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## Added mass for wave motion in density-stratified fluids

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In a density-stratified fluid the existence of a restoring force – buoyancy – alters fundamentally the dynamics of submerged bodies. This effect can be described as a modification of their added mass. We investigate it in the simplest possible configuration – the small-amplitude motion of an unbounded Boussinesq uniformly stratified fluid – and for the simplest possible bodies – a horizontal circular cylinder in two dimensions and a sphere in three dimensions.

**General case** The concept of added mass pertains to the irrotational flow of a homogeneous fluid<sup>1</sup>. It follows from the linearity of the flow, such that the translation of a rigid body at the velocity  $\mathbf{U}$  creates a velocity potential  $\phi = \phi_i U_i$ . By defining the added mass tensor  $m_{ij}$  according to

$$m_{ij} = \rho \oint_S n_i \phi_j \, d^2S,$$

with  $\rho$  the density of the fluid,  $S$  the surface of the body and  $\mathbf{n}$  the inward normal to  $S$ , we may express the pressure force on the body as

$$F_i = \oint_S p n_i \, d^2S = -\rho \frac{d}{dt} \oint_S \phi n_i \, d^2S = -m_{ij} \frac{dU_j}{dt},$$

the impulse and (kinetic) energy of the fluid as, respectively,

$$I_i = \rho \oint_S \phi n_i \, d^2S = m_{ij} U_j, \quad E = \frac{1}{2} \rho U_i \oint_S \phi n_i \, d^2S = \frac{1}{2} m_{ij} U_i U_j,$$

and the dipole strength of the body through

$$\rho \mathcal{D}_i = \rho \oint_S \left( n_i \phi - x_i \frac{\partial \phi}{\partial n} \right) \, d^2S = (m_f \delta_{ij} + m_{ij}) U_j,$$

with  $V$  the volume of the body and  $m_f = \rho V$  the mass of the displaced fluid. Accordingly, added mass characterizes the flow fully, being involved in the dynamics of the body through  $\mathbf{F}$ , in the dynamics of the fluid as a whole through  $\mathbf{I}$  and  $E$ , and in the dynamics of the distant fluid through  $\mathcal{D}$ .

The small-amplitude Boussinesq motion of a uniformly stratified fluid of buoyancy frequency  $N$  can similarly be described<sup>2–4</sup> in terms of a scalar function  $\chi$ , satisfying the internal wave equation

$$\left( \frac{\partial^2}{\partial t^2} \nabla^2 + N^2 \nabla_h^2 \right) \chi = 0,$$

with the  $z$ -axis directed vertically upwards and the subscript  $_h$  denoting a horizontal projection. The velocity  $\mathbf{u}$  and the disturbances  $p$  in pressure and  $\rho$  in density are related to the wave function through

$$\mathbf{u} = \left( \frac{\partial^2}{\partial t^2} \nabla + N^2 \nabla_h \right) \chi, \quad p = -\rho_0 \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \frac{\partial \chi}{\partial t}, \quad \rho = \rho_0 \frac{N^2}{g} \frac{\partial^2 \chi}{\partial t \partial z},$$

with  $g$  the acceleration due to gravity, while the pressure  $p_0$  and density  $\rho_0$  at rest satisfy  $dp_0/dz = -\rho_0 g$  and  $d\rho_0/dz = -\rho_0 N^2/g$ . As for irrotational flow, the linearity of the wave equation implies a linear relation between the velocity  $\mathbf{U}$  of a translating rigid body and the wave function that it creates, in the form of a temporal convolution  $\chi = \chi_i * U_i$ .

The pressure force on the body follows immediately as

$$F_i = \oint_S p n_i d^2S = -\rho_0 \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \oint_S n_i \chi d^2S.$$

Equations of conservation may be derived for the momentum and total (kinetic and potential) energy of the fluid, of respective densities  $\rho_0 u_i$  and  $\frac{1}{2}\rho_0 u_i^2 + \frac{1}{2}\rho_0 N^2 \zeta^2$ , with  $\zeta = \partial^2 \chi / \partial t \partial z$  the vertical displacement of fluid particles. By integrating the associated fluxes over the surface of the body, we obtain the momentum and energy outputs of the body as, respectively,

$$I_i = \rho_0 \oint_S \left( n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi d^2S, \quad E = \rho_0 U_i \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \oint_S n_i \chi d^2S.$$

Introduction of the Green's function of the wave equation<sup>5</sup> combined with application of standard techniques<sup>6,7</sup> yields a Kirchhoff–Helmholtz integral equation for internal waves, which corrects<sup>8</sup> and generalizes<sup>9</sup>. By expanding this equation at large distances from the body, we obtain the dipole strength of the body, such that

$$\rho_0 \mathcal{D}_i = \rho_0 \oint_S \left[ \left( n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi - x_i \left( \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial n} + N^2 \frac{\partial}{\partial n_h} \right) \chi \right] d^2S.$$

Accordingly, two distinct definitions of the added mass tensor may be proposed. One,

$$m_{ij}^{(1)} = \rho_0 \left( \frac{\partial^2}{\partial t^2} + N^2 \right) \oint_S n_i \chi_j d^2S,$$

involved in pressure and energy through

$$F_i = -m_{ij}^{(1)} * \frac{dU_j}{dt}, \quad E = U_i \left[ m_{ij}^{(1)} * U_j \right],$$

and the other

$$m_{ij}^{(2)} = \rho_0 \oint_S \left( n_i \frac{\partial^2}{\partial t^2} + n_{hi} N^2 \right) \chi_j d^2S,$$

involved in momentum and dipole strength through

$$I_i = m_{ij}^{(2)} * U_j, \quad \rho_0 \mathcal{D}_i = \left[ m_{if} \delta_{ij} \delta(t) + m_{ij}^{(2)} \right] * U_j.$$

The relation between the two is simplified in the monochromatic case, when the excitation  $\mathbf{U}$  and the responses  $\mathbf{u}$ ,  $p$  and  $\rho$  depend on time through the factor  $e^{-i\omega t}$  which is suppressed in the following. The time-dependent added masses are replaced by their temporal Fourier transforms and the above relations assume the more familiar forms

$$F_i = i\omega m_{ij}^{(1)} U_j, \quad \langle E \rangle = \frac{1}{2} \text{Re} \left[ \overline{U_i} m_{ij}^{(1)} U_j \right], \quad I_i = m_{ij}^{(2)} U_j, \quad \rho_0 \mathcal{D}_i = \left[ m_{if} \delta_{ij} + m_{ij}^{(2)} \right] U_j,$$

where  $\bar{\phantom{x}}$  denotes a complex conjugate and  $\langle \phantom{x} \rangle$  a time average. The two added mass tensors are related anisotropically through

$$m_{ij}^{(1)} = \left( 1 - \frac{N^2}{\omega^2} \delta_{i3} \right) m_{ij}^{(2)}.$$

As did<sup>10–12</sup>, in the following we will consider the first definition only and omit the superscript<sup>(1)</sup>.

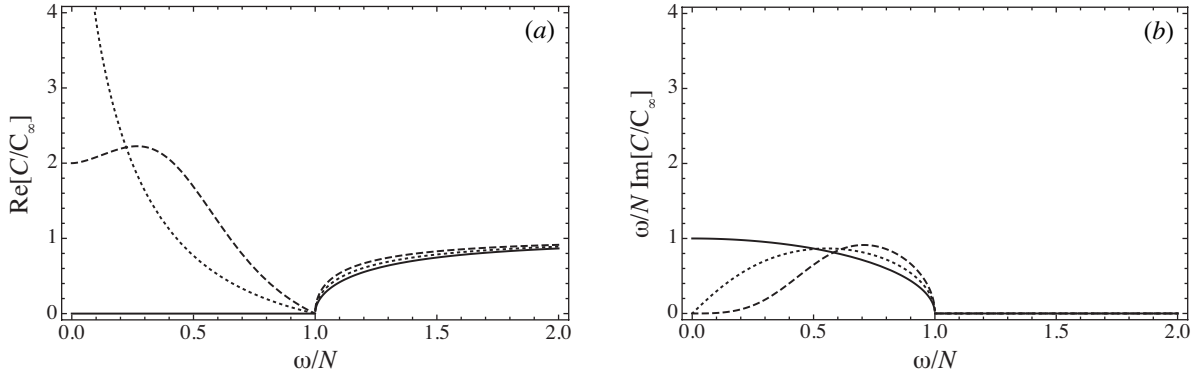


Figure 1 – Coefficients of (a) added mass and (b) damping for an oscillating circular cylinder (—) and a sphere oscillating horizontally (---) or vertically (.....).

**Oscillating bodies** We illustrate these results for two particular oscillating bodies, a horizontal circular cylinder and a sphere, represented as source terms  $q = \sigma \delta_S$  on the right-hand side of the wave equation, namely as surface distributions of singularities of density  $\sigma$ . The condition of fixed normal velocity  $U_n$  on  $S$  becomes an integral equation for  $\sigma$ , solved by use of stretched orthogonal curvilinear coordinates<sup>13</sup> and expansion into circular or spherical harmonics<sup>14</sup>. For rigid oscillation at the velocity  $\mathbf{U}$  the solution is found for the cylinder, of radius  $a$ , as

$$q(\mathbf{x}) = \left\{ \left[ 1 + \left( 1 - \frac{N^2}{\omega^2} \right)^{1/2} \right] U_x \frac{x}{a} + \left[ 1 + \left( 1 - \frac{N^2}{\omega^2} \right)^{-1/2} \right] U_z \frac{z}{a} \right\} \delta(r - a),$$

with  $r = |\mathbf{x}|$ , and for the sphere, also of radius  $a$ , as

$$q(\mathbf{x}) = \left[ \frac{2}{1 + B(\omega/N)} \mathbf{U}_h \cdot \frac{\mathbf{x}_h}{a} + \frac{1}{1 - B(\omega/N)} U_z \frac{z}{a} \right] \delta(r - a),$$

with

$$B\left(\frac{\omega}{N}\right) = \frac{\omega^2}{N^2} \left[ 1 - \left( \frac{\omega^2}{N^2} - 1 \right)^{1/2} \arcsin\left(\frac{N}{\omega}\right) \right].$$

Complex square roots are fixed, in accordance with causality, by replacing  $\omega$  by  $\omega + i0$ , in other words by adding to the real frequency  $\omega$  a positive imaginary part which is later allowed to tend to zero.

The associated added mass coefficients  $C_{ij} = m_{ij}/(\rho_0 V)$ , such that  $\mathcal{D}_i = \int x_i q(\mathbf{x}) d^3x = V[\delta_{ij} + C_{ij}/(1 - \delta_{i3} N^2/\omega^2)]U_j$ , are complex. Their real part represents added inertia and their imaginary part, only present when  $\omega < N$  namely when propagative waves are generated, represents wave damping. They are given for the cylinder by

$$C_h = C_z = \left( 1 - \frac{N^2}{\omega^2} \right)^{1/2},$$

and for the sphere by

$$C_h = \frac{1 - B(\omega/N)}{1 + B(\omega/N)}, \quad C_z = \left( 1 - \frac{N^2}{\omega^2} \right) \frac{B(\omega/N)}{1 - B(\omega/N)}.$$

Their variations, consistent with direct calculations<sup>10,12,15,16</sup> and measurements<sup>17</sup>, are represented in figure 1. In the limit  $\omega/N \rightarrow \infty$  the values  $C_\infty = 1$  for the cylinder and  $\frac{1}{2}$  for the sphere in a homogeneous fluid are recovered.

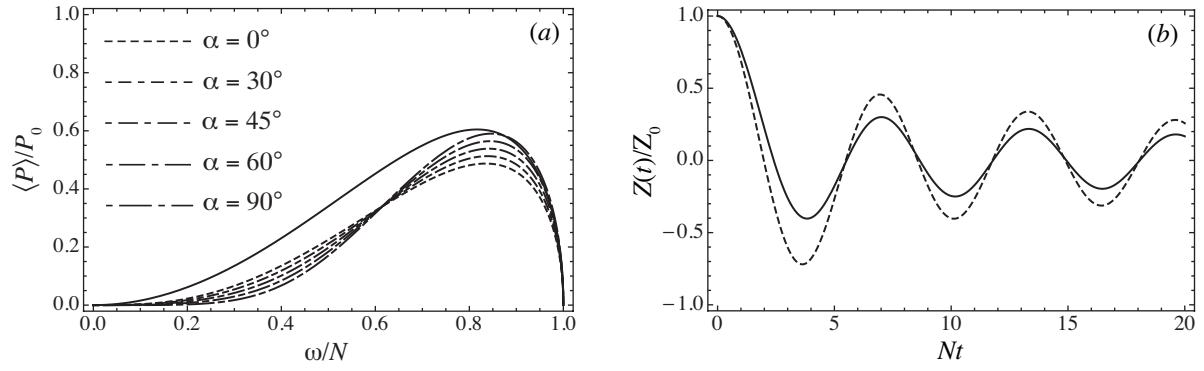


Figure 2 – For a circular cylinder (continuous line) and a sphere (dashed lines), (a) power output of forced oscillations, normalized by  $P_0 = \rho_0 N^3 a^2 A^2$  for the cylinder and  $\rho_0 N^3 a^3 A^2$  for the sphere, and (b) free buoyant oscillations.

**Two applications** A first application of the above is the average power output  $\langle P \rangle$  of a body oscillating with amplitude  $A$  at the angle  $\alpha$  to the vertical. This output, given by

$$\langle P \rangle = \frac{1}{2} \rho_0 V \omega^3 A^2 \operatorname{Im} [C_h^2 \sin^2 \alpha + C_z^2 \cos^2 \alpha],$$

is represented in figure 2(a) for the cylinder and sphere. For  $\omega > N$ , the waves are evanescent and no output is observed. For  $\omega < N$ , the waves are propagative and the output at given  $A$  is maximum at  $\omega/N = \sqrt{2/3} \approx 0.82$  for the cylinder, and at  $\omega/N$  varying weakly, between 0.84 and 0.85, with  $\alpha$  for the sphere.

A second application is the oscillation  $Z(t)$  of a body displaced slightly by  $Z_0$  from its neutral buoyancy level then released at  $t = 0$ . The temporal Fourier transform of the oscillation is

$$\frac{Z(\omega)}{Z_0} = \frac{i}{\omega} \frac{1 + C_z(\omega)}{1 + C_z(\omega) - N^2/\omega^2},$$

yielding  $Z(t)/Z_0 = J_0(Nt)$  for the cylinder and  $\frac{1}{2}\pi \mathbf{E}_1(Nt)$  for the sphere, with  $J_\nu$  a Bessel function and  $\mathbf{E}_\nu$  a Weber function. The oscillation, represented in figure 2(b), is consistent with direct calculation and measurement<sup>18</sup>. Experiments for larger initial displacements<sup>19,20</sup> have pointed out the importance of viscous damping, and the topic remains an area of active research<sup>21–23</sup>.

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