\eta)$  to the interval  $[0, \tau]$ , i.e.

$$\bar{\mathcal{C}}_\tau(\eta) := \{c(\cdot)|_{[0, \tau]} \mid c(\cdot) \in \bar{\mathcal{C}}(\eta)\}. \quad (39)$$

Also, given  $\tau > 0$ ,  $\eta \in \mathcal{D}(V)$ , we consider the following functional on  $\bar{\mathcal{C}}_\tau(\eta)$ :

$$J_\tau(\eta; c_\tau(\cdot)) := \int_0^\tau e^{-\rho t} (U_1(c_\tau(t)) + U_2(X_0(t; \eta, c(\cdot)))) dt + e^{-\rho \tau} V(X(\tau; \eta, c_\tau(\cdot))). \quad (40)$$

**Definition 4.1.** Let  $\eta \in \mathcal{D}(V)$ ,  $\tau > 0$ . We say that a control  $c_\tau(\cdot) \in \bar{\mathcal{C}}_\tau(\eta)$  is a  $\tau$ -locally optimal control for  $\eta$  if it maximizes (40).

**Remark 4.2.**

- (i) Given an optimal control  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$  for  $\eta \in \mathcal{D}(V)$ , by Dynamic Programming Principle its restriction  $c_\tau(\cdot) := c(\cdot)|_{[0, \tau]}$  to  $[0, \tau]$  is  $\tau$ -locally optimal for the same  $\eta$  for every  $\tau > 0$ .
- (ii) The Dynamic Programming Principle shows that

$$J_\tau(\eta; c_\tau(\cdot)) \leq V(\eta), \quad \forall \eta \in \mathcal{D}(V), \quad c_\tau(\cdot) \in \bar{\mathcal{C}}_\tau(\eta). \quad (41)$$

and that a control  $c_\tau(\cdot) \in \bar{\mathcal{C}}_\tau(\eta)$  is  $\tau$ -optimal if and only if it achieves equality in (38), i.e.

$$V(\eta) = \int_0^\tau e^{-\rho t} (U_1(c_\tau(t)) + U_2(X_0(t; \eta, c_\tau(\cdot)))) dt + e^{-\rho \tau} V(X(\tau; \eta, c_\tau(\cdot))). \quad (42)$$

**Proposition 4.3.** Let  $\eta \in \mathcal{D}(V)$  and  $\tau > 0$ . The functional (40) is strictly convex on  $\bar{\mathcal{C}}_\tau(\eta)$ . In particular there exists at most one maximizer of (40) over  $\bar{\mathcal{C}}_\tau(\eta)$ .

**Proof.** The claim follows from the concavity of data of the delay state equation (1), the concavity of  $U_2$  and  $V$  and the strict concavity of  $U_1$ .  $\square$

The existence of locally optimal controls in feedback form is related to the existence of local solutions to the closed loop delay equation (36). So we are going to study (36).

**Proposition 4.4.**

1. Let  $\eta \in \mathcal{D}^\circ$ . The closed loop delay state equation (36) admits a strictly positive local solution  $x^*(\cdot)$  defined on  $[0, \tau]$  for some  $\tau > 0$ .

2. Let  $\eta \in H_{++} \subset \mathcal{D}^\circ$ . The maximal interval of definition of a strictly positive local solution  $x^*(\cdot)$  of (36) is  $[0, \tau_{max})$ , where  $\tau_{max} > 0$  is implicitly defined<sup>4</sup> as

$$\tau_{max} := \inf\{t > 0 \mid \exists \lim_{s \rightarrow t^-} x^*(s) = 0\}. \quad (43)$$

**Proof.** 1. See Subsection A.4.

2. Let  $x^*(\cdot)$  be a strictly positive solution of equation (36) defined on an interval  $[0, \tau)$ ,  $\tau > 0$ . Define  $X^*(t) := (x^*(t), x^*(t + \cdot)|_{[-T, 0]})$  for  $t \in [0, \tau)$ , so that, as observed,  $X^*(t) \in \mathcal{D}^\circ$  for every  $t \in [0, \tau)$ . Since  $C(\zeta) \geq 0$  for every  $\zeta \in \mathcal{D}^\circ$ , we have  $x^*(\cdot) \leq x(\cdot; \eta, 0)$ . Therefore  $x^*(\cdot)$  is dominated from above on  $[0, \tau)$  by

$$\max_{t \in [0, \tau]} x(\cdot; \eta, 0).$$

On the other hand  $x^*(\cdot)$  is also dominated from below by 0 on  $[0, \tau)$ , since  $x^*(\cdot)$  is strictly positive on this interval. Then a standard argument of differential equations show that it must exist  $\lim_{t \rightarrow \tau^-} x^*(t) \in [0, +\infty)$  and that, if such limit is strictly positive, then the solution can be extended. So the claim is proved.  $\square$

Now we get the desired result about the existence of locally optimal control in feedback form.

**Theorem 4.5.** Let  $\tau > 0$ ,  $\eta \in H_{++}$  and let  $x^*(\cdot)$  be a strictly positive solution to the closed loop delay state equation (36) defined on some interval  $[0, \tau)$  (Proposition 4.4-(1)). Let  $c_\tau^*(\cdot)$  be the feedback control on  $[0, \tau)$  associated to  $x^*(\cdot)$  through (22), i.e.

$$c_\tau^*(t) := C((x^*(t), x^*(t + \cdot)|_{[-T, 0]})), \quad t \in [0, \tau). \quad (44)$$

Then  $c_\tau^*(\cdot) \in \bar{\mathcal{C}}_\tau(\eta)$  and it is the unique  $\tau$ -locally optimal control for  $\eta$ .

**Proof.** *Admissibility.* First of all note that, since  $\eta \in H_{++}$  and  $x^*(\cdot) > 0$  on  $[0, \tau)$ , we have  $(x^*(t), x^*(t + \cdot)|_{[-T, 0]}) \in \mathcal{D}^\circ$  for every  $t \in [0, \tau)$ . So (44) is well defined and  $c_\tau^*(t) \geq 0$  for every  $t \in [0, \tau)$ . Moreover, due to the continuity of  $x^*(\cdot)$  on  $[0, \tau)$  and of  $C$  on  $\mathcal{D}^\circ$ , we see that  $c_\tau^*(\cdot) \in L^1_{loc}([0, \tau); \mathbb{R}^+)$ .

Now we want to show that  $c_\tau^*(\cdot) \in L^1([0, \tau); \mathbb{R}^+)$ . By the uniqueness of solutions to the state equation (1),<sup>5</sup> we have the equality  $x(\cdot; \eta, c_\tau^*(\cdot)) = x^*(\cdot)$  on  $[0, \tau)$ . We also notice that it must be  $\tau \leq \tau_{max}$ , where  $\tau_{max}$  is the maximal time defined in Proposition 4.4-(2). Therefore, due to the characterization of  $\tau_{max}$  provided by the same proposition, we see that  $x^*(\cdot)$  can be extended to a continuous function on  $[0, \tau]$ . Therefore also  $x(\cdot; \eta, c_\tau^*(\cdot))$  can be extended to a continuous function on  $[0, \tau]$ . Hence, expressing  $c_\tau^*(\cdot)$  through the state equation, we see that  $c_\tau^*(\cdot)$  can be extended to a continuous function on  $[0, \tau]$  too. Therefore  $c_\tau^*(\cdot) \in L^1([0, \tau); \mathbb{R}^+)$ .

It remains to show that  $c_\tau^*(\cdot)$  defined in (44) can be extended to a control  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ . First, we note that, due to Hypothesis (2.2)-(ii) and properties of state equation (1) we have

$$\zeta \in \bar{H}_{++} \implies 0 \in \bar{\mathcal{C}}(\zeta). \quad (45)$$

Define

$$c(t) := \begin{cases} c_\tau^*(t), & \text{if } t \in [0, \tau), \\ 0, & \text{if } t \geq \tau. \end{cases} \quad (46)$$

<sup>4</sup>With the agreement  $\inf \emptyset = +\infty$

<sup>5</sup>Here we consider the state equation (1) as seen on the interval  $[0, \tau)$  for controls belonging to  $L^1_{loc}([0, \tau); \mathbb{R}^+)$ .

Note that  $c(\cdot) \in L^1([0, +\infty); \mathbb{R}^+)$ . By uniqueness of solutions to the state equation (1), we have  $x(\cdot; \eta, c(\cdot)) = x(\cdot; \eta, c_\tau^*(\cdot)) = x^*(\cdot) > 0$  on the interval  $[0, \tau)$ . Moreover, by continuity of  $x(\cdot; \eta, c_\tau^*(\cdot))$ , we have  $x(\tau; \eta, c_\tau^*(\cdot)) \geq 0$ . Thus

$$\zeta := (x(\tau; \eta, c_\tau^*(\cdot)), x^*(\tau + \cdot; \eta, c_\tau^*(\cdot))|_{[-T, 0]}) \in \bar{H}_{++}.$$

Hence, the flow property of solutions to the state equation and (45) yield  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ . By construction of  $c(\cdot)$  we have that  $c_\tau^*(\cdot)$  is the restriction over  $[0, \tau)$  of  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$  and this claim is proved.

*Optimality.* Set  $X^*(\cdot) := X(\cdot; \eta, c^*(\cdot))$ , where  $c^*(\cdot)$  is provided by (46). By Proposition 2.9 we get  $X_0^*(\cdot) = x^*(\cdot)$ . Since  $\eta \in H_{++}$  and  $x^*(\cdot) > 0$  on  $[0, \tau)$ , we see that  $X^*(t) \in \mathcal{D}^\circ$  for every  $t \in [0, \tau)$ . Moreover, since  $x^*(\cdot)$  solves the closed loop equation on  $[0, \tau)$ , we see that the pair  $(c^*(\cdot), X^*(\cdot))$  satisfies (24) for every  $t \in [0, \tau)$ . Therefore we may argue exactly as in the proof of Theorem 3.2 to get (30) with  $\tau$  in place of  $M$  that is

$$V(\eta) \leq \int_0^\tau e^{-\rho t} (U_1(c^*(t)) + U_2(X_0^*(t))) dt + e^{-\rho \tau} V(X^*(\tau)) = J_\tau(\eta, c_\tau^*(\eta)).$$

By Remark 4.2 we have the claim.

*Uniqueness.* It has already been proved in Proposition 4.3.  $\square$

As a consequence of Proposition 4.3 and Theorem 4.5 we also obtain the uniqueness of solutions to the closed loop delay state equation (36).

**Corollary 4.6.** *Let  $\eta \in H_{++}$ . The closed loop delay state equation (36) admits a unique strictly positive  $x^*(\cdot)$  solution defined on its maximal interval of definition  $[0, \tau_{max})$  provided by Proposition 4.4.*

**Proof.** The local existence has been already proved in Proposition 4.4. Let us suppose that  $x_1^*(\cdot), x_2^*(\cdot)$  are two strictly positive solutions to (36) defined in  $[0, \tau)$  for some  $\tau > 0$ . Due to Theorem 4.5, both of them would give rise to  $\tau$ -local optimal controls  $c_{1,\tau}^*(\cdot), c_{2,\tau}^*(\cdot)$ . By Proposition 4.3, we must have  $c_{1,\tau}^*(\cdot) = c_{2,\tau}^*(\cdot)$  on  $[0, \tau)$ . Therefore

$$x_1^*(t) = x(t; \eta, c_{1,\tau}^*(\cdot)) = x(t; \eta, c_{2,\tau}^*(\cdot)) = x_2^*(t), \quad \forall t \in [0, \tau).$$

This shows that the maximal interval of definition of  $x_1^*(\cdot)$  and  $x_2^*(\cdot)$  is the same and that they coincide on this interval, that is the claim.  $\square$

## 4.2 Optimal feedbacks when $U_2$ is not integrable at $0^+$

Up to now we did not make any further assumption on the functions  $a$  and  $U_2$  except Hypotheses 2.2 and 2.4. In particular,  $U_2 \equiv 0$  is allowed. However, without any further assumption we have no information on the behaviour of  $V_{\eta_0}$  when we approach the boundary of  $\mathcal{D}(V)$  and therefore we are not able to say anything about the existence of strictly positive global solutions of the closed loop equation. So basically we cannot say whether the hypothesis of Theorem 3.2 is satisfied or not. In order to give sufficient conditions for that, we need additional assumptions. We will use the following assumptions

$$\begin{cases} \text{(i)} & U_2 \text{ is not integrable at } 0^+, \\ \text{(ii)} & \int_{-\varepsilon}^0 a(\xi) d\xi > 0, \quad \forall \varepsilon > 0, \\ \text{(iii)} & f_0(0, y) > 0, \quad \forall y > 0. \end{cases} \quad (47)$$

**Remark 4.7.** We note that it is possible that a function  $U_2$  satisfies Hypothesis 2.4-(i) and is not integrable at  $0^+$ . Indeed the functions  $U_2(x) = -x^{-1}$  is not integrable and satisfies Hypothesis 2.4-(ii) thanks to (11). We also note that, due to the continuity of  $f_0$ , assumption (47)-(iii) implies (3) and is indeed just a slightly stronger when compared to (3).

The following is the key result to prove the existence of a global strictly positive solution to the closed loop delay state equation (36) under (47).

**Lemma 4.8.** Assume that (47)-(i) holds. Then

$$\lim_{\zeta \rightarrow \eta} V_{\eta_0}(\zeta) = +\infty, \quad \forall \eta \in \mathcal{B}, \quad (48)$$

where the limit is taken with respect to  $\|\cdot\|$ .

**Proof.** See Subsection A.4. □

**Proposition 4.9.** Let  $\eta \in H_{++}$  and assume (47). Then the closed loop delay state equation (36) admits a unique strictly positive global solution  $x^*(\cdot)$ .

**Proof.** Since (47)-(i) holds, by Lemma 4.8 we can extend the map  $C$  to a continuous map  $\bar{C}$  defined on the whole space  $(H, \|\cdot\|)$  defining  $\bar{C} \equiv 0$  on the complement of  $\mathcal{D}^\circ$ .

Let  $\eta \in H_{++}$  and let  $x^*(\cdot)$  be the unique local solution to closed loop delay equation (36) defined on its maximal interval of definition  $[0, \tau_{max})$  provided by Corollary 4.6. We want to prove that  $\tau_{max} = +\infty$ . Assume by contradiction that  $\tau_{max} < +\infty$ . This means that  $\lim_{t \rightarrow \tau_{max}^-} x^*(t) = 0$ . Since  $\bar{C}$  is defined on the whole space  $H$  and continuous, it is possible to extend  $x^*(\cdot)$  to a solution  $\bar{x}^*(\cdot)$  of this extended closed loop equation (with  $\bar{C}$  in place of  $C$ ) also on  $[\tau_{max}, \tau_{max} + \varepsilon)$  for some  $\varepsilon > 0$ . Since  $f_0$  and  $\bar{C}$  are continuous we have  $\bar{x}^*(\cdot) \in C^1([0, \tau_{max} + \varepsilon); \mathbb{R})$ . Let us see what happens in a neighborhood of  $\tau_{max}$ :

- (a) by definition of  $\tau_{max}$  we must have  $\bar{x}^*(\cdot) = x^*(\cdot) > 0$  in a left neighborhood of  $\tau_{max}$ ;
- (b)  $\bar{x}^*(\tau_{max}) = 0$ , therefore  $\bar{C}((\bar{x}^*(\tau_{max}), \bar{x}^*(\tau_{max} + \cdot)|_{[-T, 0]})) = 0$ ; thus, due to item (a) above and to (47)-(ii)&(iii), we must have  $\frac{d}{dt} \bar{x}^*(\tau_{max}) > 0$ .

Thus we see that the conclusion of (b) contradicts (a). Therefore  $\tau_{max} = +\infty$  and the claim is proved. □

As a corollary of Proposition 4.9, we get the existence of a unique optimal control under the assumption (47) for initial data in  $H_{++}$ . Moreover this control is given in feedback form. This is stated in the following result.

**Theorem 4.10.** Let (47) hold and  $\eta \in H_{++}$ . Let  $x^*(\cdot)$  be the unique strictly positive global solution to the closed loop delay state equation (36) provided by Proposition 4.4. Let  $c^*(\cdot)$  be the feedback control on  $[0, +\infty)$  associated to  $x^*(\cdot)$  defined by (24), that is

$$c^*(t) := C((x^*(t), x^*(t + \cdot)|_{[-T, 0]})), \quad t \geq 0. \quad (49)$$

Then  $c^*(\cdot)$  is admissible and it is the unique optimal control for  $\eta$ .

**Proof.** Proposition 4.9 and Theorem 3.2 show that the control (49) is admissible and optimal. Remark 4.2-(i) and Proposition 4.3 yield the uniqueness. □

**Remark 4.11.** When (47) holds and  $\eta \in H_{++}$ , then Theorem 4.10 provides a positive answer to the question of Remark 3.5-(iii).

### 4.3 Optimal controls in the case $U_1(c) = c^{1-\sigma}$ , $U_2 \equiv 0$

In Subsection 4.2 we have introduced the assumption of no integrability (47)-(i) of the utility function  $U_2$ . This has been necessary in order to ensure the existence of global strictly positive solutions to the closed loop equation (36). Here we use the results of Subsection 4.2 to prove, by means of an appropriate procedure of approximation, the existence of optimal control in the case

$$U_1(c) = c^{1-\sigma}, \quad \sigma \in (0, 1); \quad U_2 \equiv 0. \quad (50)$$

We also introduce the assumption

$$(x, y) \mapsto rx + f_0(x, y) \quad \text{nondecreasing in both the variables.} \quad (51)$$

The condition above is used to prove Proposition A.13, which is essential to make our approximation procedure successful. We assume it as it is coherent with the application we have in mind.

#### 4.3.1 Approximating the utility on the state

Hereafter  $\mathbb{N}^*$  denotes the set of strictly positive natural numbers that is  $\mathbb{N}^* = \{1, 2, \dots\}$ . Let us consider a sequence of functions  $(U_2^n)_{n \in \mathbb{N}^*}$  satisfying Hypothesis 2.4-(ii) and such that

$$U_2^n \uparrow 0, \quad U_2^n \text{ not integrable at } 0^+, \quad U_2^n \equiv 0 \text{ on } [1/n, +\infty). \quad (52)$$

We note that it is possible to construct such a sequence. Indeed, for any  $n \in \mathbb{N}^*$  we may find  $\varepsilon_n \in (0, 1/n)$  such that the affine function  $l_n : x \mapsto \varepsilon_n^{-2}(x - \varepsilon_n) - \varepsilon_n^{-1}$  is such that  $l_n(1/n) = 0$ . Then the functions

$$U_2^n := \begin{cases} -x^{-1}, & \text{if } x \in (0, \varepsilon_n), \\ l_n(x), & \text{if } x \in [\varepsilon_n, 1/n), \\ 0, & \text{if } x \geq 1/n, \end{cases}$$

satisfy Hypothesis 2.4-(ii) and (52).

Let us denote by  $J_n$  and  $V_n$  respectively the objective functionals and the value functions of the problems with utilities  $U_1$  and  $U_2^n$ . Let us also denote by  $J_0$  and  $V_0$  respectively the objective functionals and the value functions of the problems with utilities  $U_1$  and  $U_2 \equiv 0$ . Finally, let us denote by  $\mathcal{D}_n^\circ$  and  $\mathcal{D}_0^\circ$  the interiors in the space  $(H, \|\cdot\|_{-1})$  of  $\mathcal{D}(V_n)$  and  $\mathcal{D}(V_0)$  respectively. By Proposition 2.10-(1) we have  $H_{++} \subset \mathcal{D}_n^\circ$  for every  $n \in \mathbb{N}^*$ . We observe that, since the state equation does not change, the set of the admissible controls  $\bar{\mathcal{C}}(\eta)$  does not depend on  $n \in \mathbb{N}^*$ . Since the sequence  $(U_2^n)_{n \in \mathbb{N}^*}$  is nondecreasing, the sequence  $(V_n)_{n \in \mathbb{N}^*}$  is nondecreasing as well. Therefore  $H_{++} \subset \mathcal{D}_1^\circ \subset \mathcal{D}_2^\circ \subset \dots \subset \mathcal{D}_0^\circ$  and

$$\exists \lim_{n \rightarrow +\infty} V_n(\eta) := g(\eta) \leq V_0(\eta), \quad \forall \eta \in \bar{H}_+. \quad (53)$$

Due to Theorem 4.10, for any  $\eta \in H_{++}$  and  $n \in \mathbb{N}^*$  there exists a unique optimal control  $c_n^*(\cdot) \in \bar{\mathcal{C}}(\eta)$  (in feedback form) for the problem  $V_n$ .

**Definition 4.12** ( $\varepsilon$ -optimal control). *Let  $\eta \in \mathcal{D}(V_0)$  and  $\varepsilon > 0$ . A control  $c_\varepsilon(\cdot) \in \bar{\mathcal{C}}(\eta)$  is said to be  $\varepsilon$ -optimal for  $V_0(\eta)$  if  $J_0(\eta; c_\varepsilon(\cdot)) > V_0(\eta) - \varepsilon$ .*

**Proposition 4.13.** *Let (47)-(ii, iii) and (51) hold. Let  $\eta \in H_{++}$  and let  $(c_n^*(\cdot))_{n \in \mathbb{N}^*} \subset \bar{\mathcal{C}}(\eta)$  be the sequence of optimal controls for the problems  $V_n$  with initial datum  $\eta$  provided by Theorem 4.10. Then for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}^*$  such that  $c_n^*(\cdot)$  is  $\varepsilon$ -optimal for  $V_0(\eta)$  for every  $n \geq n_0$ . Moreover*

$$V_n(\eta) \xrightarrow{n \rightarrow \infty} V_0(\eta). \quad (54)$$

**Proof.** By Proposition A.13, for every  $\varepsilon > 0$  we can find  $n_\varepsilon > 0$  and an  $\varepsilon$ -optimal control  $c_\varepsilon(\cdot) \in \bar{\mathcal{C}}(\eta)$  for  $V_0(\eta)$  such that

$$1/n_\varepsilon \leq \inf_{t \in [0, +\infty)} x(t; \eta; c_\varepsilon(\cdot)).$$

Since  $U_2^{n_\varepsilon} \equiv 0$  on  $[1/n_\varepsilon, +\infty)$ , we have

$$\begin{aligned} V_0(\eta) - \varepsilon &\leq J_0(\eta; c_\varepsilon(\cdot)) = \int_0^{+\infty} e^{-\rho t} U_1(c_\varepsilon(t)) dt \\ &= \int_0^{+\infty} e^{-\rho t} (U_1(c_\varepsilon(t)) + U_2^{n_\varepsilon}(x(t; \eta, c_\varepsilon(\cdot)))) dt = J_{n_\varepsilon}(\eta; c_\varepsilon(\cdot)) \leq V_{n_\varepsilon}(\eta). \end{aligned}$$

Since the sequence  $(V_n)_{n \in \mathbb{N}}$  is nondecreasing, from the latter inequality we get

$$V_0(\eta) - \varepsilon \leq V_n(\eta) = J_n(\eta, c_n^*(\cdot)) \leq J_0(\eta, c_n^*(\cdot)), \quad \forall n \geq n_\varepsilon.$$

Taking also into account (53), we get both the claims.  $\square$

### 4.3.2 Existence and uniqueness of optimal controls

**Proposition 4.14.** *Let (47)-(ii, iii), (50) and (51) hold true. For each  $\eta \in H_{++}$ , there exists a unique optimal control  $c^*(\cdot) \in \bar{\mathcal{C}}(\eta)$  starting from  $\eta$ .*

**Proof.** The uniqueness is a consequence of Remark 4.2-(i) and Proposition 4.3. We prove the existence. Take a sequence of functions  $(U_2^n)_{n \in \mathbb{N}}$  satisfying Hypothesis 2.4-(ii) and (52) and let  $(c_n^*(\cdot))_{n \in \mathbb{N}} \subset \bar{\mathcal{C}}(\eta)$  be the sequence of controls considered in Proposition 4.13. Set

$$g_n^*(t) := c_n^*(t)^{1-\sigma} \geq 0, \quad t \geq 0.$$

Then, by (72) (note that this estimate does not depend on the utilities, but only on the state equation, so it is independent of  $n$ ), we have

$$\int_0^{+\infty} e^{-\rho t} g_n^*(t)^{1/1-\sigma} dt \leq c_0 + c_1 \|\eta\|.$$

This means that the sequence  $(g_n^*)_{n \in \mathbb{N}}$  is bounded in the space

$$L_\rho^{1/1-\sigma} := L^{1/1-\sigma}([0, +\infty), e^{-\rho t} dt; \mathbb{R}).$$

Therefore we may find a subsequence weakly converging in  $L_\rho^{1/1-\sigma}$  to some nonnegative function  $g^* \in L_\rho^{1/1-\sigma}$ . Without loss of generality we assume that such convergence is true for the original sequence. Define

$$c^*(t) := g^*(t)^{1/1-\sigma} \geq 0, \quad t \geq 0. \quad (55)$$

We claim that  $c^*(\cdot)$  is admissible and optimal.

*Admissibility.* By (55) we have  $c^*(\cdot) \geq 0$ . Moreover, since  $g^* \in L_\rho^{1/1-\sigma}$ , we see that  $c^*(\cdot) \in L_{loc}^1([0, +\infty), \mathbb{R}^+)$ . Now consider  $x^*(\cdot) := x(\cdot; \eta, c^*(\cdot))$ . We have to prove that  $x^*(\cdot) \geq 0$ . Set  $x_n^*(\cdot) := x(\cdot; \eta, c_n^*(\cdot))$ . We have

$$\begin{aligned} x_n^*(t) &= \eta_0 + \int_0^t r x_n^*(s) ds + \int_0^t f_0 \left( x_n^*(s), \int_{-T}^0 a(\xi) x_n^*(s + \xi) d\xi \right) ds - \int_0^t c_n^*(s) ds, \\ &= \eta_0 + \int_0^t r x_n^*(s) ds + \int_0^t f_0 \left( x_n^*(s), \int_{-T}^0 a(\xi) x_n^*(s + \xi) d\xi \right) ds - \int_0^t g_n^*(s)^{1/1-\sigma} ds, \end{aligned} \quad (56)$$

Note that the functional

$$L_\rho^{1/1-\sigma} \rightarrow \mathbb{R}, \quad g \mapsto \int_0^t |g(s)|^{1/1-\sigma} ds,$$

is convex and locally bounded from below. Therefore it is continuous, hence lower weakly semicontinuous. Since  $g_n \rightharpoonup g^*$  in  $L_\rho^{1/1-\sigma}$  and  $g_n, g^* \geq 0$ , we have

$$\int_0^t g^*(s)^{1/1-\sigma} ds \leq \liminf_{n \rightarrow \infty} \int_0^t g_n(s)^{1/1-\sigma} ds, \quad \forall t \geq 0. \quad (57)$$

Call  $\alpha^*(t) := \limsup_{n \rightarrow \infty} x_n^*(t) \geq 0$ ,  $t \geq 0$ . Due to (51)

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left( r x_n(s) + f_0 \left( x_n(s), \int_{-T}^0 a(\xi) x_n(s + \xi) d\xi \right) \right) \\ = r \alpha(s) + f_0 \left( \alpha(s), \int_{-T}^0 a(\xi) \alpha(s + \xi) d\xi \right), \quad \forall s \geq 0. \end{aligned} \quad (58)$$

Now, take the limsup in (56). Invoking the Fatou Lemma, (55), (57) and (58), we obtain

$$\begin{aligned} \alpha^*(t) &\leq \eta_0 + \int_0^t r \alpha^*(s) ds + \int_0^t f_0 \left( \alpha^*(s), \int_{-T}^0 a(\xi) \alpha^*(s + \xi) d\xi \right) ds - \left( \int_0^t g^*(s)^{1/1-\sigma} ds \right) \\ &= \eta_0 + \int_0^t r \alpha^*(s) ds + \int_0^t f_0 \left( \alpha^*(s), \int_{-T}^0 a(\xi) \alpha^*(s + \xi) d\xi \right) ds - \left( \int_0^t c^*(s) ds \right). \end{aligned}$$

From the above inequality and Lemma A.3 we deduce  $0 \leq \alpha(t) \leq x^*(t)$  for every  $t \geq 0$ , so  $c^*(\cdot) \in \bar{\mathcal{C}}(\eta)$ .

*Optimality.* First of all note that, due to the optimality of  $c_n^*(\cdot)$  for  $V_n$  and since  $(U_2^n)$  is non-decreasing we have

$$V_n(\eta) = J_n(\eta; c_n^*(\cdot)) \leq J_0(\eta; c^*(\cdot)). \quad (59)$$

The function  $h \equiv 1$  belongs to  $L_\rho^{1/\sigma} = (L_\rho^{1/1-\sigma})^*$ . Therefore, since  $g_n^* \rightharpoonup g^*$  in  $L_\rho^{1/1-\sigma}$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} J_0(\eta; c_n^*(\cdot)) &= \lim_{n \rightarrow \infty} \int_0^{+\infty} e^{-\rho t} c_n^*(t)^{1-\sigma} dt \\ &= \lim_{n \rightarrow \infty} \int_0^{+\infty} e^{-\rho t} g_n^*(t) dt = \int_0^{+\infty} e^{-\rho t} g^*(t) dt = J_0(\eta; c^*(\cdot)). \end{aligned} \quad (60)$$

From (54), (59) and (60), we obtain  $V_0(\eta) \leq J_0(\eta; c^*(\cdot))$ , the optimality of  $c^*(\cdot)$ .  $\square$

### 4.3.3 Characterization of the optimal control as feedback in the interior region

In Subsection 4.3.2 we have proved the existence and uniqueness of optimal controls for the problem with utilities (50) and initial data  $\eta \in H_{++}$ . Now we want to exploit the results of Subsection 4.1 to characterize in this case the optimal control as feedback when the current state belongs to  $\mathcal{D}^\circ$ . To this end we note that we are not able to say if the optimal state  $X^*(t) = (x^*(t), x^*(t + \cdot)|_{[-T, 0]})$  reaches the boundary  $\mathcal{B}$  (which means, in this case of initial datum  $\eta \in H_{++}$ , that  $x^*(t) = 0$  for some  $t > 0$ ). Nevertheless, we are able to characterize the optimal control as feedback when  $X^*(t) \in \mathcal{D}^\circ$ , or equivalently, due to the fact that  $\eta \in H_{++}$ , when  $x^*(t) > 0$ .

**Theorem 4.15.** *Let (47)-(ii, iii), (50) and (51) hold true. Let  $\eta \in H_{++}$  and let  $c^*(\cdot) \in \bar{\mathcal{C}}(\eta)$  be the optimal control provided by Proposition 4.14. Let  $X^*(\cdot) := X(\cdot; \eta, c^*(\cdot))$  be the associated optimal state. Then*

$$c^*(t) = C(X^*(t)), \quad \text{for a.e. } t \geq 0 \text{ such that } X^*(t) \in \mathcal{D}^\circ, \quad (61)$$

where  $C$  is the feedback map defined in (22).

**Proof.** Let  $\eta \in H_{++}$  and let  $c^*(\cdot) \in \bar{\mathcal{C}}(\eta)$  be the optimal control for  $\eta$ . Set  $X^*(\cdot) := X(\cdot; \eta, c^*(\cdot))$  and  $x^*(\cdot) := x^*(\cdot; \eta, c^*(\cdot))$ . Recall that, by Proposition 2.9, we have

$$X^*(t) = (x^*(t), x^*(t + \cdot)|_{[-T, 0]}), \quad \forall t \geq 0. \quad (62)$$

Note that, since  $\eta \in H_{++} \subset \mathcal{D}^\circ$ , we have

$$\mathcal{I} := \{t \geq 0 \mid X^*(t) \in \mathcal{D}^\circ\} = \{t \geq 0 \mid x^*(t) > 0\}.$$

Since  $x^*(\cdot)$  is continuous, the set  $\mathcal{I}$  is open in  $[0, +\infty)$ . Thus, it may be written as

$$\mathcal{I} = [a_0, b_0) \cup \bigcup_{n \geq 1} (a_n, b_n),$$

where  $a_0 = 0$  and  $a_n < b_n < a_{n+1}$  for each  $n \geq 0$ . Take  $s \in (a_n, b_n)$  for some  $n \geq 0$  and set  $\zeta := X^*(s)$ . Then we have  $\zeta \in H_{++}$ . The semigroup property of the solution  $X^*$  gives

$$X^*(t + s) = X(t; \zeta, c^*(\cdot + s)), \quad \forall t \geq 0. \quad (63)$$

As a straightforward consequence of the Dynamic Programming Principle we have the optimality of the control  $c^*(\cdot + s)$  for the initial datum  $\zeta$ . Again an application of the Dynamic Programming Principle yields the equality

$$V_0(\zeta) = \int_0^{b_n - s} e^{-\rho t} c^*(t + s)^{1 - \sigma} dt + e^{-\rho(b_n - s)} V_0(X(b_n - s; \zeta, c^*(\cdot + s))).$$

Now Theorem 4.5 (with  $\tau = b_n - s$  and  $c_\tau^*(\cdot) = c^*(\cdot + s)$ ), Remark 4.2-(ii) and (63) yield

$$c^*(t + s) = C(X(t; \zeta, c^*(\cdot + s))) = C(X^*(t + s)), \quad \text{for a.e. } t \in [0, b_n - s).$$

By the arbitrariness of  $n \geq 0$  and  $s \in (a_n, b_n)$ , we get the claim.  $\square$

**Remark 4.16.** *The main results of Sections 3 and 4 still hold without requiring the global Lipschitz continuity of  $f_0$  if we take as given the results of [14].*



- (a) In Section 3 we only use the concavity of  $V$  (which is ensured by the concavity of  $f_0$ ) to get the Verification Theorem 3.2.
- (b) In Subsection 4.1 the results on the closed loop equation only use the continuity of  $f_0$  to prove the local existence (Proposition 4.4-(1)); the monotonicity in  $y$  and the sublinear growth for  $x, y \rightarrow +\infty$  of  $f_0(x, y)$  (the latter being consequence of the concavity) to characterize the maximal interval of definition of the solution (Proposition 4.4-(2)). With regard to the latter point, we have to observe that the sublinear growth for  $x, y \rightarrow +\infty$  is needed to get the estimate (70) for initial data in  $\bar{H}_{++}$ .
- (c) The proof of existence of a global solution to the closed loop equation in Subsection 4.2 uses the global Lipschitz continuity of  $f_0$  through the proof of Lemma 4.8. However the global Lipschitz continuity in the proof is not necessary if we restrict the proof to data in  $\bar{H}_{++}$  and can be adapted by the same ingredients (monotonicity in  $y$ , sublinear growth for  $x, y \rightarrow +\infty$ ).

## 5 Economic application and an example

In this section we discuss how to apply our results to endogenous growth models with time-to-build studied in the economic literature. This kind of models are studied mainly in the case of pointwise delay (see [2, 3, 4]). Also the case of distributed delay is considered in the economic literature (see [17] for the discrete time case and [] for the continuous time case), but it is formulated in such a way that the delay appears in the control variable. Here we consider the model with pointwise delay introduced in [2] and its version with distributed delay, which is motivated as in [16, 17]<sup>6</sup>.

We start by recalling the model of [2]. Let  $x(t)$  denote the amount of capital per capita in the economy at time  $t \geq 0$  and let  $c(t)$  be the per capita consumption rate. Both are assumed to be positive for each  $t \geq 0$ . The state equation is

$$\begin{cases} x'(t) = f(x(t-T)) - \delta x(t-T) - c(t), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0), \end{cases} \quad (64)$$

where  $\delta \geq 0$  and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is the per capita production function that is continuous and such that  $\xi \mapsto f(\xi) - \delta\xi$  is strictly increasing and concave. The goal is to maximize the utility functional

$$\int_0^{+\infty} e^{-\rho t} U_1(c(t)) dt,$$

where  $U_1$  is as in Hypothesis 2.4-(i). This problem is carefully studied for linear  $f$  (the so called AK model, where  $f(x) = \alpha x$  with  $\alpha > \delta$ ) and  $U_1(c) = \frac{c^{1-\sigma}}{1-\sigma}$  in [3, 4]. The analogous model with distributed delay has the same objective functional and the following state equation

$$\begin{cases} x'(t) = f\left(\int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - \delta \int_{-T}^0 a(\xi)x(t+\xi)d\xi - c(t), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0), \end{cases}$$

Our goal is to apply, possibly under further assumptions, the results of our previous sections to these problems (with either distributed or pointwise delay) to extend the qualitative analysis

<sup>6</sup>One can see also the examples' section of [11].

performed in [3, 4] for the case of pointwise delay, linear  $f$  and power  $U_1$ . We observe that the theory that we have developed here covers the distributed delay case, but not the pointwise delay case<sup>7</sup>. So, first we show how to apply our results to the distributed delay case, and second we develop an approximation procedure that allows to get some results also for the pointwise delay case.

Concerning the distributed delay case we have the following results:

- (i) Theorem 2.13 holds and so the value function has continuous derivative along the direction  $\eta_0$  (the “present”).
- (ii) Our Verification Theorem 3.2 holds: an admissible state-control pair satisfying the feedback relation (24) is optimal.
- (iii) Proposition 4.14 holds. So, under the assumptions required in that proposition, there exists a unique optimal control. Differently from the linear case we cannot say that, for a suitable set of initial data, the closed loop equation has a unique strictly positive solution. This remains an open problem. However, due to Theorem 4.15, we can say that, when the initial datum is in  $H_{++}$ , the optimal state  $X^*$  spends some time in the interior region  $\mathcal{D}^\circ$  and, for almost every  $t$  such that  $X^*(t) \in \mathcal{D}^\circ$ , the feedback relation (61) holds.

We believe that these results are interesting from the economic point of view and may provide a starting point for numerical approximations of the optimal pair. Moreover, we notice that, when  $f$  is linear,  $\delta = 0$  (just for simplicity) and  $U_1(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma \in (0, 1)$ , we can find an explicit solution to the HJB equation (not known in the literature) that can help to get a verification theorem with an explicit feedback map. Indeed the state equation (1) takes the form

$$\begin{cases} x'(t) = \int_{-T}^0 a(\xi)x(t+\xi)d\xi - c(t), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0). \end{cases}$$

Now the delay part in the infinite dimensional representation can be inserted in the operator  $A$ , defining

$$\begin{aligned} \mathcal{D}(A) &= \left\{ \eta \in H \mid \eta_1 \in W_T^{1,2}, \eta_1(0) = \eta_0 \right\}, \\ A : \mathcal{D}(A) &\rightarrow H, \quad A\eta = \left( \int_{-T}^0 a(\xi)\eta_1(\xi)d\xi, \eta_1'(\cdot) \right). \end{aligned}$$

So the infinite dimensional equation is simply

$$\begin{cases} X'(t) = AX(t) - c(t)\hat{n}, \\ X(0) = \eta. \end{cases}$$

The HJB equation (20) takes the form

$$\rho v(\eta) = \langle \eta, A^* \nabla v(\eta) \rangle + \frac{\sigma}{1-\sigma} v_{\eta_0}(\eta)^{\frac{\sigma-1}{\sigma}}, \quad (65)$$

where

$$\mathcal{D}(A^*) = \left\{ \eta \in H \mid \eta_1 \in W_T^{1,2}, \eta_1(-T) = 0 \right\}, \quad A^*\eta = \left( \eta_1(0), a(\cdot)\eta_0 - \eta_1'(\cdot) \right).$$

---

<sup>7</sup>We cannot treat directly this case for technical reasons that are explained in [14, Rem. 4.9].

We guess a solution to this equation in the form

$$v(\eta) = \frac{1}{1-\sigma} \langle \eta, \varphi \rangle^{1-\sigma}, \quad \varphi \in D(A^*). \quad (66)$$

We easily check that the characteristic equation

$$\lambda = \int_{-T}^0 e^{\lambda s} a(s) ds$$

has a unique solution  $\lambda^* > 0$  and that, under the condition (11), the system

$$\begin{cases} \frac{\rho}{1-\sigma} = \lambda^* + \frac{\sigma}{1-\sigma} \varphi_0^{\frac{\sigma-1}{\sigma}} \\ \varphi_1(\xi) = \varphi_0 \int_{-T}^{\xi} e^{-\lambda^*(\xi-s)} a(s) ds, \end{cases}$$

has also a unique solution  $\bar{\varphi} = (\bar{\varphi}_0, \bar{\varphi}_1(\cdot))$ . Then one can see that  $\bar{\varphi} \in \mathcal{D}(A^*)$ ,  $A^* \bar{\varphi} = \lambda^* \bar{\varphi}$  and that  $v$  defined as in (66) with  $\varphi = \bar{\varphi}$  solves (65). The feedback map associated to this solution is

$$C(\eta) = \bar{\varphi}_0^{-1/\sigma} \langle \eta, \bar{\varphi} \rangle. \quad (67)$$

The infinite dimensional closed loop equation associated to this map is

$$\begin{cases} X'(t) = AX(t) - \bar{\varphi}_0^{-1/\sigma} \langle X(t), \bar{\varphi} \rangle, \\ X(0) = \eta. \end{cases} \quad (68)$$

which admits a unique mild solution  $X^*(t; \eta)$ . So, by standard arguments we can prove the following verification theorem.

**Theorem 5.1.** *Let  $\eta \in \bar{H}_{++}$ . Assume that  $X^*(t; \eta)$  is an admissible state trajectory, that is  $X_0^*(t; \eta) \geq 0$  for every  $t \geq 0$ . Then  $v(\eta) = V(\eta)$  and the feedback control*

$$c^*(t) = \bar{\varphi}_0^{-1/\sigma} \langle X^*(t; \eta), \bar{\varphi} \rangle$$

*is the unique optimal control starting from  $\eta$ .*

Given this theorem, if we want to find an optimal feedback control, we need to show that the solution of the closed loop equation (68) is admissible, i.e. satisfies the constraint  $X_0^*(t; \eta) \geq 0$ . Since we have an explicit solution of the HJB equation, this may be possible arguing as in [4], i.e. proving that for some set of initial data  $H_0 \subset \bar{H}_{++}$  this happens.

Concerning the pointwise delay case, standing the assumptions of Subsection 4.3, we approximate the problem with a suitable sequence of problems with distributed delay. We sketch the argument, sending the reader to [13, Ch. 3] for more details. Let us take a sequence  $(a_k)_{k \in \mathbb{N}} \subset W_{-2T}^{1,2}$  such that

$$\begin{cases} a_k(-2T) = 0, \\ \|a_k\|_{L_{-2T}^2} \leq 1, \\ (47)\text{-}(ii) \text{ holds true } \forall a_k, \\ a_k \xrightarrow{*} \delta_{-T} \text{ in } (C([-2T, 0]; \mathbb{R}))^*, \end{cases} \quad (69)$$

where  $\delta_{-T}$  is the Dirac measure concentrated at  $-T$ . We denote by  $x_k(\cdot; \eta, c(\cdot))$  the unique solution of (1) where  $a(\cdot)$  is replaced by  $a_k(\cdot)$  and by  $x(\cdot; \eta, c(\cdot))$  the unique solution of (1) with

$a(\cdot) = \delta_{-T}$ . Also we call  $(P_k)$ ,  $(P)$  the corresponding optimization problems, we denote by  $\bar{\mathcal{C}}_k(\eta)$ ,  $\bar{\mathcal{C}}(\eta)$  the corresponding sets of admissible controls and by  $V_k$ ,  $V$  the corresponding value functions. Proposition 4.14 holds for the optimization problems  $(P_k)$ . So, for every  $\eta \in H_{++}$  and for every  $k \in \mathbb{N}$ , there exists a unique optimal control  $c_k^* \in \bar{\mathcal{C}}_k(\eta)$  for  $(P_k)$ . Given  $\varepsilon > 0$ , starting from  $c_k^*$ , we can construct by means of the procedure used in the proof of Proposition A.13 an  $\varepsilon$ -optimal control  $c_k^\varepsilon \in \bar{\mathcal{C}}(\eta)$  for  $(P_k)$  such that  $x_k(\cdot; \eta, c_k^\varepsilon(\cdot)) \geq 1/n_\varepsilon$  for suitable  $n_\varepsilon > 0$ . Using the arguments of [13, Sec. 3.4.2], we can prove the following.

**Proposition 5.2.** *Let  $\eta \in H_{++}$ . We have  $V_k(\eta) \rightarrow V(\eta)$ , as  $k \rightarrow \infty$ . Moreover for every  $\varepsilon > 0$  we can find a constant  $k_\varepsilon$  such that the control  $c_{k_\varepsilon}^\varepsilon$  above is admissible and  $2\varepsilon$ -optimal for  $(P)$ .  $\square$*

## Appendix

### A.1 Auxiliary results on the control problem and the value function

The proofs of the following results that are not provided here can be found in [14].<sup>8</sup>

**Lemma A.3** (Comparison). *Let  $\eta \in H$  and let  $c(\cdot) \in L_{loc}^1([0, +\infty); \mathbb{R}^+)$ . Let  $x(t)$ ,  $t \geq 0$ , be an absolutely continuous function satisfying almost everywhere the differential inequality*

$$\begin{cases} x'(t) \leq rx(t) + f_0 \left( x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi \right) - c(t), \\ x(0) \leq \eta_0, \quad x(s) \leq \eta_1(s), \text{ for a.e. } s \in [-T, 0). \end{cases}$$

Then  $x(\cdot) \leq x(\cdot; \eta, c(\cdot))$ .  $\square$

**Proposition A.4.**

1. For every  $\eta \in \bar{H}_+$ , we have  $\bar{\mathcal{C}}(\eta) \neq \emptyset$  if and only if  $0 \in \bar{\mathcal{C}}(\eta)$ .
2. For every  $\eta \in \bar{H}_{++}$  we have  $x(t; \eta, 0) \geq \eta_0 e^{-C_{f_0} t}$  for all  $t \geq 0$ , where  $C_{f_0}$  is the Lipschitz constant of  $f_0$ .  $\square$

The next result is a refinement of Proposition 2.9 in [14].

**Proposition A.5.**

1. There exist  $a_0, a_1 > 0$  independent of  $\eta \in \bar{H}_+$  such that

$$x(t; \eta, 0) \leq (a_0 + a_1 \|\eta\|) e^{K_0 t}. \quad (70)$$

where  $K_0 = r + C_{f_0}(1 + T \cdot \sup_{\xi \in [-T, 0]} |a(\xi)|)$ .

2. There exist  $b_0, b_1 > 0$  independent of  $\eta \in \bar{H}_+$ ,  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$  such that

$$\int_0^{+\infty} e^{-\rho t} (U_1(c(t)) + U_2^+(x(t))) dt \leq b_0 + b_1 \|\eta\|. \quad (71)$$

In particular, the functional (2) is well defined<sup>9</sup> for every  $\eta \in \bar{H}_+$  and  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ .

<sup>8</sup>Actually the results of [14] are proved for the case of strict state constraint  $x(\cdot) > 0$ . However, the same proofs hold for the case of state large constraint  $x(\cdot) \geq 0$  and/or the results for the case of strict state constraint hold a fortiori for the case of weaker state constraint (see also Section 5 of [14]).

<sup>9</sup>Even if it may take the value  $-\infty$ .

3. There exist  $c_0, c_1 > 0$  independent of  $\eta \in \bar{H}_+$ ,  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$  such that

$$\int_0^{+\infty} e^{-\rho t} c(t) dt \leq c_0 + c_1 \|\eta\|. \quad (72)$$

**Proof.** Claims 1 and 2 are proved (in the case of strict state constraint  $x(\cdot) > 0$ ) in [14, Prop. 2.9].<sup>10</sup> We prove the third one. Let

$$\bar{a} := \sup_{\xi \in [-T, 0]} a(\xi), \quad p := f_0(0, 0) \geq 0.$$

Since  $f_0$  is Lipschitz continuous with Lipschitz constant  $C_{f_0}$ , we have

$$rx + f_0(x, y) \leq rx + C_{f_0}(x + |y|) + p := g(x, y), \quad \forall x \in \mathbb{R}^+, \forall y \in \mathbb{R}. \quad (73)$$

Let  $\eta \in H_+$ ,  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$  and set  $x(\cdot) := x(\cdot; \eta, c(\cdot))$ . Then Lemma A.3 and estimate (70) yield

$$x(t) \leq x(t; \eta, 0) \leq (a_0 + a_1 \|\eta\|) e^{K_0 t}, \quad \forall t \geq 0. \quad (74)$$

By (1) and (73) we have

$$x'(t) \leq g\left(x(t), \int_{-T}^0 a(\xi) x(t + \xi) d\xi\right) - c(t). \quad (75)$$

Set  $K_\eta := a_0 + a_1 \|\eta\|$ . Using estimates (73) and (74) in the right hand side of (75) and the monotonicity of  $g$  with respect to the second variable, we get

$$x'(s) \leq rK_\eta e^{K_0 s} + C_{f_0} \left( K_\eta e^{K_0 s} + \|a\|_{L_T^2} \|\eta_1\|_{L_T^2} + \bar{a} T K_\eta e^{K_0 s} \right) + p - c(s), \quad \text{for a.e. } s \geq 0. \quad (76)$$

Putting

$$K(s) := rK_\eta e^{K_0 s} + C_{f_0} \left( K_\eta e^{K_0 s} + \|a\|_{L_T^2} \|\eta_1\|_{L_T^2} + \bar{a} T K_\eta e^{K_0 s} \right) + p$$

and invoking (76) we obtain

$$x'(s) \leq \rho x(s) + K(s) - c(s), \quad \text{for a.e. } s \geq 0. \quad (77)$$

Since  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ , we have  $x(\cdot) \geq 0$ . Hence, (76) and the Gronwall Lemma yield

$$0 \leq x(t) \leq \eta_0 e^{\rho t} + \int_0^t e^{\rho(t-s)} K(s) ds - \int_0^t e^{\rho(t-s)} c(s) ds, \quad \forall t \geq 0. \quad (78)$$

Multiplying (78) by  $e^{-\rho t}$  we find that

$$\int_0^t e^{-\rho s} c(s) ds \leq \eta_0 + \int_0^t e^{-\rho s} K(s) ds, \quad \forall t \geq 0.$$

Due to (11) we get the claim (72).  $\square$

<sup>10</sup>Explicit dependence of the estimates on  $\|\eta\|$  is not stated in [14]. However, the sublinear dependence as stated in the above proposition can be easily deduced from the proofs in [14]. The sublinear dependence in the estimate (71) can be also obtained arguing that the value function is proper and concave.

## A.2 Characterization of interior and boundary of $\mathcal{D}(V)$

In Subsection 2.2, recalling [14], we have introduced the sets  $\mathcal{D}^\circ$  and  $\mathcal{B}$ , respectively the interior and the boundary of  $\mathcal{D}(V)$  in the space  $(H, \|\cdot\|_{-1})$ . Here we provide a useful characterization of these sets. The following result is used in [14] to prove the continuity of the value function (Proposition 2.10 here). We provide it here, because we are going to use it to characterize the sets  $\mathcal{D}^\circ$  and  $\mathcal{B}$ .

**Lemma A.6.** *Let  $X(\cdot), \bar{X}(\cdot)$  be the mild solutions to (14) starting respectively from  $\eta, \bar{\eta} \in H$  and both under the null control. Then for every  $T_0 > 0$  there exists a constant  $C_{T_0} > 0$  such that*

$$\|X(t) - \bar{X}(t)\|_{-1} \leq C_{T_0} \|\eta - \bar{\eta}\|_{-1}, \quad \forall t \in [0, T_0].$$

*In particular, for every  $T_0 \geq 0$  there exists a constant  $C_{T_0} > 0$  such that*

$$|X_0(t) - \bar{X}_0(t)| \leq rC_{T_0} \|\eta - \bar{\eta}\|_{-1}, \quad \forall t \in [0, T_0]. \quad \square$$

**Remark A.7.** *We note that, since  $\|\cdot\|_{-1}$  is dominated by  $\|\cdot\|$ , then for every  $T_0 > 0$  there exists a constant  $\tilde{C}_{T_0} > 0$  such that*

$$|X_0(t) - \bar{X}_0(t)| \leq r\tilde{C}_{T_0} \|\eta - \bar{\eta}\|, \quad \forall t \in [0, T_0].$$

**Lemma A.8.** *Let  $\eta \in H$  and*

$$g(\eta) := \min_{t \in [0, T]} x(\cdot; \eta, 0). \quad (79)$$

*Then the function  $g$  is  $\|\cdot\|_{-1}$ -continuous and*

$$\mathcal{D}^\circ = \{g > 0\}, \quad \mathcal{B} = \{g = 0\}.$$

**Proof.** The fact that  $g : H \rightarrow \mathbb{R}$  defined in (79) is  $\|\cdot\|_{-1}$ -continuous is a consequence of Lemma A.6 and Proposition 2.9.

Let us show that  $\mathcal{D}^\circ = \{g > 0\}$ . Let  $\eta \in H$  be such that  $g(\eta) > 0$ . By continuity of  $g$ , we have  $g(\zeta) > 0$  for every  $\zeta \in B(\eta, \varepsilon)$  for sufficiently small  $\varepsilon > 0$ . Therefore

$$(x(T; \zeta, 0), x(T + \cdot; \zeta, 0)|_{[-T, 0]}) \in H_{++}, \quad \forall \zeta \in B(\eta, \varepsilon).$$

Then Proposition A.4-(2) and the flow property of solutions to state equation (1) yield

$$x(t; \zeta, 0) \geq g(\zeta) \left(1 \wedge e^{-C_{f_0}(t-T)}\right), \quad \forall t \geq 0. \quad (80)$$

Recalling that  $g(\zeta) > 0$  in  $B(\eta, \varepsilon)$ , we see that  $0 \in \bar{\mathcal{C}}(\zeta)$  for every  $\zeta \in B(\eta, \varepsilon)$ . Moreover, by (9) we also have  $J(\zeta; 0) > -\infty$  for every  $\zeta \in B(\eta, \varepsilon)$ . This shows that  $B(\eta, \varepsilon) \subset \mathcal{D}(V)$ , so we have proved the inclusion  $\{g(\eta) > 0\} \subset \mathcal{D}^\circ$ .

Conversely, let  $\eta \in \mathcal{D}^\circ$  and suppose by contradiction that  $g(\eta) \leq 0$ . This means that there exists  $t \in (0, T]$  such that  $x(t; \eta, 0) = 0$ . Let  $\varepsilon > 0$  and consider  $\zeta \in H$  defined as

$$\zeta_0 = \eta_0 - \varepsilon, \quad \zeta_1(\cdot) \equiv \eta_1(\cdot),$$

so that  $\zeta \in B(\eta, \varepsilon)$ . We will to show that  $0 \notin \bar{\mathcal{C}}(\zeta)$  for every  $\varepsilon > 0$ . By Proposition A.4-(1), this will imply that  $\bar{\mathcal{C}}(\zeta) = \emptyset$  hence  $\zeta \notin \mathcal{D}(V)$ . By arbitrariness of  $\varepsilon$ , this will imply that  $\eta \notin \mathcal{D}^\circ$ , contradicting the initial assumption. Hence, also the inclusion  $\mathcal{D}^\circ \subset \{g > 0\}$  will be proved.

Define  $y(\cdot) := x(\cdot; \eta, 0)$ ,  $x(\cdot) := x(\cdot; \zeta, 0)$ . By Lemma A.3, we have  $x(\cdot) \leq y(\cdot)$ . Let  $z(\cdot), \bar{z}(\cdot)$  be respectively the solutions on  $[0, T]$  of the differential problems *without control*

$$\begin{cases} z'(t) = rz(t) + f_0 \left( z(t), \int_{-T}^0 a(\xi)x(t + \xi)d\xi \right), \\ z(0) = \zeta_0, \end{cases}$$

$$\begin{cases} \bar{z}'(t) = r\bar{z}(t) + f_0 \left( \bar{z}(t), \int_{-T}^0 a(\xi)x(t + \xi)d\xi \right), \\ \bar{z}(0) = \eta_0. \end{cases}$$

Obviously we have  $z(\cdot) \equiv x(\cdot)$ . Moreover, since  $x(\cdot) \leq y(\cdot)$  and  $f_0$  is nondecreasing with respect to the second variable, the comparison criterion for classical ODEs yields  $y(\cdot) \geq \bar{z}(\cdot)$ . Therefore, we can apply the classical results for ODEs with Lipschitz coefficients to show the uniqueness of solutions to the above problems, that yields  $z(\cdot) < \bar{z}(\cdot)$  on  $[0, T]$ . Therefore,  $x(t) = z(t) < \bar{z}(t) \leq y(t) = 0$ , hence  $0 \notin \bar{C}(\zeta)$  as claimed and the proof of the fact that  $\mathcal{D}^\circ = \{g > 0\}$  is complete.

Let us show now that  $\mathcal{B} = \{g = 0\}$ . The inclusion  $\mathcal{B} \subset \{g = 0\}$  is a consequence of the continuity of  $g$  and of the characterization  $\mathcal{D}^\circ = \{g > 0\}$ . On the other hand, let  $g(\eta) = 0$  and define for  $\varepsilon > 0$

$$\zeta_0 = \eta_0 + \varepsilon, \quad \zeta_1(\cdot) \equiv \eta_1(\cdot),$$

so that  $\zeta \in B(\eta, \varepsilon)$ . Moreover, arguing as above we get  $x(\cdot; \zeta, 0) > 0$  on  $[0, T]$  for every  $\varepsilon > 0$ . This means that  $g(\zeta) > 0$  for every  $\varepsilon > 0$ , so  $\zeta \in \mathcal{D}^\circ$  for every  $\varepsilon > 0$ . Therefore, by the arbitrariness of  $\varepsilon$  we obtain  $\eta \in \mathcal{B}$  and the proof is complete.  $\square$

**Remark A.9.** Lemma A.8, Proposition A.4-(2) and the semigroup property of the solution of state equation (1) show that

$$\eta \in \mathcal{D}^\circ \iff x(t; \eta, 0) > 0, \quad \forall t \geq 0. \quad (81)$$

**Remark A.10.** We note that the claim of Lemma A.8 holds also in the topology defined by the norm  $\|\cdot\|$ , i.e.  $g$  is  $\|\cdot\|$ -continuous and

$$\text{Int}_{(H, \|\cdot\|)}(\mathcal{D}(V)) = \{g > 0\}, \quad \text{Fr}_{(H, \|\cdot\|)}\mathcal{D}(V) = \{g = 0\}.$$

In particular we have

$$\text{Int}_{(H, \|\cdot\|)}(\mathcal{D}(V)) = \mathcal{D}^\circ, \quad \text{Fr}_{(H, \|\cdot\|)}\mathcal{D}(V) = \mathcal{B}. \quad (82)$$

### A.3 Comparison with the case of state constraint $x(\cdot) > 0$

In this subsection we investigate the relationship between the problems with strict and weak state constraint when (51) holds true. Before to proceed we need to recall a result of Convex Analysis. Let us recall that if  $\mathcal{J}$  is a concave functional from some topological vector space  $X$  to  $\mathbb{R} \cup \{-\infty\}$ , its upper semicontinuous concave regularization is the functional

$$\bar{\mathcal{J}}(x) := \inf\{\mathcal{F}(x) \mid \mathcal{F} : X \rightarrow \mathbb{R} \cup \{-\infty\}, \mathcal{F} \geq \mathcal{J}, \\ \mathcal{F} : X \rightarrow \mathbb{R} \cup \{-\infty\} \text{ concave and upper semicontinuous}\}, \quad x \in X.$$

The following result corresponds to Proposition 2.3.9-(ii) in [1].

**Lemma A.11.** *Let  $X$  be a topological vector space. Let  $\mathcal{J} : X \rightarrow \mathbb{R} \cup \{-\infty\}$  concave and let  $\bar{\mathcal{J}}$  be its upper semicontinuous concave regularization. If  $\limsup_{x \rightarrow x_0} \mathcal{J}(x) > -\infty$ , then  $\bar{\mathcal{J}}(x_0) = \limsup_{x \rightarrow x_0} \mathcal{J}(x)$ .  $\square$*

Set  $L_\rho^1 := L^1([0, +\infty), e^{-\rho t} dt; \mathbb{R})$ . Estimate (72) shows that

$$\bar{\mathcal{C}}(\eta) \text{ is a (convex) bounded subset of } L_\rho^1, \quad \forall \eta \in \mathcal{D}(V). \quad (83)$$

**Lemma A.12.** *Let  $\eta \in \mathcal{D}(V)$ ,  $(c_n)_{n \in \mathbb{N}} \subset \bar{\mathcal{C}}(\eta)$  and  $c \in \bar{\mathcal{C}}(\eta)$ . If  $c_n \rightarrow c$  weakly in  $L_\rho^1$ , then*

$$\limsup_{n \rightarrow \infty} \int_0^{+\infty} e^{-\rho t} U_1(c_n(t)) dt \geq \int_0^{+\infty} e^{-\rho t} U_1(c(t)) dt.$$

**Proof.** For  $\eta \in \mathcal{D}(V)$ , consider the functional (finiteness follows from non-negativity of  $U_1$  and (71))

$$\bar{\mathcal{C}}(\eta) \rightarrow \mathbb{R}, \quad c \mapsto \int_0^{+\infty} e^{-\rho t} U_1(c(t)) dt.$$

This functional is concave, by concavity of  $U_1$ . In order to apply Convex Analysis we have to extend this functional to the whole space  $L_\rho^1$ . Observe that  $U_1$  is defined on  $[0, +\infty)$ , so the natural way to extend this functional to  $L_\rho^1$  preserving concavity is by considering the functional

$$\mathcal{J} : L_\rho^1 \rightarrow \mathbb{R} \cup \{-\infty\}, \quad c \mapsto \begin{cases} \int_0^{+\infty} e^{-\rho t} U_1(c(t)) dt, & \text{if } c \in \bar{\mathcal{C}}(\eta), \\ -\infty, & \text{if } c \in L_\rho^1 \setminus \bar{\mathcal{C}}(\eta). \end{cases}$$

This extended functional is still concave. Consider its upper semicontinuous concave regularized envelope with respect to the weak topology of  $L_\rho^1$  that is the functional

$$\bar{\mathcal{J}}(c) := \inf\{\mathcal{F}(c) \mid \mathcal{F} : L_\rho^1 \rightarrow \mathbb{R} \cup \{-\infty\}, \mathcal{F} \geq \mathcal{J},$$

$\mathcal{F}$  concave and upper semicontinuous with respect to the weak topology of  $L_\rho^1\}$ .

Take  $(c_n)_{n \in \mathbb{N}} \subset \bar{\mathcal{C}}(\eta)$  and  $c \in \bar{\mathcal{C}}(\eta)$  such that  $c_n \rightarrow c$  in  $L_\rho^1$ . Since  $U_1 \geq 0$ , we have

$$\limsup_{n \rightarrow \infty} \mathcal{J}(c_n) = \int_0^{+\infty} e^{-\rho t} U_1(c_n(t)) dt \geq 0 > -\infty.$$

Then from Lemma A.11, we get

$$\limsup_{n \rightarrow \infty} \mathcal{J}(c_n) = \bar{\mathcal{J}}(c) \geq \mathcal{J}(c),$$

which is the claim  $\square$

**Proposition A.13.** *Let (51) hold. Let  $\eta \in H_{++}$  and  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ . Then for every  $\varepsilon > 0$ , there exists  $c^\varepsilon(\cdot) \in \bar{\mathcal{C}}(\eta)$  and  $n_\varepsilon > 0$  such that*

$$x(t; \eta, c^\varepsilon(\cdot)) \geq 1/n_\varepsilon \quad \forall t \geq 0, \quad \text{and} \quad J(\eta, c^\varepsilon(\cdot)) \geq J(\eta, c(\cdot)) - \varepsilon. \quad (84)$$

**Proof.** Let  $\eta \in \mathcal{D}^\circ$  and  $c(\cdot) \in \bar{\mathcal{C}}(\eta)$ . Let  $n \in \mathbb{N}$  be such that  $1/n \in (0, \eta_0)$  and set  $x(\cdot) := x(\cdot; \eta, c(\cdot))$ . Consider the open set

$$I_n := \{t \geq 0 \mid x(t) < 1/n\}.$$



Consider the continuous function

$$x_n(t) := x(t) \vee (1/n), \quad t \geq 0 \quad (85)$$

and the control (well defined for almost every  $t \geq 0$ )

$$c_n(t) := rx_n(t) + f_0 \left( x_n(t), \int_{-T}^0 a(\xi)x_n(t+\xi)d\xi \right) - x'_n(t). \quad (86)$$

Note that  $x'_n(\cdot) = 0$  everywhere on  $I_n$  and  $x'_n(\cdot) = x'(\cdot)$  almost everywhere on  $I_n^c$ . Moreover  $x_n(\cdot)$  is continuous. Therefore, expressing  $x'(\cdot)$  through the state equation and taking into account that  $x(\cdot)$  is continuous, we obtain  $c_n(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R})$ . Note that we do not know at this stage if  $c_n(\cdot) \geq 0$ . However, the existence and uniqueness of solution to the state equation (1) clearly holds also for controls in  $L^1_{loc}([0, +\infty); \mathbb{R})$ . So we have a solution  $x(\cdot; \eta, c_n(\cdot))$  to the state equation (1) associated to the control  $c_n(\cdot)$ . Moreover, by uniqueness of solution to this state equation

$$x(t; \eta, c_n(\cdot)) = x_n(t), \quad \forall t \geq 0. \quad (87)$$

On  $I_n$  we have  $x'_n(t) = 0$  everywhere. So (3), (51) and the definition (86) of  $c_n$  yield  $c_n(t) \geq 0$  for almost every  $t \in I_n$ . On the other hand, we have for almost every  $t \geq 0$

$$\begin{aligned} x'_n(t) &= rx_n(t) + f_0 \left( x_n(t), \int_{-T}^0 a(\xi)x_n(t+\xi)d\xi \right) - c_n(t), \\ x'(t) &= rx(t) + f_0 \left( x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi \right) - c(t). \end{aligned}$$

So, since  $x_n(\cdot) = x(\cdot)$  (and  $x'_n(\cdot) = x'(\cdot)$ ) almost everywhere on  $I_n^c$ , and recalling that  $x_n(\cdot) \geq x(\cdot)$  everywhere, we get by monotonicity of  $f_0$  with respect to the second variable that  $c_n(t) \geq c(t) \geq 0$  for almost every  $t \in I_n^c$ . Therefore  $c_n(t) \geq 0$  for almost every  $t \geq 0$ . Hence, considering also (85) and (87), we have  $c_n(\cdot) \in \bar{\mathcal{C}}(\eta)$ .

Now we claim that the sequence  $(c_n)_{n \in \mathbb{N}}$  is uniformly integrable in  $L^1_\rho$ . To this end note that

$$0 \leq x_n(t) - x(t) \leq 1/n, \quad \forall t \geq 0. \quad (88)$$

Let  $\bar{a} := \sup_{\xi \in [-T, 0]} |a(\xi)|$ . Due to the Lipschitz continuity of  $f_0$  and to (88), from the state equation we get

$$x'_n(s) - x'(s) - K_n \leq c(s) - c_n(s), \quad \text{for a.e. } s \geq 0,$$

where  $K_n := r/n + C_{f_0}(1/n + T\bar{a}/n)$ . Therefore

$$0 \leq c_n(s) \leq c(s) + |x'_n(s)| + |x'(s)| + K_n, \quad \text{for a.e. } s \geq 0.$$

Taking into account (85) and (87) we see that  $|x'_n(\cdot)| \leq |x'(\cdot)|$  almost everywhere, so

$$0 \leq c_n(s) \leq c(s) + 2|x'(s)| + K_1, \quad \text{for a.e. } s \geq 0. \quad (89)$$

By (72) we have  $c(\cdot) \in L^1_\rho$ . Moreover, expressing  $x'(\cdot)$  through the state equation (1) and using (11), (70) and Lemma A.3, we get  $x'(\cdot) \in L^1_\rho$ . Therefore, from (89) we get that  $(c_n)_{n \in \mathbb{N}}$  is uniformly integrable. Due to (83), the sequence  $(c_n)_{n \in \mathbb{N}} \subset L^1_\rho$  is bounded in  $L^1_\rho$ . Then, by

Dunford-Pettis Theorem, it is a relatively weakly compact set of  $L^1_\rho$ . Therefore, we can find a subsequence  $(c_{n_k})_{k \in \mathbb{N}}$  weakly convergent towards some  $g \in L^1_\rho$ . In particular

$$\int_0^t e^{-\rho s} c_{n_k}(s) ds \longrightarrow \int_0^t e^{-\rho s} g(s) ds, \quad \forall t \geq 0. \quad (90)$$

On the other hand, again due to the Lipschitz continuity of  $f_0$  and to (88), from the state equation (1) we get

$$x'_n(s) - x'(s) - K_n \leq c(s) - c_n(s) \leq x'_n(s) - x'(s) + K_n, \quad \text{for a.e. } s \geq 0,$$

where  $K_n := r/n + C_{f_0}(1/n + T\bar{a}/n)$ . Multiplying this inequality by  $e^{-\rho s}$  and integrating over  $[0, t]$  we get

$$\begin{aligned} [e^{-\rho s}(x_n(s) - x(s))]_0^t + \rho \int_0^t e^{-\rho s}(x_n(s) - x(s)) ds - K_n \frac{1 - e^{-\rho t}}{\rho} \\ \leq \int_0^t e^{-\rho s}(c(s) - c_n(s)) ds \\ \leq [e^{-\rho s}(x_n(s) - x(s))]_0^t + \rho \int_0^t e^{-\rho s}(x_n(s) - x(s)) ds + K_n \frac{1 - e^{-\rho t}}{\rho}. \end{aligned}$$

From the above inequality, taking into account (88) and the fact that  $x_n(0) = x(0)$  (recall that  $1/n < \eta_0$ ) we get

$$-K_n \frac{1 - e^{-\rho t}}{\rho} \leq \int_0^t e^{-\rho s}(c(s) - c_n(s)) ds \leq 1/n + K_n \frac{1 - e^{-\rho t}}{\rho}, \quad \forall t > 0,$$

with  $K_n \rightarrow 0$  when  $n \rightarrow \infty$ . This implies

$$\int_0^t e^{-\rho s} c_n(s) ds \xrightarrow{n \rightarrow \infty} \int_0^t e^{-\rho s} c(s) ds, \quad \forall t \geq 0. \quad (91)$$

From (90) and (91) we get

$$\int_0^t e^{-\rho s} g(s) ds = \int_0^t e^{-\rho s} c(s) ds, \quad \forall t \geq 0. \quad (92)$$

Deriving (92) we get  $g(t) = c(t)$  for almost every  $t \geq 0$ . Thus  $c_{n_k} \rightharpoonup c$  in  $L^1_\rho$ . Applying Lemma A.12 we get

$$\limsup_{k \rightarrow \infty} \int_0^{+\infty} e^{-\rho s} U_1(c_{n_k}(s)) ds \geq \int_0^{+\infty} e^{-\rho s} U_1(c(s)) ds. \quad (93)$$

On the other hand, since  $x(\cdot; \eta, c_{n_k}(\cdot)) \geq x(\cdot)$  for every  $k \in \mathbb{N}$ , we have by monotonicity of  $U_2$

$$\int_0^{+\infty} e^{-\rho t} U_2(x_{n_k}(t)) dt \geq \int_0^{+\infty} e^{-\rho t} U_2(x(t)) dt, \quad \forall k \in \mathbb{N}. \quad (94)$$

Therefore (85), (87), (93) and (94) yield the claim.  $\square$

From the proposition above we get the following corollary, showing that the problem with state constraint  $x(\cdot) > 0$  and the problem with state constraint  $x(\cdot) \geq 0$  have the same value functions on  $H_{++}$  when (51) holds.

**Corollary A.14.** *Let (51) hold. For every  $\eta \in H_{++}$  we have*

$$\sup_{c(\cdot) \in \bar{\mathcal{C}}(\eta)} J(\eta, c(\cdot)) = \sup_{c(\cdot) \in \mathcal{C}(\eta)} J(\eta, c(\cdot)). \quad \square$$

#### A.4 Technical proofs

**Proof of Proposition 4.4-(1).** Let  $\eta \in \mathcal{D}^\circ$  the initial datum for the equation. We set

$$G(\zeta) := r\zeta_0 + f(\zeta) - C(\zeta), \quad \zeta \in \mathcal{D}^\circ.$$

Note that  $G$  is continuous at  $\eta$ , therefore locally bounded at  $\eta$ . We have to show the local existence of a solution of

$$\begin{cases} x'(t) = G((x(t), x(t + \cdot)|_{[-T, 0]})), \\ x(0) = \eta_0, \quad x(s) = \eta_1(s), \quad s \in [-T, 0]. \end{cases}$$

By construction (recall that the map  $C$  is defined only on  $\mathcal{D}^\circ$ ) such a solution will have the property

$$(x^*(t), x^*(t + \cdot)|_{[-T, 0]}) \in \mathcal{D}^\circ,$$

therefore  $x^*(t) > 0$ .

Since  $G$  is locally bounded at  $\eta$ , there exists  $b > 0$  such that  $m := \sup_{\|\zeta - \eta\|^2 \leq b} |G(\zeta)| < +\infty$ . By continuity of translations in  $L^2(\mathbb{R}; \mathbb{R})$  we can find  $a \in [0, T]$  such that

$$\int_{-T}^{-t} |\eta_1(t + \xi) - \eta_1(\xi)|^2 d\xi \leq b/4, \quad \forall t \in [0, a].$$

Moreover, without loss of generality, we can suppose that  $\int_{-a}^0 |\eta_1(\xi)|^2 d\xi \leq b/16$ . Set

$$\alpha := \min \left\{ a, \frac{b}{2m}, \frac{b}{16} (b + 2|\eta_0|^2)^{-1} \right\}.$$

Define

$$M := \{x(\cdot) \in C([0, \alpha]; \mathbb{R}) \mid |x(\cdot) - \eta_0|^2 \leq b/2\}.$$

$M$  is a convex closed subset of the Banach space  $C([0, \alpha]; \mathbb{R})$  endowed with the sup-norm. Define

$$x(t + \xi) := \eta_1(t + \xi), \quad \text{if } t + \xi \leq 0,$$

and observe that, for  $t \in [0, \alpha]$ ,  $x(\cdot) \in M$ ,

$$\begin{aligned} \int_{-t}^0 |x(t + \xi) - \eta_1(\xi)|^2 d\xi &\leq \int_{-t}^0 (2|x(t + \xi)|^2 + 2|\eta_1(\xi)|^2) d\xi \\ &\leq 2 \left[ \int_{-t}^0 (2(|x(t + \xi) - \eta_0|^2 + 2|\eta_0|^2)) d\xi + \int_{-t}^0 |\eta_1(\xi)|^2 d\xi \right] \\ &\leq 2 \left[ 2t \left( \frac{b}{2} + |\eta_0|^2 \right) + \frac{b}{16} \right] \leq b/4 \end{aligned}$$

So, for  $t \in [0, \alpha]$ ,  $x(\cdot) \in M$ , we have

$$\begin{aligned} \|(x(t), x(t + \cdot)|_{[-T, 0]}) - \eta\|^2 &\leq |x(t) - \eta_0|^2 + \int_{-t}^0 |x(t + \xi) - \eta_1(\xi)|^2 d\xi + \int_{-T}^{-t} |\eta_1(t + \xi) - \eta_1(\xi)|^2 d\xi \\ &\leq b/2 + b/4 + b/4 = b. \end{aligned}$$

Define, for  $t \in [0, \alpha]$ ,  $x(\cdot) \in M$ ,

$$[\mathcal{J}x](t) := \bar{\eta}_0 + \int_0^t G(x(s), x(s + \cdot)|_{[-T, 0]}) ds, \quad t \in [0, \alpha].$$

We have

$$\left| [\mathcal{J}x](t) - \eta_0 \right| \leq \int_0^t |G(x(s), x(s + \xi)|_{\xi \in [-T, 0]})| ds \leq tm \leq b/2.$$

Therefore we have proved that  $\mathcal{J}$  maps the closed and convex set  $M$  in itself. We want to prove that  $\mathcal{J}$  admits a fixed point (so, by definition of  $\mathcal{J}$ , the solution we are looking for). By Schauder's Theorem it is enough to prove that  $\mathcal{J}$  is completely continuous, i.e. that  $\overline{\mathcal{J}(M)}$  is compact. For any  $x(\cdot) \in M$ , we have the estimate

$$\left| [\mathcal{J}x](t) - [\mathcal{J}x](\bar{t}) \right| \leq \int_{t \wedge \bar{t}}^{t \vee \bar{t}} |G(x(s), x(s + \cdot)|_{[-T, 0]})| ds \leq m|t - \bar{t}|, \quad t, \bar{t} \in [0, \alpha].$$

Therefore  $\mathcal{J}(M)$  is a uniformly bounded and equicontinuous family in the space  $C([0, \alpha]; \mathbb{R})$ . Thus, by the Ascoli-Arzelà Theorem,  $\overline{\mathcal{J}(M)}$  is compact.  $\square$

**Proof of Lemma 4.8.** In this proof all topological notions are referred to the topology defined by the norm  $\|\cdot\|$  and thereby (82) is satisfied. Let  $\eta \in \mathcal{B}$  and let  $(\eta^n) \subset \mathcal{D}^\circ$  be a sequence such that  $\eta^n \rightarrow \eta$ . Firstly we prove that

$$\lim_{\eta^n \rightarrow \eta} V(\eta^n) = -\infty. \quad (95)$$

We can suppose without loss of generality that  $(\eta^n) \subset B(\eta, 1)$ . Let  $g$  be the function defined in Lemma A.8. By the same lemma we have

$$\lim_{n \rightarrow \infty} g(\eta^n) = 0. \quad (96)$$

Let  $s_n \in [0, T]$  be such that

$$x(s_n; \eta^n, 0) = g(\eta^n). \quad (97)$$

For any  $n \in \mathbb{N}$ , let  $c^n(\cdot) \in \bar{\mathcal{C}}(\eta^n)$  and set

$$x^n(\cdot) := x(\cdot; \eta^n, c^n(\cdot)), \quad p^n := \sup_{\xi \in [0, 2T]} x(\xi; \eta^n, 0).$$

Since  $\eta^n \in B(\eta, 1)$ , Remark A.7 shows that there exists  $K > 0$  such that  $p^n \leq K$  for any  $n \in \mathbb{N}$ . By Lemma A.3 we have

$$x^n(t) \leq p^n \leq K, \quad \forall t \in [0, 2T]. \quad (98)$$

By Lemma A.3, (96) and (97),

$$0 \leq x^n(s_n) \leq x(s_n; \eta^n, 0) = g(\eta^n). \quad (99)$$

Since  $f_0(x, y) \leq C_{f_0}(|f_0(0, 0)| + |x| + |y|)$  and  $(\eta^n) \subset B(\eta, 1)$ , we have for the dynamics of  $x^n(\cdot)$  in the interval  $[0, 2T]$

$$\frac{d}{dt} x^n(t) \leq r x^n(t) + R,$$

where

$$R := C_{f_0} \left( |f_0(0, 0)| + K + \|a\|_{L_T^2} (\|\eta_1\|_{L_T^2} + 1) + \|a\|_{L_T^2} T^{1/2} K \right).$$

Therefore there exists  $C > 0$  such that, for any  $s \in [0, T]$ ,  $n \in \mathbb{N}$ ,

$$x^n(t) \leq x^n(s) e^{r(t-s)} + \frac{R}{r} (e^{r(t-s)} - 1) \leq x^n(s)(1 + C(t-s)) + C(t-s), \quad t \in [s, 2T]. \quad (100)$$

By (100) and (99)

$$\begin{aligned} \int_{s_n}^{2T} e^{-\rho t} U_2^-(x^n(t)) dt &\leq e^{-2\rho T} \int_{s_n}^{2T} U_2^-(x^n(s_n)(1 + C(t-s_n)) + C(t-s_n)) dt \\ &\leq e^{-2\rho T} \int_{s_n}^{2T} U_2^-(g(\eta^n)(1 + C(t-s_n)) + C(t-s_n)) dt \\ &= \frac{e^{-2\rho T}}{C(x^n(s_n) + 1)} \int_{x_n(s_n)}^{x^n(s_n)(1+C(2T-s_n))+C(2T-s_n)} U_2^-(x) dt. \end{aligned} \quad (101)$$

Since  $0 \leq x^n(\cdot) \leq K$  on  $[0, 2T]$ , from (101) and (99) we get

$$\int_{s_n}^{2T} e^{-\rho t} U_2^-(x^n(t)) dt \leq \frac{e^{-2\rho T}}{C(K+1)} \int_{g(\eta^n) \wedge (TC)}^{TC} U_2^-(x) dt. \quad (102)$$

On the other hand, since  $\eta^n \in B(\eta, 1)$ , Proposition 2.10-(2) shows that there exists  $C_0 > 0$  such that

$$\int_0^{+\infty} e^{-\rho t} (U_1(c^n(t)) + U_2^+(x^n(t))) dt \leq C_0 \quad \forall n \in \mathbb{N}, \quad \forall c^n \in \bar{\mathcal{C}}(\eta^n). \quad (103)$$

For every fixed  $n \in \mathbb{N}$ , the estimate (102) and (103) are uniform with respect to  $c^n(\cdot) \in \bar{\mathcal{C}}(\eta^n)$ . This means that

$$V(\eta^n) \leq C_0 + \frac{e^{-2\rho T}}{C(K+1)} \int_{g(\eta^n) \wedge (TC)}^{TC} U_2^-(x) dt.$$

On the other hand (47)-(i) and (96) yield

$$\int_{g(\eta^n) \wedge (TC)}^{TC} U_2^-(x) dt \rightarrow -\infty, \quad (104)$$

so we have proved (95). Now we prove (48) as a consequence of (95). Let  $\eta \in \mathcal{B}$  and  $(\eta^n) \subset \mathcal{D}^\circ$  be such that  $\eta^n \rightarrow \eta$ . Without loss of generality we can assume that  $(\eta_n) \subset B(\eta, 1)$ . Let  $\alpha > 0$  and set

$$x_\alpha^n(\cdot) := x(\cdot; (\eta_0^n + \alpha, \eta_1^n), 0).$$

Since  $\eta^n \in \mathcal{D}^\circ$  for every  $\mathbb{N}$ , we have in particular  $\bar{\mathcal{C}}(\eta^n) \neq \emptyset$  for every  $n \in \mathbb{N}$ . Then, by Proposition A.4-(1), we have  $x(\cdot; \eta^n, 0) \geq 0$  for every  $n \in \mathbb{N}$ . By Lemma A.3,  $x_\alpha^n(\cdot) \geq x(\cdot; \eta^n, 0)$ , so also  $x_\alpha^n(\cdot) \geq 0$  for every  $n \in \mathbb{N}$ . Let

$$\bar{a} = \sup_{\xi \in [-T, 0]} |a(\xi)|.$$

Since  $f_0$  is nondecreasing on the second variable and Lipschitz continuous and  $\eta^n \in B(\eta, 1)$ , there exists some  $R > 0$  independent of  $n$  such that

$$f_0 \left( x_\alpha^n(t), \int_{-T}^0 a(\xi) x_\alpha^n(t + \xi) d\xi \right) \geq f_0 \left( x_\alpha^n(t), -\bar{a} \|\eta_1^n\|_{L_T^1} \right) \geq -C_{f_0} x_\alpha^n(t) - R.$$

Then we have for the dynamics of  $x_\alpha^n$

$$\begin{cases} \frac{d}{dt}x_\alpha^n(t) \geq (r - C_{f_0})x_\alpha^n(t) - R, \\ x_\alpha^n = \eta_0^n + \alpha. \end{cases}$$

Since  $\eta^n \in \mathcal{D}^\circ$  for every  $n \in \mathbb{N}$ , we have  $\eta_0^n > 0$  for every  $n \in \mathbb{N}$ . Thus, if we take  $\alpha = R/(r + C_{f_0})$ , we have  $x_\alpha^n(\cdot) \geq R/(r + C_{f_0}) > 0$  for every  $n \in \mathbb{N}$ . This implies that, for such  $\alpha$ , there exists a constant  $K > 0$  independent of  $n \in \mathbb{N}$  such that  $J((\eta_0^n + \alpha, \eta_1^n); 0) \geq K$  for every  $n \in \mathbb{N}$ . Therefore also  $V(\eta_0^n + \alpha, \eta_1^n) \geq K$  for every  $n \in \mathbb{N}$ . Due to the concavity of  $V$  we have the estimate

$$V_{\eta_0}(\eta^n) \geq \frac{1}{\alpha} [V(\eta_0^n + \alpha, \eta_1^n(\cdot)) - V(\eta_0^n, \eta_1^n(\cdot))] \geq \frac{1}{\alpha} [K - V(\eta_0^n, \eta_1^n(\cdot))], \quad \forall n \in \mathbb{N}.$$

From (95) we have the claim. □

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## References

- [1] Acquistapace P. *Appunti di Analisi Convessa*, available at <http://www.dm.unipi.it/acquistp/anacon.pdf>, 2009.
- [2] Asea P.K., Zak P.J., *Time-to-build and cycles*, Journal of Economic Dynamics and Control, Vol. 23, No.8, pp. 1155–1175, 1999.
- [3] Bambi M., *Endogenous growth and time to build: the AK case*, Journal of Economic Dynamics and Control, Vol. 32, pp. 1015–1040, 2008.
- [4] Bambi M., Fabbri G., Gozzi F., *Optimal policy and consumption smoothing effects in the time-to-build AK model*, Economic Theory, to appear.
- [5] Bardi M., Capuzzo Dolcetta I., *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhauser, Boston, 1997.
- [6] Benhabib J., Rustichini A., *Vintage capital, investment, and growth*, Journal of Economic Theory, Vol. 55, No.1, pp.323–339, 1991.
- [7] Boucekkine R., De la Croix D., Licandro O., *Modelling vintage structures with DDE's: Principles and applications*, Mathematical Population Studies, Vol.11, No. 3/4, pp.151–179, 2004.
- [8] Bruckner A., *Differentiation of real functions*, Lecture Notes in Mathematics, Vol. 659, Springer-Verlag, Berlin-Heidelberg-New York, 1978.  
Bruckner A., *Differentiation of real functions*. Second edition. CRM Monograph Series, 5. American Mathematical Society, Providence, RI, 1994. xii+195 pp. ISBN: 0-8218-6990-6 26-01
- [9] Cannarsa P., *Regularity properties of solutions to Hamilton-Jacobi-Bellman equations in infinite dimensions and nonlinear optimal control*, Differential and Integral Equations, Vol.2, No.4, pp. 479–493, 1989.

- [10] Carlier G., Taharoui R., *On some optimal control problems governed by a state equation with memory*, ESAIM Control, Optimisation and Calculus of Variations, Vol.14, No.4, pp.725-743, 2008.
- [11] Carlson D., *The Existence of Catching-Up Optimal Solutions for a Class of Infinite Horizon Optimal Control Problems with Time Delay*, SIAM Journal on Control and Optimization, Vol.28, No.2, pp. 402–422, 1990.
- [12] Fabbri G., Gozzi F., Swiech A., *Verification theorem and construction of  $\varepsilon$ -optimal controls for control of abstract evolution equations*, Journal of Convex Analysis, 17(2), pp. 611-642, 2010.
- [13] Federico S., *Stochastic optimal control problems for pension funds management*, PhD Thesis, Scuola Normale Superiore (Pisa), 2009. Available at [sfederico.altervista.org](http://sfederico.altervista.org).
- [14] Federico S., Goldys B., Gozzi F., *HJB equations for the optimal control of differential equations with delays and state constraints, I: regularity of viscosity solutions*, SIAM Journal on Control and Optimization, Vol. 48, No. 8, pp. 4821-5653, 2009.
- [15] Hale J.K., Verduyn Lunel S.M., *Introduction to Functional Differential Equations*, Springer-Verlag, Applied Mathematical Sciences, Vol. 99, 1993.
- [16] Ioannides Y., Taub B., *On dynamics with time-to-build investment technology and non-time-separable leisure*, Journal of Economic Dynamics and Control, Vol.16, No.2, pp.225–241, 1992.
- [17] Kydland F.E., Prescott E.C., *Time-to-build and aggregate fluctuations*, Econometrica, Vol. 50, No. 6, pp. 1345-1370, 1982.
- [18] Li X., Yong J., *Optimal Control Theory for Infinite-Dimensional Systems*, Birkhauser Boston, 1995.
- [19] Li X., Yong J., Zhou X.Y., *Stochastic Verification theorems within the framework of viscosity solutions*, SIAM Journal Control Optimization, 37, pp. 243–253, 1997.
- [20] Saks S., *Theory of the integral*, Dover Publications, 1964.
- [21] Yong J., Zhou X.Y., *Stochastic Controls - Hamiltonian Systems and HJB equations*, Springer-Verlag, Berlin-New York, 1999.