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Separable solutions of quasilinear Lane-Emden equations

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Abstract For $0 < p-1 < q$ and $\epsilon = \pm 1$, we prove the existence of solutions of $-\Delta_p u = \epsilon u^q$ in a cone C_S , with vertex 0 and opening S , vanishing on ∂C_S , under the form $u(x) = |x|^\beta \omega(\frac{x}{|x|})$. The problem reduces to a quasilinear elliptic equation on S and existence is based upon degree theory and homotopy methods. We also obtain a non-existence result in some critical case by an integral type identity.

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Key words. quasilinear elliptic equations; Leray-Schauder degree; homotopy invariance; excision.

1 Introduction

It is well established that the description of the boundary behavior of singular solutions of Lane-Emden equations

$$-\Delta u = \epsilon u^q \tag{1.1}$$

with $q > 1$ in a domain $\Omega \subset \mathbb{R}^N$ is greatly helped by using specific separable solutions of the same equation. This was performed in 1991 by Gmira-Véron [7] in the case $\epsilon = -1$ and more recently by Bidaut-Véron-Ponce-Véron [3] in the case $\epsilon = 1$. If the domain is assumed to be a cone $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : x/|x| \in S\}$ with vertex 0 and opening $S \subset S^{N-1}$, separable solutions of (1.1) vanishing on $\partial C_S \setminus \{0\}$ were of the form

$$u(x) = |x|^{-\frac{2}{q-1}} \omega(x/|x|), \tag{1.2}$$

with ω satisfying

$$-\Delta' \omega - \ell_{q,N} \omega - \epsilon \omega^q = 0 \quad \text{in } S, \tag{1.3}$$

vanishing on ∂S and where $\ell_{q,N} = \left(\left(\frac{2}{q-1} \right) \left(\frac{2q}{q-1} - N \right) \right)$ and Δ' is the Laplace-Beltrami operator on S^{N-1} . To this equation is associated the functional

$$J(\phi) := \int_S \left(\frac{1}{2} |\nabla' \phi|^2 - \frac{\ell_{q,N}}{2} \phi^2 - \frac{\epsilon}{q+1} |\phi|^{q+1} \right) dv_g, \tag{1.4}$$

where ∇' is the covariant derivative on S^{N-1} . In the case $\epsilon = 1$, non-existence of a non-trivial solution of (1.3) when $\ell_{q,N} \leq \lambda_S$ (the first eigenvalue of Δ' in $W_0^{1,2}$) follows by multiplying the equation by the first eigenfunction and integrating over S ; existence holds when $\ell_{q,N} > \lambda_S$ and $q < \frac{N+1}{N-3}$ by classical variational methods, and again non-existence holds when $q \geq \frac{N+1}{N-3}$ by using an integral identity. When $\epsilon = -1$, non-existence of a non-trivial solution of (1.3) when $\ell_{q,N} \leq \lambda_S$ is obtained by multiplying the equation by ω and integrating over S , while existence when $\ell_{q,N} > \lambda_S$ follows by minimizing J over $W_0^{1,2}(S)$.

In this paper we investigate similar questions for the the quasilinear Lane-Emden equations

$$- \operatorname{div} (|\nabla u|^{p-2} \nabla u) = \epsilon u^q \quad \text{in } C_S, \quad (1.5)$$

where $q > p - 1 > 0$ and $\epsilon = \pm 1$ and we look for positive solutions u , vanishing on $\partial C_S \setminus \{0\}$, under the separable form

$$u(x) = |x|^{-\beta} \omega(x/|x|). \quad (1.6)$$

It is straightforward to check that u is a solution of (1.5) provided

$$\beta = \beta_q := \frac{p}{q + 1 - p} \quad (1.7)$$

and ω is a positive solution of

$$- \operatorname{div} \left((\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{(p-2)/2} \nabla' \omega \right) - \beta_q \lambda(\beta_q) (\beta_q^2 \omega^2 + |\nabla' \omega|^2)^{(p-2)/2} \omega = \epsilon \omega^q \quad (1.8)$$

in S vanishing on ∂S , where $\operatorname{div}(\cdot)$ is the divergence operator defined according to the intrinsic metric g and where we have set

$$\lambda(\beta) = \beta(p-1) + p - N. \quad (1.9)$$

If $\epsilon = 0$, it is now well-known that positive p -harmonic functions in C_S vanishing on ∂S exist under the form (1.6), and either they are regular at 0 and $\beta = -\tilde{\beta}_S < 0$, or $\beta = \beta_S > 0$. In this case $\omega = \tilde{\omega}_S$ or ω_S is a solution of

$$- \operatorname{div} \left((\beta^2 \omega^2 + |\nabla' \omega|^2)^{(p-2)/2} \nabla' \omega \right) - \beta \lambda(\beta) (\beta^2 \omega^2 + |\nabla' \omega|^2)^{(p-2)/2} \omega = 0 \quad (1.10)$$

in S , where $\beta = \tilde{\beta}_S$ or β_S . The existence of $(\tilde{\beta}_S, \tilde{\omega}_S)$ is due to Tolksdorf in a pioneering work [16]. Tolksdorf's method has been adapted by Véron [18] in order to prove the existence (β_S, ω_S) . Later on Porretta and Véron [11] obtained a more general proof of the existence of such couples. Notice that β is uniquely determined and ω is unique up to homothety. In both cases the proofs rely on strict maximum principle. When $p \neq 2$, existence of a nontrivial solution in the case $\epsilon = 1$ is obtained when $N = 2$ and $\beta_q < \beta_S$ by a dynamical system approach [2]; while if $\epsilon = -1$ and $\beta_q > \beta_S$, such an existence is proved in [18] by a suitable adaptation of Tolksdorf's construction. No functional can be associated to (1.8), excepted in the

case $q = q^* = \frac{Np}{N-p} - 1$. In such a case (1.8) is the Euler-Lagrange equation for the functional

$$J_q(\phi) := \int_S \left(\frac{1}{p} (\beta_{q^*}^2 \phi^2 + |\nabla' \phi|^2)^{\frac{p}{2}} - \frac{\epsilon}{q^* + 1} |\phi|^{q^* + 1} \right) dv_g, \quad (1.11)$$

and existence of a non-trivial solution of (1.8) with $\epsilon = 1$ is derived from the mountain pass theorem. In all the other cases variational techniques cannot be used and have to be replaced by topological methods based upon Leray-Schauder degree. Define q_c by

$$q_c = q_{c,p} = \begin{cases} \frac{(N-1)p}{N-1-p} - 1 & \text{if } p < N - 1 \\ \infty & \text{if } p \geq N - 1, \end{cases}$$

then we prove the following results:

I Let $\epsilon = 1$. Assume $p > 1$, $q < q_c$ and $\beta_q < \beta_S$, then (1.8) admits a positive solution in S vanishing on ∂S .

II Let $\epsilon = -1$. Assume $p > 1$ and $\beta_q > \beta_S$, then (1.8) admits a unique positive solution in S vanishing on ∂S .

The result **I** is based upon sharp Liouville theorems for solutions of (1.5) in \mathbb{R}^N or \mathbb{R}_+^N respectively due to Serrin-Zou [15] and Zou [21]. In the case of **II**, the existence part is already known, but we give here a simpler form than the one in [18], using a topological deformation acting on the exponent p . In the case $\epsilon = 1$, the result is optimal in the case $q = q_c$; indeed, using an integral identity, we also prove

III Let $\epsilon = 1$, $S \subset S_+^{N-1}$ be a starshaped domain and $1 < p < N - 1$. If $q = q_c$, then (1.8) admits no positive solution in S vanishing on ∂S .

Notice that when $p = 2$ an integral identity was used in [3] to prove non existence for all $q \geq q_{c,2}$. The form which is derived in the case $p \neq 2$ is much more complicated and we prove non-existence only in the case $q = q_{c,p}$. Other non-existence results for positive solutions of (1.8), $\epsilon = \pm 1$, have been obtained in [2] and [20] by using different form of maximum principle and they all depend upon the sign $\beta_q - \beta_S$.

2 Nonexistence for the reaction problem

Let us consider the positive solutions of

$$- \operatorname{div} \left((\beta^2 \omega^2 + |\nabla' \omega|^2)^{(p-2)/2} \nabla' \omega \right) - \beta \lambda(\beta) (\beta^2 \omega^2 + |\nabla' \omega|^2)^{(p-2)/2} \omega = \epsilon \omega^q \quad (2.1)$$

vanishing on ∂S . Recall that $\lambda(\beta)$ is given by (1.9) and that, in connection with problem (1.5), we have interest in the special case where $\beta = \beta_q$ is given by (1.7). The following Pohozaev-type identity, which is valid for any β , is the key for non-existence.

Proposition 2.1 *Let $S \subset S^{N-1}$, $S \neq S^{N-1}$, be a C^1 domain and ϕ the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S)$. If $\omega \in W_0^{1,p}(S) \cap C(\bar{S})$ is a positive solution in S of (2.1), and if we set $\Omega = (\beta^2\omega^2 + |\nabla'\omega|^2)^{1/2}$, then the following identity holds*

$$\left(1 - \frac{1}{p}\right) \int_{\partial S} |\omega_\nu|^p \phi_\nu dS = \epsilon A \int_S \omega^{q+1} \phi d\sigma + B \int_S \Omega^{p-2} |\nabla'\omega|^2 \phi d\sigma + C \int_S \Omega^{p-2} \omega^2 \phi d\sigma, \quad (2.2)$$

with

$$A = A(\beta) := -\frac{N-1}{q+1} - \beta(p\beta + p - N) \quad (2.3)$$

$$B = B(\beta) := \frac{N-1-p}{p} + \beta(p\beta + p - N), \quad (2.4)$$

$$C = C(\beta) := \beta^2 \left(\frac{N-1}{p} - (p\beta + p - N)\lambda(\beta) \right). \quad (2.5)$$

In order to prove Proposition 2.1, we start with the following lemma.

Lemma 2.1 *Let $\omega \in W_0^{1,p}(S) \cap C(\bar{S})$ be a positive solution in S of (2.1). Then we have*

$$\begin{aligned} \left(1 - \frac{1}{p}\right) \int_{\partial S} |\omega_\nu|^p \phi_\nu dS &= \int_S \left(\frac{\Delta'\phi}{q+1} - \beta(p\beta + p - N)\phi \right) \omega^{q+1} d\sigma - \frac{1}{p} \int_S \Omega^p \Delta'\phi d\sigma \\ &\quad + \int_S \Omega^{p-2} D^2\phi(\nabla'\omega, \nabla'\omega) d\sigma + \beta(p\beta + p - N) \int_S \Omega^{p-2} |\nabla'\omega|^2 \phi d\sigma \\ &\quad - \beta^2(p\beta + p - N)((p-1)\beta + p - N) \int_S \Omega^{p-2} \omega^2 \phi. \end{aligned} \quad (2.6)$$

Proof. By the regularity theory of p -Laplace type equations (see e.g. ...and the Appendix in [11]) it turns out that $\omega \in C^{1,\gamma}(\bar{S})$ for some $\gamma \in (0, 1)$, and since $(\beta^2\omega^2 + |\nabla'\omega|^2) > 0$ in the interior, by elliptic regularity we have $\omega \in C^2(S)$. Let $\phi \in C^2(S)$ be a given function and $\zeta \in C_c^1(S)$; since ζ is compactly supported we can multiply (2.1) by the test function $\langle \nabla'\omega, \nabla'\phi \rangle \zeta$. Integrating by parts we get (using the notation $\Omega := (\beta^2\omega^2 + |\nabla'\omega|^2)^{1/2}$)

$$\begin{aligned} \int_S \Omega^{p-2} \left(\frac{1}{2} \langle \nabla'|\nabla'\omega|^2, \nabla'\phi \rangle + D^2\phi(\nabla'\omega, \nabla'\omega) \right) \zeta d\sigma + \int_S \Omega^{p-2} \langle \nabla'\omega, \nabla\zeta \rangle \langle \nabla'\omega, \nabla'\phi \rangle d\sigma \\ = \beta\lambda(\beta) \int_S \Omega^{p-2} \omega \langle \nabla'\omega, \nabla'\phi \rangle \zeta d\sigma + \frac{1}{q+1} \int_S \langle \nabla'\omega^{q+1}, \nabla'\phi \rangle \zeta d\sigma. \end{aligned}$$

Since

$$\Omega^{p-2} \frac{1}{2} \langle \nabla'|\nabla'\omega|^2, \nabla'\phi \rangle = \frac{1}{p} \langle \nabla'\Omega^p, \nabla\phi \rangle - \beta^2 \Omega^{p-2} \omega \langle \nabla'\omega, \nabla'\phi \rangle$$

we obtain, due to (1.9),

$$\begin{aligned} \frac{1}{p} \int_S \langle \nabla'\Omega^p, \nabla\phi \rangle \zeta d\sigma + \int_S \Omega^{p-2} D^2\phi(\nabla'\omega, \nabla'\omega) \zeta d\sigma + \int_S \Omega^{p-2} \langle \nabla'\omega, \nabla\zeta \rangle \langle \nabla'\omega, \nabla'\phi \rangle d\sigma \\ = \beta(p\beta + p - N) \int_S \Omega^{p-2} \omega \langle \nabla'\omega, \nabla'\phi \rangle \zeta d\sigma + \frac{1}{q+1} \int_S \langle \nabla'\omega^{q+1}, \nabla'\phi \rangle \zeta d\sigma. \end{aligned}$$

Integrating by parts the first and last term we get

$$\begin{aligned}
& -\frac{1}{p} \int_S \Omega^p \langle \nabla \phi, \nabla \zeta \rangle d\sigma + \frac{1}{q+1} \int_S \omega^{q+1} \langle \nabla' \phi, \nabla' \zeta \rangle d\sigma + \int_S \left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p} \right) \Delta' \phi \zeta d\sigma \\
& + \int_S \Omega^{p-2} D^2 \phi (\nabla' \omega, \nabla' \omega) \zeta d\sigma + \int_S \Omega^{p-2} \langle \nabla' \omega, \nabla \zeta \rangle \langle \nabla' \omega, \nabla' \phi \rangle d\sigma \\
& = \beta(p\beta + p - N) \int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle \zeta d\sigma.
\end{aligned} \tag{2.7}$$

Now we choose $\zeta = \zeta_\delta$, where ζ_δ is a sequence of C^1 compactly supported functions such that $\zeta_\delta(\sigma) \rightarrow 1$ for every $\sigma \in S$ and $|\nabla' \zeta_\delta|$ is bounded in $L^1(S)$. It is easy to see by integration by parts that we have for every continuous vector field $F \in C(\overline{S})$

$$\int_S \langle F, \nabla' \zeta_\delta \rangle d\sigma \rightarrow - \int_{\partial S} \langle F, \nu(\sigma) \rangle d\sigma$$

where ν is the outward unit normal on ∂S . We take $\zeta = \zeta_\delta$ in (2.7) and we let $\delta \rightarrow 0$. Using that $\omega \in C^1(\overline{S})$ and that, by Hopf lemma, $\omega_\nu := \langle \nabla' \omega, \nu(\sigma) \rangle < 0$ we can actually pass to the limit in the integrals containing $\nabla \zeta_\delta$. Recalling that $\omega = 0$ and $\nabla' \omega = -|\omega_\nu| \nu$ on ∂S we obtain

$$\begin{aligned}
\left(1 - \frac{1}{p}\right) \int_{\partial S} |\omega_\nu|^p \phi_\nu dS &= \int_S \left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p} \right) \Delta' \phi d\sigma + \int_S \Omega^{p-2} D^2 \phi (\nabla' \omega, \nabla' \omega) d\sigma \\
&\quad - \beta(p\beta + p - N) \int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle d\sigma.
\end{aligned} \tag{2.8}$$

Multiplying (2.1) by $\omega \phi$ we derive

$$\int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle d\sigma = - \int_S \Omega^{p-2} |\nabla' \omega|^2 \phi d\sigma + \beta \lambda(\beta) \int_S \Omega^{p-2} \omega^2 \phi d\sigma + \int_S \omega^{q+1} \phi d\sigma,$$

therefore (2.8) becomes, replacing its last term,

$$\begin{aligned}
\left(1 - \frac{1}{p}\right) \int_{\partial S} |\omega_\nu|^p \phi_\nu dS &= \int_S \left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p} \right) \Delta' \phi d\sigma + \int_S \Omega^{p-2} D^2 \phi (\nabla' \omega, \nabla' \omega) d\sigma \\
&\quad - \beta(p\beta + p - N) \int_S \omega^{q+1} \phi d\sigma + \beta(p\beta + p - N) \int_S \Omega^{p-2} |\nabla' \omega|^2 \phi \\
&\quad - \beta^2(p\beta + p - N) \lambda(\beta) \int_S \Omega^{p-2} \omega^2 \phi.
\end{aligned}$$

which is (2.6). □

Proof of Proposition 2.1. We use Lemma 2.1 choosing in (2.6) ϕ to be the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$. Since $\Delta'\phi = (1-N)\phi$, $D^2\phi = -\phi g_0$, we get

$$\begin{aligned} \left(1 - \frac{1}{p}\right) \int_{\partial S} |\omega_\nu|^p \phi_\nu dS &= - \int_S \left(\frac{N-1}{q+1} + \beta(p\beta + p - N) \right) \omega^{q+1} \phi d\sigma \\ &+ \frac{N-1}{p} \int_S \Omega^p \phi d\sigma - \int_S \Omega^{p-2} |\nabla' \omega|^2 \phi d\sigma \\ &+ \beta(p\beta + p - N) \int_S \Omega^{p-2} |\nabla' \omega|^2 \phi d\sigma \\ &- \beta^2(p\beta + p - N) \lambda(\beta) \int_S \Omega^{p-2} \omega^2 \phi. \end{aligned} \quad (2.9)$$

Then (2.2) follows, with A , B and C given by (2.3)-(2.5). \square

We recall that a C^1 domain $S \subset S_+^{N-1}$ is starshaped if there exists a spherical harmonic ϕ of degree 1 such that $\phi > 0$ on S and for any $a \in \partial S$,

$$\langle \nabla \phi, \nu_a \rangle \leq 0 \quad (2.10)$$

where ν_a is the unit outward normal vector to ∂S at a in the tangent plane T_a to S^{N-1} . It means also that there exists some $x_0 \in S$ such that the geodesic (or great circle) connecting x_0 and a remains inside S .

Theorem 2.1 *Assume $1 < p < N - 1$, $q = q_c$ and $S \subset S_+^{N-1}$ is starshaped with respect to some point. Then (1.8) with $\epsilon = 1$ admits no positive solution in S vanishing on ∂S .*

Proof. Recall that in (1.8) we have $\beta_q = \frac{p}{q-(p-1)}$, hence different values of q are in one-to-one correspondence with different values of β . We first notice that, if $q = q_c$ the corresponding critical β is given by

$$\beta_c := \frac{p}{q_c - (p-1)} = \frac{N-1-p}{p} \quad (2.11)$$

We use now Proposition 2.1 with $\beta = \beta_q$ and we analyze the values of the coefficients A , B , C given by (2.3)-(2.5) as functions of β . First of all, since $q+1 = \frac{p(1+\beta)}{\beta}$, we have

$$A = -\frac{(N-1)\beta}{p(1+\beta)} - \beta(p\beta + p - N) = -\frac{\beta}{(\beta+1)} \left(\frac{N-1}{p} + p(\beta+1)^2 - N(\beta+1) \right)$$

and since from (2.11) we have $\beta_c + 1 = \frac{N-1}{p}$ we deduce

$$A = -\frac{\beta}{(\beta+1)} p \left(\beta + 1 - \frac{1}{p} \right) (\beta - \beta_c)$$

Still using (2.11), we also get

$$B = \beta_c + \beta(p(\beta - \beta_c) - 1) = (\beta - \beta_c)(\beta p - 1)$$

Finally, using (1.9) and (2.11) we have

$$\begin{aligned} C &= \beta^2 \left(\frac{N-1}{p} - (p\beta + p - N)((p-1)\beta_q + p - N) \right) \\ &= \beta^2 (\beta_c + 1 - (p(\beta - \beta_c) - 1)(p(\beta - \beta_c) - (\beta + 1))) \\ &= \beta^2 (\beta - \beta_c)(1 - p) \left(p\beta - 1 - \frac{p(N-p)}{p-1} \right). \end{aligned} \quad (2.12)$$

Therefore $A \geq 0$, $B \geq 0$ and $C \geq 0$ can be obtained only if $q = q_c$, i.e. $\beta = \beta_c$, in which case $A = B = C = 0$. Since $\phi_\nu \leq 0$, we deduce from (2.2) that $|\omega_\nu|^p \phi_\nu = 0$ on ∂S . Unless ω is identically zero, we have $\omega_\nu < 0$ by Hopf boundary lemma. Then $\phi_\nu \equiv 0$, and using the equation satisfied by ϕ and Gauss formula, we derive

$$\lambda_S \int_S \phi d\sigma = 0 \implies \phi \equiv 0,$$

which is a contradiction. This proves the first assertion. \square

Remark. If $p = 2$, it is proved in [3] that the nonexistence result of Theorem 2.1 holds for every $q \geq q_c$, which suggests that our result above is not optimal. The proof in [3] cannot be applied here since the term $\int_S \Omega^{p-2} \omega \langle \nabla' \omega, \nabla' \phi \rangle d\sigma$ is completely integrable only if $p = 2$. However, we conjecture that, even when $p \neq 2$, the conclusion of Theorem 2.1 holds under the more general condition $q \geq q_c$.

Remark. If we assume that $p \neq 2$, the proof of Theorem 2.1 relies on the existence of a positive function ϕ in S , satisfying (2.10) on ∂S and

$$\frac{\Delta' \phi}{(q+1)\phi} - \beta(p\beta + p - N) \geq 0, \quad (2.13)$$

$$\frac{pD^2\phi(\xi, \xi) - \Delta' \phi}{p\phi} + \beta(p\beta + p - N) \geq 0 \quad \forall \xi \in S^{N-1}, \quad (2.14)$$

and

$$-\frac{\Delta' \phi}{p\phi} - (p\beta + p - N)((p-1)\beta + p - N) \geq 0. \quad (2.15)$$

Remark 2.1 For the sake of completeness, we recall the following previous non-existence results:

[6, Th 1] Let $\epsilon = 1$ and $0 < p - 1 < q$. If $\beta_q \geq \beta_S$ there exists no positive solution of (1.8) in S which vanishes on ∂S .

[20, Th 3.4] Let $\epsilon = -1$ and $0 < p - 1 < q$. If $\beta_q \leq \beta_S$ there exists no positive solution of (1.8) in S which vanishes on ∂S .

3 Existence for the reaction problem

Concerning the problem with reaction we consider a more general statement than Theorem I, replacing the sphere by a complete d -dimensional Riemannian manifold (M, g) and suppose that S is a relatively compact smooth open domain of M . We denote by $\nabla := \nabla_g$ the gradient of a function identified with its covariant derivatives and by $div := div_g$ the divergence operator acting on vector fields. The following result is proved in [11].

Theorem 3.1 *For any $\beta > 0$ there exists a unique $\Lambda_\beta > 0$ and a unique (up to an homothety) positive function $\omega_\beta \in C^2(S) \cap C^1(\overline{S})$ solution of*

$$\begin{cases} -div \left(\left(\beta^2 \omega_\beta^2 + |\nabla \omega_\beta|^2 \right)^{p-2} \nabla \omega_\beta \right) = \beta \Lambda_\beta \left(\beta^2 \omega_\beta^2 + |\nabla \omega_\beta|^2 \right)^{p-2} \omega_\beta & \text{in } S \\ \omega_\beta = 0 & \text{on } \partial S. \end{cases} \quad (3.1)$$

The mapping $\beta \mapsto \Lambda_\beta$ is continuous and decreasing, and the spectral exponent β_S is the unique $\beta > 0$ such that $\Lambda_{\beta_S} = \beta_S(p-1) + p - d - 1$.

Remark 3.1 *Let us notice that the monotone character of $\beta \mapsto \Lambda_\beta$ implies that*

$$0 < \beta < \beta_S \iff \Lambda_\beta - \beta(p-1) > \Lambda_{\beta_S} - \beta_S(p-1) = p - d - 1$$

Therefore, if we set $\lambda(\beta) = \beta(p-1) + p - d - 1$, we deduce that

$$0 < \beta < \beta_S \iff \Lambda_\beta > \lambda(\beta). \quad (3.2)$$

Let us now prove the existence of solutions for the reaction problem.

Theorem 3.2 *Assume $1 < p < d$ and $p-1 < q < q_c := pd/(d-p) - 1$. Then for any $0 < \beta < \beta_S$, there exists a positive function $\omega \in C(\overline{S}) \cap C^2(S)$ satisfying*

$$\begin{cases} -div \left(\left(\beta^2 \omega^2 + |\nabla \omega|^2 \right)^{\frac{p}{2}-1} \nabla \omega \right) = \beta \lambda(\beta) \left(\beta^2 \omega^2 + |\nabla \omega|^2 \right)^{\frac{p}{2}-1} \omega + \omega^q & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \quad (3.3)$$

where $\lambda(\beta) = \beta(p-1) + p - d - 1$.

The proof will be based upon the following variant of Schauder fixed point theorem due to de Figuereido, Lions and Nussbaum [5].

Theorem 3.3 *Let X be a Banach space and $K \subset X$ a closed cone with non empty interior. Let $F : K \times \mathbb{R}_+ \rightarrow K$ be a compact mapping, and let $\Phi(u) = F(u, 0)$ (compact mapping from K into K). Assume the following holds: there exist $R_1 < R_2$ and $T > 0$ such that*

(i) $u \neq s\Phi(u)$ for every $s \in [0, 1]$ and every u : $\|u\| = R_1$.

(ii) $F(u, t) \neq u$ for every (u, t) : $\|u\| \leq R_2$ and $t \geq T$.

(iii) $F(u, t) \neq u$ for every u : $\|u\| = R_2$ and every $t \geq 0$.

Then, the mapping Φ has a fixed point u such that $R_1 < \|u\| < R_2$.

We also recall the following non-existence results due to Serrin and Zou [15], and Zou [21].

Theorem 3.4 *Assume $1 < p < d$ and $p - 1 < q < q_c$. Then there exists no C^1 positive solution of*

$$-\Delta_p u = u^q \quad (3.4)$$

in \mathbb{R}^d .

Theorem 3.5 *Assume $1 < p < d$ and $p - 1 < q < q_c$. Then there exists no C^1 positive solution of*

$$-\Delta_p u = u^q \quad (3.5)$$

in $\mathbb{R}_+^d := \{x = (x_1, \dots, x_d) : x_d > 0\}$ vanishing on $\partial\mathbb{R}_+^d := \{x = (x_1, \dots, x_d) : x_d = 0\}$.

Proof of Theorem 3.2. Define the operator \mathcal{A} in $W_0^{1,p}(S)$ as

$$\mathcal{A}(\omega) := -\operatorname{div}_g \left((\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \nabla \omega \right) + \beta^2 \omega (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1}.$$

Note that \mathcal{A} is the derivative of the functional

$$J(\omega) = \frac{1}{p} \int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}} dv_g$$

Since J is strictly convex, then \mathcal{A} is a strictly monotone and operator from $W_0^{1,p}(S)$ into $W^{-1,p'}(S)$, henceforth its inverse is well defined and continuous [8]. In order to apply Theorem 3.3, we denote by $X = C_0^1(\overline{S})$, the closure of $X = C_0^1(S)$ in $C^1(\overline{S})$. Clearly $X \subset W_0^{1,p}(S)$, with continuous imbedding, if it is endowed with its natural norm $\|\cdot\|_X := \|\cdot\|_{C^1(\overline{S})}$. Furthermore, since ∂S is C^2 , $C^1(\overline{S}) \cap W_0^{1,p}(S) = C_0^1(\overline{S})$. If K is the cone of nonnegative functions in S , it has a nonempty interior. For $t > 0$, we set

$$F(\omega, t) := \mathcal{A}^{-1} \left(\beta (\lambda(\beta) + \beta + t) \omega (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} + (\omega + t)^q \right).$$

Note that

$$\Phi(\omega) := F(\omega, 0) = \mathcal{A}^{-1} \left(\beta (\lambda(\beta) + \beta) \omega (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} + \omega^q \right);$$

henceforth any nontrivial fixed point for Φ would solve problem. (3.3).

We have to verify the assumptions of Theorem 3.3. First of all, the compactness of $F(\omega, t)$. If we set $F(\omega, t) = \phi$, then it means that $\phi \in W_0^{1,p}(S)$ satisfies

$$\begin{aligned} & -\operatorname{div}_g \left((\beta^2 \phi^2 + |\nabla \phi|^2)^{\frac{p}{2}-1} \nabla \phi \right) + \beta^2 \phi (\beta^2 \phi^2 + |\nabla \phi|^2)^{\frac{p}{2}-1} \\ & = \left(\beta (\lambda(\beta) + \beta + t) \omega (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} + (\omega + t)^q \right) \end{aligned} \quad (3.6)$$

Thus, if we assume that ω belongs to a bounded set in $K \cap X$, the right-hand side of (3.6) is bounded in $C(\overline{S})$. Thus, by standard regularity estimates up to the boundary for p -Laplace type operators (see [11, Appendix] and [17]), ϕ remains bounded in $C^{1,\alpha}(\overline{S})$ and therefore relatively compact in $C^1(\overline{S})$. It remains to show that conditions (i)–(iii) of Theorem 3.3 hold.

Step 1: Condition (i) holds. We proceed by contradiction in supposing that there exists a sequence $\{s_n\} \subset [0, 1]$ such that for any $n \in \mathbb{N}$ the following problem

$$\begin{cases} -\operatorname{div}_g \left((\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \nabla \omega \right) + \beta^2 (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega \\ \quad = s^{p-1} \beta (\lambda(\beta) + \beta) (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega + s_n^{p-1} \omega^q & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \quad (3.7)$$

admits a positive solution ω_n , and that there holds

$$\|\omega_n\|_X \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Set $w_n = \omega_n / \|\omega_n\|$, then w_n solves

$$\begin{cases} -\operatorname{div}_g \left((\beta^2 w_n^2 + |\nabla w_n|^2)^{\frac{p}{2}-1} \nabla w_n \right) + \beta^2 w_n (\beta^2 w_n^2 + |\nabla w_n|^2)^{\frac{p}{2}-1} \\ \quad = s_n^{p-1} \beta (\lambda(\beta) + \beta) (\beta^2 w_n^2 + |\nabla w_n|^2)^{\frac{p}{2}-1} w_n + s_n^{p-1} w_n^q \|w_n\|_X^{q-(p-1)} & \text{in } S \\ w_n = 0 & \text{on } \partial S \end{cases}$$

Up to subsequences, we assume that $s_n \rightarrow s$ for some $s \in [0, 1]$. Using compactness arguments we deduce that w_n will converge strongly in $C^1(\overline{S})$ to some positive function w such that $\|w\|_X = 1$ and which solves

$$\begin{cases} -\operatorname{div}_g \left((\beta^2 w^2 + |\nabla w|^2)^{\frac{p}{2}-1} \nabla w \right) \\ \quad = \beta (s^{p-1} \lambda(\beta) + (s^{p-1} - 1) \beta) (\beta^2 w^2 + |\nabla w|^2)^{\frac{p}{2}-1} w & \text{in } S \\ w = 0 & \text{on } \partial S \end{cases} \quad (3.8)$$

Using Theorem 3.1, we derive $\Lambda_\beta = s^{p-1} \lambda(\beta) + (s^{p-1} - 1) \beta$. Since $\beta < \beta_s$, $\lambda(\beta) < \Lambda_\beta$ by (3.2). Therefore, as $s \leq 1$, we get

$$s^{p-1} \lambda(\beta) + (s^{p-1} - 1) \beta \leq s^{p-1} \lambda(\beta) < \Lambda_\beta,$$

which is a contradiction. Consequently, there exists $R_1 > 0$ such that for any $s \in [0, 1]$, there holds $\omega \neq s\Phi(\omega)$ for any ω such that $\|\omega\|_X = R_1$.

Step 2: Condition (ii) holds. Consider the first eigenvalue $\lambda_{1,\beta}$ associated with the operator \mathcal{A} , i.e.

$$\lambda_{1,\beta} = \min \left\{ \int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}} dv_g : \omega \in W_0^{1,p}(S), \int_S |\omega|^p dv_g = 1 \right\} \quad (3.9)$$

Note that for t large enough, we have $\lambda(\beta) + \beta + t \geq 0$, hence, using that $q > p - 1$, we can find $T > 0$ such that

$$\beta(\lambda(\beta) + \beta + t)\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} + (\omega + t)^q \geq (\lambda_1 + \delta)\omega^{p-1} \quad \forall t \geq T, \forall \omega \geq 0.$$

Therefore, if $t \geq T$ and $F(\omega, t) = \omega$ we deduce that $\omega \neq 0$ and satisfies

$$\begin{cases} -\operatorname{div}_g \left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} \nabla\omega \right) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} \geq (\lambda_{1,\beta} + \delta)\omega^{p-1} & \text{in } S \\ \omega = 0 & \text{on } \partial S \end{cases}$$

The existence of a positive super-solution with $\lambda_{1,\beta} + \delta$ would make it possible to construct a positive solution as well. But since $\lambda_{1,\beta}$ is an isolated eigenvalue (see Appendix) this yields a contradiction. Therefore, for $t \geq T$ the equation $F(\omega, t) = \omega$ has no solution at all. Note that T only depends on λ_1, β .

Step 3: Condition (iii) holds. Note that, since we proved that (ii) holds independently on the choice of R_2 , it is enough to show that (iii) holds for every $t \leq T$.

This is done if we have the existence of universal a priori estimates, i.e. if we can prove the existence of a constant R_2 such that for any $t \leq T$ every positive solution of

$$\begin{cases} -\operatorname{div}_g \left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} \nabla\omega \right) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} = \\ \quad \beta(\lambda(\beta) + \beta + t)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + (\omega + t)^q & \text{in } S \\ \omega = 0 & \text{on } \partial S \end{cases}$$

satisfies $\|\omega\| \leq R_2$.

The crucial step is to prove that there exist universal a priori estimates for the L^∞ -norm (a bound for the $W_0^{1,p}$ -norm would follow immediately). A standard procedure is to reach this result reasoning by contradiction and using a blow-up argument. Indeed, if a universal bound does not exist, there exist a sequence of solutions ω_n and $t_n \leq T$ such that

$$\|\omega_n\|_\infty \rightarrow \infty.$$

Let σ_n be the (local coordinates of) maximum points of ω_n . Setting $M_n = \|\omega_n\|_\infty^{-\frac{q-(p-1)}{p}}$, define

$$v_n(y) = \frac{\omega_n(\sigma_n + M_n y)}{\|\omega_n\|_\infty} = M_n^{\frac{p}{q-(p-1)}} \omega_n(\sigma_n + M_n y)$$

Then v_n is a sequence of uniformly bounded solutions, which will be locally compact in the C^1 -topology. Rescaling the equation and passing to the limit in n we find out that the limit function v is positive and satisfies the equation

$$-\Delta_p v = c_0 v^q$$

for some constant c_0 (coming out from the local expression of Laplace-Beltrami operator). The equation would take place in either \mathbb{R}^d or in the half space \mathbb{R}_+^d ,

where $d = N - 1$, in which case v vanishes on $\partial\mathbb{R}_+^d$. Since $p - 1 < q < q_c$, this contradicts either Theorem 3.4, or Theorem 3.5 because, by construction, we have $v(0) = 1$. \square

Remark. In the case $p = 2$, existence is proved in [3] using a standard variational method. It is also proved that, if $(M, g) = (S^d, g_0)$ (the standard sphere), and if S is a spherical cap with center a , any positive solution of

$$\begin{cases} \Delta'\omega + \beta(\beta + 1 - d)\omega + \omega^q = 0 & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \quad (3.10)$$

depends only on the angle θ from a . Furthermore uniqueness is proved by a delicate analysis of the non-autonomous second order O.D.E. satisfied by ω . In the case $p \neq 2$ and assuming always that S is a spherical cap of (S^d, g_0) , it is still possible to construct a radial (i.e. depending only on θ) positive solution of (3.3): it suffices to restrict the functional analysis framework to radial functions. However, there are two interesting open questions the answer to which would be important:

- (i) *Are all positive solution of (3.3) radial ?*
- (ii) *Is any positive radial solution of (3.3) unique ?*

4 Existence for the absorption problem

Let us now consider the absorption problem, namely (1.8) with $\epsilon = -1$. We give an existence result which extends the previous ones obtained in [18], with a simpler proof.

Theorem 4.1 *Assume $0 < p - 1 < q$. Then for any $\beta > \beta_S$, there exists a unique positive function $\omega \in C(\bar{S}) \cap C^2(S)$ satisfying*

$$\begin{cases} -\operatorname{div}_g \left((\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \nabla \omega \right) = \beta \lambda(\beta) (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega - |\omega|^{q-1} \omega & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \quad (4.1)$$

where $\lambda(\beta) = \beta(p - 1) + p - d - 1$.

Before proving Theorem 4.1, we will need the following lemma.

Lemma 4.2 *For $\beta > 0$ and $p > 1$, let Λ_β and β_S be defined by Theorem 3.1. Then both Λ_β and β_S are continuous functions of p , varying in $(1, \infty)$.*

Proof. By Theorem 3.1, Λ_β is uniquely defined for any fixed $p > 1$. To emphasize the dependence of Λ_β on p , let us denote it now by $\Lambda_{\beta,p}$. The continuity of $\Lambda_{\beta,p}$ with respect to p can be proved in the same way as we proved (see Proposition 2.4 in [11]) the continuity of $\Lambda_{\beta,p}$ with respect to β . Thus, we only sketch the argument,

which relies on the construction itself of $\Lambda_{\beta,p}$. Indeed, we proved in [11] that $\Lambda_{\beta,p}$ is the unique constant such that there exists a function $v \in C^2(S)$ satisfying

$$\begin{cases} -\Delta_g v - (p-2) \frac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2} + \beta(p-1) |\nabla v|^2 = -\Lambda_{\beta,p} & \text{in } S \\ \lim_{\sigma \rightarrow \partial S} v(\sigma) = \infty. \end{cases} \quad (4.2)$$

If we normalize v by setting, for example, $v(\sigma_0) = 0$ for some $\sigma_0 \in S$, then v is unique. Moreover $v \in C^2(S)$ and v satisfies estimates in $W_{loc}^{1,\infty}(S)$ which are uniform as $\beta \in (0, \infty)$ and $p \in (1, \infty)$ vary in compact sets. It is also easy to check (see [11]) that $\Lambda_{\beta,p}$ remains bounded whenever β varies in a compact set of $(0, \infty)$ and p vary in a compact set of $(1, \infty)$. The estimates obtained on v and ∇v imply that, whenever β_n or p_n are convergent sequences, the sequence of corresponding solutions v_n of (4.2) (such that $v_n(\sigma_0) = 0$) is relatively compact (locally uniformly in C^1). The equation (4.2) turns out then to be stable (including the boundary estimates); finally, the uniqueness property of $\Lambda_{\beta,p}$, and of the associated (normalized) solution v , implies the continuity of $\Lambda_{\beta,p}$ with respect to both β and p .

Let now $\beta_{S,p}$ be the spectral exponent defined by the equation

$$\Lambda_{\beta,p} = \beta(p-1) + p - d - 1 \quad (4.3)$$

First of all note that when p lies in a compact set in $(1, \infty)$, then necessarily $\beta_{S,p}$ is bounded. Indeed, since $\Lambda_{\beta,p} \leq \Lambda_{1,p}$ whenever $\beta \geq 1$, we have that

$$\beta_S(p-1) + p - d - 1 \leq \Lambda_{1,p} \quad \text{if } \beta_S \geq 1,$$

so that

$$\beta_S \leq 1 + \frac{1}{p-1} (\Lambda_{1,p} - (p-d-1)).$$

Therefore, if p belongs to a compact set in $(1, \infty)$, then β_S remains also in a bounded set. Now, if $p_n \rightarrow p_0$, setting $\beta_n = \beta_{S,p_n}$, we have that β_n is bounded and, up to subsequences, it is convergent to some β_0 . From (4.3), we deduce that Λ_{β_n, p_n} is bounded, which implies that β_n cannot converge to zero, hence $\beta_0 > 0$. Then, using the continuity of $\Lambda_{\beta,p}$, we can pass to the limit in (4.3) and we deduce that β_0 is the spectral exponent with $p = p_0$, i.e. $\beta_0 = \beta_{S,p_0}$. This proves that $\beta_{S,p}$ is continuous with respect to p . \square

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1.

Step 1: construction of a solution. We use similar ideas as in the proof of Theorem 3.2, i.e. a topological degree argument. On the Banach space $X = C_0^1(\bar{S})$ (endowed with its natural norm) with positive cone K , we set

$$\mathcal{B}(\omega) = -\text{div}_g \left((\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{q}{2}-1} \nabla \omega \right) + \beta^2 (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{q}{2}-1} \omega + |\omega|^{q-1} \omega \quad (4.4)$$

$$\Psi(\omega) := \mathcal{B}^{-1} \left(\beta (\lambda(\beta) + \beta) (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega_+ \right).$$

Clearly, $\Psi(\omega) = \omega$ implies that $\omega \geq 0$ and solves (4.1). Then, it is enough to prove the existence of a non trivial fixed point for Ψ . Observe that, as in Theorem 3.2, Ψ is a continuous compact operator in X thanks to the $C^{1,\alpha}$ estimates for p -Laplace operators, and $\Psi(K) \subset K$.

We now wish to compute the fixed point index of $I - \Psi$. First of all we consider, if R is sufficiently large, $\deg(I - \Psi, B_R^+, 0)$ where $B_R^+ = B_R \cap K$. To this purpose, define, for $t \in [0, 1]$, $\Psi^*(\omega, t) = t\Psi(\omega)$. Then Ψ^* is a compact map on $X \times [0, 1]$ and if $\Psi^*(\omega, t) = \omega$, we have

$$\begin{aligned} -\operatorname{div}_g \left((\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \nabla \omega \right) + \beta^2 (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega + \frac{1}{t^{q-(p-1)}} \omega^q \\ = t^{p-1} \beta (\lambda(\beta) + \beta) (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega. \end{aligned} \quad (4.5)$$

We get, by the maximum principle,

$$\left\| \frac{\omega}{t} \right\|_{\infty}^{q-(p-1)} \leq t^{p-1} \beta^{p-1} (\lambda(\beta) + \beta) \leq \beta^{p-1} (\lambda(\beta) + \beta).$$

Since $t \leq 1$, we deduce in particular that $\|\omega\|_{\infty}$ is bounded independently on t . Then, we have

$$\frac{1}{t^{q-(p-1)}} \omega^q \leq \left\| \frac{\omega}{t} \right\|_{\infty}^{q-(p-1)} \|\omega\|_{\infty}^{p-1} \leq C \|\omega\|_{\infty}^{p-1} \leq C.$$

Multiplying by ω we obtain a similar bound for $\|\omega\|_{W_0^{1,p}(S)}$, and the regularity theory for p -Laplace type equations yields a further estimate on $\|\nabla \omega\|_{\infty}$. Therefore, we conclude that there exists a constant M , independent on $t \in [0, 1]$, such that $t\Psi(\omega) = \omega$ implies $\|\omega\|_X \leq M$. As a consequence, if R is sufficiently large we have $t\Psi(\omega) \neq \omega$ on ∂B_R . We deduce that $\deg(I - t\Psi, B_R^+, 0)$ is constant, Therefore

$$\deg(I - \Psi, B_R^+, 0) = \deg(I - t\Psi, B_R^+, 0) = \deg(I, B_R^+, 0) = 1 \quad (4.6)$$

Next, we compute $\deg(I - \Psi, B_r^+, 0)$ for small r . We set

$$\mathcal{B}_t(\omega) = -\operatorname{div}_g \left((\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \nabla \omega \right) + \beta^2 (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \omega + t|\omega|^{q-1} \omega \quad (4.7)$$

and

$$F(\omega, t) := \mathcal{B}_t^{-1} \left(\beta (\lambda(\beta) + \beta) \omega_+ (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}-1} \right)$$

Again, we have $\Psi(\cdot) = F(\cdot, 1)$. We claim that there exists a small $r > 0$ such that $F(\omega, t) \neq \omega$ for every $t \in [0, 1]$ and $\omega \in \partial B_r$. Indeed, reasoning by contradiction, if this were not true there would exist a nonnegative sequence ω_n such that $0 \neq \|\omega_n\| \rightarrow 0$, and $t_n \in [0, 1]$ such that $F(\omega_n, t_n) = \omega_n$, which means that

$$\begin{aligned} -\operatorname{div}_g \left((\beta^2 \omega_n^2 + |\nabla \omega_n|^2)^{\frac{p}{2}-1} \nabla \omega_n \right) + \beta^2 (\beta^2 \omega_n^2 + |\nabla \omega_n|^2)^{\frac{p}{2}-1} \omega_n + t_n \omega_n^q \\ = \beta (\lambda(\beta) + \beta) \omega_n (\beta^2 \omega_n^2 + |\nabla \omega_n|^2)^{\frac{p}{2}-1} \end{aligned}$$

Dividing by $\|\omega_n\|^{p-1}$ and letting $n \rightarrow \infty$, we find that $\frac{\omega_n}{\|\omega_n\|}$ would converge to some function \hat{w} such that $\hat{w} \geq 0$, $\|\hat{w}\| = 1$ and

$$\begin{aligned} -\operatorname{div}_g \left((\beta^2 \hat{\omega}^2 + |\nabla \hat{\omega}|^2)^{\frac{p}{2}-1} \nabla \hat{\omega} \right) + \beta^2 (\beta^2 \hat{\omega}^2 + |\nabla \hat{\omega}|^2)^{\frac{p}{2}-1} \hat{\omega} \\ = \beta (\lambda(\beta) + \beta) \hat{\omega} (\beta^2 \hat{\omega}^2 + |\nabla \hat{\omega}|^2)^{\frac{p}{2}-1} \end{aligned}$$

By Theorem 3.1 this means that $\lambda(\beta) = \Lambda_\beta$, which is not possible since $\lambda(\beta) > \Lambda_\beta$ because $\beta > \beta_S$ (see Remark 3.1). Therefore, we conclude that $F(\omega, t) \neq \omega$ for every $t \in [0, 1]$ and $\omega \in \partial B_r$ provided r is sufficiently small. We deduce that $\deg(I - F(\cdot, t), B_r, 0)$ is constant and in particular

$$\deg(I - \Psi, B_r^+, 0) = \deg(I - F(\cdot, 0), B_r^+, 0).$$

In order to compute this degree, we perform an homotopy acting on p and β by setting $p_t = 2t + (1-t)p$ and by taking β_t so that $t \mapsto \beta_t$ is continuous on $[0, 1]$, $\beta_0 = \beta$, $\beta_t > \beta_{S, p_t}$ for every $t \in [0, 1]$ (where β_{S, p_t} is the spectral exponent for S with $p = p_t$) and $\beta_1 > 0$ is large enough. It follows from Lemma 4.2 that β_{S, p_t} is a continuous function of t and remains bounded as $t \in [0, 1]$. Therefore, a similar choice of function β_t is possible. In the space $C_0^1(\bar{S})$ we define the mapping \mathcal{C}_t by

$$\mathcal{C}_t(\omega) := -\operatorname{div}_g \left((\beta_t^2 \omega^2 + |\nabla \omega|^2)^{\frac{p_t}{2}-1} \nabla \omega \right) + \beta_t^2 (\beta_t^2 \omega^2 + |\nabla \omega|^2)^{\frac{p_t}{2}-1} \omega. \quad (4.8)$$

We set

$$\tilde{F}(\omega, t) = \mathcal{C}_t^{-1} \left(\beta_t (\lambda(\beta_t) + \beta_t) (\beta_t^2 \omega^2 + |\nabla \omega|^2)^{\frac{p_t}{2}-1} \omega \right). \quad (4.9)$$

Combining the Tolksdorf's construction [17] which shows the uniformity with respect to p_t of the $C^{1, \alpha}$ estimates (with $\alpha = \alpha_t \in (0, 1)$), with the perturbation method of [11, Th A1], we obtain that $(\omega, t) \mapsto \tilde{F}(\omega, t)$ is compact in $C_0^1(\bar{S}) \times [0, 1]$. Clearly $I - \tilde{F}(\cdot, t)$ does not vanish on $\|\omega\|_X = r$ for any $r > 0$ which implies that

$$\deg(I - \Psi, B_r^+, 0) = \deg(I - \tilde{F}(\cdot, 0), B_r^+, 0) = \deg(I - \tilde{F}(\cdot, 1), B_r^+, 0).$$

But

$$I - \tilde{F}(\cdot, 1) = I - \beta_1 (\lambda(\beta_1) + \beta_1) (-\Delta_g + \beta_1^2)^{-1}. \quad (4.10)$$

Since $I - \beta_1 (\lambda(\beta_1) + \beta_1) (-\Delta_g + \beta_1^2)^{-1}$ has only one eigenvalue with positive eigenfunction and multiplicity one, it follows that

$$\deg(I - \tilde{F}(\cdot, 1), B_r^+, 0) = -1 = \deg(I - \Psi, B_r^+, 0).$$

To conclude, since we have

$$\deg(I - \Psi, B_R^+ \setminus \bar{B}_r^+, 0) = \deg(I - \Psi, B_R^+, 0) - \deg(I - \Psi, B_r^+, 0) \neq 0$$

we deduce the existence of some ω such that $r < \|\omega\| < R$ which is a solution of (4.1).

Step 2: uniqueness. If ω is any positive solution, then $\beta^2\omega^2 + |\nabla\omega^2|$ is always positive. This is obvious in S and it is a consequence of Hopf boundary lemma on ∂S . Let $\bar{\omega}$ and ω be two positive solutions. Either the two functions are ordered or their graphs intersect. Since all the solutions are positive in S and satisfies Hopf boundary lemma, and up to permuting ω and $\bar{\omega}$, we can define

$$\theta := \inf\{s \geq 1 : s\omega \geq \bar{\omega}\},$$

and denote $\omega^* := \theta\omega$. Either the graphs of $\bar{\omega}$ and $\omega^* := \theta\omega$ are tangent at some interior point $\alpha \in S$, or $\omega^* > \bar{\omega}$ in S and there exists $\alpha \in \partial S$ such that $\bar{\omega}_\nu(\alpha) = \omega^*_\nu(\alpha) < 0$. We put $w = \bar{\omega} - \omega^*$ and use local coordinates $(\sigma_1, \dots, \sigma_d)$ on M near σ_0 . We denote by $g = (g_{ij})$ metric tensor on M and g^{jk} and its contravariant components. Then, for any $\varphi \in C^1(S)$,

$$|\nabla\varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial\varphi}{\partial\sigma_j} \frac{\partial\varphi}{\partial\sigma_k} = \langle \nabla\varphi, \nabla\varphi \rangle_g.$$

If $X = (X^1, \dots, X^d) \in C^1(TM)$ is a vector field, if we lower indices by setting $X^\ell = \sum_i g^{\ell i} X_i$.

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \sum_\ell \frac{\partial}{\partial\sigma_\ell} \left(\sqrt{|g|} X^\ell \right) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell} \left(\sqrt{|g|} g^{\ell i} X_i \right).$$

By the mean value theorem applied to

$$t \mapsto \Phi(t) = \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2 \right)^{\frac{p}{2}-1} (\omega^* + tw) \quad t = 0, 1,$$

we have, for some $t \in (0, 1)$,

$$\left(\beta^2\bar{\omega}^2 + |\nabla\bar{\omega}|^2 \right)^{\frac{p}{2}-1} \bar{\omega} - \left(\beta^2\omega^{*2} + |\nabla\omega^*|^2 \right)^{(p-2)/2} \omega^* = \sum_j a_j \frac{\partial w}{\partial\sigma_j} + bw,$$

where

$$b = \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2 \right)^{\frac{p}{2}-2} \left((p-1)\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2 \right)$$

and

$$a_j = (p-2) \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2 \right)^{\frac{p}{2}-2} (\omega^* + tw) \sum_k g^{jk} \frac{\partial(\omega^* + tw)}{\partial\sigma_k}$$

Considering now

$$t \mapsto \Phi_i(t) = \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2 \right)^{\frac{p}{2}-1} \frac{\partial(\omega^* + tw)}{\partial\sigma_i} \quad t = 0, 1,$$

we see that there exists some $t_i \in (0, 1)$ such that

$$(\beta^2 \bar{w}^2 + |\nabla \bar{w}|^2)^{\frac{p}{2}-1} \frac{\partial \bar{w}}{\partial \sigma_i} - (\beta^2 \omega^{*2} + |\nabla \omega^*|^2)^{\frac{p}{2}-1} \frac{\partial \omega^*}{\partial \sigma_i} = \sum_j a_{ij} \frac{\partial w}{\partial \sigma_j} + b_i w,$$

where

$$b_i = (p-2) \left(\beta^2 (\omega^* + t_i w)^2 + |\nabla (\omega^* + t_i w)|^2 \right)^{(p-4)/2} (\omega^* + t_i w) \frac{\partial (\omega^* + t_i w)}{\partial \sigma_i}$$

and

$$a_{ij} = (p-2) \left(\beta^2 (\omega^* + t_i w)^2 + |\nabla (\omega^* + t_i w)|^2 \right)^{(p-4)/2} \frac{\partial (\omega^* + t_i w)}{\partial \sigma_i} \sum_k g^{jk} \frac{\partial (\omega^* + t_i w)}{\partial \sigma_k} + \delta_i^j \left(\beta^2 (\omega^* + t_i w)^2 + |\nabla (\omega^* + t_i w)|^2 \right)^{(p-2)/2}.$$

Set $P = \omega^*(\alpha) = \bar{w}(\alpha)$ and $Q = \nabla \omega^*(\alpha) = \nabla \bar{w}(\alpha)$. Then $P^2 + |Q|^2 > 0$ and

$$b_i(\alpha) = (p-2) \left(\beta^2 P^2 + |Q|^2 \right)^{\frac{p}{2}-2} P Q_i,$$

and

$$a_{ij}(\alpha) = \left(\beta^2 P^2 + |Q|^2 \right)^{\frac{p}{2}-2} \left(\delta_i^j (\beta^2 P^2 + |Q|^2) + (p-2) Q_i \sum_k g^{jk} Q_k \right).$$

Because ω^* is a supersolution for (4.1), the function w satisfies

$$-\frac{1}{\sqrt{|g|}} \sum_{\ell, j} \frac{\partial}{\partial \sigma_\ell} \left(A_{j\ell} \frac{\partial w}{\partial \sigma_j} \right) + \sum_i C_i \frac{\partial w}{\partial \sigma_i} + D w \leq 0 \quad (4.11)$$

where the C_i and D are continuous functions and

$$A_{j\ell} = \sqrt{|g|} \sum_i g^{\ell i} a_{ij}.$$

The matrix $(a_{ij})(a)$ is symmetric definite and positive since it is the Hessian of

$$x = (x_1, \dots, x_d) = \frac{1}{p} (P^2 + |x|^2)^{\frac{p}{2}} = \frac{1}{p} \left(P^2 + \sum_{j,k} g^{jk} x_j x_k \right)^{\frac{p}{2}}$$

Therefore the matrix $(A_{j\ell})$ keeps the same property in a neighborhood of a . Since w is nonpositive and vanishes at some $a \in S$ or $w < 0$ and $w_\nu = 0$ at some boundary point, it follows from the strict maximum principle or Hopf boundary lemma (see [12]) that $w \equiv 0$. This implies $\theta = 1$ and $w = \bar{w}$. \square

5 Appendix

We prove here the following result

Theorem 5.1 *Let S be a subdomain of a complete d -dimensional Riemannian manifold (M, g) . If $\beta > 0$ and $p > 1$, the first eigenvalue $\lambda_{1,\beta}$ of the operator $\omega \mapsto -\operatorname{div}(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega$ in $W_0^{1,p}(S)$ is isolated. Furthermore any corresponding eigenfunction has constant sign.*

Proof. The proof is an adaptation of the original one due to Anane and Lindqvist when $\beta = 0$. We recall that

$$\lambda_{1,\beta} = \inf \left\{ \int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}} dv_g : \omega \in W_0^{1,p}(S), \int |\omega|^p dv_g = 1 \right\}. \quad (5.1)$$

and that there exists $\omega \in W_0^{1,p}(S) \cap C^{1,\alpha}(S)$ such that

$$-\operatorname{div}(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega = \lambda_{1,\beta}|\omega|^{p-2}\omega \quad \text{in } S. \quad (5.2)$$

The function $|\omega|$ is also a minimizer for $\lambda_{1,\beta}$, thus it is a positive solution of (5.2). By Harnack inequality [13], for any compact subset K of S , there exists C_K such that

$$\frac{|\omega|(\sigma_1)}{|\omega|(\sigma_2)} \leq C_K \quad \forall \sigma_i \in K, i = 1, 2.$$

Thus ω must keep a constant sign. If $\lambda_{1,\beta}$ is not isolated, there exists a decreasing sequence $\{\mu_n\}$ of real numbers converging to $\lambda_{1,\beta}$ and a sequence of functions $\omega_n \in W_0^{1,p}(S)$, solutions of

$$-\operatorname{div}(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{\frac{p}{2}-1}\nabla\omega_n = \mu_n|\omega_n|^{p-2}\omega_n \quad \text{in } S. \quad (5.3)$$

such that $\|\omega_n\|_{L^p(S)} = 1$. By standard compactness and regularity results we can assume that $\omega_n \rightarrow \omega$ weakly in $W_0^{1,p}(S)$ and strongly in $L^p(S)$. Thus

$$\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}} dv_g \leq \liminf_{n \rightarrow \infty} \int_S (\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{\frac{p}{2}} dv_g = \lambda_{1,\beta}$$

which implies that ω is an eigenfunction associated with $\lambda_{1,\beta}$. Set $S_n^+ = \{\sigma \in S : \omega_n(\sigma) > 0\}$ and $S_n^- = \{\sigma \in S : \omega_n(\sigma) < 0\}$. Therefore

$$\begin{aligned} \int_{S_n^+} (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}} dv_g &\geq (1-\theta)\beta^p \int_{S_n^+} |\omega|^p dv_g + \theta \int_{S_n^+} |\nabla\omega|^p dv_g \\ &\geq (1-\theta)\beta^p \int_{S_n^+} |\omega|^p dv_g + \theta C(d, q) \left(\int_{S_n^+} |\omega|^{p^*} dv_g \right)^{p/p^*} \\ &\geq \left((1-\theta)\beta^p + \theta C(d, q) |S_n^+|^{-p/d} \right) \int_{S_n^+} |\omega|^p dv_g \end{aligned}$$

It follows from (5.2) that

$$\int_{S_n^+} (\beta^2 \omega^2 + |\nabla \omega|^2)^{\frac{p}{2}} dv_g = \lambda_{1,\beta} \int_{S_n^+} |\omega|^p dv_g.$$

We obtain

$$\lambda_{1,\beta} \geq (1 - \theta)\beta^p + \theta C(d, q) |S_n^+|^{-p/d}.$$

Similarly

$$\lambda_{1,\beta} \geq (1 - \theta)\beta^p + \theta C(d, q) |S_n^-|^{-p/d}.$$

The two sets

$$S^\pm = \limsup_{n \rightarrow \infty} S_n^\pm$$

have positive measure. Since $\omega \geq 0$ on S^+ and $\omega \leq 0$ on S^- , we derive a contradiction with the fact that any eigenfunction corresponding to $\lambda_{1,\beta}$ has constant sign. \square

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