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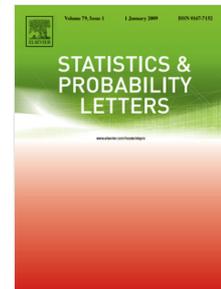
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On the Number of Renewals in Random Time

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Abstract

For the renewal counting process $M(t) = \min\{k : S_k > t\}$ and the independent of it nonnegative random variable T we investigate the asymptotic behaviour of $P(M(t) < T)$ and $P(M(t) \leq K(t)x \mid M(t) < T)$ in case when the interarrival times have an infinite mean. These quantities appear in a natural way when considering limiting behaviour of random time changed branching processes and shock models.

Key words: regular variation, (weighted) renewal theory, domains of attraction, shock models, branching processes.

AMS (2000) Subject Classification: 60K05, 60K99.

1 Introduction

Consider the discrete time stochastic process $\mathbf{Z} = \{Z(n), 0 \leq n < T\}$ where the random variable T is a positive integer valued. The variable T is called the life period of the process. In fact, the process evolves during the life period T and then dies out.

Consider also, an independent of \mathbf{Z} , sequence of nonnegative i.i.d. random variables $\mathbf{X} = \{X_1, X_2, \dots\}$ with distribution function (d.f.) $F(x)$. Let $S_0 = 0$ and for $n \geq 1$, let $S_n = X_1 + X_2 + \dots + X_n$. Define the renewal counting process

$$M(t) = \min\{k : S_k > t\}.$$

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Using these two processes one can define a new process $\xi(t)$ as follows:

$$\xi(t) = Z(M(t)), 0 \leq M(t) < T, t \geq 0.$$

Clearly the subordinated process has a lifetime depending on T and $M(t)$.

In the present paper we are interested in the following problems.

Q1. What is the life period of the subordinated process? To this end we shall determine the probability of the event

$$\{\xi(t) \text{ is alive at time } t\} = \{M(t) < T\}.$$

Later we will determine the asymptotic behavior of $P(M(t) < T)$ as $t \rightarrow \infty$. Note that

$$P(M(t) < T) = E(\bar{F}_T(M(t))),$$

where $\bar{F}_T(t) = 1 - F_T(t) = P(T > t)$.

Q2. What happens with the subordinator during the life period? In this case we study probabilities of the form

$$P(M(t) \leq x \mid M(t) < T).$$

For an appropriate normalizing function $K(t)$, we will obtain the detailed asymptotic behaviour of

$$U_t(x) = P(M(t) \leq K(t)x \mid M(t) < T).$$

Clearly we have

$$1 - U_t(x) = \frac{P(M(t) > K(t)x, M(t) < T)}{P(M(t) < T)},$$

and we see that problem *Q2* is related to problem *Q1*.

In particular, one can consider $Z(n)$ as a Galton-Watson branching process in subcritical or critical case. (See e.g. Athreya and Ney (1972)). In these two cases the branching process evolves until reaching the state zero, i.e. it degenerates with probability one at random time T . The asymptotic of the probability for non-extinction and the limiting distributions are of the main interest for the branching processes. Let us remember that the limiting distributions are non degenerate only conditionally on the event $\{T > n\} = \{\text{the process does not degenerate at time } n\}$. In case of the random time changed branching process $Z(M(t))$ this event becomes $\{T > M(t)\}$ and the solutions of the questions *Q1* and *Q2* relate to the probability of non extinction and the limiting distribution of this process.

The paper is organized as follows. Section 2 contains some results from weighted renewal theory which are used later. In section 3 we study the event $\{M(t) > T\}$ and use results from weighted renewal theory to obtain the asymptotic behaviour of $P(M(t) > T)$ under various conditions on (the distribution of) \mathbf{X} and T .

In section 4 we study the asymptotic behaviour of $U_t(x) = P(M(t) \leq K(t)x \mid M(t) < T)$. We finish the paper with some concluding remarks.

In this paper we shall assume that $\mu = E(X) = \infty$. In the case where $\mu < \infty$, it is well-known that $M(t)/t \rightarrow 1/\mu$ and for this case there are many papers cf. Alsmeyer (1992), Omey and Teugels (2002), Mallor and Omey (2006) that give conditions under which

$$P(M(t) < T) = E(\bar{F}_T(M(t))) \sim \bar{F}_T(t/\mu).$$

We treat the more complicated case where $\mu = \infty$. As for the random variable T , we consider the two cases where either $E(T) < \infty$ or $E(T) = \infty$.

2 Weighted renewal theory

2.1 Notations

Let X_1, X_2, \dots denote i.i.d. positive r.v. with distribution function $F(x) = P(X_i \leq x)$. Let $a_k, k \geq 0$ denote a sequence of weights.

Now consider the weighted renewal function (wrf) $G(x)$ defined as follows:

$$G(x) = \sum_{k=0}^{\infty} a_k F^{*k}(x),$$

where, as usual, $F^{*k}(\cdot)$ means the k -fold convolution of the d.f. $F(\cdot)$.

The wrf $G(x)$ has a useful interpretation as an expectation. To see this, we define the function $A(x)$ as follows: $A(x) = a_0 + a_1 + \dots + a_{[x]}$, $x \geq 0$. In this case we have $G(x) = E(A(M(x)))$, where $M(x)$ denotes the renewal counting measure defined above. Conversely, for any function $A(x)$ we have

$$E(A(M(x))) = \sum_{k=0}^{\infty} a_k F^{*k}(x),$$

where $a_0 = A(0)$ and $a_k = A(k) - A(k-1)$, $k \geq 1$.

2.2 Asymptotic behaviour of $G(t)$

In the case where $\mu = \infty$, it makes sense to assume that the tail distribution $\bar{F}(x)$ is regularly varying. The motivation is in the following result. To formulate the result, define the integrated tail

$$m(x) = \int_0^x \bar{F}(t) dt,$$

and let $K(x) = x/m(x)$. Note that if $\mu < \infty$, we have $K(x) \sim x/\mu$.

Proposition 1 (Mallor and Omey (2006), Theorems 6 and 9) (i) If $m(x) \in RV(1-\alpha)$, with $0 < \alpha \leq 1$, then

$$\frac{M(t)}{K(t)} \Longrightarrow L^{-\alpha}(\alpha),$$

where for $0 < \alpha < 1$, $L(\alpha)$ is an α -stable law with $E(\exp(-sL(\alpha))) = \exp(-\Gamma(2-\alpha)s^\alpha)$ and where $L(1)$ is degenerate at 1.

(ii) If $m(x) \in RV(1-\alpha)$, with $0 < \alpha < 1$, then for all $r \geq 0$, $(M(t)/K(t))^r$ is uniformly integrable and

$$E\left(\frac{M(t)}{K(t)}\right)^r \rightarrow E(L^{-\alpha r}(\alpha)).$$

Here and later \Longrightarrow means convergence in distribution.

In Mallor and Omey (2006), the asymptotic behaviour of $G(t) = E(A(M(t)))$ is determined in case where $A(\infty) = G(\infty) = \infty$. (See also Embrechts and Omey (1983)).

Proposition 2 (Mallor and Omey (2006), Theorem 43) Suppose $0 < \alpha \leq 1$ and $\rho > 0$. Each two of the following statements (a, b, c) implies the third statement and (d).

(a) $A(x) \in RV(\rho)$;

(b) $m(x) \in RV(1-\alpha)$;

(c) $G(x) \in RV(\alpha\rho)$;

(d) $G(x)/A(K(x)) \rightarrow D(\alpha, \rho) = (\Gamma(2-\alpha))^{-\rho}\Gamma(1+\rho)/\Gamma(1+\alpha\rho)$.

Moreover we have the following implications

(i) (a) with $\rho \geq 0$ and (b) with $0 \leq \alpha \leq 1$ imply (c) and (d).

(ii) (a) with $\rho > 0$ and (c) imply (b) and (d).

(iii) (b) with $0 < \alpha \leq 1$ and (c) with $\rho \geq 0$ imply (a) and (d).

Remark 1 (1) In view of Proposition 1, we have $D(\alpha, \rho) = E(L^{-\alpha\rho}(\alpha))$, $\rho \geq 0$.

(2) If $A(x) \in RV(0)$, then $m(x) \in RV(1-\alpha)$, $0 < \alpha \leq 1$, implies that $G(x) \in RV(0)$ and $G(x) \sim W(K(x))$.

Remark 2 Proposition 2 concerns the extension of the Elementary Renewal Theorem for the weighted renewal functions in the infinite mean case. It was proved by Omey in his PhD dissertation Omey (1982). An extension of the Blackwell's Theorem for the weighted renewal functions in the infinite mean case was proved by Anderson and Athreya (1988), Theorem 2. Their proof used an extension of the Key Renewal Theorem

in the infinite mean case (Theorem 1) and the assertions of Proposition 2, given there as Theorem D. Let us note that the strongest form of the Key Renewal Theorem in the infinite mean case was proved also by Anderson and Athreya (1987).

3 Asymptotic behaviour of $P(M(t) < T)$

In this section we discuss the asymptotic behaviour of $G(t) = P(M(t) < T)$ assuming the more general situation where T is a positive, but not mandatory integer valued, r.v. independent of $\{X_k\}$. Clearly we have the following identity:

$$G(t) = E(\bar{F}_T(M(t))).$$

We suppose that $\bar{F}_T(x) \in RV(-\beta)$ and consider several cases.

Using the lemma of Fatou and Proposition 1, we see that

$$\liminf \frac{G(t)}{\bar{F}_T(K(t))} \geq E(L^{\alpha\beta}(\alpha)).$$

Using $P(L(\alpha) > x) \sim x^{-\alpha}$, we have $P(L^{\alpha\beta}(\alpha) > x) \sim x^{-1/\beta}$ and we obtain for $\beta \geq 1$ that

$$\lim \frac{G(t)}{\bar{F}_T(K(t))} = \infty. \quad (1)$$

Clearly we should compare $G(t)$ with another function.

3.1 The case $\beta \geq 1$ and $E(T) < \infty$

If $\beta \geq 1$ and $E(T) < \infty$, we proceed as follows. We clearly have

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} P(M(t) < T, T = n) + \sum_{n=0}^{\infty} P(M(t) < T, n < T < n+1) \\ &= \sum_{n=2}^{\infty} P(M(t) \leq n-1, T = n) + \sum_{n=1}^{\infty} P(M(t) \leq n, n < T < n+1) \\ &= \sum_{n=2}^{\infty} P(M(t) \leq n-1)P(T = n) + \sum_{n=1}^{\infty} P(M(t) \leq n)P(n < T < n+1) \\ &= \sum_{n=2}^{\infty} P(S_{n-1} > t)P(T = n) + \sum_{n=1}^{\infty} P(S_n > t)P(n < T < n+1). \end{aligned}$$

Now $\bar{F}(x) \in RV(-\alpha)$ implies that $P(S_n > t)/\bar{F}(t) \rightarrow n$, $n \geq 1$. We need the following result

Lemma 1 (i) For each $n \geq 1$, $\overline{F}^{*n}(x) \leq nm(x)/x$.

(ii) If $\overline{F}(x) \in RV(-\alpha)$, $0 \leq \alpha < 1$ then $m(x) \sim x\overline{F}(x)/(1-\alpha)$.

Proof.

(i) See Daley et al. (2007), Lemma 5).

(ii) This is a standard result in regular variation, cf. Bingham et al. (1987). ■

From Lemma 1, it follows that for $0 \leq \alpha < 1$ we have $P(S_n > t) \leq Cn\overline{F}(t)$. If $E([T]) < \infty$, Lebesgue's theorem can be applied twice, and we obtain that

$$\begin{aligned} \frac{G(t)}{\overline{F}(t)} &\rightarrow \sum_{n=2}^{\infty} (n-1)P(T=n) + \sum_{n=1}^{\infty} nP(n < T < n+1) \\ &= E([T]) - P(T \in \{1, 2, \dots\}) \\ &= E([T]) - P(T \in \mathbb{N}_0). \end{aligned}$$

As usual $[T]$ is the integer part of T .

Remark 3 (1) If $0 \leq \alpha < 1$ we have $\overline{F}(x)K(x) = \overline{F}(x)x/m(x) \sim 1 - \alpha$. Using $\overline{F}_T(x) \in RV(-\beta)$, we obtain that

$$\overline{F}_T(K(t)) \sim c\overline{F}_T(1/\overline{F}(t)).$$

Since $E(T) < \infty$, it follows that

$$\lim_{t \rightarrow \infty} \frac{\overline{F}_T(K(t))}{\overline{F}(t)} = \lim_{x \rightarrow \infty} x\overline{F}_T(x) = 0.$$

This explains why (1) holds.

(2) In case where T is positive, integer valued one gets

$$\frac{G(t)}{\overline{F}(t)} \rightarrow E(T-1), \quad t \rightarrow \infty.$$

3.2 The case $\beta \leq 1$ and $E(T) = \infty$

In this case we proceed in a different way. We write $G(t) = \overline{U}(t) = 1 - U(t) := 1 - P(T \leq M(t))$. Using the definition of $M(t)$, we see that

$$\begin{aligned}
U(t) &= \sum_{k=1}^{\infty} P(T \leq k)P(M(t) = k) \\
&= \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} P(i < T \leq i+1)P(M(t) = k) \\
&= \sum_{i=0}^{\infty} P(i < T \leq i+1)P(M(t) > i) \\
&= \sum_{i=0}^{\infty} a_i P(S_i \leq t),
\end{aligned}$$

where $a_i = P(i < T \leq i+1)$. Since (cf. section 2) $A(x) \uparrow 1$, the results of Section 2 are not directly applicable. We proceed as follows.

Using Laplace transforms, we have (similar notations are used for U):

$$\widehat{F}(s) = \int_0^{\infty} e^{-sx} dF(x), \quad m_F(x) = \int_0^x \overline{F}(t) dt, \quad \widehat{m}_F(s) = \frac{1 - \widehat{F}(s)}{s}.$$

Note that since \overline{F} is not increasing, we always have $x\overline{F}(x) \leq m_F(x) \leq x$. Using these notations, we obtain that

$$\widehat{U}(s) = \sum_{k=0}^{\infty} a_k (\widehat{F}(s))^k.$$

Since $\{a_k, k = 0, 1, 2, \dots\}$ is a probability distribution, we also have

$$1 - \widehat{U}(s) = \sum_{k=1}^{\infty} a_k \left(1 - (\widehat{F}(s))^k\right)$$

and then

$$\frac{\widehat{m}_U(s)}{\widehat{m}_F(s)} = \frac{1 - \widehat{U}(s)}{1 - \widehat{F}(s)} = \sum_{k=0}^{\infty} w_k (\widehat{F}(s))^k = \widehat{H}(s),$$

where

$$w_k = \sum_{i=k+1}^{\infty} a_i = P(T > k+1) \text{ and } H(x) = \sum_{k=0}^{\infty} w_k F^{*k}(x).$$

It follows that

$$\widehat{m}_U(s) = \widehat{H}(s)\widehat{m}_F(s) \tag{2}$$

where $\widehat{H}(s) = \sum_{k=0}^{\infty} w_k (\widehat{F}(s))^k$. Since $E(T) = \infty$, we can treat $H(x)$ as in Proposition 2. To this end, let $W(x) = \sum_{k=0}^{\lfloor x \rfloor} w_k$ and as before, let $K(x) = x/m_F(x)$.

Note that if $\overline{F}_T(t) \in RV(-\beta)$, $\beta < 1$, then

$$W(x) \sim x\overline{F}_T(x)/(1 - \beta) \in RV(\rho),$$

where $\rho = 1 - \beta$.

If $\overline{F}_T(t) \in RV(-1)$ then

$$W(x) \in RV(0) \text{ and } x\overline{F}_T(x) = o(W(x)) \text{ as } x \rightarrow \infty.$$

The following result follows from Proposition 2.

Proposition 3 *Suppose $0 < \alpha < 1$ and $\rho = 1 - \beta \geq 0$. If $W(x) \in RV(\rho)$ and $\overline{F}(x) \in RV(-\alpha)$, then $H(x) \in RV(\rho\alpha)$ and $H(x) \sim D(\alpha, \rho)W(K(x))$.*

In our application we need to find the asymptotic behaviour of $\overline{U}(x)$. Relation (2) plays a central role. Proposition 3 shows that

$$H(x) \sim D(\alpha, 1 - \beta)W(K(x)) \in RV(\alpha(1 - \beta))$$

and by Karamata's theorem we have

$$\widehat{H}(1/x) \sim H(x)\Gamma(1 + (1 - \beta)\alpha).$$

Also $\overline{F}(x) \in RV(-\alpha)$ implies that $m_F(x) \sim x\overline{F}(x)/(1 - \alpha) \in RV(1 - \alpha)$ and then

$$\widehat{m}_F(1/x) \sim m_F(x)\Gamma(2 - \alpha) \sim x\overline{F}(x)\Gamma(1 - \alpha).$$

Using (2) one gets $\widehat{m}_U(1/x) \in RV(2 - \alpha(1 - \rho))$ and then also

$$\widehat{m}_U\left(\frac{1}{x}\right) \sim m_U(x)\Gamma(2 - \alpha\beta).$$

Therefore

$$m_U(x)\Gamma(2 - \alpha\beta) \sim \Gamma(1 - \alpha)\Gamma(1 + (1 - \beta)\alpha)x\overline{F}(x)H(x).$$

Using the monotone density theorem, we find that $m_U(x) \sim x\overline{U}(x)/(1 - \alpha\beta)$ and then

$$\overline{U}(x)\Gamma(1 - \alpha\beta) \sim \Gamma(1 - \alpha)\Gamma(1 + (1 - \beta)\alpha)\overline{F}(x)H(x).$$

Using the relation $H(x) \sim D(\alpha, 1 - \beta)W(K(x))$ one gets

$$\overline{U}(x)\Gamma(1 - \alpha\beta) \sim \Gamma(1 - \alpha)\Gamma(1 + (1 - \beta)\alpha)D(\alpha, 1 - \beta)\overline{F}(x)W(K(x)). \quad (3)$$

To simplify further, we have to consider the cases $0 < \beta < 1$ and $\beta = 1$ separately.

If $0 < \beta < 1$ then from $W(x) \sim x\overline{F}_T(x)/(1 - \beta)$ we obtain

$$H(x) \sim D(\alpha, 1 - \beta)\frac{1}{1 - \beta}K(x)\overline{F}_T(K(x))$$

and then

$$\begin{aligned}
& \bar{U}(x)\Gamma(1-\alpha\beta) \\
& \sim D(\alpha, 1-\beta)\frac{1}{1-\beta}\Gamma(1-\alpha)\Gamma(1+\alpha(1-\beta))\bar{F}(x)K(x)\bar{F}_T(K(x)) \\
& \sim D(\alpha, 1-\beta)\frac{1}{1-\beta}\Gamma(2-\alpha)\Gamma(1+\alpha(1-\beta))\bar{F}_T(K(x)).
\end{aligned}$$

Using the expression for $D(\alpha, 1-\beta)$, one obtains that

$$\frac{\bar{U}(t)}{\bar{F}_T(K(t))} \rightarrow \frac{(\Gamma(2-\alpha))^\beta\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)} = E(L^{\alpha\beta}(\alpha)).$$

If $\beta = 1$ then we have from (3) that

$$\bar{U}(x)\Gamma(1-\alpha) \sim \Gamma(1-\alpha)\Gamma(1)D(\alpha, 0)\bar{F}(x)W(K(x)).$$

Using that $D(\alpha, 0) = 1$, $K(x) \sim (1-\alpha)/\bar{F}(x)$ and $W(x) \in RV(0)$ we get

$$\bar{U}(x) \sim D(\alpha, 0)\bar{F}(x)W(K(x)) \sim \bar{F}(x)W(1/\bar{F}(x)).$$

Therefore

$$\frac{\bar{U}(t)}{\bar{F}(t)W(1/\bar{F}(t))} \rightarrow 1.$$

3.3 Limit Theorem

We summarize our findings in the following theorem.

Theorem 1 Suppose that $\bar{F}(x) \in RV(-\alpha)$, $0 \leq \alpha < 1$ and $\bar{F}_T(x) \in RV(-\beta)$, $\beta \geq 0$.

(i) If $\beta \geq 1$ and $E(T) < \infty$, then

$$\frac{G(t)}{\bar{F}(t)} \rightarrow E([T]) - P(T \in \mathbb{N}_0).$$

(ii) If $\beta < 1$ and $E(T) = \infty$, then

$$\frac{G(t)}{\bar{F}_T(K(t))} \rightarrow \frac{(\Gamma(2-\alpha))^\beta\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)} = E(L^{\alpha\beta}(\alpha)).$$

(iii) If $\beta = 1$ and $E(T) = \infty$, then

$$\frac{G(t)}{\bar{F}(t)W(1/\bar{F}(t))} \rightarrow 1.$$

4 Asymptotic behaviour of $U_t(x)$

In this section we discuss the asymptotic behaviour of $U_t(x)$, where

$$U_t(x) = P(M(t) \leq K(t)x \mid M(t) < T).$$

Denote by $I\{\cdot\}$ the indicator of the event $\{\cdot\}$. The following theorem holds.

Theorem 2 Suppose $\bar{F}(t) \in RV(-\alpha)$, $0 < \alpha < 1$ and $\bar{F}_T(t) \in RV(-\beta)$.

(i) If $\beta < 1$ and $E(T) = \infty$, we have

$$U_t(x) \rightarrow \frac{E(L^{\alpha\beta}(\alpha)I\{L^{-\alpha}(\alpha) \leq x\})}{E(L^{\alpha\beta}(\alpha))}.$$

(ii) If $\beta \geq 1$ we have $U_t(x) \rightarrow 1$. Moreover, if $P(T \in \mathbb{N}_0) < E([T]) < \infty$ then

$$\frac{\bar{F}(t)}{\bar{F}_T(K(t))}(1 - U_t(x)) \rightarrow \frac{1}{(E([T]) - P(T \in \mathbb{N}_0))} E(L^{\alpha\beta}(\alpha)I\{L^{-\alpha}(\alpha) > x\}).$$

Proof. Clearly we have

$$1 - U_t(x) = \frac{P(M(t) > K(t)x, M(t) < T)}{P(M(t) < T)} = \frac{A}{B}.$$

The numerator A can be written in the following form

$$A = E(\bar{F}_T(M(t))I\{M(t) > K(t)x\}).$$

Since $\bar{F}_T \in RV(-\beta)$ is non-increasing, we have

$$\bar{F}_T(M(t))I\{M(t) > K(t)x\} \leq \bar{F}_T(xK(t))I\{M(t) > K(t)x\}$$

and

$$\frac{\bar{F}_T(K(t))(M(t)/K(t))}{\bar{F}_T(K(t))} \implies L^{\alpha\beta}(\alpha).$$

Lebesgues theorem on dominated convergence shows that

$$\frac{A}{\bar{F}_T(K(t))} \rightarrow E(L^{\alpha\beta}(\alpha)I\{L^{-\alpha}(\alpha) > x\}).$$

Using Theorem 1, the result now easily follows. ■

5 Concluding remarks

- If $\rho = 0$, Proposition 2 shows that $G(x) \sim W(K(x))$. Using the class II, Theorem 45 of Mallor and Omey (2006) can be used to obtain an estimate for the difference $G(x) - W(K(x))$. Using (2) this result, in the case $\beta = 0$, leads to an estimate for the difference $m_U(x) - H(x)m_F(x)$. It would be interesting to see how this leads to an estimate of $G(t) - \bar{F}_T(K(t))$.
- Some of the results can be extended into the multivariate case. In the bivariate case we consider weighted renewal function of the form

$$G(x, y) = \sum_{n=0}^{\infty} a_n F^{*n}(x, y)$$

and its interpretation in terms of renewal counting processes. In the finite means case, Mallor et al. (2007) extended the univariate results of Omey and Teugels (2002).

- The results of this paper can be used in the context of shock models. Imagine a system that is subject to shocks that appear with i.i.d. inter-shock times X_1, X_2, \dots . The shocks cause i.i.d. damage D_1, D_2, \dots . The system fails if the total damage exceeds a critical value K , i.e. the system fails at time T given by

$$T = \min \{k : D_1 + D_2 + \dots + D_k > K\}$$

Since the number of shocks up to time t is given by $M(t)$, we can use our results to discuss $P(M(t) < T)$ and related quantities.

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