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# HUA OPERATORS AND RELATIVE DISCRETE SERIES ON LINE BUNDLE OVER BOUNDED SYMMETRIC DOMAINS

KHALID KOUFANY AND GENKAI ZHANG

ABSTRACT. Let  $D = G/K$  be a bounded symmetric domain and  $S = K/L$  be its Shilov boundary. We consider the action of  $G$  with weight  $\nu \in \mathbb{Z}$  on functions on  $D$  viewed as sections of the line bundle and the corresponding eigenspace of  $G$ -invariant differential operators. The Poisson transform maps hyperfunctions on  $S$  to the eigenspaces. We characterize the image in terms of Hua operators on the sections of the line bundle. For some special parameter the Poisson transform is of Szegő type mapping into the relative discrete series; we compute the corresponding elements in the discrete series.

## 1. INTRODUCTION

Let  $X = G/K$  be a Riemannian symmetric space of the non compact type. It is nowadays known that the Poisson transform maps certain parabolically induced representations into the null space of some system of differential equations.

For the minimal parabolic subgroup  $P_{\min} \subset G$ , Kashiwara *et al.* proved [10], that the Poisson transform gives a  $G$ -isomorphism from the set of hyperfunctions on the maximal boundary  $G/P_{\min}$  onto the joint eigenspace of invariant differential operators on  $X$ .

When  $X = G/K$  is a bounded symmetric domain of tube type, it is proved by Shimeno the analogue of Kashiwara *et al.* theorem for the maximal parabolic subgroup  $P_{\max} \subset G$ . More precisely, he proved in [22] that the Poisson transform is a  $G$ -isomorphism from the set of hyperfunctions on the minimal boundary  $G/P_{\max}$  (which coincides with the Shilov boundary) onto the space of eigenfunctions of the Hua operator of the second order.

The generalization to the non-tube bounded symmetric domains has been given in our earlier paper [12].

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*Key words and phrases.* Bounded symmetric domains, Shilov boundary, invariant differential operators, eigenfunctions, Poisson transform, Hua systems.

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In the harmonic case, *i.e.* for a certain special parameter, the image of the Poisson transform is the set of Hua-harmonic functions studied earlier by Hua [7], Johnson [8], Johnson and Korányi [9], Lassalle [13]. The harmonic case for non-tube bounded symmetric domains was studied by Berline and Vergne [1].

A more interesting problem is to consider homogeneous line bundles over  $\Omega$  with the corresponding weighted action of  $G$ . In this setting Shimeno [21] generalized the Kashiwara *et al.* theorem to homogeneous line bundles over Hermitian symmetric spaces of tube type  $G/K$ . That is for a given a line bundle  $E$  and a homomorphism  $\chi$  of the algebra of  $G$ -invariant differential operators on  $E$ , the Poisson transform maps hyperfunction-valued sections of a line bundle over the maximal boundary onto the set of eigenfunctions with eigenvalue  $\chi$ .

Our purpose in this paper is to prove the characterization of the Poisson integrals of hyperfunction-valued sections of a line bundle over the Shilov boundary of a bounded symmetric domain.

Let  $\Omega = G/K$  be a bounded symmetric domain. Let  $r$  and  $p$  denote respectively the rank and genus of  $\Omega$ . For  $\nu \in p\mathbb{Z}$  we consider the (unique) character  $\tau_\nu$  of  $K$  and the corresponding homogeneous line bundle  $E_\nu$  over  $\Omega$ . We identify  $C^\infty$ -sections of  $E_\nu$  with the space  $C^\infty(G/K, \tau_\nu)$  of  $C^\infty$ -functions on  $G$  such that  $f(gk) = \tau_\nu(k)^{-1}f(g)$ .

Let  $\mathcal{D}_\nu(G/K)^K$  denote the algebra of  $G$ -invariant differential operators on  $\Omega$  acting on  $C^\infty(G/K, \tau_\nu)$  and  $\gamma_\nu$  the corresponding Harich-Chandra homomorphism. The characters of this algebra are given by  $\chi_{\lambda, \nu}(D) = \gamma_\nu(D)(\lambda)$  with  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ .

Finally let  $\mathcal{B}(G/P; L_{\lambda, \nu})$  be the  $G$ -submodule of  $\mathcal{B}(G)$  identified with the space of hyperfunction-valued sections of the line bundle  $L_{\lambda, \nu}$  on the maximal boundary  $G/P$  associated with the character of  $P$  given by  $man \mapsto e^{(\rho - \lambda)(\log a)}\tau_\nu(m)$ .

The main result in [22] states that, for generic  $\lambda$  and  $\nu$ , the Poisson transform  $(\mathcal{P}_{\lambda, \nu}f) = \int_K f(gk)\tau_\nu(k)dk$  is a  $G$ -isomorphism of the space  $\mathcal{B}(G/P; L_{\lambda, \nu})$  onto the space  $\mathcal{A}(G/K, \mathcal{M}_{\lambda, \nu})$  of functions  $\varphi$  on  $C^\infty(G/K, \tau_\nu)$  satisfying the system of differential equations  $\mathcal{M}_{\lambda, \nu} : D\varphi = \chi_{\lambda, \nu}(D)\varphi$ ,  $D \in \mathcal{D}_\nu(G/K)^K$ .

Now, following Hua, we can consider the generalized Poisson transform  $(\mathcal{P}_{s, \nu}f)(z) = \int_S P_{s, \nu}(z, u)f(u)du$  where  $P_{s, \nu}$  is the generalized Shilov kernel and  $S = G/P_1$  the Shilov boundary of  $\Omega$ . The subgroup  $P_1$  is the maximal parabolic subgroup of  $G$ .

For  $s \in \mathbb{C}$ , let  $\mathcal{B}(S, s, \nu)$  be the space of hyperfunction-valued sections on  $S = G/P_1$  associated with the character of  $P_1$  given by  $man \mapsto e^{(s\rho_0 - \rho_1)(\log a)} \tau_\nu(m)$ . Then for  $\lambda_s = \rho + 2n(s-1)\xi_e^* - \nu r \xi_e^*$ , the space  $\mathcal{B}(S, s, \nu)$  can be considered as a subspace of  $\mathcal{B}(G/P; L_{\lambda_s, \nu})$ , then using the Shimeno theorem we can prove (Theorem 5.2) that for generic values of  $s$  the Poisson transform  $\mathcal{P}_{s, \nu}$  is a  $G$ -isomorphism of  $\mathcal{B}(S, s, \nu)$  of the space of eigenfunctions of the Hua operator of the second order.

A similar theorem for non-tube bounded symmetric of domains of type one is also considered (Theorem 8.1).

For singular value of  $s$  we prove (Theorem 9.3) that the Poisson transform is a Szegő type map of principal series representation onto the relative discrete series representation and we compute the exact image in the discrete series.

The paper is organized as follows. In §2 we briefly recall the Jordan algebraic characterization of bounded symmetric domains. In §3 we introduce the line bundle over the bounded symmetric domain  $\Omega$ . The generalized Poisson transform of hyperfunction-valued sections on the maximal and the Shilov boundaries are studied in §4. The characterization of Poisson integrals of hyperfunction-valued sections on the Shilov boundary is given in §5. In this section we also recall our geometric construction of the Hua operator. The necessary condition is proved in §6. In §7 we compute the radial part of the Hua system and prove sufficiency condition. Finally in §8 we show a relationship between the Poisson transform, Hua operator and the relative discrete series representation.

After this paper was finished we were informed by Professor T. Oshima that he and N. Shimeno have obtained in [25] some similar results about Poisson transforms and Hua operators in §5.

## 2. BOUNDED SYMMETRIC DOMAINS AND JORDAN TRIPLES

We begin with a review of basic fact about bounded symmetric domains and Jordan triple systems. We apologize the tedium, but we will need the details.

Let  $V$ ,  $\dim V = n$  be a complex vector space,  $\Omega \subset V$  a circular bounded symmetric domain. Let  $Aut(\Omega)$  be the group of all biholomorphic automorphisms of  $\Omega$ , let  $G$  be the connected component of the identity of  $Aut(\Omega)$ , and let  $K$  be the isotropy subgroup of  $G$  at the

point  $0 \in \Omega$ . Elements of  $K$  are linear,  $K$  is maximal compact and equals the fixed group of the Cartan involution  $\theta = \text{Ad}(-\text{id}_\Omega)$ . As a Hermitian symmetric space,  $\Omega = G/K$ .

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and consider the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . There exists a quadratic map  $Q : V \rightarrow \text{End}(\bar{V}, V)$  (where  $\bar{V}$  is the complex conjugate of  $V$ ), such that  $\mathfrak{p} = \{\xi_v ; v \in V\}$ , where  $\xi_v(z) := v - Q(z)\bar{v}$ . Define  $2Q_{z,w} = Q_{z+w} - Q_z - Q_w$ , then  $\{z \bar{v} w\} = Q_{z,w}(v)$  is linear in  $z$  and  $w$ , conjugate linear in  $v$ , and  $D(z, \bar{v})w := \{z \bar{v} w\}$  satisfies

$$(1) \quad \{z \bar{v} w\} = \{w \bar{v} z\}, \quad [D(u, \bar{v}), D(z, \bar{w})] = D(\{u \bar{v} z\}, \bar{w}) - D(z, \overline{\{w \bar{u} v\}})$$

so  $V$  is a Jordan triple system. Furthermore we have  $[X, \xi_z] = \xi_{Xz}$  for  $X \in \mathfrak{k}$ ,  $z \in V$  and  $[\xi_z, \xi_v] = D(z, \bar{v}) - D(v, \bar{z})$  for all  $z, v \in V$ .

The trace form  $\text{tr} D(z, \bar{v})$  (where  $\text{tr}$  is the trace functional on  $\text{End}(V)$ ) defines a positive Hermitian inner product (which is  $K$ -invariant), so the Jordan triple  $V$  is Hermitian.

The group  $K$  acts on  $V$  by unitary transformations. The domain  $\Omega$  is realized as the open unit ball of  $V$  with respect to the spectral norm,

$$(2) \quad \Omega = \{z \in V : \|D(z, \bar{z})\|^2 < 1\}$$

where  $\|D(z, \bar{z})\|$  is the operator norm of  $D(z, \bar{z})$  on the Hilbert space  $V$ . This sets up a bijection between isomorphic classes of Hermitian Jordan triple systems and of bounded symmetric domains.

An element  $e \in V$  is a tripotent if  $\{e \bar{e} e\} = e$ . The subspaces  $V_\lambda(e) = \ker(D(e, \bar{e}) - \lambda \text{id})$  are called Pierce  $\lambda$ -spaces. Then we have  $V = V_0(e) \oplus V_1(e) \oplus V_2(e)$ , orthogonal sum with respect to the trace form. We have the Pierce rules

$$\begin{aligned} \{V_\alpha(e), \overline{V_\beta(e)}, V_\gamma(e)\} &\subset V_{\alpha-\beta+\gamma}(e) \\ \{V_0(e), \overline{V_1(e)}, V\} &= \{V_1(e), \overline{V_0(e)}, V\} = 0 \end{aligned}$$

In particular,  $V_1(e)$  and  $V_0(e)$  are sub-triples. Two tripotents  $e$  and  $c$  are orthogonal if  $D(e, \bar{c}) = 0$ . Orthogonality is a symmetric relation. A tripotent  $e$  is minimal if it cannot be written as the sum of two non-zero orthogonal tripotents. This is equivalent to  $V_1(e) = \mathbb{C}e$ . The tripotent  $e$  is maximal if  $V_0(e) = 0$ .

A frame is a maximal family of pairwise orthogonal, minimal tripotents. It is known that the group  $K$  acts transitively on frames. In particular, the cardinality of all frames is the same, and it is equal to the rank  $r$  of  $\Omega$ .

Let us choose and fix a frame  $\{e_j\}_{j=1}^r$  in  $V$ . Then, by transitivity of  $K$  on the frames, each element  $z \in V$  admits a polar decomposition

$z = k \sum_{j=1}^r s_j e_j$ , where  $k \in K$  and  $s_j = s_j(z)$  are the singular numbers of  $z$ .

Let  $e = e_1 + \dots + e_r$ , then  $e$  is maximal tripotent, i.e. the only tripotent which is orthogonal to  $e$  is 0. The Shilov boundary of  $\Omega$  is  $S = K/K_1$  where  $K_1 = \{k \in K : k e = e\}$ . It is known that  $S$  coincides with the set of maximal tripotents of  $V$ .

The joint Peirce spaces are

$$(3) \quad V_{ij} = \{z \in Z : D(e_k, \bar{e}_k)z = (\delta_{ik} + \delta_{jk})z, \forall k\}, \quad 0 \leq i \leq j \leq r$$

Then  $V = \bigoplus_{0 \leq i \leq j \leq r} V_{ij}$ ,  $V_{00} = 0$ , and  $V_{ii} = \mathbb{C}e_i$  ( $i > 0$ ).

Suppose  $V$  simple. Then the triple of integers  $(r, a, b)$  with

$$(4) \quad a := \dim V_{jk} \quad (1 \leq j < k \leq r); \quad b := \dim V_{0j} \quad (1 \leq j \leq r)$$

is independent of the choice of the frame and uniquely determines the Jordan triple. Notice that  $b = 0$  exactly if  $V$  is a Jordan algebra which is equivalent to say that  $\Omega$  is of tube type.

The Peirce decomposition associated with  $e$  is then  $V = V_2 \oplus V_1$  where

$$(5) \quad V_2 = \sum_{1 \leq j < k \leq r} V_{jk} \quad V_1 = \sum_{j=1}^r V_{0j}.$$

Let  $n_1 = \dim V_1$  and  $n_2 = \dim V_2$ , then

$$n_1 = rn, \quad n_2 = r + \frac{r(r-1)}{2}a \quad \text{and} \quad n = n_1 + n_2.$$

The genus of  $\Omega$  is

$$p = p(\Omega) = \frac{1}{r} \operatorname{tr} D(e, \bar{e}) = (r-1)a + b + 2.$$

We define

$$(6) \quad \langle z, w \rangle = \frac{1}{p} \operatorname{tr} D(z, \bar{w}),$$

Then  $\langle \cdot, \cdot \rangle$  is a  $K$ -invariant Hermitian product on  $V$ , and for any  $j = 1, \dots, r$  we have

$$\langle e_j, e_j \rangle = \frac{1}{p} \operatorname{tr} D(e_j, \bar{e}_j) = \frac{1}{rp} \operatorname{tr} D(e, \bar{e}) = 1.$$

Let  $\mathfrak{a} = \mathbb{R}\xi_{e_1} + \dots + \mathbb{R}\xi_{e_r}$ . Then  $\mathfrak{a}$  is a maximal abelian subspace of  $\mathfrak{p}$  with basis vectors  $\{\xi_{e_1}, \dots, \xi_{e_r}\}$ . Let  $\{\beta_j\}_{j=1}^r \subset \mathfrak{a}^*$  be the basis of  $\mathfrak{a}^*$  determined by

$$(7) \quad \beta_j(\xi_{e_k}) = 2\delta_{jk}, \quad 1 \leq j, k \leq r,$$

and define an ordering on  $\mathfrak{a}^*$  via

$$(8) \quad \beta_r > \beta_{r-1} > \cdots > \beta_1 > 0.$$

It is known that the restricted roots system  $\Sigma(\mathfrak{g}, \mathfrak{a})$  of  $\mathfrak{g}$  relative to  $\mathfrak{a}$  is of type  $C_r$  or  $BC_r$ , it consists of the roots  $\pm\beta_j$  ( $1 \leq j \leq r$ ) with multiplicity 1, the roots  $\pm\frac{1}{2}\beta_j \pm \frac{1}{2}\beta_k$  ( $1 \leq j \neq k \leq r$ ) with multiplicity  $a$ , and possibly the roots  $\pm\frac{1}{2}\beta_j$  ( $1 \leq j \leq r$ ) with multiplicity  $2b$ . The set of positive roots  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  consists of  $\frac{1}{2}(\beta_j \pm \beta_k)$  ( $1 \leq j < k \leq r$ ),  $\beta_j$  and  $\frac{1}{2}\beta_j$  ( $1 \leq j \leq r$ ).

We will write an element  $\underline{\lambda} \in (\mathfrak{a}^*)_{\mathbb{C}}$  as

$$(9) \quad \underline{\lambda} = \sum_{j=1}^r \lambda_j \beta_j,$$

and identify  $\underline{\lambda}$  with  $(\lambda_1, \lambda_2, \dots, \lambda_r)$ . The half sum of positive roots is given by

$$(10) \quad \underline{\rho} = \sum_{j=1}^r \rho_j \beta_j = \sum_{j=1}^r \frac{b+1+a(j-1)}{2} \beta_j.$$

Let  $A$  be the analytic subgroup of  $G$  corresponding to  $\mathfrak{a}$ . Let  $\mathfrak{n}^+ = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$  and  $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$ . Let  $N^+$  and  $N^-$  be the corresponding analytic subgroups of  $G$ . Let  $M$  be the centralizer of  $\mathfrak{a}$  in  $K$ . The subgroup  $P = MAN^+$  is a minimal parabolic subgroup of  $G$ .

The set  $\Lambda = \{\alpha_1, \dots, \alpha_{r-1}, \alpha_r\}$  of simple roots in  $\Sigma^+$  is such that

$$\alpha_j = \frac{1}{2}(\beta_{r-j+1} - \beta_{r-j}), \quad 1 \leq j \leq r-1$$

and

$$\alpha_r = \begin{cases} \beta_1 & \text{for the tube case} \\ \frac{1}{2}\beta_1 & \text{for the non-tube case.} \end{cases}$$

Let  $\Lambda_1 = \{\alpha_1, \dots, \alpha_{r-1}\}$  and  $P_1$  the corresponding standard parabolic subgroup of  $G$  with the Langlands decomposition  $P_1 = M_1 A_1 N_1^+$  such that  $A_1 \subset A$ . Then the Lie algebra of  $A_1$  is  $\mathfrak{a}_1 = \mathbb{R}\xi_e$  where  $\xi_e = \xi_{e_1} + \dots + \xi_{e_r}$ .

The spaces  $G/P = K/M$  is the Furstenberg boundary and  $G/P_1 = K/K_1$ , with  $K_1 = M_1 \cap K$ , is the Shilov boundary of  $G/K$ .

### 3. LINE BUNDLE OVER $\Omega$

Let  $Z_0$  be the element in the center of  $k$  which defines the complex structure on  $\mathfrak{p}^+ = V$ , see [20]. Let

$$Z = \frac{p}{n} Z_0.$$

Let  $K_s$  be the analytic subgroup of  $K$  with Lie algebra  $\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]$  and  $K = \exp(\mathbb{R}Z)K_s$ .

Let  $\nu \in p\mathbb{Z}$  and consider the character  $\tau_\nu$  of  $K$  defined by

$$\tau_\nu(k) = \begin{cases} e^{it\nu} & \text{if } k = \exp(tZ) \in \exp(\mathbb{R}Z) \\ 1 & \text{if } k \in K_s \end{cases}$$

see [20] and [2].

Let  $E_\nu$  be the homogeneous line bundle on  $\Omega = G/K$  associated with the one dimensional representation  $\tau_\nu$  of  $K$ . We identify the space of  $C^\infty$ -sections of  $E_\nu$  with the space  $C^\infty(G/K; \tau_\nu)$  of  $C^\infty$ -functions on  $G$  such that

$$f(gk) = \tau_\nu(k)^{-1}f(g).$$

Let  $D_\nu(G/K)$  denote the set of  $G$ -invariant differential operators on  $G/K$  acting on  $C^\infty(G/K; \tau_\nu)$ . We have the Harish-Chandra isomorphism, see [21]

$$\gamma_\nu : D_\nu(G/K) \simeq U(\mathfrak{a})^W$$

where  $U(\mathfrak{a})^W$  denote the set of  $W$ -invariant elements in the symmetric algebra  $S(\mathfrak{a})$ .

The characters of  $D_\nu(G/K)$  are given by

$$\chi_{\lambda, \nu}(D) = \gamma_\nu(D)(\lambda) \quad D \in D_\nu(G/K), \quad \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

For  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  we define  $\mathcal{A}(G/K, \mathcal{M}_{\lambda, \nu})$  to be the space of functions  $\varphi \in C^\infty(G/K; \tau_\nu)$  satisfying the system of differential equations

$$(11) \quad \mathcal{M}_{\lambda, \nu} : D\varphi = \chi_{\lambda, \nu}(D)\varphi \quad D \in D_\nu(G/K).$$

Finally, let  $\mathcal{B}(G/P; L_{\lambda, \nu})$  be the space of hyperfunctions  $f$  on  $G$  satisfying

$$f(gman) = e^{(\lambda - \rho)(\log a)} \tau_\nu(m)^{-1} f(g)$$

for all  $g \in G$ ,  $m \in M$ ,  $a \in A$ ,  $n \in N^+$ . The space  $\mathcal{B}(G/P; L_{\lambda, \nu})$  is a  $G$ -submodule of  $\mathcal{B}(G)$  and can be identified with the space of hyperfunction valued sections of the line bundle  $L_{\lambda, \nu}$  on  $G/P$  associated with the character of  $P$  given by  $man \mapsto e^{(\rho - \lambda)(\log a)} \tau_\nu(m)$ ,  $m \in M$ ,  $a \in A$ ,  $n \in N^+$ .

#### 4. POISSON TRANSFORM

**4.1. Poisson integrals on the Furstenberg boundary.** For  $g \in G$  let  $\kappa(g) \in K$  and  $H(g) \in \mathfrak{a}$  be elements uniquely determined by

$$g \in \kappa(g) \exp(H(g))N^+ \subset KAN^+ = G.$$



Let  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  and  $\nu \in \mathbb{C}$ . We define the Poisson transform  $\mathcal{P}_{\lambda, \nu}$  by

$$(12) \quad (\mathcal{P}_{\lambda, \nu} f)(g) = \int_K f(gk) \tau_{\nu}(k) dk \quad g \in G$$

for any  $f \in \mathcal{B}(G/P, L_{\lambda, \nu})$ .

The Poisson transform  $\mathcal{P}_{\lambda, \nu}$  can be also written, for  $f \in \mathcal{B}(G/P, L_{\lambda, \nu})$ , as

$$(13) \quad (\mathcal{P}_{\lambda, \nu} f)(g) = \int_K f(k) \tau_{\nu}(\kappa(g^{-1}k)) e^{-(\lambda + \rho)(H(g^{-1}k))} dk$$

and maps  $\mathcal{B}(G/P, L_{\lambda, \nu})$  into  $\mathcal{A}(G/K, \mathcal{M}_{\lambda, \nu})$ , see [21, Theorem 5.2]

Let  $e_{\nu}(\lambda)$  denote the denominator of the  $c$ -function

$$c_{\nu}(\lambda) = \int_{N^-} e^{(\lambda + \rho)(H(\bar{n}))} \tau_{\nu}(\kappa(\bar{n})) d\bar{n},$$

see [21, page 227]. In our normalization this function is given by

$$e_{\nu}(\lambda) = \prod_{j>k} \Gamma\left(\frac{1}{2}a + \lambda_j \pm \lambda_k\right) \prod_j \Gamma\left(\frac{1}{2}(b+1+2\lambda_j+\nu)\right) \Gamma\left(\frac{1}{2}(b+1+2\lambda_j-\nu)\right).$$

The following theorem characterize the rang of the

**Theorem 4.1** ([21], Theorem 8.1). *If  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$  and  $\nu \in \mathbb{C}$  satisfy the conditions*

$$(14) \quad -2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{1, 2, 3, \dots\} \quad \text{for any } \alpha \in \Sigma^+$$

$$(15) \quad e_{\nu}(\lambda) \neq 0,$$

then the Poisson transform  $\mathcal{P}_{\lambda, \nu}$  is a  $G$ -isomorphism of  $\mathcal{B}(G/P, L_{\lambda, \nu})$  onto  $\mathcal{A}(G/K, \mathcal{M}_{\lambda, \nu})$ .

**4.2. Poisson integrals on the Shilov boundary.** Let  $h$  be the unique  $K$ -invariant polynomial on  $\mathfrak{p}^+ = V$  whose restriction on  $\mathbb{R}c_1 \oplus \dots \oplus \mathbb{R}c_r$  is given by

$$h\left(\sum_{j=1}^r t_j c_j\right) = \prod_{j=1}^r (1 - t_j^2).$$

Let  $h(z, w)$  be its polarization such that  $h(z, z) = h(z)$ . The function  $h(z, w)$  is holomorphic on  $z$  and antiholomorphic in  $w$  and is related to the Bergman operator via

$$(16) \quad \det b(z, \bar{z}) = h(z, \bar{z})^p.$$

For any complex number  $s$  and for any  $\nu$  we define the generalized Poisson kernel

$$(17) \quad P_{s,\nu}(z, u) = \left( \frac{h(z, z)}{|h(z, u)|^2} \right)^{\frac{s}{r}} h(z, u)^{-\nu}, \quad z \in \Omega, u \in S.$$

We also define the generalized Poisson transform

$$(18) \quad (\mathcal{P}_{s,\nu}f)(z) = \int_S P_{s,\nu}(z, u) f(u) du, \quad \text{for } f \in \mathcal{B}(S)$$

Let

$$\mathfrak{a}_1^\top = \sum_{j=1}^r \mathbb{R}(\xi_j - \xi_{j+1})$$

be the orthocomplement of  $\mathfrak{a}_1 = \mathbb{R}\xi_e$  in  $\mathfrak{a}$  with respect to the Killing form. We denote  $\xi_e^*$  the dual vector such that  $\xi_e^*(\xi_e) = 1$  and we extend  $\xi_e^*$  to  $\mathfrak{a}$  by the orthogonal projection defined above. Then  $\rho_1 = \rho|_{\mathfrak{a}_1}$  the restriction of  $\rho$  to  $\mathfrak{a}_1$  is given by  $\rho_1 = n\xi_e^*$ . Define  $\rho_0$  to be the linear form on  $\mathfrak{a}_1$  such that  $\rho_0 = r\xi_e^*$ .

Consider the following representation of  $P_1 = M_1A_1N_1$  given by  $\sigma_{s,\nu} = \tau_\nu|_{M_1} \otimes e^{s\rho_0 - \rho} \otimes 1$  and let  $\mathcal{B}(G/P_1, s, \nu)$  be the space of hyperfunction valued sections of the line bundle on  $S = G/P_1$  corresponding to  $\sigma_{s,\nu}$ , i.e., the space of hyperfunctions  $f$  on  $G$  satisfying

$$f(gman) = e^{(s\rho_0 - \rho_1)(\log a)} \tau_\nu(m)^{-1} f(g)$$

for all  $g \in G$ ,  $m \in M_1$ ,  $a \in A_1$ ,  $n \in N_1^+$ .

For  $s \in \mathbb{C}$  we define the linear form  $\lambda_s$  on  $\mathfrak{a}$  by

$$(19) \quad \lambda_s = \rho + 2n(s-1)\xi_e^* - \nu r\xi_e^*$$

Let  $\nu \in p\mathbb{Z}$ , then we have

$$(20) \quad \mathcal{B}(G/P_1; s, \nu) \subset \mathcal{B}(G/P; L_{\lambda_s, \nu})$$

therefore

$$(21) \quad \mathcal{P}_{s,\nu}(\mathcal{B}(G/P_1; s, \nu)) \subset \mathcal{A}(G/K; \mathcal{M}_{\lambda_s, \nu}).$$

## 5. HUA OPERATORS

The bounded symmetric domain  $\Omega = G/K$  is a Kähler manifold with the Bergman metric

$$(u, v)_z = \partial_u \bar{\partial}_v \log h(z, \bar{z})^{-1} = \langle B(z, \bar{z})^{-1} u, v \rangle$$

where  $B(z, \bar{z}) = I - D(z, \bar{z}) + Q(z)Q(\bar{z})$  is the Bergman operator of  $\Omega$ .

Let  $\tau$  be a finite-dimensional holomorphic representation of  $K_{\mathbb{C}}$ . Let  $E$  be the Hermitian vector bundle over  $\Omega$  associated with  $\tau$ . Then there exists a unique connection operator  $\nabla : C^\infty(\Omega, E) \rightarrow C^\infty(\Omega, E \otimes T')$  compatible with the Hermitian structure and the anti-holomorphic differentiation, where  $T' = T'\Omega$  is the cotangent bundle over  $\Omega$ . That is, under the splitting in holomorphic and antiholomorphic parts,  $T' = T'^{(1,0)} \oplus T'^{(0,1)}$ , we have  $\nabla = \mathcal{D} + \bar{\partial}$  where  $\mathcal{D}$  is the differentiation operator on  $E$ ,

$$(22) \quad \mathcal{D} : C^\infty(\Omega, E) \rightarrow C^\infty(\Omega, E \otimes T'^{(1,0)})$$

Let  $\bar{D}$  be the invariant Cauchy-Reimann operator on  $E$  defined in [3, 18, 19], by

$$(23) \quad \bar{D}f = B(z, \bar{z})\bar{\partial}f.$$

Then

$$(24) \quad \bar{D} : C^\infty(\Omega, E) \rightarrow C^\infty(\Omega, E \otimes T'^{(1,0)})$$

We will use the following identifications,  $T_z'^{(1,0)} = \mathfrak{p}^- = V' = \bar{V}$  and  $T_z'^{(0,1)} = T'^{(0,1)} = \mathfrak{p}^+ = V$ .

We now specialize the above to the line bundle  $E_\nu$  associated with the one-dimensional representation  $\tau_\nu$ . The Hua operator is then defined as the resulting operator of the following diagram

$$\begin{array}{ccc} C^\infty(\Omega, E_\nu \otimes \mathfrak{p}^+) & \xrightarrow{\mathcal{D}} & C^\infty(\Omega; E_\nu \otimes \mathfrak{p}^+ \otimes \mathfrak{p}^-) \\ \bar{D} \uparrow & & \downarrow \text{Ad}_{\mathfrak{p}^+ \otimes \mathfrak{p}^-} \\ C^\infty(\Omega, E_\nu) & \xrightarrow{\mathcal{H}} & C^\infty(\Omega, E_\nu \otimes \mathfrak{k}_{\mathbb{C}}) \end{array}$$

**Remarks 5.1.** (1) *We can also define by the same way on other Hua operator,*

$$\mathcal{H}'f = \text{Ad}_{\mathfrak{p}^+ \otimes \mathfrak{p}^-} \bar{D}\mathcal{D}f,$$

*and one can prove, by direct computations, the following relation*

$$(25) \quad (\mathcal{H} - \mathcal{H}')F = -\frac{2n}{r}\nu F \text{Id}.$$

(2) *On other hand, following Johnson-Korányi [9], the Hua operator can also be described using the enveloping algebra : Let  $\{E_\alpha\}$  be a basis of  $\mathfrak{p}^+$  and  $\{E_\alpha^*\}$  be the dual basis of  $\mathfrak{p}^+$  with*

respect to the Killing form. Then the Hua operator is defined as element of  $U(\mathfrak{g})_{\mathbb{C}} \otimes \mathfrak{k}_{\mathbb{C}}$  by

$$(26) \quad \mathcal{H} = \sum_{\alpha, \beta} E_{\alpha} E_{\beta}^* \otimes [E_{\beta}, E_{\alpha}^*].$$

Notice that this definition does not depend on the choice of the basis.

**Theorem 5.2.** *Let  $\Omega$  be a bounded symmetric domain of tube type. Suppose  $s \in \mathbb{C}$  satisfies the following condition*

$$(27) \quad \frac{4n(1-s)}{r} \notin \Lambda_1 \cup \Lambda_2$$

where  $\Lambda_1 = \mathbb{Z}_+ - 2\nu + 2$ ,  $\Lambda_2 = 2\mathbb{Z}_{\geq} - 4\nu + 4$ . Then the Poisson transform  $\mathcal{P}_{\lambda, \nu}$  is a  $G$ -isomorphism of  $\mathcal{B}(S; s, \nu)$  onto the space of analytic functions  $F$  on  $\Omega$  such that

$$(28) \quad \mathcal{H}F = 2\frac{n}{r}s\left(\frac{n}{r}(s-1) + \nu\right)FId.$$

## 6. THE NECESSARY CONDITION OF THE HUA EQUATIONS

For any  $f \in \mathcal{B}(S; s, \nu)$ , the function  $\mathcal{P}_{s, \nu}f$  satisfies the condition (??). This can be proved using the following theorem.

**Theorem 6.1.** *For any  $u \in S$  the function  $z \mapsto P_{s, \nu}(z, \bar{u})$  satisfies the following system*

$$(29) \quad \mathcal{H}P_{s, \nu} = 2\frac{n}{r}s\left[\frac{n}{r}(s-1) + \nu\right]P_{s, \nu}Id$$

*Proof.* Recall the following formulas in [12, Lemma 5.2] : for any  $\bar{w} \in \bar{V}$  and any complex number  $s$ ,

$$(30) \quad \partial h(z, \bar{w})^s = -sh(z, \bar{w})^s \bar{w}^z, \quad \bar{\partial} h(z, \bar{w})^s = -sh(z, \bar{w})^s w^{\bar{z}}$$

where

$$z^{\bar{w}} = B(z, \bar{w})^{-1}(z - Q(z)\bar{w})$$

is the quazi-inverse of  $z$  with respect to  $\bar{w}$ .

Then we have

$$\bar{\partial} P_{s, \nu}(z, \bar{u}) = \bar{\partial} \left[ h(z, u)^{-\nu} \left( \frac{h(z, z)}{|h(z, u)|^2} \right)^{\frac{n}{r}s} \right] = -\left(\frac{n}{r}s\right)P_{s, \nu}(z, \bar{u})[z^{\bar{z}} - u^{\bar{z}}],$$

and

$$\begin{aligned} -\left(\frac{n}{r}s\right)\partial P_{s, \nu}(z, \bar{u})[z^{\bar{z}} - u^{\bar{z}}] &= -\left(\frac{n}{r}s\right)(\nu + \frac{n}{r}s)P_{s, \nu}(z, \bar{u}) \bar{u}^z \otimes [z^{\bar{z}} - u^{\bar{z}}] \\ &\quad + \left(\frac{n}{r}s\right)^2 P_{s, \nu}(z, \bar{u}) \bar{z}^z \otimes [z^{\bar{z}} - u^{\bar{z}}] \\ &\quad - \left(\frac{n}{r}s\right)P_{s, \nu}(z, \bar{u}) \partial [z^{\bar{z}} - u^{\bar{z}}]. \end{aligned}$$

Using  $\square$  the last derivative can be written as

$$\partial[z^{\bar{z}} - u^{\bar{z}}] = \partial(z^{\bar{z}}) = \partial\bar{\partial} \log h(z, \bar{z})^{-1} = B(z, \bar{z})^{-1} Id$$

where  $Id$  is the identity form in  $(\mathfrak{p}^+)' \otimes (\mathfrak{p}^-)'$ .

Now to continue the proof, it is sufficient to prove the formula at  $z = 0$ . Specifying the above to  $z = 0$  and choosing an orthonormal basis  $\{e_\alpha\}$  of  $V$  consisting of the fixed frame  $\{e_j\}_{1 \leq j \leq r}$  and orthonormal basis of each subspaces  $V_{jk}$ , then we get

$$\begin{aligned} \mathcal{H}P_{s,\nu}(0, \bar{u}) &= +\binom{n}{r}s(\nu + \frac{n}{r}s)D(u, \bar{u}) \\ &+ 0 \\ &- \binom{n}{r}s \sum_{\alpha=1}^n D(e_\alpha, \bar{e}_\alpha) \end{aligned}$$

Since  $D(u, \bar{u}) = 2I$  and  $\sum_{\alpha=1}^n D(e_\alpha, \bar{e}_\alpha) = 2\frac{n}{r}I$ , the proof is completed.  $\square$

## 7. THE SUFFICIENCY CONDITION OF THE HUA EQUATIONS

The aim of this section is to prove that, using the radial part of the Hua operator, each solution  $F$  of the system (28) satisfies the equation (11)  $\mathcal{M}_{\lambda_s, \nu}$ . Then under the condition (27), using the induced equations, the boundary value of  $F$  is contained in  $\mathcal{B}(S; s, \nu)$ .

We start to show that eigenfunctions of the Hua operator (28) are all  $\tau_\nu$ -spherical functions. We need only to prove the claim for  $K$ -invariant functions  $F$  on  $\Omega$ , i.e.,  $F(kz) = F(z)$ ,  $k \in K$ .

For that purpose we compute the radial part of the Hua operator. The functions  $F(z)$  will be identified as permutation invariant function  $F(t)$  on the diagonal  $z = \sum_{j=1}^r t_j e_j$ . The operator  $\mathcal{H}$  has the form

$$\mathcal{H}F(z) = \sum_{j=1}^r \mathcal{H}_j F(z) D(e_j, e_j),$$

for some operators  $\mathcal{H}_j$  in  $t = (t_1, \dots, t_r)$ . It's convenient to find the radial part of  $4\mathcal{H}$  (due to the usual convention that  $\bar{\partial}\partial = \frac{1}{4}(\partial_x^2 + \partial_y^2) = \frac{1}{4}(\partial_r^2 + \frac{1}{r}\partial_r)$  for radial functions in  $z$ ).

**Theorem 7.1.** *Let  $\Omega$  be the tube type domain Let  $F$  be  $\mathcal{C}^2$  and  $K$ -invariant function, then for  $a = \sum_{j=1}^r t_j c_j$ ,*

$$(31) \quad 4\mathcal{H}F(a) = \sum_{j=1}^r \mathcal{H}_j F(t_1, \dots, t_r) D(c_j, \bar{c}_j)^{(1)},$$

where the scalar-valued operators  $\mathcal{H}_j$  are given by

$$\begin{aligned} \mathcal{H}_j = & (1 - t_j^2)^2 \left( \frac{\partial^2}{\partial t_j^2} + \frac{1}{t_j} (1 + 2\nu) \frac{\partial}{\partial t_j} \right) + \\ & + \frac{a}{2} \sum_{k \neq j} (1 - t_j^2)(1 - t_k^2) \left[ \frac{1}{t_j - t_k} \left( \frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_k} \right) + \frac{1}{t_j + t_k} \left( \frac{\partial}{\partial t_j} + \frac{\partial}{\partial t_k} \right) \right] + \\ & + (-2\nu)(1 - t_j^2) \frac{1}{t_j} \frac{\partial}{\partial t_j}. \end{aligned}$$

*Proof.* Recall that the operator  $\mathcal{D}$  acting on  $T^{(1,0)}$ -valued function takes the form

$$\mathcal{D}F(z) = h^{-\nu}(z, z)B(z, z) \sum_k \partial_k (h^\nu(z, z)B(z, z)^{-1}F(z)) \otimes \bar{v}_k$$

which is a  $\mathfrak{p}^+ \otimes \mathfrak{p}^-$ -valued function. Thus

$$\mathcal{H}F(z) = \text{Ad}_{\mathfrak{p}^+ \otimes \mathfrak{p}^-} \sum_{k,l} h^{-\nu}(z, z)B(z, z) \sum_k \partial_k (h^\nu(z, z)(\bar{\partial}_l F(z))v_l) \otimes \bar{v}_k,$$

where we have performed that cancellation of the  $B^{-1}$ -term in  $\mathcal{D}$  with  $B$  in  $\bar{D}$ . Performing the Leibniz rule for  $\partial_k$  we get the above as sum of two terms, say  $I$  and  $II$  where

$$II = \text{Ad}_{\mathfrak{p}^+ \otimes \mathfrak{p}^-} \sum_{k,l} B(z, z) \sum_k \partial_k ((\bar{\partial}_l F(z))v_l) \otimes \bar{v}_k$$

is a second order term and

$$I = \text{Ad}_{\mathfrak{p} \otimes \bar{\mathfrak{p}}} \sum_{k,l} h^{-\nu}(z, z)B(z, z) \sum_k (\partial_k h^\nu(z, z))((\bar{\partial}_l F(z))v_l) \otimes \bar{v}_k$$

The second order term  $II$  is computed as in [12] or [5] and we need only to treat  $I$ . Recall that

$$\partial_v h^\nu(z, z) = \partial_v e^{\nu \log h(z, z)} = -\nu e^{\nu \log h(z, z)} \partial_v (-\log h(z, z)) = -\nu h^\nu(z, z)(v, z^{\bar{z}})$$

where  $z^{\bar{z}} = \bar{\partial}(-\log h(z, z))$  is quasi-inverse of  $z$  (and generally the primitive of the Kähler potential); see [14], [31]. We have then

$$I = -\nu \sum_{k,l} (v_k, z^{\bar{z}})(\bar{\partial}_l F(z))[B(z, z)v_l, \bar{v}_k].$$

For  $z = \sum_{j=1}^r t_j e_j$  we have  $z^{\bar{z}} = \sum_{j=1}^r \frac{t_j}{1-t_j^2} e_j$ , and

$$B(z, z) = \sum_{j=1}^r (1 - t_j^2)^2 D(e_j, e_j)$$

which is diagonalized under the Peirce decomposition of  $V$ . In particular the above sum reduces to

$$I = -\nu \sum_{j=1}^r \frac{t_j}{1-t_j^2} (\bar{\partial}_{e_j} F(z)) (1-t_j^2)^2 D(e_j, e_j).$$

Being a  $K$ -invariant function  $F(z)$  is rotation invariant on the plane  $\mathbb{C}e_j$  and we have then  $\bar{\partial}_{e_j} F(z) = \frac{1}{2} \partial_{t_j} F(t)$ . Finally we have

$$I = -\frac{\nu}{2} \sum_{j=1}^r t_j (1-t_j^2) \partial_{t_j} F(t) D(e_j, e_j).$$

To put  $I+II$  in a better form we write  $t_j(1-t_j^2) = (1-t_j^2) \left( \frac{1-t_j^2}{t_j} - \frac{(1-t_j^2)^2}{t_j} \right)$  and we get then the form  $\mathcal{H}_j$  as claimed.  $\square$

Note that this formula is consistent with the formula for Laplace-Beltrami operator on line bundle with parameter  $\nu$  where the root multiplicity 1 for the root  $\gamma_j$  is replaced by  $1+2\nu$  and the multiplicity  $2b$  of  $\frac{\gamma_j}{2}$  by  $2b-2\nu$ ; see e.g. [22].

We come back to the sufficiency condition of the Hua equation in Theorem (??). Let  $\varphi_{\lambda_s, \nu}$  the unique elementary spherical function of type  $\tau_\nu$  in  $E_\nu$ ,

$$\varphi_{\lambda_s, \nu}(g) = \int_K e^{(\lambda_s + \rho)H} \tau_\nu(k^{-1} \kappa(g^{-1}k)) dk, \quad g \in G$$

Suppose  $F$  be a  $K$ -invariant analytic eigenfunction of the Hua equation (??). Let  $g \in G$ . Then the function

$$\Phi_g(z) = \int_K F(gkz) \tau_\nu(k) dk$$

is  $K$ -biinvariant solution of the differential system

$$\mathcal{H}_j \Phi = 2 \frac{n}{r} s \left( \frac{n}{r} (s-1) + \nu \right) \Phi, \quad j = 1, \dots, r.$$

Therefore by a result of Yan [28],  $\Phi_g$  is proportional to  $\varphi_{\lambda_s, \nu}$  and thus,

$$\int_K F(gkz) \tau_\nu(k) dk = \varphi_{\lambda_s, \nu}(z) F(g \cdot 0).$$

Following Shimeno [22, Theorem 3.2], it is easy to see that this integral formula characterizes the joint eigenfunctions in  $E_\lambda$ , that is  $F \in \mathcal{A}(G/K, \mathcal{M}_{\lambda_s, \nu})$ .

Now applying Theorem 4.1 we can find  $f \in \mathcal{B}(G/P, L_{\lambda_s, \nu})$  such that  $F = \mathcal{P}_{s, \nu}(f)$ . Then we closely follow the method of Berline and Vergne

in [12] to prove that  $f \in \mathcal{B}(S, s, \nu)$  under the condition (27). We leave details to the reader.

### 8. TYPE ONE DOMAINS

Let  $V = M_{r,r+b}(\mathbb{C})$  be the vector space of complex  $r \times (r+b)$ -matrices.  $V$  is a Jordan triple system for the following triple product  $\{x\bar{y}z\} = xy^*z + zy^*x$ , and the endomorphisms  $D(z, \bar{v})$  are given by

$$D(z, \bar{v})w = \{z\bar{v}w\} = zv^*w + wv^*z.$$

Let

$$\mathbf{I}_{r,r+b} = \{z \in M_{r,r+b}(\mathbb{C}) : I_r - z^*z \gg 0\}$$

where  $I_r$  denote the unit matrix of rank  $r$ . Then  $\mathbf{I}_{r,r+b}$  is a bounded symmetric domain of dimension  $r(r+b)$ , rank  $r$  and genus  $2r+b$ . The multiplicities are  $2b$  and  $a = 2$  if  $2 \leq r$ ,  $a = 0$  if  $r = 1$ . The domain  $\mathbf{I}_{r,r+b}$  is of tube type if and only if  $b = 0$ . Its Shilov boundary is

$$S_{r,r+b} = \{z \in M_{r,r+b}(\mathbb{C}) : z^*z = I_r\}.$$

As a homogeneous space, the bounded domain  $\mathbf{I}_{r,r+b}$  can be identified with  $SU(r, r+b)/S(U(r) \times U(r+b))$ .

The complex Lie algebra  $\mathfrak{k}_{\mathbb{C}}$  is given by the set of all matrices

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{d} \end{pmatrix}, \quad \mathbf{a} \in M_{r,r}(\mathbb{C}), \quad \mathbf{d} \in M_{r+b,r+b}(\mathbb{C}), \quad \mathbf{tr}(\mathbf{a}) + \mathbf{tr}(\mathbf{d}) = 0.$$

Hence,  $\mathfrak{k}_{\mathbb{C}}$  can be written as the sum

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{(1)} \oplus \mathfrak{k}_{\mathbb{C}}^{(2)},$$

where  $\mathfrak{k}_{\mathbb{C}}^{(1)}$  and  $\mathfrak{k}_{\mathbb{C}}^{(2)}$  are the following ideals

$$\mathfrak{k}_{\mathbb{C}}^{(1)} = \left\{ \begin{pmatrix} \mathbf{a} & 0 \\ 0 & -\frac{\mathbf{tr}(\mathbf{a})}{r+b} I_{r+b} \end{pmatrix}, \quad \mathbf{a} \in M_{r,r}(\mathbb{C}) \right\},$$

$$\mathfrak{k}_{\mathbb{C}}^{(2)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{d} \end{pmatrix}, \quad \mathbf{d} \in M_{r+b,r+b}(\mathbb{C}), \quad \mathbf{tr}(\mathbf{d}) = 0 \right\}.$$

Let  $\mathcal{H}^{(1)}$  be the projection of the Hua operator  $\mathcal{H}$  on  $\mathfrak{k}_{\mathbb{C}}^{(1)}$ . It maps  $C^\infty(G/K, \tau_\nu)$  into  $C^\infty(G/K, \tau_\nu \otimes \mathfrak{k}_{\mathbb{C}}^{(1)})$ .

The main result of this section is the following theorem which is the analogue of Theorem 5.2 for non tube domains of type one.

**Theorem 8.1.** *Let  $\Omega = \mathbf{I}_{r,r+b}$  be a bounded symmetric domain of type one. Suppose  $s \in \mathbb{C}$  satisfies the condition (27). Then the Poisson*



transform  $\mathcal{P}_{\lambda,s,\nu}$  is a  $SU(r, r+b)$ -isomorphism of  $\mathcal{B}(S_{r,r+b}; s, \nu)$  onto the space of analytic functions  $F$  on  $I_{r,r+b}$  such that

$$(32) \quad \mathcal{H}^{(1)}F = (r+b)s((r+b)(s-1) + \nu)F \otimes I_r.$$

The bounded domain  $I_{r,r+b}$  is of non tube type, however the characterization of Poisson integrals involves a Hua operator of the second order. As a consequence, the proof of Theorem 8.1 is similar to Theorem 5.2 and [12, Theorem 6.1].

The necessarily condition is guaranteed by the following

**Proposition 8.2.** *For any  $u \in S_{r,r+b}$  the function  $z \mapsto P_{s,\nu}(z, \bar{u})$  satisfies*

$$\mathcal{H}P_{s,\nu}(z, \bar{u}) = (r+b)s[(r+b)(s-1) + \nu]P_{s,\nu}(z, \bar{u})I_r.$$

Indeed, using the computations of the proof of Theorem 6.1 and some arguments in the proof of [12, Theorem 5.3] we can extend the formula (29) to any (not necessarily tube) bounded symmetric domain:

$$\begin{aligned} \mathcal{H}^1 P_{s,\nu}(z, \bar{u}) = & \left[ \left(\frac{n}{r}s\right)^2 D(B(z, \bar{z})(\bar{z}^z - \bar{u}^z), z^{\bar{z}} - u^{\bar{z}}) - \left(\frac{n}{r}s\right)pZ_0 \right. \\ & \left. - \left(\frac{n}{r}s\right)\nu D(B(z, \bar{z})(\bar{u}^z - \bar{u}^z), z^{\bar{z}} - u^{\bar{z}}) \right] P_{s,\nu}(z, \bar{u})I_r. \end{aligned}$$

Specifying this formula to the domain  $I_{r,r+b}$  we get proposition.

On the other hand, to prove the sufficiency condition, the key point is to compute the radial part of the Hua operator  $\mathcal{H}^{(1)}$ . This follows immediately from the proof of Theorem 7.1.

**Theorem 8.3.** *Let  $f$  be  $\mathcal{C}^2$  and  $K$ -invariant function, then for  $a = \sum_{j=1}^r t_j c_j$ ,*

$$(33) \quad 4\mathcal{H}^{(1)}f(a) = \sum_{j=1}^r \mathcal{H}_j F(t_1, \dots, t_r) D(c_j, \bar{c}_j)^{(1)},$$

where the scalar-valued operators  $\mathcal{H}_j$  are given by

$$\begin{aligned} \mathcal{H}_j = & (1 - t_j^2)^2 \left( \frac{\partial^2}{\partial t_j^2} + \frac{1}{t_j} (1 + 2\nu) \frac{\partial}{\partial t_j} \right) + \\ & + \sum_{k \neq j} (1 - t_j^2)(1 - t_k^2) \left[ \frac{1}{t_j - t_k} \left( \frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_k} \right) + \frac{1}{t_j + t_k} \left( \frac{\partial}{\partial t_j} + \frac{\partial}{\partial t_k} \right) \right] + \\ & + (2b - 2\nu)(1 - t_j^2) \frac{1}{t_j} \frac{\partial}{\partial t_j}. \end{aligned}$$

## 9. RELATIVE DISCRETE SERIES

The aim of this section is to prove, for some special parameter, that the Poisson transform is Szegő type map into the relative discrete series. We will compute explicitly the corresponding element in the discrete series.

Let  $\alpha > -1$  such that  $\nu = p + \alpha$ , and consider the weighted probability measure on  $\Omega$

$$(34) \quad d\mu_\alpha(z) = c_\alpha h(z, \bar{z})^\alpha dm(z)$$

The group  $G$  acts unitarily on  $L^2(\Omega, \mu_\alpha)$  via

$$(35) \quad \pi_\nu(g)f(z) = f(g^{-1}z)(J_{g^{-1}}(z))^{\frac{\nu}{p}} \quad g \in G$$

where  $J_g$  is the Jacobian determinant of  $g \in G$ .

The irreducible decomposition of  $L^2(\Omega, \mu_\alpha)$  under the  $G$ -action has been given by Shimeno in [22, Theorem 5.10] where he proved abstractly that all discrete parts called relative discrete series appearing in the decomposition are holomorphic discrete series. In this section we need their explicit realization given by the second author in [31].

Let us introduce the conical functions, see [5]. Let  $c_1, \dots, c_r$  be the fixed Jordan frame. Put  $e_j = c_1 + \dots + c_j$ , for  $j = 1, \dots, r$ . Let  $U_j = \{z \in V : D(c_j, c_j)z = z\}$ . Then  $U_j$  is a Jordan subalgebra of  $V_1 = V_1(e)$  with a determinant polynomial  $\Delta_j$ . We extend the principal minors  $\Delta_j$  to all  $V$  via  $\Delta_j(z) := \Delta_j(P_{U_j}(z))$ , where  $P_{U_j}$  is the orthogonal projection onto  $U_j$ . Notice that  $\Delta_r(z) = \Delta(z) = \mathbf{det}(z)$ .

For any  $\underline{m} = (m_1, \dots, m_r) \in \mathbb{C}^r$ , consider the associated conical function

$$\Delta_{\underline{m}}(z) := \Delta_1^{m_1 - m_2}(z) \Delta_2^{m_2 - m_3}(z) \cdots \Delta_r^{m_r}(z).$$

If  $z = \sum_{j=1}^r z_j c_j$  then  $\Delta_{\underline{m}}(z) = \prod_{j=1}^r z_j^{m_j}$ .

Let  $q(z)$  be the following function

$$q(z) = \frac{1}{p} \partial \log \mathbf{det} B(z, \bar{z})^{-1}$$

where  $B(z, \bar{w}) = I - Q(z)Q(\bar{w}) + D(z, \bar{w})$  is the Bergman operator. The function  $q(z)$  is also given, using the quasi-inverse, by

$$q(z) = \bar{z}^z = B(\bar{z}, z)^{-1}(\bar{z} - Q(\bar{z})z),$$

see [31].

Let  $\nu = \alpha + p$  and denote

$$\ell = \begin{cases} \frac{\alpha+1}{2} - 1 = \frac{\nu-p-1}{2} & \text{if } \alpha \text{ is an odd integer} \\ \lceil \frac{\alpha+1}{2} \rceil = \lceil \frac{\nu-p+1}{2} \rceil & \text{otherwise} \end{cases}$$

Here  $[t]$  stands for the integer part of  $t \in \mathbb{R}$ . Define

$$(36) \quad D_\nu = \{\underline{m} = \sum_{j=1}^r m_j \gamma_j, \quad 0 \leq m_1 \leq \cdots \leq m_r \leq \ell, \quad m_j \in \mathbb{Z}\}.$$

We let  $A_{\underline{m}}^{2,\alpha}$  to be the subspace of  $L^2(\Omega, \mu_\alpha)$  generated by the function  $\bar{\Delta}_{\underline{m}}(q(z))$ , for  $\underline{m} \in D_\nu$ .

We reformulate [22, Theorem 5.10] and [31, Theorem 4.7, remark 4.8] in the following.

**Theorem 9.1.** *The relative discrete series representations appearing in  $L^2(\Omega, \mu_\alpha)$  are all holomorphic discrete series of the form  $A_{\underline{m}}^{2,\alpha}$  with  $\underline{m} \in D_\nu$ . The highest weight vector of  $A_{\underline{m}}^{2,\alpha}$  is given by  $\bar{\Delta}_{\underline{m}}(q(z))$ .*

We can now state the main theorem in this section.

**Theorem 9.2.** *Let  $\underline{\delta} = (\delta, \delta, \dots, \delta)$  such that  $s_{\underline{r}}^{\underline{n}} = \frac{\underline{n}}{r} + \delta - \nu$  and  $0 \leq \delta < \lceil \frac{\nu-p}{2} \rceil$ . Then the Poisson transform  $\mathcal{P}_{\lambda_{s,\nu}}$  is a  $G$ -equivariant Szegő type map that carries the space  $\mathcal{B}(S, s, \nu)$  onto the relative discrete series representation  $A_{\underline{\delta}}^{2,\alpha}$ .*

This theorem is consequence of the following proposition.

**Proposition 9.3.** *Under the same conditions as in Theorem 9.3 we have*

$$(37) \quad (\mathcal{P}_{s,\nu} \bar{\Delta}_{\underline{\delta}})(z) = \int_S P_{s,\nu}(z, u) \bar{\Delta}_{\underline{\delta}}(u) du = \frac{\binom{s_{\underline{r}}^{\underline{n}}}{\underline{n}}_\delta}{\binom{\underline{n}}{r}_\delta} \bar{\Delta}_{\underline{\delta}}(q(z)).$$

Let us first prove the formula below

**Lemma 9.4.** *For any highest weight  $\underline{m}$  the following formula holds*

$$(38) \quad \Delta(z)^\delta \bar{\Delta}(w)^\delta K_{\underline{m}}(z, w) = c(\underline{m}, \delta) K_{\underline{m}+\delta}(z, w)$$

where  $c(\underline{m}, \delta) = \binom{\underline{n}}{r} + \underline{m}_\delta = \prod_{j=1}^r (\frac{\alpha}{2}(r-j) + 1 + m_j)_\delta$

*Proof.* Since  $f(z) \mapsto \Delta(z)^\delta f(z)$  is a  $K$ -intertwining map from  $\mathcal{P}_{\underline{m}}$  onto  $\mathcal{P}_{\underline{m}+\delta}$  we have that  $\Delta(z)^\delta \bar{\Delta}(w)^\delta K_{\underline{m}}(z, w)$  is equal to  $K_{\underline{m}+\delta}(z, w)$  up to

a constant  $c(\underline{\mathbf{m}}, \delta)$ . Now taking  $z = w = e$  and using [5, Lemma 3.1, Theorem 3.4] we find that the constant is

$$c(\underline{\mathbf{m}}, \delta) = \frac{K_{\underline{\mathbf{m}}}(e, e)}{K_{\underline{\mathbf{m}+\delta}}(e, e)} = \frac{d_{\underline{\mathbf{m}}}}{(n/r)_{\underline{\mathbf{m}}}} \frac{(n/r)_{\underline{\mathbf{m}+\delta}}}{d_{\underline{\mathbf{m}+\delta}}} = \left(\frac{n}{r} + \underline{\mathbf{m}}\right)_{\delta}.$$

□

*Proof of Proposition 9.3.* Put  $\sigma = s\frac{n}{r}$ . We compute the image  $\mathcal{P}_{s,\nu}\bar{\Delta}_{\delta}$ ,

$$\begin{aligned} (\mathcal{P}_{s,\nu}\bar{\Delta}_{\delta})(z) &= \int_S h(z, u)^{-\nu} h(z, z)^{\sigma} h(z, u)^{-\sigma} h(u, z)^{-\sigma} \bar{\Delta}_{\delta}(u) du \\ &= h(z, z)^{\sigma} \int_S h(z, u)^{-\nu-\sigma} h(u, z)^{-\sigma} \bar{\Delta}_{\delta}(u) du \end{aligned}$$

We use the Faraut-Koranyi expansion [4, Theorem 3.8] of the reproducing kernels  $h(z, u)^{-\nu-\sigma}$  and  $h(u, z)^{-\sigma}$  so that,

$$\begin{aligned} (\mathcal{P}_{s,\nu}\bar{\Delta}_{\delta})(z) &= h(z, z)^{\sigma} \int_S \left[ \sum_{\underline{\mathbf{m}} \geq 0} (\nu + \sigma)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, u) \times \right. \\ &\quad \left. \times \sum_{\underline{\mathbf{m}'} \geq 0} (\sigma)_{\underline{\mathbf{m}'}} K_{\underline{\mathbf{m}'}}(u, z) \right] \bar{\Delta}_{\delta}(u) du \end{aligned}$$

which by Lemma 9.4 is

$$\begin{aligned} (\mathcal{P}_{s,\nu}\bar{\Delta}_{\delta})(z) &= h(z, z)^{\sigma} \int \left[ \sum_{\underline{\mathbf{m}} \geq 0} (\nu + \sigma)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}+\delta}(z, u) \Delta_{\delta}(z)^{-1} c(\underline{\mathbf{m}}, \delta) \times \right. \\ &\quad \left. \times \sum_{\underline{\mathbf{m}'} \geq 0} (\sigma)_{\underline{\mathbf{m}'}} K_{\underline{\mathbf{m}'}}(u, z) \right] du \\ &= h(z, z)^{\sigma} \sum_{\underline{\mathbf{m}} \geq 0} \left[ (\nu + \sigma)_{\underline{\mathbf{m}}} (\sigma)_{\underline{\mathbf{m}+\delta}} c(\underline{\mathbf{m}}, \delta) \Delta_{\delta}(z)^{-1} \times \right. \\ &\quad \left. \times \int K_{\underline{\mathbf{m}+\delta}(z, u) K_{\underline{\mathbf{m}+\delta}}(u, z) du \right] \end{aligned}$$

The last equality follows from the Schur orthogonality relation. Furthermore, since the ratio of the Fischer inner product and the standard  $K$ -invariant inner product of  $L^2(S)$  is constant, see [5, Corollary 3.5] or [27], we obtain

$$\begin{aligned} (\mathcal{P}_{s,\nu}\bar{\Delta}_{\delta})(z) &= h(z, z)^{\sigma} \sum_{\underline{\mathbf{m}} \geq 0} \left[ (\nu + \sigma)_{\underline{\mathbf{m}}} (\sigma)_{\underline{\mathbf{m}+\delta}} c(\underline{\mathbf{m}}, \delta) \Delta_{\delta}(z)^{-1} \times \right. \\ &\quad \left. \times \left(\frac{n}{r}\right)_{\underline{\mathbf{m}+\delta}}^{-1} K_{\underline{\mathbf{m}+\delta}}(z, z) \right] \end{aligned}$$

which again by Lemma 9.4 is

$$\begin{aligned} (\mathcal{P}_{s,\nu}\bar{\Delta}_{\delta})(z) &= h(z, z)^{\sigma} \bar{\Delta}_{\delta}(z) \sum_{\underline{\mathbf{m}} \geq 0} (\nu + \sigma)_{\underline{\mathbf{m}}} (\sigma)_{\underline{\mathbf{m}+\delta}} \left(\frac{n}{r}\right)_{\underline{\mathbf{m}+\delta}}^{-1} \times \\ &\quad \times \Delta_{\delta}(z) K_{\underline{\mathbf{m}}}(z, z) \\ &= h(z, z)^{\sigma} \frac{(\sigma)_{\delta}}{\left(\frac{n}{r}\right)_{\delta}} \bar{\Delta}_{\delta}(z) \sum_{\underline{\mathbf{m}} \geq 0} (\sigma + \delta)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, z) \\ &= h(z, z)^{\sigma} \frac{(\sigma)_{\delta}}{\left(\frac{n}{r}\right)_{\delta}} \bar{\Delta}_{\delta}(z) h(z, z)^{-(\sigma+\delta)} \\ &= \frac{(\sigma)_{\delta}}{\left(\frac{n}{r}\right)_{\delta}} \frac{\bar{\Delta}_{\delta}(z)}{h(z, z)^{\delta}} \end{aligned}$$

Finally since  $\frac{\bar{\Delta}(z)}{h(z, z)} = \bar{\Delta}(q(z))$ , see [31, Corollary 4.4], the proof is completed.

□

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