Explicit constructions of operator scaling Gaussian fields

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Abstract

We propose an explicit way to generate a large class of Operator scaling Gaussian random fields (OSGRF). Such fields are anisotropic generalizations of self-similar fields. More specifically, we are able to construct any Gaussian field belonging to this class with given Hurst index and exponent. Our construction provides - for simulations of texture as well as for detection of anisotropies in an image - a large class of models with controlled anisotropic geometries and structures.

Key words: Operator scaling Gaussian random field, anisotropy, pseudo-norms, harmonizable representation.
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1. Introduction

Random fields are a useful tool for modelling spatial phenomena such as environmental fields, including for example, hydrology, geology, oceanography and medical images. Particularly important is the fact that in many cases these random fields have an anisotropic nature in the sense that they have different geometric characteristics along different directions (see, for example, Davies and Hall ([9]), Bonami and Estrade ([4]) and Benson, et al.([3])).

Moreover, many times the model chosen has to include some statistical dependence structure that might be present across the scales. For this purpose, the usual assumption of self-similarity is formulated. Unfortunately, the classical notion of self-similarity (see [15]), defined for a field $\{X(x)\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$\{X(ax)\}_{x\in\mathbb{R}^d} \overset{\mathcal{L}}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d}$$

for some $H \in \mathbb{R}$ (called the Hurst index), is genuinely isotropic and therefore has to be changed to fit anisotropic situations.

For this reason, there has been an increasing interest in defining a suitable concept for anisotropic self-similarity. Many authors have developed techniques to handle anisotropy in the scaling. The main papers that have to be mentioned in this context are those of Hudson and Mason, Schertzer and Lovejoy (see [12, 17, 18]).

This motivated the introduction by Biermé, Meerschaert and Scheffler of operator scaling random fields (OSRF) in [6]. These fields satisfy the following scaling property :

$$\{X(a^E x)\}_{x\in\mathbb{R}^d} \overset{\mathcal{L}}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d} , \tag{1.1}$$

for some $d\times d$ matrix E with positive real parts of the eigenvalues.

A large class of random fields obeys this property. For example the Fractional Brownian Field (FBM) and the Fractional Brownian Sheet (FBS) are both Operator Scaling Gaussian Random Fields (OSGRF) with exponent $E = Id$. Denote $<\cdot,\cdot>$ the Euclidean scalar product of $\mathbb{R}^d$ defined for any $x = (x_1,\cdots,x_d)\in\mathbb{R}^d$ and $y = (y_1,\cdots,y_d)\in\mathbb{R}^d$ as $<x,y>=\sum_{i=1}^d x_i y_i$. Recall that the FBS is the Gaussian field $\{B_{H_1,\cdots,H_d}(x)\}_{x\in\mathbb{R}^d}$ defined for some $(H_1,\cdots,H_d)\in(0,1)^d$ as :

$$B_{H_1,\cdots,H_d}(x) = \int_{\mathbb{R}^d} \frac{(\mathrm{e}^{\mathrm{i}<x,\xi>}-1)\mathrm{d}\widehat{W}(\xi)}{|\xi_1|^{H_1+1/2}\dots|\xi_d|^{H_d+1/2}} ,$$

where $\mathrm{d}\widehat{W}$ is the Fourier transform of white noise on $\mathbb{R}^d$. This Gaussian field enjoys with the following scaling property : for all $(a_1,\cdots,a_d)\in(\mathbb{R}_+)^d$

$$\{B_{H_1,\cdots,H_d}(a_1x_1,\cdots,a_dx_d)\}_{x=(x_1,\cdots,x_d)\in\mathbb{R}^d} \overset{\mathcal{L}}{=} \{a_1^{H_1}\cdots a_d^{H_d} B_{H_1,\cdots,H_d}(x_1,\cdots,x_d)\}_{x\in\mathbb{R}^d} .$$

In particular, if we set $a = a_1 = \cdots = a_d$, we recover that

$$\{B_{H_1,\cdots,H_d}(ax_1,\cdots,ax_d)\}_{x=(x_1,\cdots,x_d)\in\mathbb{R}^d} \overset{\mathcal{L}}{=} \{a^{H_1+\cdots+H_d} B_{H_1,\cdots,H_d}(x_1,\cdots,x_d)\}_{x\in\mathbb{R}^d} ,$$

that is $B_{H_1,\cdots,H_d}$ satisfies Property (2.1) with $E = Id$ and $H = H_1 + \cdots + H_d$.

In [6] the existence of OSRF with stationary increments in the stable case for any $d \times d$ matrix E with positive real parts of the eigenvalues is proved. A special class of OSRF is defined through its harmonizable representation. For Gaussian models, which is here the case of interest, it reduces to consider an integral representation of the form

$$\int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1) f^{1/2}(\xi) \mathrm{d}\widehat{W}(\xi) ,$$

where f is a positive valued function defined on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} (1 \wedge \|\xi\|^2) f(\xi) \mathrm{d}\xi \ < \infty ,$$

for any norm $\|\cdot\|$ on $\mathbb{R}^d$. Such a function f is called **a spectral density**. In order to recover the scaling property (2.1), the spectral density f is required to satisfy additional specific homogeneity properties (see Section 2 below). In [6], such spectral densities are defined by an integral formula. This is a non explicit definition in the sense that actual computations require numerical approximations. However, these calculations are, in practice, quite difficult to implement. Nevertheless, a simpler and explicit formula is furnished in the particular case of diagonalizable matrices.

In this paper, we mainly aim at providing **a complete description through explicit formulae** for the spectral densities in the model defined in [6]. We focus on a specific case : The **Gaussian** model. The motivation of this restriction is twofold. On the one hand, it is a reasonable assumption in many applications; on the other hand, to improve the model, it is necessary to understand and classify its geometrical properties which is easier in the Gaussian case.

Our main results are stated and proved in Section 3. The first ones, Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4

1. reduces the construction of an explicit example of OSGRF for a fixed matrix E and an admissible Hurst exponent H (as defined in Section 2) to four particular cases related to specific geometries,
2. provides an explicit example in each of these four specific cases.

Thus, we are able to provide an explicit example of OSGRF satisfying Equation (1.1) and then extend the already existing results. Moreover, our second result Theorem 3.2, gives a very simple relationship existing between all possible spectral densities associated to a given exponent E. This result is not formal and can also be turned into an algorithm which generates different fields – with different geometries – satisfying Equation (1.1) for the same matrix of anisotropy E.

These results have important consequences. Firstly, it allows to define the studied class of OSGRF from four specific cases. Furthermore we give a complete description of the whole class of spectral densities of these fields. Finally, since our construction is explicit, the numerical simulations of OSGRF become much easier. Thus, our approach provides an explicit definition of an interesting and large class of fields for simulations of textures with new geometries. There is actually a practical motivation to be able to compare natural/real images (of clouds, bones,...) and models with controlled anisotropy.

In the following pages, we are given $d \in \mathbb{N} \setminus \{0\}$ and E a $d \times d$ matrix with positive real parts of the eigenvalues. We define

$$\lambda_{\min}(E) = \min_{\lambda \in Sp(E)} (\mathrm{Re}(\lambda)) \ .$$

For any $a > 0$ recall that a^E is defined as follows

$$a^E = \exp(E \log(a)) = \sum_{k \geq 0} \frac{\log^k(a)\ E^k}{k!} \ .$$

As usual, E^t denotes the transpose of the matrix E.
We denote $|\cdot|$ the Euclidean norm defined for any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ as

$$|x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2} \ .$$

2. Presentation of the model : Operator Scaling Random Fields (OSRF)

Let us recall some preliminary facts about Operator Scaling Random Fields (OSRF) and Operator Scaling Gaussian Random Fields (OSGRF). We refer to [6] for all the material of this section.

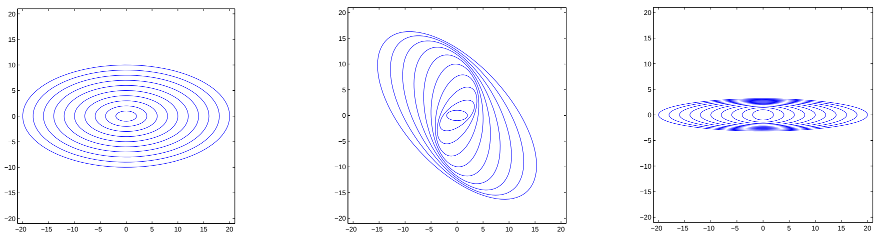
Definition 2.1. *A scalar–valued random field $\{X(x)\}_{x \in \mathbb{R}^d}$ is called operator–scaling if there exists a $d \times d$ matrix E with positive real parts of the eigenvalues and some $H > 0$ such that*

$$\{X(a^E x)\}_{x \in \mathbb{R}^d} \stackrel{\mathcal{L}}{=} \{a^H X(x)\}_{x \in \mathbb{R}^d} \ , \tag{2.1}$$

where $\stackrel{(\mathcal{L})}{=}$ denotes equality of all finite-dimensional marginal distributions. Matrix E and real number H are respectively called an exponent (of scaling) or an anisotropy, and an Hurst index of the field.

Remark 2.1. *In general, the exponent E and the Hurst index H of an OSRF are not unique.*

Thus the usual notion of self-similarity is extended replacing **usual scaling**, (corresponding to the case where $E = Id$) by **a linear scaling** involving matrix E (see Figure 1 below). It allows to define new classes of random fields with new geometry and structure.



$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \{1, \cdots, 10\}$$

$$E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \lambda \in \{1, \cdots, 10\}$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \lambda \in \{1, \cdots, 10\}$$

Figure 1:
Action of a linear scaling $x \mapsto \lambda^E x$ on a ellipsis.

As said in the introduction, when the matrix E is given, the class of OSRF with exponent E may be very general. In [6], for any given admissible matrix E, the existence of OSRF with stationary increments is proved using a harmonisable representation.

Recall that according to [19] or [11], given a stochastically continuous Gaussian field with stationary increments $\{X(x)\}_{x \in \mathbb{R}^d}$, its covariance can be represented as

$$\mathbb{E}(X(x)X(y)) = \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1)(\mathrm{e}^{-\mathrm{i}<y,\xi>} - 1)\mathrm{d}\mu(\xi) + < x, Qy > ,$$

where Q is a $d \times d$ non–negative definite matrix and μ a non–negative symmetric measure μ such that

$$\int_{\mathbb{R}^d} (1 \wedge \|\xi\|^2)\mathrm{d}\mu(\xi) \ .$$

for any norm $\| \cdot \|$ on $\mathbb{R}^d$. Measure μ and matrix Q are proved to be unique. Measure μ is called the spectral measure of $\{X(x)\}_{x \in \mathbb{R}^d}$. In the case where this measure is absolutely continuous with respect to Lebesgue measure, the density function of μ is called the spectral density of the Gaussian field $\{X(x)\}_{x \in \mathbb{R}^d}$. In this case, the Gaussian field $\{X(x)\}_{x \in \mathbb{R}^d}$ can thus be represented as

$$X(x) \stackrel{\mathcal{L}}{=} \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1) f^{1/2}(\xi)\mathrm{d}\widehat{W}(\xi) \ , \tag{2.2}$$

with

$$\int_{\mathbb{R}^d} (1 \wedge \|\xi\|) f(\xi) \mathrm{d}\xi < +\infty \ , \tag{2.3}$$

for any norm $\|\cdot\|$ on $\mathbb{R}^d$. This representation is then called the harmonisable representation of the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$.

To prove the existence of OSRGF with stationary increments for any admissible matrix E, a quite natural approach is then to use a harmonisable representation of the form (2.2). In [6], the following result is proved. We state it only in the Gaussian case :

Theorem 2.1. *Let ρ a continuous function defined on $\mathbb{R}^d$ with non–negative values such that for all $x \in \mathbb{R}^d\backslash\{0\}$, $\rho(x) \neq 0$. Assume that ρ is E^t-homogeneous that is :*

$$\forall a > 0,\, \forall \xi \in \mathbb{R}^d,\, \rho(a^{E^t}\xi) = a\rho(\xi) \ .$$

Then the Gaussian field $\{X_\rho(x)\}_{x\in\mathbb{R}^d}$ defined as follows

$$X_\rho(x) = \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1)\rho(\xi)^{-H-\frac{Tr(E)}{2}} \mathrm{d}\widehat{W}(\xi), \tag{2.4}$$

exists and is stochastically continuous if and only if $H \in (0, \lambda_{\min}(E))$. Moreover, this field has the following properties :

1. *Stationary increments, that is for any $h \in \mathbb{R}^d$*

$$\{X_\rho(x+h) - X_\rho(h)\}_{x\in\mathbb{R}^d} \overset{(fd)}{=} \{X_\rho(x)\}_{x\in\mathbb{R}^d} \ .$$

2. *Operator scaling : The scaling relation (2.1) is satisfied.*

Remark 2.2. *Through this new class of Gaussian fields, even if it is a quite general model, we do not describe the whole class of OSRGF with stationary increments.*

Remark 2.3. *If $H \in (0, \lambda_{\min}(E))$, $f(\xi) = \rho(\xi)^{-2H-Tr(E)}$ is proved to be a spectral density in the sense that (2.3) holds. Moreover, the spectral density of the Gaussian field $\{X_\rho(x)\}_{x\in\mathbb{R}^d}$ is f. Observe that f is continuous and satisfies a specific homogeneity assumption. This homogeneity assumption is necessary for the operator scaling property of the Gaussian field $\{X_\rho(x)\}_{x\in\mathbb{R}^d}$ whereas the continuity assumption ensures that the field $\{X_\rho(x)\}_{x\in\mathbb{R}^d}$ being defined is stochastically continuous.*

The main difficulty to overcome is to define **suitable spectral densities** of this new class of Gaussian fields using continuous, E^t-homogeneous functions with positive values. In [16] such functions are called $(\mathbb{R}^d, E^t)$ **pseudo-norms**. They can be defined using an integral formula (see Theorem 2.11 of [6]) :

Proposition 2.2. *Function ρ defined as*

$$\rho(\xi) = \int_{S_0}\int_0^{\infty} (1 - \cos(< x, r^{E^t}\theta >))\frac{\mathrm{d}r}{r^2}\mathrm{d}\mu(\theta) \ ,$$

is continuous with positive values and E^t-homogeneous. Here S_0 denotes the unit sphere of $\mathbb{R}^d$ for a well chosen norm defined from E and μ a finite measure on S_0.

Remark 2.4. *Remark that this formula is not the most appropriate for numerical simulations since we need to approximate an integral. In what follows, we will give simpler examples of $(\mathbb{R}^d, E^t)$ pseudo-norms in the sense that these examples lead to* **exact** *numerical computations. We then conclude that we give* **explicit** *examples of $(\mathbb{R}^d, E^t)$ pseudo-norms (in the numerical sense).*

Remark 2.5. *We also refer to P.G.Lemarie (see [16]) whose definition of $(\mathbb{R}^d, E^t)$ pseudo-norms is slightly different (see Remark 3.1 below).*

Finally, in the special case where matrix E is diagonalizable, an explicit expression is given (Corollary 2.12 of [6]) :

Proposition 2.3. *Let E a diagonalizable matrix with positive eigenvalues*

$$0 < \lambda_1 \leq \cdots \leq \lambda_d,$$

with associated eigenvectors

$$\theta_1, \cdots, \theta_d,$$

and $C_1, \cdots, C_d > 0$. Then for any $\tau < 2\lambda_{\min}(E)$

$$\rho(x) = \left(\sum_{j=1}^{d} C_j | < x, \theta_j > |^{\tau/\lambda_j}\right)^{1/\tau} ,$$

is a continuous, E^t-homogeneous function with positive values.

In this paper, we aim at extending these results and then describing for any given admissible matrix E and Hurst index H, all the spectral densities of this model of OSGRF with stationary increments in an **explicit** way.

3. Definition of explicit spectral densities of the model

As has already been said in Section 2, the main difficulty is to define appropriate spectral densities of the model. To this end, we note that the class of the spectral density used in [6] is intimately related to the class of the so-called **pseudo-norms** defined in [16]. We then explicit the link between two $(\mathbb{R}^d, E)$ pseudo-norms when the matrix E is given. Thereafter using a Jordan reduction, for each matrix E with positive real parts of the eigenvalues, we give an explicit example of a suitable spectral density of the studied model. Combining these two results, we entirely describe **in a explicit way** the class of spectral densities of the Gaussian fields considered in [6].

3.1. More about pseudo-norms

Let us first recall some well known facts about pseudo-norms which can be found with more details in [16]. This concept is fundamental when defining anisotropic functional spaces since using pseudo-norms allows to introduce anisotropic topology on $\mathbb{R}^d$. Thus, even if the introduction of this concept is not necessary to the definition of spectral densities, it is of great importance to relate the notion of anisotropic spectral densities to the concept of pseudo-norms. This gives us indeed all the tools of "anisotropic functional analysis" to study, for example, the sample paths properties of the fields in anisotropic spaces (see [8]) and to better understand the inherent topology of these spaces.

Definition 3.1. *A function ρ defined on $\mathbb{R}^d$ is a $(\mathbb{R}^d, E)$ pseudo-norm if it satisfies the three following properties :*

1. *ρ is continuous on $\mathbb{R}^d$,*
2. *ρ is E-homogeneous, i.e. $\rho(a^E x) = a\rho(x) \quad \forall x \in \mathbb{R}^d$, $\forall a > 0$,*
3. *ρ is positive on $\mathbb{R}^d \setminus \{0\}$.*

Remark 3.1. *Our definition of $(\mathbb{R}^d, E)$ pseudo-norm is a slightly modified version of the concept of pseudo-norm on $(\mathbb{R}^d, A)$ defined by P.G.Lemarié in [16]. In [16], A denotes a matrix with eigenvalues having a modulus greater than one. A pseudo-norm on $(\mathbb{R}^d, A)$ is a function satisfying properties 1 and 3 of the previous definition and the following property :*

$$\rho(Ax) = |\det(A)|\rho(x), \text{ for any } x \text{ in } \mathbb{R}^d \ .$$

Further for any $d \times d$ matrix A with eigenvalues having modulus greater than one and any compactly supported smooth function ϕ, an example of pseudo-norm on $(\mathbb{R}^d, A)$ is provided by

$$\rho_\phi(x) = \sum_{j \in \mathbb{Z}} |\det(A)|^j \phi(A^j x) \ .$$

Remark that if ρ is a $(\mathbb{R}^d, E)$ pseudo-norm then $\rho(\cdot)^{1/Tr(E)}$ is a pseudo-norm on $(\mathbb{R}^d, A)$ in the sense of [16] with $A = a^E$ for any given $a > 0$. The properties satisfied by $(\mathbb{R}^d, E)$ pseudo-norms are very similar to those of pseudo-norms on $(\mathbb{R}^d, A)$, as proved in [16]. Moreover, the example of pseudo-norm on $(\mathbb{R}^d, A)$ given in [16] can be adapted to our case. Indeed for any compactly supported smooth function ϕ

$$\rho_\phi(x) = \int_0^{+\infty} \phi(a^{-E} x) \mathrm{d}a \ ,$$

is a $(\mathbb{R}^d, E)$ pseudo-norm. This formula also leads to numerical approximations and thus is a non explicit one.

The term of pseudo-norm is justified by the following proposition which is proved for instance in [16] or [6] :

Proposition 3.1. *Let ρ a $(\mathbb{R}^d, E)$ pseudo-norm. There exists $C > 0$ such that*

$$\rho(x + y) \le C(\rho(x) + \rho(y)), \quad \forall x, y \in \mathbb{R}^d \ .$$

3.2. Relationship between two given pseudo–norms

The main result of this section is the description of all the $(\mathbb{R}^d, E)$ pseudo-norms for a given matrix E :

Theorem 3.2. *Let ρ_1 be a $(\mathbb{R}^d, E)$ pseudo-norm. Then ρ_2 is a $(\mathbb{R}^d, E)$ pseudo-norm if and only if there exists a continuous and positive function g defined on $\mathbb{R}^d \setminus \{0\}$ such that*

$$\rho_2(\xi) = g(\rho_1(\xi)^{-E}\xi)\rho_1(\xi) \ . \tag{3.1}$$

Proof. Let ρ_1 and ρ_2 be two $(\mathbb{R}^d, E)$ pseudo-norms. Then the function $g = \frac{\rho_2}{\rho_1}$ is continuous, positive on $\mathbb{R}^d \setminus \{0\}$ and satisfies for all $a > 0$,

$$g(a^E\xi) = g(\xi).$$

In particular, for a fixed ξ and $a = \rho_1(\xi)^{-1}$, it follows that

$$\rho_2(\xi) = g(\xi)\rho_1(\xi) = g(\rho_1(\xi)^{-E}\xi)\rho_1(\xi) \ .$$

The converse is straightforward. ∎

Consider now the special case $E = Id$. Theorem 3.2 implies the following corollary

Corollary 3.1. *Let $\{X(x)\}_{x\in\mathbb{R}^d}$ be a Gaussian field with stationary increments admitting a continuous spectral density. Assume that X is self–similar with Hurst index H. Then, there exists a continuous function S defined on the unit sphere $\{\xi \in \mathbb{R}^d,\ |\xi| = 1\}$ with positive values such that*

$$\{X(x)\}_{x\in\mathbb{R}^d} \stackrel{\mathcal{L}}{=} \left\{\int_{\mathbb{R}^d} \left(\frac{e^{i<x,\xi>}-1}{|\xi|^{H+d/2}}\right) S\left(\frac{\xi}{|\xi|}\right) \mathrm{d}\widehat{W}(\xi)\right\}_{x\in\mathbb{R}^d} \ . \tag{3.2}$$

Proof. By assumption the Gaussian field with stationary increments $\{X(x)\}_{x\in\mathbb{R}^d}$ admits a continuous spectral density denoted f. Then

$$X(x) \stackrel{\mathcal{L}}{=} \int_{\mathbb{R}^d} (e^{i<x,\xi>}-1)f^{1/2}(\xi)\mathrm{d}\widehat{W}(\xi) \ .$$

Since X is self–similar with Hurst index H,

$$\{X(ax)\}_{x\in\mathbb{R}^d} \stackrel{\mathcal{L}}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d} \ .$$

By assumption,

$$X(ax) \stackrel{\mathcal{L}}{=} \int_{\mathbb{R}^d} (e^{i<ax,\xi>}-1)f^{1/2}(\xi)\mathrm{d}\widehat{W}(\xi) \ .$$

Set now $\zeta = a\xi$ in the harmonizable representation of X and deduce that

$$X(ax) \stackrel{\mathcal{L}}{=} a^{-d/2} \int_{\mathbb{R}^d} (e^{i<x,\zeta>}-1)f^{1/2}(a^{-1}\zeta)\mathrm{d}\widehat{W}(\zeta) \ .$$

We now identify the two spectral densities of the two Gaussian fields $\{X(ax)\}_{x\in\mathbb{R}^d}$, $\{a^H X(x)\}_{x\in\mathbb{R}^d}$ which are equal in law . It implies that

$$a^{-d/2} f(a^{-1}\xi) = a^H f(\xi) \; , \tag{3.3}$$

that is $\rho(\xi) = f(\xi)^{-1/(H+d/2)}$ is a $(\mathbb{R}^d, Id)$ pseudo–norm.
We now apply Theorem 3.2 with $E = Id$. Then any $(\mathbb{R}^d, Id)$ pseudo–norm ρ can be written

$$\rho(\xi) = g(|\xi|^{-1}\xi)|\xi| \; , \tag{3.4}$$

since the Euclidean norm $|\cdot|$ is a $(\mathbb{R}^d, Id)$ pseudo–norm.
We deduce that any continuous spectral density can be written as

$$f(\xi) = \left(g(|\xi|^{-1}\xi)|\xi|\right)^{-H-d/2} \; .$$

Set now $S(\xi) = g(|\xi|^{-1}\xi)^{-H-d/2}$ to deduce the required result. ■

Thus we recover well–known results of Dobrushin (see [10]). Indeed, in [10] a complete description of self-similar generalized Gaussian fields with stationary r–th increments is given. It implies in particular Corollary . The class of anisotropic Gaussian field defined by the representation (3.2) has been widely studied (see [5, 4]). Recently in [14], Istas has defined an estimator of S using shifted generalized quadratic variations. Let us emphasize that if an anisotropy E may be known, using Theorem 3.2 and a fixed $(\mathbb{R}^d, E^t)$ pseudo–norm ρ_1 (see Section 3.3 below), one can probably define in a similar way an estimator of function g defined in (3.1).

In next section, we now define explicit examples of $(\mathbb{R}^d, E^t)$ pseudo-norms.

3.3. Explicit construction of $(\mathbb{R}^d, E^t)$ *pseudo-norms*

The result of this section is based on the real Jordan decomposition of any $d \times d$ matrix E.

Proposition 3.3. *Any $d \times d$ matrix E can be written, using the real Jordan reduction as*

$$E = P \begin{pmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_{m_1+m_2} \end{pmatrix} P^{-1},$$

where $(m_1, m_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$, with

1. *For all $\ell_1 \in \{1, \cdots, m_1\}$,*

$$E_{\ell_1} = \lambda_{\ell_1} Id \text{ or } E_{\ell_1} = \begin{pmatrix} \lambda_{\ell_1} & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_{\ell_1} \end{pmatrix},$$

where $\lambda_{\ell_1} \in \mathbb{R}$,

2. *For all* $\ell_2 \in \{1, \cdots, m_2\}$,

$$E_{m_1+\ell_2} = \begin{pmatrix} A_{\ell_2} & & 0 \\ & \ddots & \\ 0 & & A_{\ell_2} \end{pmatrix} \text{ or } E_{m_1+\ell_2} = \begin{pmatrix} A_{\ell_2} & I_2 & & 0 \\ & \ddots & \ddots & \\ & & & I_2 \\ 0 & & & A_{\ell_2} \end{pmatrix},$$

$$\text{with } A_{\ell_2} = \begin{pmatrix} \alpha_{\ell_2} & \beta_{\ell_2} \\ -\beta_{\ell_2} & \alpha_{\ell_2} \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ where } (\alpha_{\ell_2}, \beta_{\ell_2}) \in \mathbb{R}^2$$

As a consequence of the real Jordan decomposition, we state the following proposition :

Proposition 3.4. *The notations are those of Proposition 3.3. For any ℓ, denote d_ℓ the size of the matrix E_ℓ. Assume that for each $1 \leq \ell \leq m_1 + m_2$, we are given a $(\mathbb{R}^{d_\ell}, E_\ell^t)$ pseudo–norm τ_ℓ. Define the function φ for any $\xi = (\xi_1, \cdots, \xi_{m_1+m_2}) \in \prod_{\ell=1}^{m_1+m_2} \mathbb{R}^{d_\ell}$ as*

$$\varphi(\xi) = \left(\tau_1^2(\xi_1) + \cdots + \tau_{m_1+m_2}^2(\xi_{m_1+m_2})\right)^{1/2} .$$

Then, the function ρ defined for any $\xi \in \mathbb{R}^d$ as

$$\rho(\xi) = \varphi(P^t \xi)$$

is a $(\mathbb{R}^d, E^t)$ pseudo-norm. Further $f = \rho^{-(2H+Tr(E))}$ is a suitable spectral density of an operator scaling Gaussian random field with stationary increments.

Proof. Let $F = P^{-1}EP$. Then for any $\zeta = (\zeta_1, \cdots, \zeta_{m_1+m_2}) \in \prod_{\ell=1}^{m_1+m_2} \mathbb{R}^{d_\ell}$:

$$\begin{aligned} \varphi(a^{F^t}\zeta) &= \left(\tau_1^2(a^{E_1^t}\zeta_1) + \cdots + \tau_{m_1+m_2}^2(a^{E_{m_1+m_2}^t}\zeta_{m_1+m_2})\right)^{\frac{1}{2}} \\ &= \left(a^2\tau_1^2(\zeta_1) + \cdots + a^2\tau_{m_1+m_2}^2(\zeta_{m_1+m_2})\right)^{\frac{1}{2}} \\ &= a\varphi(\zeta) \end{aligned}$$

It follows that

$$\rho(a^{E^t}\xi) = \varphi(P^t a^{E^t}\xi) = \varphi(a^{F^t}P^t\xi) = a\varphi(P^t\xi) = a\rho(\xi)$$

The conclusion is then straightforward. ■

Let us illustrate Proposition 3.4 through an example :

Example 3.5. *Set*

$$E = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} .$$

Note that E is a diagonalizable matrix since it has two different eigenvalues. One has $E = PDP^{-1}$ with

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} .$$

A $(\mathbb{R}^d, D)$ pseudo-norm can be defined as

$$\rho_D(\xi) = |\xi_1|^{1/2} + |\xi_2| \ .$$

Hence Proposition 3.4 allows to give an explicit expression of a $(\mathbb{R}^d, E^t)$ pseudo-norm :

$$\rho_E(\xi) = \rho_D(P^t\xi) = |\xi_1|^{1/2} + |\xi_2 - \xi_1| \ .$$

Remark that in this case, Corollary 2.12 *of [6] exactly yields the same result since it gives an explicit example of $(\mathbb{R}^d, E^t)$ pseudo-norm in the special where matrix E is diagonalizable.*

Thus, it is sufficient to define an explicit pseudo-norm for the four following matrices.

1. $E_1(\lambda) = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}, \ \lambda \in \mathbb{R}_+^*.$
2. $E_2(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, \lambda \in \mathbb{R}_+^*.$
3. $E_3(\alpha,\beta) = \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix}$ with $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $(\alpha,\beta) \in \mathbb{R}_+^* \times \mathbb{R}$.
4. $E_4(\alpha,\beta) = \begin{pmatrix} A & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & A \end{pmatrix}$ with $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $(\alpha,\beta) \in \mathbb{R}_+^* \times \mathbb{R}$.

We emphasize that Proposition 3.4 above has two important consequences :

- The first consequence is that, Lemmas 3.1, 3.2, 3.3, 3.4, Proposition 3.4 and Theorem 3.2 give a complete description of the spectral densities and then of the class of Gaussian fields introduced in [6].
- Moreover, it implies that all the Gaussian fields belonging to the class being studied can be generated from **four generic cases** corresponding to four specific geometries.

In the four following lemmas, we define a $(\mathbb{R}^d, E)$ pseudo-norm in each generic case. Recall that we denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$.

Let us first consider the case $E = E_1(\lambda)$ for some $\lambda \in \mathbb{R}_+^*$:

Lemma 3.1. *The function* ρ_1*, defined for* $\xi \in \mathbb{R}^d$ *by*

$$\rho_1(\xi) = |\xi|^{1/\lambda} \ , \tag{3.5}$$

is a $(\mathbb{R}^d, E_1^t(\lambda))$ *pseudo-norm.*

Proof. The conclusion is straightforward.

■

We now consider the case $E = E_2(\lambda)$ for some $\lambda \in \mathbb{R}_+^*$:

Lemma 3.2. *Let us define the functions* τ_i *and* Φ_i *for any* $i \in \{1, \cdots, d\}$ *as follows*

- *If* $i = 1$*, for any* $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$*,* $\tau_1(\xi) = \Phi_1(\xi) = |\xi_1|$*.*
- *If* $i \geq 2$*, for any* $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$

$$\tau_i(\xi) = \begin{cases} |\xi_i| & \textit{if} \quad \xi_1 = \xi_2 = \cdots = \xi_{i-1} = 0 \ , \\ \Phi_{i-1}(\xi) \left(\Phi_{i-1}(\xi)^{-\lambda^{-1} E^t} \xi \right)_i & \textit{otherwise} \ . \end{cases}$$

and

$$\Phi_i(\xi) = |\tau_1(\xi)| + \cdots + |\tau_i(\xi)| \ .$$

Then, the function ρ_2 *defined for* $\xi \in \mathbb{R}^d$ *by*

$$\rho_2(\xi) = \Phi_d(\xi)^{1/\lambda} \ , \tag{3.6}$$

is a $(\mathbb{R}^d, E_2^t(\lambda))$ *pseudo-norm.*

Proof. Let us first prove that the function ρ_2 is well–defined and positive on $\mathbb{R}^d \setminus \{0\}$. It is clear that $\Phi_d \geq 0$. Further $\Phi_d(\xi) = 0$ if and only if for any $i \in \{1, \cdots, d\}$, $\tau_i(\xi) = 0$. By induction and by definition of τ_i, it implies that

$$|\xi_1| = \cdots = |\xi_d| = 0 \ ,$$

that is $\xi = 0$. Therefore, ρ_2 is well–defined and positive on $\mathbb{R}^d \setminus \{0\}$.
To show that ρ_2 is continuous, the only point to verify is that for all $1 \leq i \leq d$, τ_i is continuous. To this end, observe that for all $\xi \in \mathbb{R}^d$

$$\Phi_{i-1}(\xi) \Phi_{i-1}(\xi)^{-\lambda^{-1} E_2^t(\lambda)} \xi = \Phi_{i-1}(\xi)^{Id - \lambda^{-1} E_2^t(\lambda)} \xi = \ .$$

By definition of the exponential of a matrix, one has

$$\Phi_{i-1}(\xi)^{Id - \lambda^{-1} E_2^t(\lambda)} \xi = \xi + \sum_{k=1}^{+\infty} \frac{(-1)^k \log^k(\Phi_{i-1}(\xi)) N^k \xi}{k!} \ ,$$

where $N = \lambda^{-1} E_2^t(\lambda) - Id$. Since $N = \lambda^{-1} \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$, one has

$$\Phi_{i-1}(\xi) \left(\Phi_{i-1}(\xi)^{-\lambda^{-1} E^t} \xi \right)_i = \xi_i + \sum_{\ell=1}^{i-1} \frac{(-1)^{i-\ell} \log^{i-\ell}(\Phi_{i-1}(\xi))}{(i-\ell)! \lambda^{i-\ell}} \xi_\ell \ .$$

Further one has by induction on $i \in \{2, \cdots, d\}$, that, for all $1 \le \ell \le i-1$

$$\lim_{\xi \to 0} \Big(\xi_\ell \log^{i-\ell}(\Phi_{i-1}(\xi)) \Big) = 0 \ .$$

Then the continuity of τ_i follows.
We now verify that ρ_2 satisfies the homogeneity condition. It can be done by induction on i, showing that, for all $1 \le i \le d$, one has

$$\tau_i(a^{-E_2^t(\lambda)}\xi) = a^{-\lambda}\tau_i(\xi) \quad \text{and} \quad \rho_i(a^{-E_2^t(\lambda)}\xi) = a^{-\lambda}\Phi_i(\xi)$$

Indeed, assume that the result holds for $i-1$, then for any $a > 0$ and any ξ such that $\xi_1, \cdots, \xi_i$ are not both equal to 0 (the other case being trivial), one has

$$\begin{aligned} \tau_i(a^{-E_2^t(\lambda)}\xi) &= |\Phi_{i-1}(a^{-E_2^t(\lambda)}\xi)|(|\rho_{i-1}(a^{-E_2^t(\lambda)}\xi)|^{-\frac{E_2^t(\lambda)}{\lambda}} a^{-E_2^t(\lambda)}\xi)_i \\ &= a^{-\lambda}|\Phi_{i-1}(\xi)|((a^{-\lambda}|\Phi_{i-1}(\xi)|)^{-\frac{E_2^t(\lambda)}{\lambda}} a^{-E_2^t(\lambda)}\xi)_i \\ &= a^{-\lambda}\tau_i(\xi). \end{aligned}$$

We have then proved the homogeneity property of function ρ_2.
■
We now consider the case $E = E_3(\alpha, \beta)$ for some $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}$:

Lemma 3.3. *The function ρ_3 defined for $\xi \in \mathbb{R}^d$ by*

$$\rho_3(\xi) = |\xi|^{1/\alpha} \ , \tag{3.7}$$

is a $(\mathbb{R}^d, E_3^t(\alpha, \beta))$ pseudo-norm.

Remark 3.2. *The function ρ_3 defined by (3.7) is an* **isotropic** $(\mathbb{R}^d, E_3^t(\alpha, \beta))$ *pseudo–norm, that is invariant by any isometry T of $\mathbb{R}^d$. Up to a multiplicative constant, it is the unique one. Indeed, let ρ be another isotropic $(\mathbb{R}^d, E_3^t(\alpha, \beta))$ pseudo–norm. Observe that for any $a > 0$, $a^{E_3^t(\alpha,\beta)} = a^\alpha T$ with*

$$T = \begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix} \quad \textit{where } R = \begin{pmatrix} \cos(\beta\log(a)) & -\sin(\beta\log(a)) \\ \sin(\beta\log(a)) & \cos(\beta\log(a)) \end{pmatrix} \ .$$

Remark that T is an isometry. By assumptions on ρ_3

$$\rho_3(a^\alpha\xi) = \rho_3(T^{-1}a^{E_3^t}\xi) = \rho_3(a^{E_3^t}\xi) = a\rho_3(\xi) \quad \forall a > 0 \ .$$

Then ρ_3 is a $(\mathbb{R}^d, \alpha Id)$ pseudo-norm. Now, consider the function g defined for any $\xi \in \mathbb{R}^d \setminus \{0\}$ by $g = |\xi|^{-1/\alpha}\rho$. Since ρ_3 and $|\cdot|^{1/\alpha}$ are two isotropic $(\mathbb{R}^d, \alpha Id)$ pseudo-norms, g is isotropic and we have, for all $\xi \in \mathbb{R}^d$ and $a > 0$,

$$g(a^\alpha\xi) = g(\xi) \ .$$

that is setting $b = a^\alpha$, for all $b > 0$

$$g(b\xi) = g(\xi) \ . \tag{3.8}$$

Hence g is constant on $\mathbb{R}^d \setminus \{0\}$.

Remark 3.3. *Using Theorem 3.2, with g non trivial (i.e. non constant on the isotropic unit ball) and* $\rho_1 = |\cdot|^{1/\alpha}$*, we are able to define non–isotropic* $(\mathbb{R}^d, E_3^t(\alpha,\beta))$ *pseudo-norms.*

Proof. Observe that for any $a > 0$

$$a^{E_3^t(\alpha,\beta)} = a^\alpha \begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix} \text{ with } R = \begin{pmatrix} \cos(\beta\log(a)) & -\sin(\beta\log(a)) \\ \sin(\beta\log(a)) & \cos(\beta\log(a)) \end{pmatrix} .$$

Since the Euclidean norm $|\cdot|$ is invariant by any isometry, and in particular by

$$T = \begin{pmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{pmatrix} ,$$

one has for any $\xi \in \mathbb{R}^d$

$$|a^\alpha T\xi| = |a^\alpha \xi| = a^\alpha |\xi| .$$

The conclusion is then straightforward.
■

We now consider the case $E = E_4(\alpha,\beta)$ for some $(\alpha,\beta) \in \mathbb{R}_+^* \times \mathbb{R}$

Lemma 3.4. *Let us define the functions* τ_i *and* Φ_i *for any* $1 \le i \le d$ *as ,*

- *If* $i = 1$*, for all* $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$*,* $\Phi_1(\xi) = \tau_1(\xi) = (|\xi_1|^2 + |\xi_2|^2)^{\frac{1}{2}}$.
- *If* $i \ge 2$*, for all* $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$

$$\tau_i(\xi) = \begin{cases} (|\xi_{2i-1}|^2 + |\xi_{2i}|^2)^{\frac{1}{2}} & \text{if } \xi_1 = \xi_2 = \cdots = \xi_{2i-2} = 0 \\ \Phi_{i-1}(\xi)\left[\left(\Phi_{i-1}(\xi)^{-\alpha^{-1}E^t}\xi\right)_{2i-1}^2 + \left(\Phi_{i-1}(\xi)^{-\alpha^{-1}E^t}\xi\right)_{2i}^2\right]^{\frac{1}{2}} & \text{otherwise} \end{cases} ,$$

and

$$\Phi_i(\xi) = |\tau_1(\xi)| + \cdots + |\tau_i(\xi)| .$$

Then, the function ρ_4 *defined for all* $\xi \in \mathbb{R}^d$ *by*

$$\rho_4(\xi) = \Phi_d(\xi)^{1/\alpha} , \tag{3.9}$$

is a $(\mathbb{R}^d, E_4(\alpha,\beta))$ *pseudo-norm.*

Proof. The proof is similar to this of Lemma 3.2. Indeed, set for all $1 \le i \le d/2$

$$r_i(\xi) = \left(|\xi_{2i-1}|^2 + |\xi_{2i}|^2\right)^{\frac{1}{2}} ,$$

and observe that for all $\xi \in \mathbb{R}^d$

$$\rho_4(\xi) = \rho_2(r_1(\xi), \cdots, r_{d/2}(\xi)) ,$$

where ρ_2 is defined by (3.6) with $\lambda = \alpha$. ■

3.4. Two dimensional examples

We now focus on the two dimensional case. Up to a change of basis, E is a matrix of the form :

1. $E_1(\lambda_1,\lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $(\lambda_1,\lambda_2) \in (\mathbb{R}_+^*)^2$.
2. $E_2(\lambda) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ with $\lambda \in \mathbb{R}_+^*$.
3. $E_3(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ with $(\alpha,\beta) \in \mathbb{R}^2$.

Let us remark that in dimension 2 there is not four generic cases but three since the matrix E cannot be equivalent to $\begin{pmatrix} A & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & A \end{pmatrix}$ with $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ for some $(\alpha,\beta) \in \mathbb{R}^2$.

We now give an explicit example in each of the case above using the results of Lemma 3.1, Lemma 3.2 and Lemma 3.3:

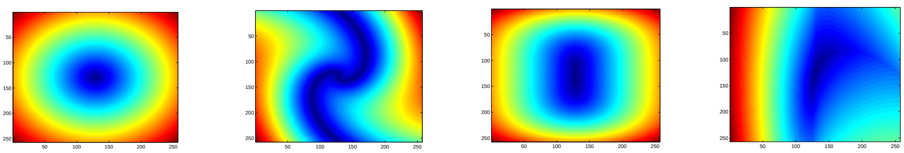
1. If $E = E_1(\lambda_1,\lambda_2)$ for some $(\lambda_1,\lambda_2) \in (\mathbb{R}_+^*)^2$, the function $\rho_1(\xi_1,\xi_2) = (|\xi_1|^{2/\lambda_1} + |\xi_2|^{2/\lambda_2})^{1/2}$ is a $(\mathbb{R}^2, E_1^t)$ pseudo-norm.

2. If $E = E_2(\lambda)$ for some $\lambda \in \mathbb{R}_+^*$, the function $\rho_2(\xi_1,\xi_2) = (|\xi_1| + |\xi_2 - \frac{\xi_1}{\lambda}\ln|\xi_1|)^{1/\lambda}$ is a $(\mathbb{R}^2, E_2^t)$ pseudo-norm.

3. If $E = E_3(\alpha,\beta)$ for some $(\alpha,\beta) \in \mathbb{R}^2$, the function $\rho(\xi_1,\xi_2) = |\xi|^{1/\alpha}$ is a $(\mathbb{R}^2, E_3^t)$ pseudo-norm. More interesting is the fact that the function
$$\rho_3 = \frac{|\xi_1 \cos(\beta/\alpha \ln(r(\xi))) - \xi_2 \sin(\beta/\alpha \ln(r(\xi))|}{r(\xi)^{2/\alpha}} ,$$
with
$$r(\xi) = (|\xi_1|^2 + |\xi_2|^2)^{\frac{1}{2}}$$
is also a $(\mathbb{R}^2, E_3^t)$ pseudo-norm.

Combining these results with Proposition 3.4 yields us to an explicit example of $(\mathbb{R}^2, E^t)$ pseudo-norms for any 2×2 matrix E whose eigenvalues have positive real parts (see Figure 2 below). Thereafter Theorem 3.2 brings us a complete description of the whole class of $(\mathbb{R}^2, E^t)$ pseudo-norms for any matrix E and thus for spectral densities of the class of OSGRF defined in [6].

In Figure 2 just below we represented the pseudo–norms ρ_1, ρ_2, ρ_3 for some values of the matrix E. Remark that the cases $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$ belong to the same generic case (the first one).



$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Figure 2:

Four pseudo-norms corresponding to the three generic two–dimensional cases.

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