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Mass redistribution method for finite element contact problems in elastodynamics

Houari Boumediène KHENOUS [1], Patrick LABORDE[2], Yves RENARD[1].

Abstract

This paper is devoted to a new method dealing with the semi-discretized finite element unilateral contact problem in elastodynamics. This problem is ill-posed mainly because the nodes on the contact surface have their own inertia. We introduce a method based on an equivalent redistribution of the mass matrix such that there is no inertia on the contact boundary. This leads to a mathematically well-posed and energy conserving problem. Finally, some numerical tests are presented.

keywords: elasticity, unilateral contact, time integration schemes, energy conservation, stability, mass redistribution method.

Introduction

Many works have been devoted to the numerical solution of contact problems in elastodynamics *e.g.* [21, 14, 13, 6]. In this paper, we are interested in numerical instabilities caused by the space approximation. For simplicity, we limit ourself to the small deformations.

Concerning the continuous purely elastodynamic contact problems (hyperbolic problems), as far as we know, an existence result has been proved in a scalar two dimensional case by Lebeau-Schatzman [15], Kim [12], in the vector case with a modified contact law by Renard-Paumier [24], but no uniqueness result is known. This ill-posedness leads to numerical instabilities of time integration schemes. Thus, many researchers adapted different approaches to overcome this difficulty.

To recover uniqueness in the discretized case, one of the approaches well adapted to rigid bodies [21] is to introduce an impact law with a restitution coefficient. However, this approach seems not satisfactory for deformable bodies. On the other hand the unilateral contact condition leads to some difficulties in the construction of energy conserving schemes [21] [14] [13] [6] because of presence of important oscillations of the displacement and a very noisy contact stress on the contact boundary. To deal with this last difficulty, in [25, 3] the contact force is implicited which consists in fixing nodes being in contact but the drawback of this method is the loss of energy because of the annulation of kinetic energy. An other well known approach is the penality method which introduce a very important oscillations that we have to reduce using a damping parameter [25]. However, in [2] authors propose a conservative scheme with a posteriori velocity correction in the same way as it is done in [7, 14] by introducing a jump on velocity

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during impact which permits the verification of the contact condition. The price to pay is a supplemetary problem to solve on velocity.

Even this correction leads to a conserved physical properties, the contact condition is not well approximated because we need the verification of contact condition either on displacement and or on velocity and acceleration which is very difficult to obtain. We can find an idea to verify the last two complementarity conditions in [26] with introducing a force in the discret lagrangian. Also, [16] who propose a conservative iterative processus to correct the contact stress. An other way, is to impose the persistency condition (complementarity on velocity) [13] with a good choice of approximation. But this allows a small interpenetrations going to zero when mesh parameter goes to zero. This interpenetration is eliminated by doing the same thing and using a penality method [6].

In this paper, we perform an analysis of the ill-posedness encountered in this kind of discretization and conclude that the main cause is the fact that the nodes on contact boundary have their own inertia. We propose a new method which consists in the redistribution of the mass near the contact boundary. We prove that the well-posedness of the semi-discretized problem is recovered and that the unique solution is energy conserving. Numerical simulations show that the quality of the contact stress is greatly improved by this technique.

In section 1, we give the strong and week formulations of the contact problem in elastodynamics. The next section is devoted to the corresponding evolutionary finite element problem. In section 3, the ill-posedness and energy conserving characteristics of the finite element semi-discretized problem is illustrated for a one degree of freedom system.

In section 4, we introduce a new distribution of the body mass with conservation of the total mass, the coordinates of the center of gravity and the inertia momenta. This distribution of the mass is done so that there is no inertia for the contact nodes (similarly to what happens in the continuous case). Using this method, we prove existence and uniqueness of the semi-discrete solution. Numerical tests are presented in a last section. In particular, the propagation of the impact wave is exhibited. Finally, we compare the evolution of the energy and the normal stress with and without the mass redistribution method.

1 Contact problems in elastodynamics

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded Lipschitz domain representing the reference configuration of a linearly elastic body. It is assumed that this body is submitted to a Neumann condition on Γ_N, a Dirichlet condition on Γ_D and a unilateral contact with Coulomb friction condition on Γ_C between the body and a flat rigid foundation, where Γ_N, Γ_D and Γ_C form a partition of $\partial\Omega$, the boundary of Ω.

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1.1 Strong formulation

Let us denote ρ, $\boldsymbol{\sigma}(u)$, $\varepsilon(u)$ and $\mathcal{A}$ the mass density, the stress tensor, the linearized strain tensor and the elasticity tensor, respectively.

The problem consists in finding the displacement field $u(t,x)$ satisfying

$$\begin{cases} \rho\ddot{u} - \ \mathrm{div}\ \boldsymbol{\sigma}(u) = f & \text{in } [0,T]\times\Omega, \\ \boldsymbol{\sigma}(u) = \mathcal{A}\ \varepsilon(u) & \text{in } [0,T]\times\Omega, \\ u = 0 & \text{on } [0,T]\times\Gamma_{D}, \\ \sigma(u)\nu = g & \text{on } [0,T]\times\Gamma_{N}, \\ u(0) = u_0, \dot{u}(0) = u_1 & \text{in } \Omega, \end{cases} \tag{1}$$

where ν is the outward unit normal to Ω on $\partial\Omega$. Finally, g and f are the given external loads.

On Γ_C, we decompose the displacement and the stress vector in normal and tangential components as follows:

$$u_N = u.\nu, \quad u_T = u - u_N\nu,$$
$$\boldsymbol{\sigma}_N(u) = (\boldsymbol{\sigma}(u)\nu).\nu, \quad \boldsymbol{\sigma}_T(u) = \boldsymbol{\sigma}(u)\nu - \boldsymbol{\sigma}_N(u)\nu.$$

To give a clear sense to this decomposition, we assume Γ_C to have the $\mathcal{C}^1$ regularity. To simplify, there is no initial gap between the solid and the rigid foundation. The unilateral contact frictionless condition is expressed thanks to the complementarity condition

$$u_N \le 0,\ \boldsymbol{\sigma}_N(u) \le 0,\ u_N\boldsymbol{\sigma}_N(u) = 0 \text{ and } \boldsymbol{\sigma}_T(u) = 0 \quad \text{on } [0,T]\times\Gamma_C. \tag{2}$$

1.2 Weak formulation

Let us define the following vector spaces:

$$V = \{v \in H^1(\Omega;\mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad X_N = \{v_N|_{\Gamma_C} : v \in V\},$$

their topological dual spaces V' and X'_N and the following maps:

$$a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v)dx, \quad l(v) = \int_\Omega f.vdx + \int_{\Gamma_N} g.vd\Gamma.$$

Now, let us denote

$$K_N = \{v_N \in X_N : v_N \le 0\}$$

and

$$N_{K_N}(v_N) = \begin{cases} \{\mu_N \in X'_N : \langle \mu_N, w_N - v_N\rangle_{X'_N,X_N} \le 0,\ \ \forall\ w_N \in K_N\}, & \text{if } v_N \in K_N, \\ \varnothing, & \text{if } v_N \notin K_N, \end{cases}$$

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the cone of admissibles normal displacements and its normal cone.

Formally, the weak formulation of Problem (1)(2) can be expressed as follows:

$$\begin{cases} \text{find } u: \ [0,T] \longrightarrow V \text{ and } \ \lambda_{_N}: \ [0,T] \longrightarrow X'_{_N} \text{ satisfying, for a.e. } t \in [0,T]: \\ \langle \rho \ddot{u}(t), v \rangle_{V',V} + a(u(t),v) = l(v) + \langle \lambda_{_N}(t), v_{_N} \rangle_{X'_{_N},X_{_N}} \qquad \forall v \in V, \\ -\lambda_{_N}(t) \in N_{K_{_N}}(u_{_N}(t)), \\ u(0) = u^0, \qquad \dot{u}(0) = u^1. \end{cases} \tag{3}$$

Remark. For simplicity, we denote $u = u(t)$. More details about the contact problems in elasticity can be found in [11, 4, 9].

2 The finite element approximation of contact problems in elastodynamics

We consider a Lagrange finite element method for the contact problem in elastodynamics (3). Let $a_1, ..., a_n$ be the finite element nodes, $\varphi_1, ..., \varphi_{n.d}$ the (vector) basis functions of the finite element displacement space and $I_{_C} = \{i : a_i \in \Gamma_{_C}\}$. We denote m the number of nodes on $\Gamma_{_C}$, d the space dimension and n the number of nodes.

Let U be the vector of degrees of freedom of the finite element displacement field $u_h(x)$:

$$u_h(x) = \sum_{1 \le i \le nd} u_i \varphi_i \quad \text{and} \quad U = (u_i) \in \mathbb{R}^{n.d}.$$

With a nodal contact condition, the elastodynamic problem (3) can be approximated as follows.

$$\begin{cases} \text{Find } U : [0,T] \mapsto \mathbb{R}^{nd} \text{ satisfying, at each time in [0,T]:} \\ M\ \ddot{U} + KU = L + \sum_{i \in I_C} \lambda^i_{_N} N_i, \\ N_i^T U \le 0, \ \ \lambda^i_{_N} \le 0, \ \ (N_i^T U)\lambda^i_{_N} = 0 \quad \ \forall\ i \in I_{_C}, \\ U(0) = U^0, \qquad \dot{U}(0) = U^1, \end{cases} \tag{4}$$

where

$$K_{ij} = a(\varphi_i, \varphi_j)$$

are the components of the stiffness matrix K and the components of the finite element mass matrix M are equal to

$$M_{ij} = \int_\Omega \rho \ \varphi_i.\varphi_j \ dx \qquad (1 \le i, j \le n.d)\,. \tag{5}$$

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The elasticity coefficients obey the usual symmetry and uniform ellipticity conditions. The external loads vector $L = (L_i)$ is written

$$L_i = \int_\Omega f.\varphi_i dx + \int_{\Gamma_C} g.\varphi_i dx.$$

The vectors $N_i \in \mathbb{R}^n$ are chosen such that the normal displacement on the contact surface is equal to

$$u^h_{_N}(a_i) = N_i^T U \quad \forall i \in I_{_C}. \tag{6}$$

The multipliers $\lambda^i_{_N}$ define the nodal equivalent contact forces vector:

$$\Lambda_{_N} = \left(\lambda^i_{_N}\right) \in \mathbb{R}^m, \ m = Card(I_{_C}).$$

Problem (4) represents a differential inclusion with mesure solution (see [**?**], [21]). For more details on the discretization of contact with friction problems, see [9].

3 Ill-posedness of elastodynamic frictionless contact problem

It is known that Problem (4) is ill-posed [18, 19, 22, 23]. For instance, we can exhibit an infinite number of solutions for the one degree of freedom (d.o.f) system represented in Fig. 1. In fact, this very simple system appears on the normal component for each contact node in Problem (4) with a supplementary right hand side corresponding to the remaining terms.

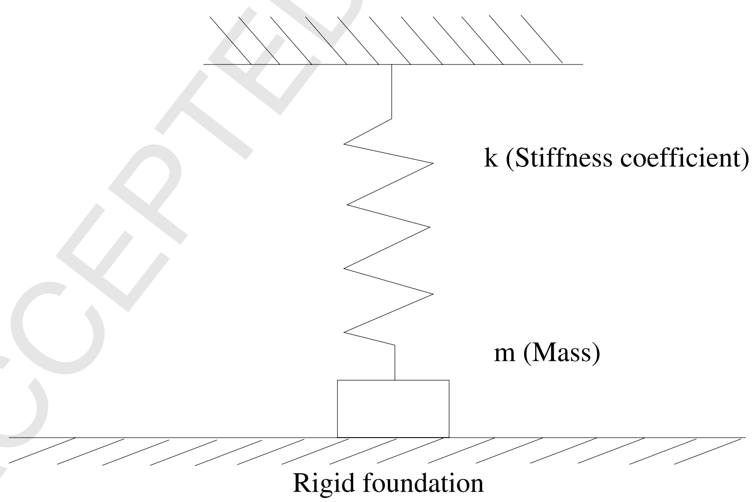

Figure 1: *system with one d.o.f.*

The vertical motion $U \in \mathbb{R}$ of the simple mechanical system represented in Fig. 1 is governed

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by the following set of equations

$$\begin{cases} m\ddot{U} + kU = \Lambda_{_N}, \\ U \leq 0,\ \Lambda_{_N} \leq 0,\ \Lambda_{_N}\ U = 0, \\ U(0) \text{ and } \dot{U}(0) \text{ given,} \end{cases} \tag{7}$$

where k is the stiffness coefficient of the spring, m is the mass placed in its extremity and $\Lambda_{_N}$ is the reaction of the rigid formulation. With the particular initial data $U(0) = -1$ and $\dot{U}(0) = 0$, and for any $\alpha \geq 0$, we obtain a solution to Problem (7) given by

$$\begin{aligned} U(t) &= -cos\left(t\sqrt{\frac{k}{m}}\right),\ 0 \leq t < \frac{\pi}{2}\sqrt{\frac{m}{k}}, \\ U(t) &= \alpha\ cos\left(t\sqrt{\frac{k}{m}}\right),\ \frac{\pi}{2}\sqrt{\frac{m}{k}} < t < \pi\sqrt{\frac{m}{k}}. \end{aligned}$$

Hence, the space semi-discretized elastodynamic frictionless contact problem admits an infinite number of solutions and is ill-posed in that sense.

4 Mass redistribution method (MRM)

As we just saw, the finite element semi-discretization of the elastodynamic contact problem is ill-posed. To recover the uniqueness, one of the approaches well adapted to rigid bodies is to introduce an impact law with a restitution coefficient [18, 19, 22, 23]. This seems not to be completely satisfactory for deformable bodies because, whatever is the restitution coefficient, the system tends to a global restitution of energy when the mesh parameter goes to zero.

The aim now is to present a new method which permits to recover the uniqueness for the finite element semi-discretized elastodynamic contact problem and also its energy conservation. Some of the results presented bellow were annouced in [10].

4.1 Construction of the redistributed mass matrix

The ill-posedness of Problem (4) comes from the fact that the nodes on the contact boundary have their own inertia. This leads to instabilities even for energy conserving schemes. An explanation of those instabilities is that if a node is stopped on the contact boundary, its kinetik energy is definitively lost. Thus, energy schemes make the node on the contact boundary oscillate in order to keep this kenetic energy.

We propose here to introduce a new distribution of the mass which conserves the total mass, the center of gravity and the inertia momenta. This distribution of the mass is done so that there is no inertia for the contact nodes (similarly to what happens in the continuous case).

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Let us denote M_r the redistributed mass matrix. The elimination of the mass on the contact boundary leads to the following condition:

$$N_i{}^T M_r N_j = 0, \forall\ i, j \in I_C, \tag{8}$$

where N_i is defined by (6).

The construction of the matrix M_r (10) is done with the same sparsity than M (*i.e* without adding non-zero elements).

The total mass can be expressed from M as follows (for a Lagrange finite element method):

$$\int_\Omega \rho\ dx = X^T M\ X,$$

where $X = 1/\sqrt{d}\ \ (1...1)^T \in \mathbb{R}^n$ $(d = 2, 3)$. The k^{th}-coordinate of the center of gravity is written

$$\int_\Omega \rho\ x_k\ dx = X^T M\ Y_k \quad (1 \le k \le d),$$

denoting $Y_k = (y_i) \in \mathbb{R}^n$ the vector such that

$$1/\sqrt{d}\ \sum_{i,j} y_i\ \varphi_i\ .\varphi_j = x_k.$$

Finally, the moment of inertia matrix is derived from the quantities

$$\int_\Omega \rho\ x_k\ x_l\ dx = Y_k^T M\ Y_l \quad (1 \le k, l \le d).$$

The matrix M_r will be said to be *equivalent* to M if the following equality constraints are satisfied:

$$\begin{cases} X^T(M_r - M)\ X = 0, \\ X^T(M_r - M)\ Y_k = 0 \ \ (1 \le k \le d), \\ Y_k^T(M_r - M)\ Y_l = 0 \ \ (1 \le k, l \le d). \end{cases} \tag{9}$$

Moreover, for a reason of computational cost, the considered matrices M_r have the same form than M, *i.e.* the zeros are in the same position for the two matrices.

Finally, the new mass matrix M_r is subjected to the above-mentioned constraints and minimizes the distance to the standard finite element matrix M (Fröbenius norm). This choice leads to a very simple system (6×2 in 2D and 10×10 in 3D) to be solved with Lagrange formulation in order to compute M_r.

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4.2 Elastodynamic contact problem with redistributed mass matrix

If we number the degrees of freedom such that the last ones are the nodes on the contact boundary, hypothesis (8) leads to a new mass matrix having the following pattern

$$M_r = \begin{pmatrix} \overline{M} & 0 \\ 0 & 0 \end{pmatrix}. \tag{10}$$

We can also split each matrix and vector into interior part and contact boundary part as follows:

$$K = \begin{pmatrix} \overline{K} & C^T \\ C & \widetilde{K} \end{pmatrix}, \quad N_i = \begin{pmatrix} 0 \\ \widetilde{N}_i \end{pmatrix}, \quad L = \begin{pmatrix} \overline{L} \\ \widetilde{L} \end{pmatrix} \quad \text{and } U = \begin{pmatrix} \overline{U} \\ \widetilde{U} \end{pmatrix}.$$

Replacing M by M_r, Problem (4) becomes

$$\begin{cases} \begin{pmatrix} \overline{M} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\overline{U}} \\ \ddot{\widetilde{U}} \end{pmatrix} + \begin{pmatrix} \overline{K} & C^T \\ C & \widetilde{K} \end{pmatrix} \begin{pmatrix} \overline{U} \\ \widetilde{U} \end{pmatrix} = \begin{pmatrix} \overline{L} \\ \widetilde{L} \end{pmatrix} + \sum_{i \in I_C} \lambda_{_N}^i \begin{pmatrix} 0 \\ \widetilde{N}_i \end{pmatrix}, \\ \widetilde{N}_i^T \widetilde{U} \leq 0, \quad \lambda_{_N}^i \leq 0, \quad \lambda_{_N}^i (\widetilde{N}_i^T \widetilde{U}) = 0 \quad \forall\, i \in I_{_C}, \\ U(0) = U^0, \dot{U}(0) = U^1. \end{cases} \tag{11}$$

4.3 Stability analysis

Theorem 1 *Let us assume that the load vector L is a Lipschitz continuous function on $[0,T]$. Then, there exists one and only one Lipschitz continuous function $t \longrightarrow (U(t), \Lambda_{_N}(t))$ solution to the discretized Problem (11).*

Proof. Problem (11) is equivalent to:

$$\begin{cases} \overline{M}\, \ddot{\overline{U}} + \overline{K}\, \overline{U} = \overline{L} - C^T \widetilde{U}, \\ C\overline{U} + \widetilde{K}\widetilde{U} = \widetilde{L} + \sum_{i \in I_C} \lambda_{_N}^i \widetilde{N}_i, \\ \widetilde{N}_i^T \widetilde{U} \leq 0, \quad \lambda_{_N}^i \leq 0, \quad \lambda_{_N}^i (\widetilde{N}_i^T \widetilde{U}) = 0 \quad \forall\, i \in I_{_C}, \\ U(0) = U^0, \dot{U}(0) = U^1. \end{cases} \tag{12}$$

The following sub-system of (12):

$$\begin{cases} \widetilde{K}\widetilde{U} + C\overline{U} = \widetilde{L} + \sum_{i \in I_C} \lambda_{_N}^i \widetilde{N}_i, \\ \widetilde{N}_i^T \widetilde{U} \leq 0, \quad \lambda_{_N}^i \leq 0, \quad \lambda_{_N}^i (\widetilde{N}_i^T \widetilde{U}) = 0 \quad \forall\, i \in I_{_C} \end{cases} \tag{13}$$

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can be expressed as follows :

$$a(\widetilde{U}, \widetilde{V} - \widetilde{U}) \geq l_{\overline{U}}(\widetilde{V} - \widetilde{U}) \quad \forall\, \widetilde{V} \in Q, \tag{14}$$

where $a(\widetilde{U}, \widetilde{V}) = \widetilde{V}^T \widetilde{K} \widetilde{U}$, $l_{\overline{U}}(\widetilde{V}) = \widetilde{V}^T \widetilde{L} - \widetilde{V}^T C \overline{U}$ and $Q = \{V : \widetilde{N}_i^T \widetilde{V} \leq 0,\ i \in I_C\}$.

The standard assumptions of the elasticity problem imply on the one hand that $\widetilde{U}$ is uniquely defined from the variational inequality (14) for given $\overline{U}$ and $\widetilde{L}$, and on the other hand that $\widetilde{U}$ and Λ_N are Lipschitz continuous functions with respect to $\overline{U}$ and $\widetilde{L}$. It follows that the first equation in the system (12) is a second order Lipschitz ordinary differential equation with respect to the unknown $\overline{U}$. Such an equation, with the initial conditions, has a unique solution $\overline{U}$ with a Lipschitz continuous derivative.

Since $\overline{U}$ and $\widetilde{L}$ are Lipschitz continuous functions in time, Λ_N is Lipschitz in time too.

■

Proposition 1 *The solution (U, Λ_N) to Problem (11) satisfies the following persistency condition at each node on Γ_C:*

$$\forall i \in I_C, \quad \lambda_N^i (N_i^T \dot{U}) = 0 \quad a.e.\ on\ [0, T].$$

Proof. Thanks to the fact that the solution (U, Λ_N) to Problem (11) is Lipschitz continuous, we have:

$$\lambda_N^i = 0 \ \text{ on } \ \mathrm{Supp}(N_i^T U) = \omega_i \subset [0, T] \quad (i \in \Gamma_C),$$

where $\mathrm{Supp}(\psi)$ denotes the support of the function $\psi(t)$. The continuity of λ_N^i on $[0, T]$ implies

$$\lambda_N^i = 0 \ \text{ on } \ \overline{\omega_i}.$$

On the other hand,

$$N_i^T \dot{U} = 0 \quad \text{a.e. on } \ \theta_i,$$

where θ_i is the complementary part in $[0, T]$ of the interior of ω_i. Hence

$$\lambda_N^i (N_i^T \dot{U}) = 0, \quad \text{a.e. on } \ [0, T].$$

■

Remark. The previous statement is a transcription in a finite element framework of the so-called persistency contact condition in elastodynamics (see [13]).

Theorem 2 *Assuming that the load vector L is constant in time, the finite element elastodynamic system with unilateral contact (11) is energy conserving.*

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Proof. The discrete energy of system (11) is given by:

$$E(t) = \frac{1}{2}\,\dot{U}^T M \dot{U} + \frac{1}{2}\,U^T K U - U^T L.$$

The first equation in (11) implies:

$$\dot{U}^T M_r \ddot{U} + \dot{U}^T K U = \dot{U}^T L + \sum_{i \in I_C} \lambda_{_N}^i \dot{U}^T N_i.$$

Integrating from 0 to t, it follows:

$$\frac{1}{2}\,\dot{U}^T M \dot{U} + \frac{1}{2}\,U^T K U - U^T L = \sum_{i \in I_C} \int_0^t \lambda_{_N}^i\,\dot{U}^T N_i\,dt + E(0).$$

In others words, one has

$$E(t) = \sum_{i \in I_C} \int_0^t \lambda_{_N}^i\,\dot{U}^T N_i\,dt + E(0) \quad \forall t \in [0, T].$$

Thanks to Proposition 1, we finally obtain

$$E(t) = E(0) \quad \forall t \in [0, T].$$

■

5 Numerical results

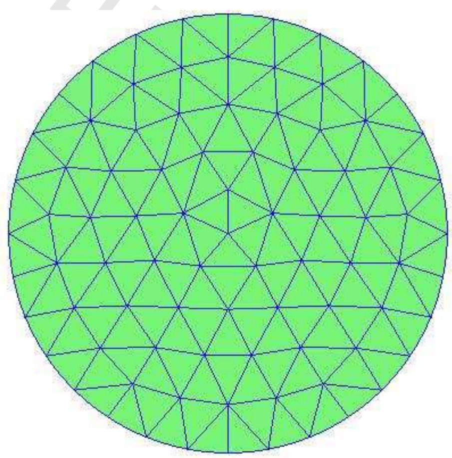

Figure 2: *the mesh of the disc (isoparametric P_2 elements).*

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Disc property	Values	Property of the resolution method	Values
ρ, diameter	$6\ 10^3 kg/m^3$, $0.2\ m$	Time step	$10^{-3}s$
Lamé coefficients	$\lambda = 10^6\ P$, $\mu = 5\ 10^5\ P$	Simulation time	$0.3\ s$
u^0, v^0	$0.01\ m$, $0.\ m/s$	Mesh parameter	$\simeq 0.02\ cm$

Table 1: *characteristics of the elastic disc and the resolution method*

In this section, we study the dynamic contact of an elastic disc (see Fig. 2) the properties of which are summarized in Tab. 1. We denote A the lowest point of the disc (the first point which will be in contact with the foundation). The numerical tests were performed with the finite element library Getfem [27]. The test program is available on the website of Getfem.

The semi-discretization in time is done with two time integration schemes: Crank-Nicholson scheme and Newmark scheme.

Crank-Nicholson scheme

The Crank-Nicholson scheme is defined by

$$\begin{cases} U^{n+1} = U^n + \dfrac{\Delta t}{2}\left(V^n + V^{n+1}\right), \\[2ex] V^{n+1} = V^n + \dfrac{\Delta t}{2}\left(A^n + A^{n+1}\right), \\[2ex] M_r A^{n+1} + KU^{n+1} = L + \displaystyle\sum_{i \in I_C} (\lambda_{_N}^i)^{n+1} N_i, \\[2ex] N_i^T U^{n+1} \leq 0, \quad (\lambda_{_N}^i)^{n+1} \leq 0, \quad (N_i^T U^{n+1})(\lambda_{_N}^i)^{n+1} = 0 \quad \forall\, i \in I_{_C}, \\[2ex] U(0) = U^0, \dot{U}(0) = U^1, \end{cases} \tag{15}$$

where U^n, V^n and A^n approximate $U(t_n), \dot{U}(t_n)$ and $\ddot{U}(t_n)$ respectively.

The energy evolution for the Crank-Nicholson scheme with and without mass redistribution method is shown on Fig. 3. We remark that the energy is blowing up with the standard mass matrix. Whereas, there are very small fluctuations in the energy evolution which is quasi-conserved with the mass redistribution method.

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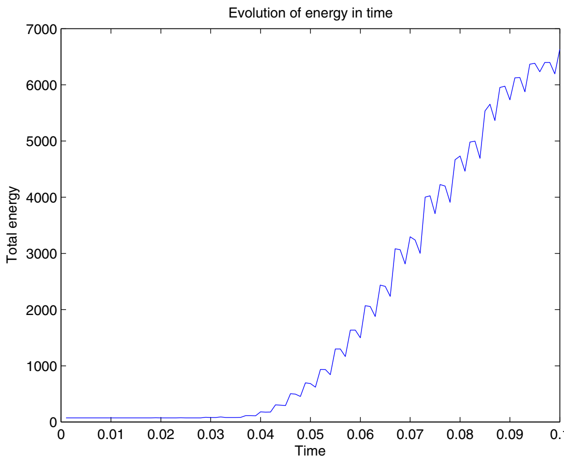
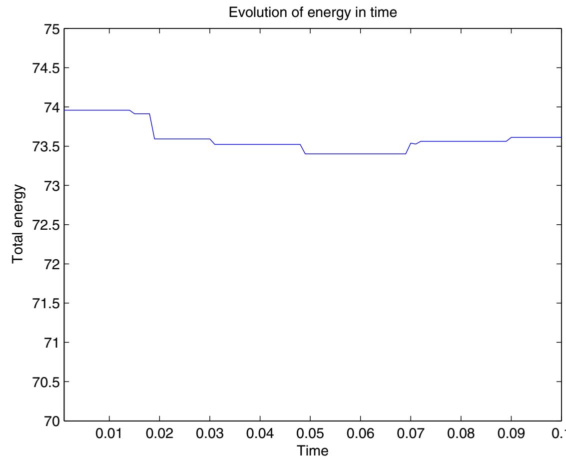


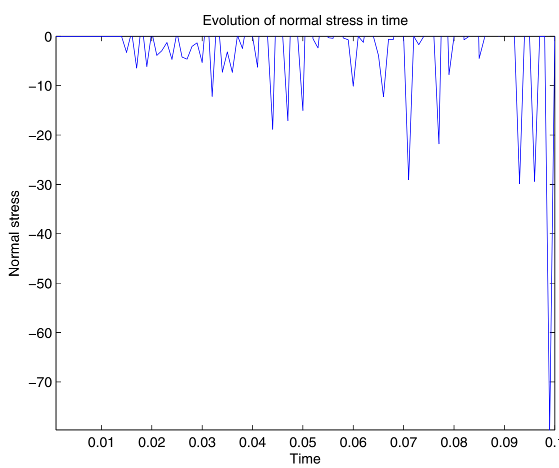
With the standard mass matrix



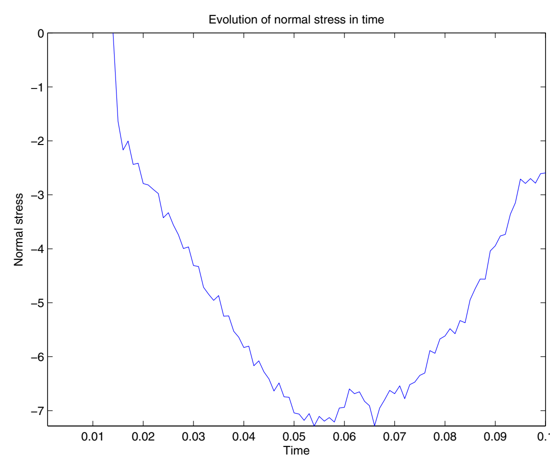
With the redistributed mass matrix

Figure 3: *energy evolution for the elastodynamic contact problem* ($\Delta t = 10^{-3}$).

The evolution of the numerical normal stress at point A is presented on Fig. 4. The normal stress is rather smooth with the mass redistribution method unlike with the standard finite element mass matrix where it is completely unexploitable.



With the standard mass matrix



With the redistributed mass matrix

Figure 4: *normal stress evolution of the lowest point of the disc* ($\Delta t = 10^{-3}$).

The numerical behaviour of the energy and normal stress evolution is stabilized using the redistributed mass matrix. The discretized elastodynamic contact problem is then equivalent to a

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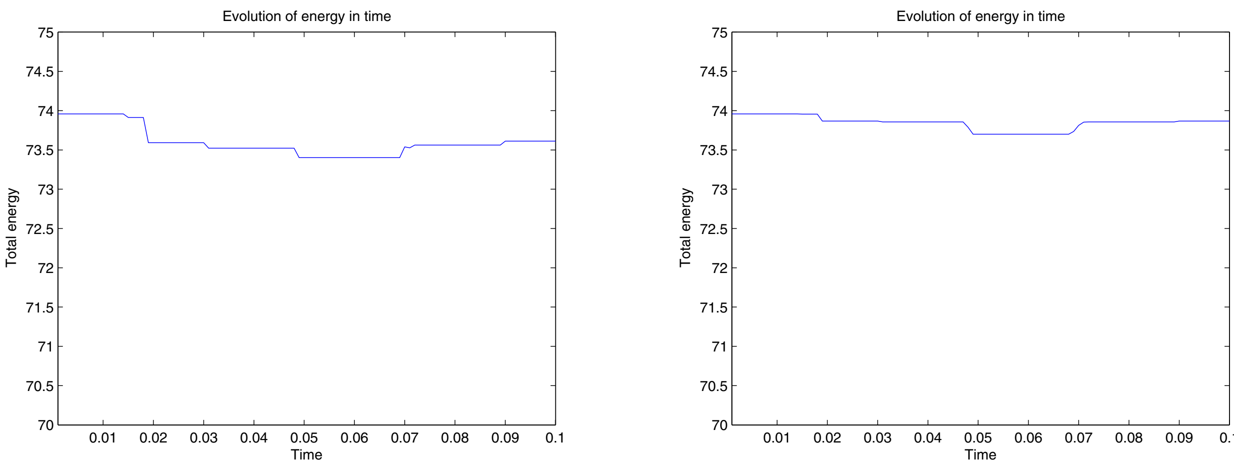
Lipschitz ODE in time, allowing the convergence of classical schemes when the time step goes to zero. It is illustrated with the use of a time step equal to 10^{-4} for the Cranck Nicholson scheme. The result on Fig. 5 clearly shows that the energy tends to be conserved when the time step goes to zero.



$\Delta t \;= 0.001$ $\qquad\qquad$ $\Delta t \;= 0.0001$

Figure 5: *influence of the time step on the energy evolution.*

Newmark scheme

Let us consider the following Newmark scheme

$$
\begin{cases}
U^{n+1} = U^n + \Delta t \;\; V^n + \dfrac{\Delta t^2}{2} A^{n+1}, \\[2ex]
V^{n+1} = V^n + \dfrac{\Delta t}{2}\; (A^n + A^{n+1}), \\[2ex]
M_r A^{n+1} + K U^{n+1} = L + \displaystyle\sum_{i \in I_C} (\lambda_{_N}^i)^{n+1} N_i, \\[2ex]
N_i^T U^{n+1} \le 0, \;\; (\lambda_{_N}^i)^{n+1} \le 0, \;\; (N_i^T U^{n+1})(\lambda_{_N}^i)^{n+1} = 0 \quad \forall\, i \in I_{_C}, \\[2ex]
U(0) = U^0, \dot{U}(0) = U^1.
\end{cases}
\tag{16}
$$

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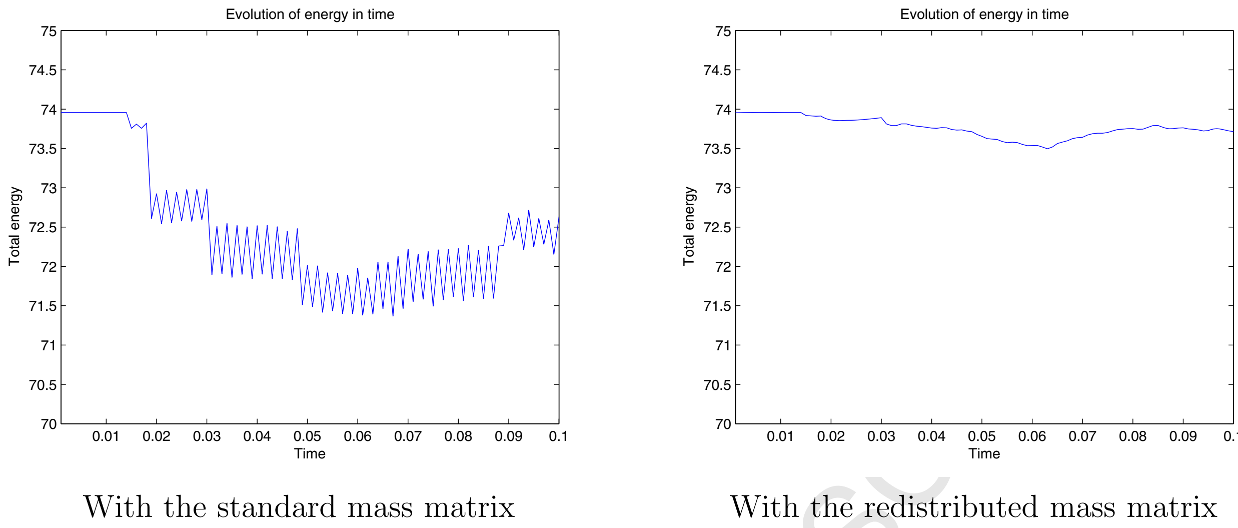

Figure 6: *energy evolution for the elastodynamic contact problem* ($\Delta t = 10^{-3}$).

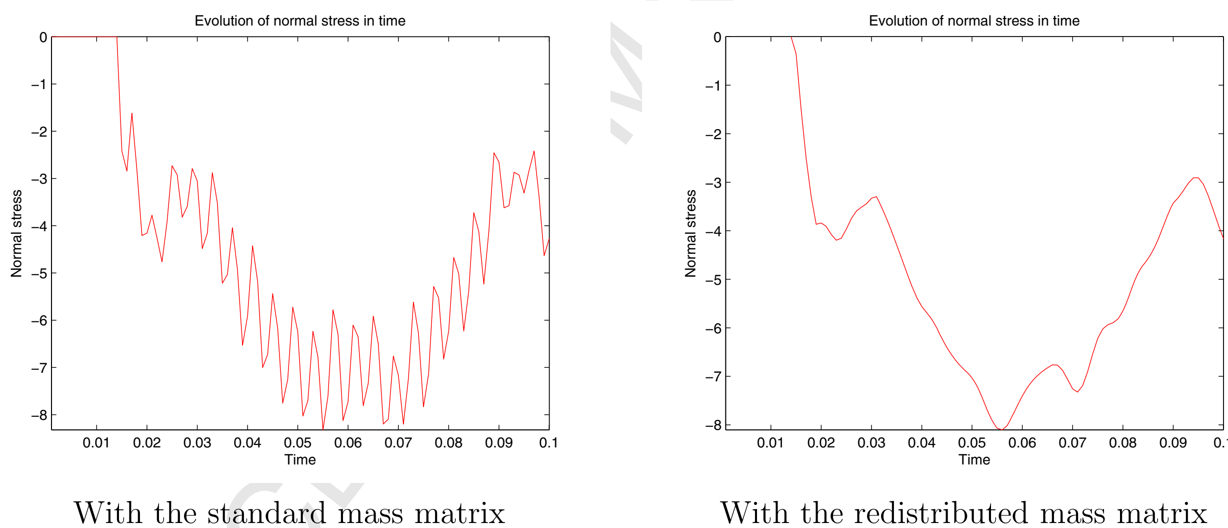

Figure 7: *normal stress evolution on the lowest point of the disc* ($\Delta t = 10^{-3}$).

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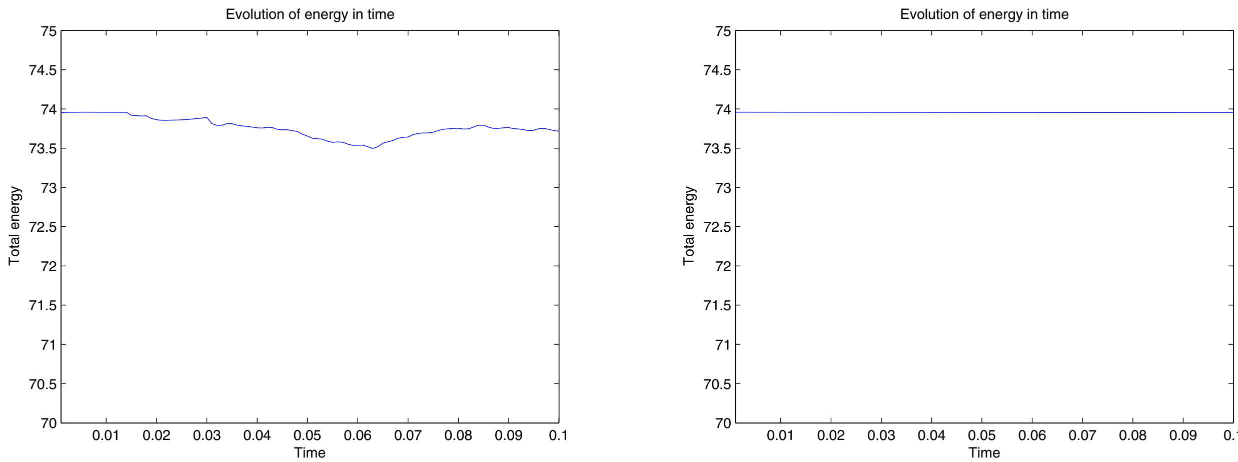

$\Delta t = 0.001$ $\Delta t = 0.0001$

Figure 8: *influence of time step on energy evolution.*

Concerning the Newmark scheme, we remark that the mass redistribution method stabilizes also the scheme and improves the behaviour of the normal stress (see Fig. 6, Fig.7 and Fig.8). Furthermore, when time step goes to zero, the discrete solution tends also to be energy conserving.

Propagation of the impact wave

In order to show that the method does not change the behaviour of the solution and that in particular we conserve the propagation of the impact wave, we give the following test. Fig.9 represents the evolution of the Von Mises stress during the first impact.

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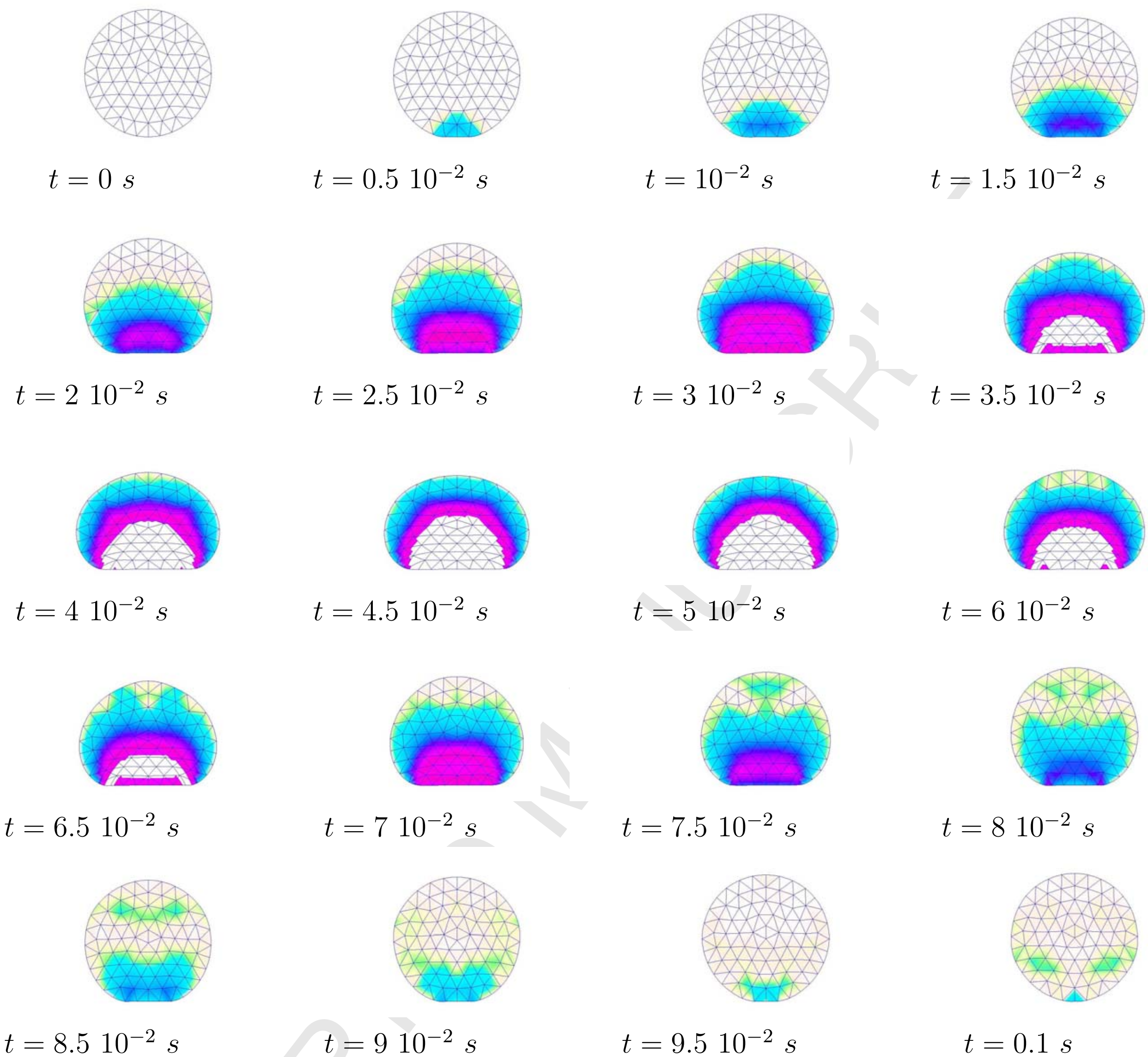

Figure 9: *Von Mises stress evolution during the first impact (Cranck-Nicholson scheme with MRM). The return of the shear wave causes the rebound of the ball.*

One can see how the return of the shear wave causes the rebound of the ball. Despite the roughness of the mesh, the behavior of the shear wave seems to be quite healthy. This reinforce the confidence on the MRM which does not qualitatively affect the propagation of the shear wave and allows to have non-oscillatory constraints.

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Other comparisons

5.1 Comparison with Paoli-Schatzman scheme

The Paoli-Schatzman scheme is well adapted to rigid bodies because it is based on introducing a restitution parameter (see [18]) to define the jump of velocity during the impact [21]. This idea is not true for deformable bodies. We adapted this scheme for those bodies and we inroduce the following formulation:

$$U^{n+1} = U^n + \Delta t\ V^{n+\frac{1}{2}}, \qquad U^{n+\frac{1}{2}} = \frac{U^{n+1} + U^n}{2}, \tag{17}$$

$$\begin{cases} U^0 \text{ and } V^0 \text{ given}, U^1 = U^0 + \Delta t\ V^0 + \Delta t\ z(\Delta t\ ),\ \text{avec} \lim\limits_{\Delta t\ \longrightarrow 0} z(\Delta t\ ) = 0, \\ \forall n \geq 2, \\ M\left(\dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2}\right) + K\left(\dfrac{U^{n+1} + 2U^n + U^{n-1}}{4}\right) = L + B_{_N}^T \Lambda_{_N}^n + B_{_T}^T \Lambda_{_T}^n, \\ -\Lambda_{_N}^n \in N_{K_N}\left(\dfrac{B_{_N} U^{n+1} + e B_{_N} U^{n-1}}{1+e}\right). \end{cases} \tag{18}$$

Let us notice that the contact condition is verifyed for displacement on the proximal point defined by

$$\frac{B_{_N} U^{n+1} + e B_{_N} U^{n-1}}{1+e},$$

where e is the restitution coeffcient ($e \in [0,1]$). For more details of the stability of this scheme please see [8].

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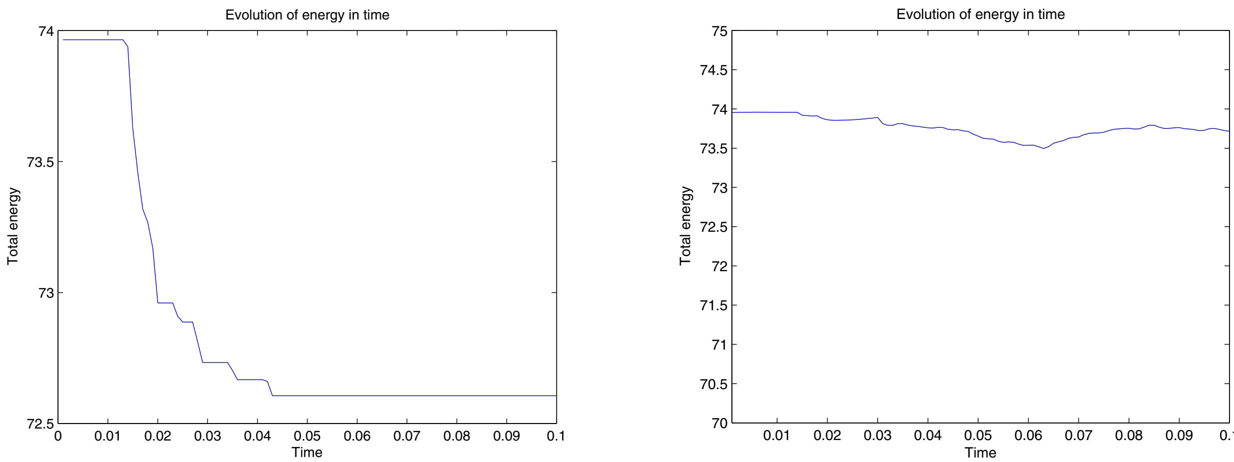
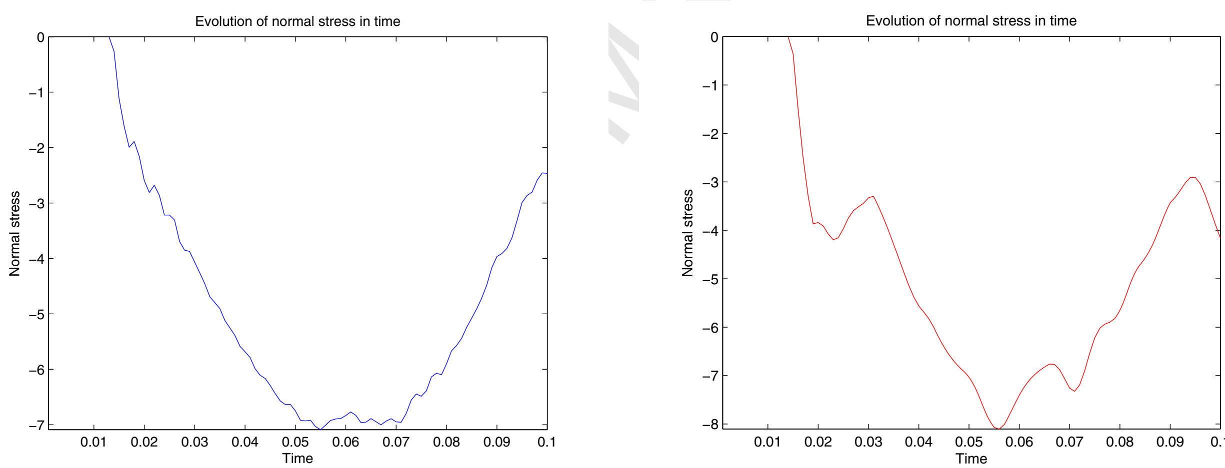


Paoli-Schatzman scheme   Newmark ($\beta = \gamma = 0.5$) scheme with MRM

Figure 10: *energy evolution for the elastodynamic contact problem* ($\Delta t = 10^{-3}$).



Paoli-Schatzman scheme   Newmark ($\beta = \gamma = 0.5$) scheme with MRM

Figure 11: *normal stress evolution of the lowest point of the disc* ($\Delta t = 10^{-3}$).

5.2 Comparison with Laursen-Chawla scheme

This scheme is based on idea to verify the contact condition on velocity (persistency condition) [13]. Here, we will do only the comparison between this scheme with and the Newmark scheme ($\beta = \gamma = 0.5$) using the mass redistribution method of section (4.2). A comparison can also be found in [10] with an other scheme close to Laursen-Chawla scheme.

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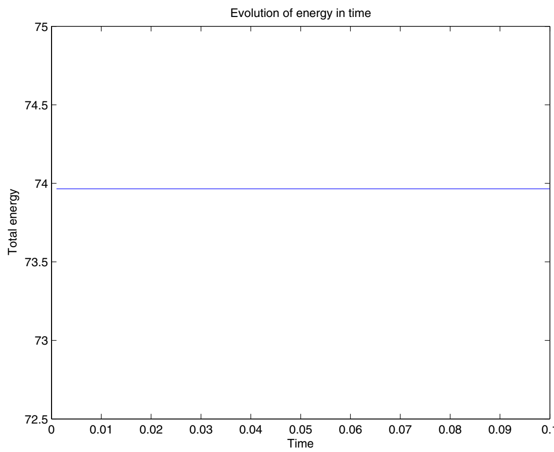
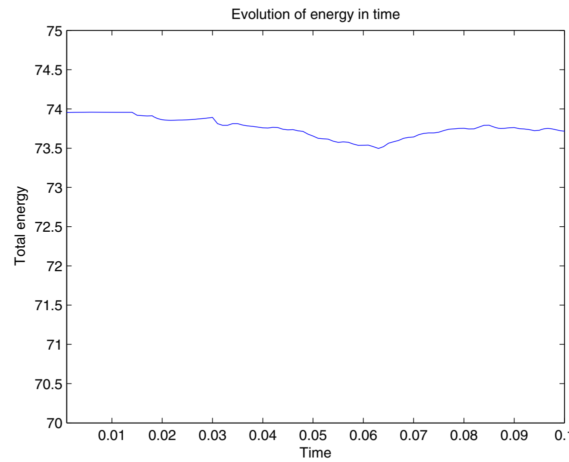
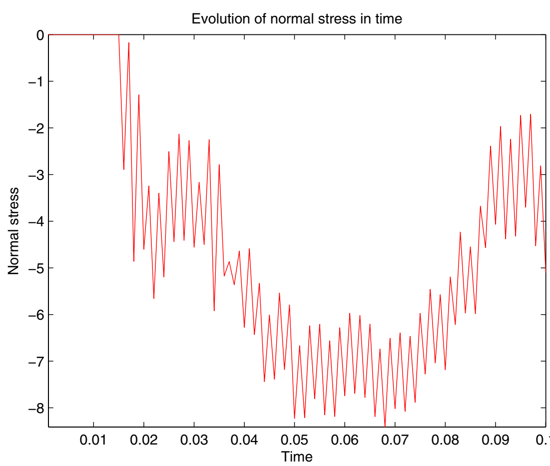


Laursen-Chawla scheme

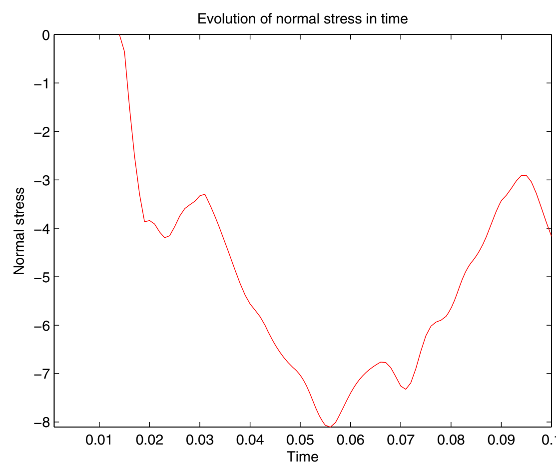


Newmark ($\beta = \gamma = 0.5$) scheme with MRM

Figure 12: *energy evolution for the elastodynamic contact problem ($\Delta t = 10^{-3}$).*



Laursen-Chawla scheme



Newmark ($\beta = \gamma = 0.5$) scheme with MRM

Figure 13: *normal stress evolution of the lowest point of the disc ($\Delta t = 10^{-3}$).*

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Concluding remarks

In [10], a comparison is done between the approach presented above and an energy conserving scheme similar to the one introduced in [13]. The advantage of the mass redistribution method is first to lead to a well-posed finite element contact problem in elastodynamics which is energy conserving. Secondly, such a method allows to improve the behaviour, on the contact surface, of the numerical normal stress (which is very noisy in the other approach). Moreover, the mass redistribution method does not affect the propagation of the impact wave. Let us also note that adding a Coulomb friction condition is not a difficulty from a stability point of view. This work can easily be extended to large deformations [6, 5]. We can confirm also that the mass redistribution method is consistent when the mesh parameter goes to zero and this leads to a very small redistributed mass tending to zero.

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