



## Ample thoughts

Daniel Palacin, Frank Olaf Wagner

### ► To cite this version:

| Daniel Palacin, Frank Olaf Wagner. Ample thoughts. 2011. hal-00581795v3

**HAL Id: hal-00581795**

**<https://hal.science/hal-00581795v3>**

Preprint submitted on 28 Nov 2011 (v3), last revised 28 Sep 2012 (v5)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# AMPLE THOUGHTS

DANIEL PALACÍN AND FRANK O. WAGNER

ABSTRACT. Non- $n$ -ampleness as defined by Pillay [18] and Evans [6] is preserved under analysability. Generalizing this to a more general notion of  $\Sigma$ -ampleness, this gives an immediate proof for all simple theories of a weakened version of the Canonical Base Property (CBP) proven by Chatzidakis [5] for types of finite SU-rank. This is then applied to the special case of groups.

## 1. INTRODUCTION

Recall that a type  $p$  over a set  $A$  in a simple theory is *one-based* if for any tuple  $\bar{a}$  of realizations of  $p$  and any  $B \supseteq A$  the canonical base  $\text{Cb}(\bar{a}/B)$  is contained in  $\text{bdd}(\bar{a}A)$ . One-basedness implies that the forking geometry is particularly well-behaved; for instance one-based groups are bounded-by-abelian-by-bounded. The principal result in [24] is that one-basedness is preserved under analyses: a type analysable in one-based types is itself one-based. This generalized earlier results of Hrushovski [12] and Chatzidakis [5]. One-basedness is the first level in a hierarchy of possible geometric behaviour of forking independence first defined by Pillay [18] and slightly modified by Evans [6],  $n$ -ampleness. Not 1-ample means one-based; not 2-ample is equivalent to a notion previously introduced by Hrushovski [13], CM-triviality. In [18] Pillay defines  $n$ -ampleness locally for a single type and shows that a superstable theory of finite Lascar rank is  $n$ -ample if and only if all its types of rank 1 are; his proof implies that in such a theory, a type analysable in non- $n$ -ample types is again non- $n$ -ample.

---

*Date:* 24 November 2011.

*2000 Mathematics Subject Classification.* 03C45.

*Key words and phrases.* stable; simple; one-based; CM-trivial;  $n$ -ample; internal; analysable; closure; level; flat; ultraflat; canonical base property.

The first author was partially supported by research project MTM 2008-01545 of the Spanish government and research project 2009SGR 00187 of the Catalan government. He also would like to thank Amador Martín Pizarro for interesting discussions around the Canonical Base Property.

The second author was partially supported by ANR-09-BLAN-0047 Modig.

We shall give a definition of  $n$ -ampleness for invariant families of partial types, and generalize Pillay's result to arbitrary simple theories. Note that for  $n = 1$  this gives an alternative proof of the main result in [24]. Since for types of infinite rank the algebraic (bounded) closure used in the definition is not necessarily appropriate (for a regular type  $p$  one might, for instance, replace it by  $p$ -closure), we also generalize the notion to  $\Sigma$ -closure for some  $\emptyset$ -invariant collection of partial types (thought of as small), giving rise to the notion of  $n$ - $\Sigma$ -ample. This may for instance be applied to consider ampleness modulo types of finite SU-rank, or modulo superstable types. Readers not interested in this additional generality are invited to simply replace  $\Sigma$ -closure by bounded closure. However, this will only marginally shorten the proofs. As an immediate Corollary of the more general version, we shall derive a weakened version of the Canonical Base Property CBP [21] shown by Chatzidakis [5], where analysability replaces internality in the definition. We also give a version appropriate for supersimple theories. Finally, we deduce that in a simple theory with enough regular types, a hyperdefinable group modulo its approximate centre is analysable in the family of non one-based regular types; the group modulo a normal nilpotent subgroup is almost internal in that family.

Our notation is standard and follows [23]. Throughout the paper, the ambient theory will be simple, and we shall be working in  $\mathfrak{M}^{heq}$ , where  $\mathfrak{M}$  is a sufficiently saturated model of the ambient theory. Thus tuples are tuples of hyperimaginaries, and  $\text{dcl} = \text{dcl}^{heq}$ .

## 2. INTERNALITY AND ANALYSABILITY

For the rest of the paper  $\Sigma$  will be an  $\emptyset$ -invariant family of partial types. Recall first the definitions of internality, analysability and foreignness.

**Definition 2.1.** Let  $\pi$  be a partial type over  $A$ . Then  $\pi$  is

- (almost)  $\Sigma$ -internal if for every realization  $a$  of  $\pi$  there is  $B \downarrow_A a$  and a tuple  $\bar{b}$  of realizations of types in  $\Sigma$  based on  $B$ , such that  $a \in \text{dcl}(B\bar{b})$  (or  $a \in \text{bdd}(B\bar{b})$ , respectively).
- $\Sigma$ -analysable if for any realization  $a$  of  $\pi$  there are  $(a_i : i < \alpha) \in \text{dcl}(Aa)$  such that  $\text{tp}(a_i/A, a_j : j < i)$  is  $\Sigma$ -internal for all  $i < \alpha$ , and  $a \in \text{bdd}(A, a_i : i < \alpha)$ .

A type  $\text{tp}(a/A)$  is *foreign* to  $\Sigma$  if  $a \not\downarrow_{AB} \bar{b}$  for all  $B \downarrow_A a$  and  $\bar{b}$  realizing types in  $\Sigma$  over  $B$ .

The following lemmas and their corollaries are folklore, but we add some precision about non-orthogonality.

**Lemma 2.2.** *Suppose  $a \perp b$  and  $a \not\perp_b c$ . Let  $(b_i : i < \omega)$  be an indiscernible sequence in  $\text{tp}(b)$  and put  $p_b = \text{tp}(c/b)$ . Then  $p_{b_i}$  is non-orthogonal to  $p_{b_j}$  for all  $i, j < \omega$ .*

*Proof:* We prolong the sequence to have length  $\alpha$ . As  $a \perp b$  and  $(b_i : i < \alpha)$  is indiscernible, by [23, Theorem 2.5.4] we may assume  $ab \equiv ab_i$  for all  $i < \alpha$  and  $a \perp (b_i : i < \alpha)$ . Let  $B = (b_i : i < \omega)$ , so  $(b_i : \omega \leq i < \alpha)$  is independent over  $B$  and  $a \perp B$ . Choose  $c_i$  with  $b_i c_i \equiv_a bc$  and

$$c_i \perp_{ab_i} (b_j : j < \alpha)$$

for all  $\omega \leq i < \alpha$ . Then  $ab_i c_i \perp_B (b_j : j \neq i)$  for all  $\omega \leq i < \alpha$ . By indiscernability, if  $p_{b_i}$  were orthogonal to  $p_{b_j}$  for some  $i \neq j$ , then they would be orthogonal for all  $i \neq j$ . As  $c_i \perp_{ab_i} (b_j : j \neq i)$ , the sequence  $(b_i c_i : \omega \leq i < \alpha)$  would be independent over  $B$ . However,  $a \not\perp_B b_i c_i$  for all  $\omega \leq i < \alpha$ , contradicting the boundedness of weight of  $\text{tp}(a/B)$ .  $\square$

**Lemma 2.3.** *Suppose  $a \perp b$  and  $a' = \text{Cb}(bc/a)$ . Let  $\mathcal{P}$  be the family of  $\text{bdd}(\emptyset)$ -conjugates of  $\text{tp}(c/b)$  non-orthogonal to  $\text{tp}(c/b)$ . Then  $a' \in \text{bdd}(a)$  is  $\mathcal{P}$ -internal and  $\text{bdd}(ab) \cap \text{bdd}(bc) \subseteq \text{bdd}(a'b)$ .*

*Proof:* If  $a \perp bc$  then  $a' \in \text{bdd}(\emptyset)$  and  $\text{bdd}(ab) \cap \text{bdd}(bc) = \text{bdd}(b)$ , so there is nothing to show. Assume  $a \not\perp_b c$ . Clearly  $a' \in \text{bdd}(a)$ ; as  $bc \perp_{a'} a$  we get  $c \perp_{a'b} a$  and hence  $\text{bdd}(ab) \cap \text{bdd}(bc) \subseteq \text{bdd}(a'b)$ . Let  $(b_i c_i : i < \omega)$  be a Morley sequence in  $\text{lstp}(bc/a)$  with  $b_0 c_0 = bc$ . Then  $a' \in \text{dcl}(b_i c_i : i < \omega)$ ; since  $b \perp a$  we get  $(b_i : i < \omega) \perp a$ , whence  $(b_i : i < \omega) \perp a'$ . So  $a'$  is internal in  $\{\text{tp}(c_i/b_i) : i < \omega\}$ . Finally,  $\text{tp}(c_i/b_i)$  is non-orthogonal to  $\text{tp}(c/b)$  for all  $i < \omega$  by Lemma 2.2.  $\square$

**Corollary 2.4.** *If  $a \perp b$  and  $\text{tp}(c/b)$  is (almost)  $\Sigma$ -internal, then  $\text{Cb}(bc/a)$  is (almost)  $\Sigma$ -internal. The same statement holds with analysable instead of internal.*

*Proof:* Let  $d \perp_b c$  and  $\bar{e}$  realize partial types in  $\Sigma$  over  $bd$  such that  $c \in \text{dcl}(bd\bar{e})$  (or  $c \in \text{bdd}(bd\bar{e})$ , respectively). We may take  $d\bar{e} \perp_{bc} a$ . Then  $d \perp_b ac$ , whence  $a \perp bd$ . So  $\text{Cb}(bd\bar{e}/a)$  is  $\Sigma$ -internal by Lemma 2.3. But  $a \perp_{bc} d\bar{e}$  and  $c \in \text{dcl}(bd\bar{e})$  implies  $\text{Cb}(bc/a) \in \text{dcl}(\text{Cb}(bd\bar{e}/a))$ ; similarly  $c \in \text{bdd}(bd\bar{e})$  implies  $\text{Cb}(bc/a) \in \text{bdd}(\text{Cb}(bd\bar{e}/a))$ .

The proof for  $\Sigma$ -analysability is analogous.  $\square$

**Definition 2.5.** Two partial types  $\pi_1$  and  $\pi_2$  are *perpendicular*, denoted  $\pi_1 \perp \pi_2$ , if for any set  $A$  containing their domains and any tuple  $\bar{a}_i \models \pi_i$  for  $i = 1, 2$  we have  $\bar{a}_1 \perp_A \bar{a}_2$ .

For instance, orthogonal types of rank 1 are perpendicular.

**Corollary 2.6.** *Suppose  $a \perp b$ , and  $a_0 \in \text{bdd}(ab)$  is (almost)  $\Pi$ -internal over  $b$  for some  $b$ -invariant family  $\Pi$  of partial types. Let  $\Pi'$  be the family of  $\text{bdd}(\emptyset)$ -conjugates  $\pi'$  of partial types  $\pi \in \Pi$  with  $\pi' \not\perp \pi$ . Then there is (almost)  $\Pi'$ -internal  $a_1 \in \text{bdd}(a)$  with  $a_0 \in \text{bdd}(a_1 b)$ . The same statement holds with analysable instead of internal.*

*Proof:* If  $\text{tp}(a_0/b)$  is  $\Pi$ -internal, there is  $c \perp_b a_0$  and  $\bar{e}$  realizing partial types in  $\Pi$  over  $bc$  such that  $a_0 \in \text{dcl}(bc\bar{e})$ ; we choose them with  $c\bar{e} \perp_{ba_0} a$ . So  $c \perp_b a$ , whence  $a \perp bc$ . Furthermore, we may assume that  $e \not\perp_{bc} a$  for all  $e \in \bar{e}$ , since otherwise  $ec \perp_b a_0$  and we may just include  $e$  in  $c$ , reducing the length of  $\bar{e}$ . Now  $a_0 \in \text{bdd}(abc) \cap \text{bdd}(bc\bar{e})$ , so by Lemmas 2.2 and 2.3 there is  $\Pi'$ -internal  $a_1 \in \text{bdd}(a)$  with  $a_0 \in \text{bdd}(bca_1)$ . Since  $a \perp_b c$  implies  $a_0 \perp_{a_1 b} c$ , we get  $a_0 \in \text{bdd}(a_1 b)$ .

For the almost internal case, we replace definable by bounded closure; for the analysability statement we iterate, adding  $a_1$  to the parameters.  $\square$

To finish this section, a decomposition lemma for almost internality.

**Lemma 2.7.** *Let  $\Sigma = \bigcup_{i < \alpha} \Sigma_i$ , where  $(\Sigma_i : i < \alpha)$  is a collection of pairwise perpendicular  $\emptyset$ -invariant families of partial types. If  $\text{tp}(a/A)$  is almost  $\Sigma$ -internal, then there are  $(a_i : i < \alpha)$  interbounded over  $A$  with  $a$  such that  $\text{tp}(a_i/A)$  is  $\Sigma_i$ -internal for  $i < \alpha$ .*

Clearly, if  $a$  is a finite imaginary tuple, we only need finitely many  $a_i$ .

*Proof:* By assumption there is  $B \perp_A a$  and some tuples  $(b_i : i < \alpha)$  such that  $b_i$  realizes partial types in  $\Sigma_i$  over  $B$ , with  $a \in \text{bdd}(B, b_i : i < \alpha)$ . Let  $a_i = \text{Cb}(Bb_i/Aa)$ . Then  $a_i \in \text{bdd}(Aa)$  and  $\text{tp}(a_i/A)$  is  $\Sigma_i$ -internal by Corollary 2.4.

Put  $\bar{a} = (a_i : i < \alpha)$ . Then  $a \perp_{Aa_i} Bb_i$  implies  $a \perp_{B\bar{a}} b_i$ ; since  $b_i \perp_{Ba} (b_j : j \neq i)$  by perpendicularity we obtain  $b_i \perp_{B\bar{a}} (a, b_j : j \neq i)$  for all  $i < \alpha$ . Hence  $(a, b_i : i < \alpha)$  is independent over  $B\bar{a}$ , and in particular

$$a \perp_{B\bar{a}} (b_i : i < \alpha).$$

Since  $a \in \text{bdd}(B, b_i : i < \alpha)$  we get  $a \in \text{bdd}(B\bar{a})$ ; as  $a \perp_A B$  implies  $a \perp_{A\bar{a}} B$  we obtain  $a \in \text{bdd}(A\bar{a})$ .  $\square$

3.  $\Sigma$ -CLOSURE,  $\Sigma_1$ -CLOSURE AND A THEORY OF LEVELS

In his proof of Vaught's conjecture for superstable theories of finite rank [3], Buechler defines the first level  $\ell_1(a)$  of an element  $a$  of finite Lascar rank as the set of all  $b \in \text{acl}^{eq}(a)$  internal in the family of all types of Lascar rank one; higher levels are defined inductively by  $\ell_{n+1}(a) = \ell_1(a/\ell_n(a))$ . The notion has been studied by Prerna Bihani Juhlin in her thesis [1] in connection with a reformulation of the canonical base property. We shall generalise the notion to arbitrary simple theories.

**Definition 3.1.** The *first  $\Sigma$ -level of  $a$  over  $A$*  is given by

$$\ell_1^\Sigma(a/A) = \{b \in \text{bdd}(aA) : \text{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Inductively,  $\ell_{\alpha+1}^\Sigma(a/A) = \ell_1^\Sigma(a/\ell_\alpha^\Sigma(a/A))$ , and  $\ell_\lambda^\Sigma(a/A) = \bigcup_{\alpha < \lambda} \ell_\alpha^\Sigma(a/A)$  for limit ordinals  $\lambda$ . Finally, we shall write  $\ell_\infty^\Sigma(a/A)$  for the set of all hyperimaginaries  $b \in \text{bdd}(aA)$  such that  $\text{tp}(b/A)$  is  $\Sigma$ -analysable.

**Remark 3.2.** Clearly,  $\text{tp}(a/A)$  is  $\Sigma$ -analysable if and only if  $\ell_\infty^\Sigma(a/A) = \text{bdd}(aA)$  if and only if  $\ell_\alpha^\Sigma(a/A) = \text{bdd}(aA)$  for some ordinal  $\alpha$ , and the minimal such  $\alpha$  is the minimal length of a  $\Sigma$ -analysis of  $a$  over  $A$ .

**Lemma 3.3.** If  $a \perp b$ , then  $\ell_\alpha^\Sigma(ab) = \text{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b))$  for any  $\alpha$ .

*Proof:* Let  $c = \ell_\alpha^\Sigma(ab)$ . Clearly,  $\ell_\alpha^\Sigma(a)\ell_\alpha^\Sigma(b) \subseteq c$ . Conversely, put  $a_0 = \text{Cb}(bc/a)$ . Then  $\text{tp}(a_0)$  is internal in the family of  $\text{bdd}(\emptyset)$ -conjugates of  $\text{tp}(c/b)$  by Corollary 2.4; since even  $\text{tp}(c)$  is  $\Sigma$ -analysable in  $\alpha$  steps, so is  $\text{tp}(a_0)$ . Thus  $a_0 \subseteq \ell_\alpha^\Sigma(a)$ . Now  $bc \perp_{a_0} a$  implies

$$c \perp_{\ell_\alpha^\Sigma(a)b} a,$$

whence  $c \subseteq \text{bdd}(\ell_\alpha^\Sigma(a), b)$ . By symmetry,  $c \subseteq \text{bdd}(\ell_\alpha^\Sigma(b), a)$ , that is,

$$\ell_\alpha^\Sigma(ab) \subseteq \text{bdd}(\ell_\alpha^\Sigma(a), b) \cap \text{bdd}(\ell_\alpha^\Sigma(b), a).$$

On the other hand,  $a \perp b$  yields  $a \perp_{\ell_\alpha^\Sigma(a)\ell_\alpha^\Sigma(b)} b$ . Thus,

$$\text{bdd}(\ell_\alpha^\Sigma(a), b) \cap \text{bdd}(\ell_\alpha^\Sigma(b), a) = \text{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b)),$$

whence the result.  $\square$

We shall see that the first level governs domination-equivalence.

**Definition 3.4.** An element  $a$   $\Sigma$ -dominates an element  $b$  over  $A$ , denoted  $a \succeq_A^\Sigma b$ , if for all  $c$  such that  $\text{tp}(c/A)$  is  $\Sigma$ -analysable,  $a \perp_A c$  implies  $b \perp_A c$ . Two elements  $a$  and  $b$  are  $\Sigma$ -domination-equivalent over  $A$ , denoted  $a \sqsubseteq_A^\Sigma b$ , if  $a \succeq_A^\Sigma b$  and  $b \succeq_A^\Sigma a$ . If  $\Sigma$  is the set of all types, it is omitted.

The following generalizes a theorem by Buechler [3, Proposition 3.1] for finite Lascar rank.

**Theorem 3.5.** *Let  $\Sigma'$  be an  $\emptyset$ -invariant family of partial types.*

- (1)  *$a$  and  $\ell_1^\Sigma(a/A)$  are  $\Sigma$ -domination-equivalent over  $A$ .*
- (2) *If  $\text{tp}(a/A)$  is  $\Sigma$ -analysable, then  $a$  and  $\ell_1^\Sigma(a/A)$  are domination-equivalent over  $A$ .*
- (3) *If  $\text{tp}(a/A)$  is  $\Sigma \cup \Sigma'$ -analysable and foreign to  $\Sigma'$ , then  $a$  and  $\ell_1^\Sigma(a/A)$  are domination-equivalent over  $A$ .*

*Proof:* Since  $\ell_1^\Sigma(a/A) \in \text{bdd}(Aa)$ , clearly  $a$  dominates (and  $\Sigma$ -dominates)  $\ell_1^\Sigma(a/A)$  over  $A$ .

For the converse, suppose  $\text{tp}(b/A)$  is  $\Sigma$ -analysable and  $b \not\downarrow_A a$ . Consider a sequence  $(b_i : i < \alpha)$  in  $\text{bdd}(Ab)$  such that  $\text{tp}(b_i/A, b_j : j < i)$  is  $\Sigma$ -internal for all  $i < \alpha$  and  $b \in \text{bdd}(A, b_i : i < \alpha)$ . Since  $a \not\downarrow_A b$  there is a minimal  $i < \alpha$  such that  $a \not\downarrow_{A, (b_j : j < i)} b_i$ . Put  $a' = \text{Cb}(b_j : j \leq i/Aa)$ . Then  $a' \in \text{bdd}(Aa)$  is  $\Sigma$ -internal by Corollary 2.4, and  $a' \subseteq \ell_1^\Sigma(a/A)$ . Clearly  $a' \not\downarrow_A (b_j : j \leq i)$ , whence  $a' \not\downarrow_A b$  and finally  $\ell_1^\Sigma(a/A) \not\downarrow_A b$ . This shows (1).

If  $\text{tp}(a/A)$  is  $\Sigma$ -analysable and  $b \not\downarrow_A a$ , we first consider  $b' = \text{Cb}(a/Ab)$ . Then  $\text{tp}(b'/A)$  is  $\Sigma$ -analysable,  $b' \in \text{bdd}(Ab)$  and  $a \not\downarrow_A b'$ . Hence  $\ell_1^\Sigma(a/A) \not\downarrow_A b'$ , whence  $\ell_1^\Sigma(a/A) \not\downarrow_A b$ . This shows (2).

For (3), suppose  $b \not\downarrow_A a$ . We may assume that  $b = \text{Cb}(a/Ab)$ , so  $\text{tp}(b/A)$  is  $(\Sigma \cup \Sigma')$ -analysable. Let  $(b_i : i < \alpha)$  be a  $(\Sigma \cup \Sigma')$ -analysis of  $b$  over  $A$ ; we can choose it such that for all  $i < \alpha$  the type  $\text{tp}(b_i/A, b_j : j < i)$  is internal in either  $\Sigma$  or in  $\Sigma'$ . Consider a minimal  $i < \alpha$  such that  $(b_j : j \leq i) \not\downarrow_A a$ . Then  $(b_j : j < i) \perp_A a$ , so  $\text{tp}(a/A, b_j : j < i)$  is foreign to  $\Sigma'$ . Hence  $\text{tp}(b_i/A, b_j : j < i)$  must be  $\Sigma$ -internal, and  $a' = \text{Cb}(b_j : j \leq i/Aa)$  is also  $\Sigma$ -internal over  $A$ . Thus  $a' \in \ell_1^\Sigma(a/A)$ ; since  $(b_j : j \leq i) \not\downarrow_A a'$  we get  $b \not\downarrow_A \ell_1^\Sigma(a/A)$ .  $\square$

**Remark 3.6.** If  $\text{tp}(a/A)$  is  $\Sigma_0$ -analysable and  $\Sigma_1$  is a subfamily of  $\Sigma_0$  such that  $\text{tp}(a/A)$  remains  $\Sigma_1$ -analysable, then

$$\ell_1^{\Sigma_1}(a/A) \subseteq \ell_1^{\Sigma_0}(a/A) \subseteq \text{bdd}(aA)$$

and  $\ell_1^{\Sigma_1}(a/A)$  and  $\ell_1^{\Sigma_0}(a/A)$  are both domination-equivalent to  $a$  over  $A$ . In fact it would be sufficient to have  $\Sigma_1$  such that  $\text{tp}(\ell_1^{\Sigma_0}(a/A)/A)$  is  $\Sigma_1$ -analysable.

**Question 3.7.** When is there a minimal (boundedly closed)  $a_0 \in \text{bdd}(aA)$  domination-equivalent with  $a$  over  $A$ ?

If  $T$  has finite SU-rank, one can take  $a_0 \in \text{bdd}(aA) \setminus \text{bdd}(A)$  with  $SU(a_0/A)$  minimal possible.

**Definition 3.8.** • A type  $\text{tp}(a/A)$  is  $\Sigma$ -flat if  $\ell_1^\Sigma(a/A) = \ell_2^\Sigma(a/A)$ . It is flat if it is  $\Sigma$ -flat for all  $\Sigma$ .  $T$  is flat if all its types are.  
• A type  $p$  is ultraflat if it is almost internal in any type it is non-orthogonal to.

Note that both notions are preserved under non-forking extensions and non-forking restrictions.

**Remark 3.9.** If  $\text{tp}(a/A)$  is  $\Sigma$ -flat, then  $\ell_\alpha^\Sigma(a/A) = \ell_1^\Sigma(a/A)$  for all  $\alpha > 0$ . Clearly, ultraflat implies flat.

**Example.** • Generic types of fields or definably simple groups interpretable in a simple theory are ultraflat.  
• If there is no boundedly closed set between  $\text{bdd}(A)$  and  $\text{bdd}(aA)$ , then  $\text{tp}(a/A)$  is ultraflat. In particular, this applies to types of Lascar rank one.  
• In a small simple theory there are many ultraflat types over finite sets, as the lattice of boundedly closed subsets of  $\text{bdd}(aA)$  is scattered for finitary  $aA$ .

Next we shall prove that any type internal in a family of Lascar rank one types is also flat.

**Lemma 3.10.** *If  $\text{tp}(a/A)$  is flat (ultraflat), then so is  $\text{tp}(a_0/A)$  for any  $a_0 \in \text{bdd}(aA)$ .*

*Proof:* The flat case is clear since  $\ell_\alpha^\Sigma(a_0/A) = \ell_\alpha^\Sigma(a/A) \cap \text{bdd}(Aa_0)$  for any  $\alpha > 0$ . Assume now  $\text{tp}(a/A)$  is ultraflat. Let  $B \downarrow_A a_0$  and  $b \not\downarrow_B a_0$  and let  $\mathcal{P}$  be the family of  $\text{bdd}(\emptyset)$ -conjugates of  $\text{tp}(b/B)$ . We may assume  $Bb \downarrow_{Aa_0} a$ . Thus  $a \downarrow_A B$  and  $a \not\downarrow_B b$ , so  $\text{tp}(a/A)$  is almost  $\mathcal{P}$ -internal by ultraflatness, and so is  $\text{tp}(a_0/A)$ .  $\square$

**Corollary 3.11.** *If  $\text{tp}(a/A)$  is a real type which is almost internal in a family of Lascar rank one real types, then it is flat.*

*Proof:* Assume there is some  $B \downarrow_A a$  and some tuple  $\bar{b}$  of realizations of types of Lascar rank one over  $B$  such that  $a \subseteq \text{bdd}(B, \bar{b})$ . By assumption,  $a \subseteq \text{acl}(B, \bar{b})$  where  $B$  and  $\bar{b}$  live in the imaginary universe; so, we may assume  $\bar{b}$  is a finite tuple. Moreover, we also may assume  $\bar{b}$  is an independent sequence over  $B$  since all its elements have SU-rank one, whence  $\text{tp}(\bar{b}/B)$  is flat by Lemma 3.3. Thus,  $\text{tp}(a/B)$  is flat by Lemma 3.10 and so is  $\text{tp}(a/A)$ .  $\square$



**Question 3.12.** Is every (finitary) type in a small simple theory non-orthogonal to a flat type?

**Question 3.13.** Is every type in a supersimple theory non-orthogonal to a flat type?

**Problem 3.14.** Construct a flat type which is not ultraflat.

We shall now recall the definitions and properties of  $\Sigma$ -closure from [22, Section 4.0] in the stable and [23, Section 3.5] in the simple case (where it is called  $P$ -closure: our  $\Sigma$  corresponds to the collection of all  $P$ -analysable types which are co-foreign to  $P$ ). Buechler and Hoover [2, Definition 1.2] redefine such a closure operator in the context of superstable theories and reprove some of the properties [2, Lemma 2.5].

**Definition 3.15.** The  $\Sigma$ -closure  $\Sigma\text{cl}(A)$  of a set  $A$  is the collection of all hyperimaginaries  $a$  such that  $\text{tp}(a/A)$  is  $\Sigma$ -analysable.

**Remark 3.16.** We think of  $\Sigma$  as small. We always have  $\text{bdd}(A) \subseteq \Sigma\text{cl}(A)$ ; equality holds if  $\Sigma$  is the family of all bounded types. Other useful example for  $\Sigma$  are the family of all types of  $SU$ -rank  $< \omega^\alpha$  for some ordinal  $\alpha$ , the family of all supersimple types in a properly simple theory, or the family of  $p$ -simple types of  $p$ -weight 0 for some regular type  $p$ , giving rise to Hrushovski's  $p$ -closure [10].

**Fact 3.17.** *The following are equivalent:*

- (1)  $\text{tp}(a/A)$  is foreign to  $\Sigma$ .
- (2)  $a \downarrow_A \Sigma\text{cl}(A)$ .
- (3)  $a \downarrow_A \text{dcl}(aA) \cap \Sigma\text{cl}(A)$ .
- (4)  $\text{dcl}(aA) \cap \Sigma\text{cl}(A) \subseteq \text{bdd}(A)$ .

*Proof:* The equivalence of (1), (2) and (3) is [23, Lemma 3.5.3]; the equivalence (3)  $\Leftrightarrow$  (4) is obvious.  $\square$

Unless it equals bounded closure,  $\Sigma$ -closure has the size of the monster model and thus violates the usual conventions. The equivalence (2)  $\Leftrightarrow$  (3) can be used to cut it down to some small part.

**Fact 3.18.** *Suppose  $A \downarrow_B C$ . Then  $\Sigma\text{cl}(A) \downarrow_{\Sigma\text{cl}(B)} \Sigma\text{cl}(C)$ . More precisely, for any  $A_0 \subseteq \Sigma\text{cl}(A)$  we have  $A_0 \downarrow_{B_0} \Sigma\text{cl}(C)$ , where  $B_0 = \text{dcl}(A_0 B) \cap \Sigma\text{cl}(B)$ . In particular,  $\Sigma\text{cl}(AB) \cap \Sigma\text{cl}(BC) = \Sigma\text{cl}(B)$ .*

*Proof:* This is [23, Lemma 3.5.5]; the second clause follows from Fact 3.17.  $\square$

**Lemma 3.19.** *Suppose  $C \subseteq A \cap B \cap D$  and  $AB \downarrow_C D$ .*

- (1) *If  $\Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(C)$ , then  $\Sigma\text{cl}(AD) \cap \Sigma\text{cl}(BD) = \Sigma\text{cl}(D)$ .*
- (2) *If  $\text{bdd}(A) \cap \Sigma\text{cl}(B) = \text{bdd}(C)$ , then  $\text{bdd}(AD) \cap \Sigma\text{cl}(BD) = \text{bdd}(D)$ .*

*Proof:*

- (1) This is [23, Lemma 3.5.6], which in turn adapts [16, Fact 2.4].
- (2) This is similar to (1). By Fact 15

$$\Sigma\text{cl}(BD) \downarrow_{\Sigma\text{cl}(B) \cap \text{dcl}(AB)} AB ;$$

since  $AD \downarrow_A AB$  we obtain

$$\text{Cb}(\text{bdd}(AD) \cap \Sigma\text{cl}(BD)/AB) \subseteq \text{bdd}(A) \cap \Sigma\text{cl}(B) = \text{bdd}(C).$$

Hence

$$\text{bdd}(AD) \cap \Sigma\text{cl}(BD) \downarrow_C AB$$

and by transitivity

$$\text{bdd}(AD) \cap \Sigma\text{cl}(BD) \downarrow_D ABD,$$

whence the result.  $\square$

The following lemma tells us that we can actually find a set  $C$  with  $\Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(C)$  as in Lemma 3.19(1), even though the  $\Sigma$ -closures have the size of the monster model.

**Lemma 3.20.** *Let  $C = \text{bdd}(AB) \cap \Sigma\text{cl}(A) \cap \Sigma\text{cl}(B)$ . Then  $\Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(C)$ .*

*Proof:* Consider  $e \in \Sigma\text{cl}(A) \cap \Sigma\text{cl}(B)$  and put  $f = \text{Cb}(e/AB)$ . Then  $e \downarrow_f AB$ ; since  $\text{tp}(e/A)$  is  $\Sigma$ -analysable, so is  $\text{tp}(e/f)$ , and  $e \in \Sigma\text{cl}(f)$ . If  $I$  is a Morley sequence in  $\text{tp}(e/AB)$ , then  $f \in \text{dcl}(I)$ . However, since  $e$  is  $\Sigma$ -analysable over  $A$  and over  $B$ , so is  $I$ , whence  $f$ . Hence

$$f \in \text{bdd}(AB) \cap \Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = C.$$

The result follows.  $\square$

However, for considerations such as the canonical base property, one should like to work with the first level of the  $\Sigma$ -closure rather than with the full closure operator.

**Definition 3.21.** The  $\Sigma_1$ -closure of  $A$  is given by

$$\Sigma_1\text{cl}(A) = \ell_1^\Sigma(\Sigma\text{cl}(A)/A) = \{b : \text{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Unfortunately, unless  $\text{tp}(\Sigma\text{cl}(A)/A)$  is  $\Sigma$ -flat,  $\Sigma_1$ -closure is not a closure operator, as  $\Sigma_1\text{cl}(\Sigma_1\text{cl}(A)) \supset \Sigma_1\text{cl}(A)$ .

**Lemma 3.22.** Suppose  $A \downarrow_B C$  with  $B \subseteq A \cap C$ . Then

$$\Sigma_1\text{cl}(A) \downarrow_{\Sigma_1\text{cl}(B)} C.$$

More precisely,  $\Sigma_1\text{cl}(A) \downarrow_{\Sigma_1\text{cl}(B) \cap \text{bdd}(C)} C$ .

*Proof:* Consider  $a \in \Sigma_1\text{cl}(A)$  and put  $c = \text{Cb}(Aa/C)$ . Then  $\text{tp}(c/B)$  is almost  $\Sigma$ -internal by Corollary 2.4, and  $c \in \text{bdd}(C) \cap \Sigma_1\text{cl}(B)$ .  $\square$

**Question 3.23.** If  $A \downarrow_B C$ , is  $\Sigma_1\text{cl}(A) \downarrow_{\Sigma_1\text{cl}(B)} \Sigma_1\text{cl}(C)$  ?

#### 4. $\Sigma$ -AMPLENESS AND WEAK $\Sigma$ -AMPLENESS

Let  $\Phi$  and  $\Sigma$  be  $\emptyset$ -invariant families of partial types.

**Definition 4.1.**  $\Phi$  is  $n$ - $\Sigma$ -ample if there are tuples  $a_0, \dots, a_n$ , with  $a_n$  a tuple of realizations of partial types in  $\Phi$  over some parameters  $A$ , such that

- (1)  $a_n \not\downarrow_{\Sigma\text{cl}(A)} a_0$ ;
- (2)  $a_{i+1} \downarrow_{\Sigma\text{cl}(Aa_i)} a_0 \dots a_{i-1}$  for  $1 \leq i < n$ ;
- (3)  $\Sigma\text{cl}(Aa_0 \dots a_{i-1}a_i) \cap \Sigma\text{cl}(Aa_0 \dots a_{i-1}a_{i+1}) = \Sigma\text{cl}(Aa_0 \dots a_{i-1})$  for  $0 \leq i < n$ .

**Remark 4.2.** Pillay [18] requires  $a_n \downarrow_{Aa_i} a_0 \dots a_{i-1}$  for  $1 \leq i < n$  in item (2). We follow the variant proposed by Evans and Nübling [6] which seems more natural and which implies

$$a_n \dots a_{i+1} \downarrow_{\Sigma\text{cl}(Aa_i)} a_0 \dots a_{i-1}.$$

**Remark 4.3.** [18, Remark 3.7] In Definition 4.1 one may require  $a_0, \dots, a_{n-1}$  to lie in  $\Phi^{\text{heq}}$ , replacing  $a_i$  by  $a'_i = \text{Cb}(a'_{i+1}/\Sigma\text{cl}(a_iA))$  for  $i < n$ , where  $a'_n = a_n$ .

Similarly, if  $\Sigma'$  is  $\Sigma$ -analysable and  $a_0, a_1, \dots, a_n$  witness  $n$ - $\Sigma$ -ampleness over  $A$ , the same  $a'_0, a'_1, \dots, a'_n$  as above witness  $n$ - $\Sigma'$ -ampleness over  $A' = \Sigma\text{cl}(A) \cap \text{bdd}(Aa_0 \dots a_n)$ . In particular  $n$ - $\Sigma$ -ample implies  $n$ -ample.

**Remark 4.4.** [18, Lemma 3.2 and Corollary 3.3] If  $a_0, \dots, a_n$  witness  $n$ - $\Sigma$ -ampleness over  $A$ , then  $a_n \not\downarrow_{\Sigma\text{cl}(Aa_0 \dots a_{i-1})} a_i$  for all  $i < n$ . Hence  $a_i, \dots, a_n$  witness  $(n - i)$ - $\Sigma$ -ampleness over  $Aa_0 \dots a_{i-1}$ . Thus  $n$ - $\Sigma$ -ample implies  $i$ - $\Sigma$ -ample for all  $i \leq n$ .

**Remark 4.5.** It is clear from the definition that even though  $\Phi$  might be a complete type  $p$ , if  $p$  is not  $n$ - $\Sigma$ -ample, neither is any extension of  $p$ , not only the non-forking ones.

For  $n = 1$  and  $n = 2$  there are alternative definitions of non- $n$ - $\Sigma$ -ampleness:

**Definition 4.6.** (1)  $\Phi$  is  $\Sigma$ -based if  $\text{Cb}(a/\Sigma\text{cl}(B)) \subseteq \Sigma\text{cl}(aA)$  for any tuple  $a$  of realizations of partial types in  $\Phi$  over some parameters  $A$  and any  $B \supseteq A$ .  
 (2)  $\Phi$  is  $\Sigma$ -CM-trivial if  $\text{Cb}(a/\Sigma\text{cl}(AB)) \subseteq \Sigma\text{cl}(A, \text{Cb}(a/\Sigma\text{cl}(AC)))$  for any tuple  $a$  of realizations of partial types in  $\Phi$  over some parameters  $A$  and any  $B \subseteq C$  such that  $\Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) = \Sigma\text{cl}(AB)$ .

**Lemma 4.7.** (1)  $\Phi$  is  $\Sigma$ -based if and only if  $\Phi$  is not 1- $\Sigma$ -ample.  
 (2)  $\Phi$  is  $\Sigma$ -CM-trivial if and only if  $\Phi$  is not 2- $\Sigma$ -ample.

*Proof:*

- (1) Suppose  $\Phi$  is  $\Sigma$ -based and consider  $a_0, a_1, A$  with  $\Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \Sigma\text{cl}(A)$ . Put  $a = a_1$  and  $B = Aa_0$ . By  $\Sigma$ -basedness

$$\text{Cb}(a/\Sigma\text{cl}(B)) \subseteq \Sigma\text{cl}(Aa) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(A).$$

Hence  $a \downarrow_{\Sigma\text{cl}(A)} \Sigma\text{cl}(B)$ , whence  $a_1 \downarrow_{\Sigma\text{cl}(A)} a_0$ , so  $\Phi$  is not 1- $\Sigma$ -ample.

Conversely, if  $\Phi$  is not  $\Sigma$ -based, let  $a, A, B$  be a counterexample. Put  $a_0 = \text{Cb}(a_1/\Sigma\text{cl}(B))$  and  $a_1 = a$ . Then  $a_0 \notin \Sigma\text{cl}(Aa_1)$ . Now take

$$A' = \text{bdd}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1).$$

Then  $\Sigma\text{cl}(A'a_0) \cap \Sigma\text{cl}(A'a_1) = \Sigma\text{cl}(A')$  by Lemma 3.20.

Suppose  $a_1 \downarrow_{\Sigma\text{cl}(A')} a_0$ . Since  $\Sigma\text{cl}(A') \subseteq \Sigma\text{cl}(Aa_0) \subseteq \Sigma\text{cl}(B)$  we have  $a_1 \downarrow_{a_0} \Sigma\text{cl}(A')$ . As  $a_0 = \text{Cb}(a_1/\Sigma\text{cl}(B))$ , this implies

$$a_0 \subseteq \Sigma\text{cl}(A') \subseteq \Sigma\text{cl}(Aa_1),$$

a contradiction. Hence  $a_0, a_1, A'$  witness 1- $\Sigma$ -ampleness of  $\Phi$ .

(2) Suppose  $\Phi$  is  $\Sigma$ -CM-trivial and consider  $a_0, a_1, a_2, A$  with

$$\begin{aligned}\Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1) &= \Sigma\text{cl}(A), \\ \Sigma\text{cl}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0a_2) &= \Sigma\text{cl}(Aa_0), \quad \text{and} \\ a_2 &\downarrow_{\Sigma\text{cl}(Aa_1)} a_0.\end{aligned}$$

Put  $a = a_2$ ,  $B = a_0$  and  $C = a_0a_1$ . Then

$$a_2 \downarrow_{\Sigma\text{cl}(Aa_1)} \Sigma\text{cl}(Aa_0a_1),$$

so  $\text{Cb}(a/\Sigma\text{cl}(AC)) \subseteq \Sigma\text{cl}(Aa_1)$ . Moreover

$$\Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) = \Sigma\text{cl}(AB),$$

whence by  $\Sigma$ -CM-triviality

$$\begin{aligned}\text{Cb}(a/\Sigma\text{cl}(AB)) &\subseteq \Sigma\text{cl}(A, \text{Cb}(a/AC)) \cap \Sigma\text{cl}(AB) \\ &\subseteq \Sigma\text{cl}(Aa_1) \cap \Sigma\text{cl}(Aa_0) = \Sigma\text{cl}(A).\end{aligned}$$

Hence  $a_2 \downarrow_{\Sigma\text{cl}(A)} a_0$ , so  $\Phi$  is not 2- $\Sigma$ -ample.

Conversely, if  $\Phi$  is not  $\Sigma$ -CM-trivial, let  $a, A, B, C$  be a counterexample. Put

$$a_0 = \text{Cb}(a/\Sigma\text{cl}(AB)), \quad a_1 = \text{Cb}(a/\Sigma\text{cl}(AC)), \quad a_2 = a,$$

$$A' = \text{bdd}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1).$$

Then  $a_2 \downarrow_{\Sigma\text{cl}(A'a_1)} a_0$  and  $a_0 \notin \Sigma\text{cl}(Aa_1)$ ; by Lemma 3.20

$$\Sigma\text{cl}(A'a_0) \cap \Sigma\text{cl}(A'a_1) = \Sigma\text{cl}(A').$$

Moreover,  $a_2 \downarrow_{a_0} \Sigma\text{cl}(AB)$  implies

$$\Sigma\text{cl}(A'a_0a_2) \downarrow_{\Sigma\text{cl}(A'a_0)} \Sigma\text{cl}(AB).$$

Thus

$$\begin{aligned}\Sigma\text{cl}(A'a_0a_2) \cap \Sigma\text{cl}(A'a_0a_1) &\subseteq \Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) \\ &= \Sigma\text{cl}(AB) \cap \Sigma\text{cl}(A'a_0a_2) = \Sigma\text{cl}(A'a_0).\end{aligned}$$

Suppose  $a_2 \downarrow_{\Sigma\text{cl}(A')} a_0$ . As above  $a_2 \downarrow_{a_0} \Sigma\text{cl}(A')$  and

$$a_0 \subseteq \Sigma\text{cl}(A') \subseteq \Sigma\text{cl}(Aa_1),$$

a contradiction. Hence  $a_0, a_1, a_2, A'$  witness 2- $\Sigma$ -ampleness of  $\Phi$ .  $\square$

In our definition of  $\Sigma$ -ampleness, we only consider the type of  $a_n$  over a  $\Sigma$ -closed set, namely  $\Sigma\text{cl}(A)$ . This seems natural since the idea of  $\Sigma$ -closure is to work *modulo*  $\Sigma$ . However, sometimes one needs a stronger notion which takes care of all types. Let us first look at  $n = 1$  and  $n = 2$ .

**Definition 4.8.** •  $\Phi$  is *strongly  $\Sigma$ -based* if  $\text{Cb}(a/B) \subseteq \Sigma\text{cl}(aA)$  for any tuple  $a$  of realizations of partial types in  $\Phi$  over some  $A$  and any  $B \supseteq A$ .  
 •  $\Phi$  is *strongly  $\Sigma$ -CM-trivial* if  $\text{Cb}(a/AB) \subseteq \Sigma\text{cl}(A, \text{Cb}(a/\Sigma\text{cl}(AC)))$  for any tuple  $a$  of realizations of partial types in  $\Phi$  over some  $A$  and any  $B \subseteq C$  with  $\Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) = \Sigma\text{cl}(AB)$ .

**Remark 4.9.**  $\text{Cb}(a/\Sigma\text{cl}(B)) \subseteq \text{bdd}(\text{Cb}(a/B), a) \cap \Sigma\text{cl}(\text{Cb}(a/B))$ .

*Proof:* By Fact 3.18 the independence  $a \downarrow_{\text{Cb}(a/B)} B$  implies

$$\begin{array}{ccc} a & \downarrow & \Sigma\text{cl}(B). \\ \text{dcl}(a, \text{Cb}(a/B)) \cap \Sigma\text{cl}(\text{Cb}(a/B)) & & \end{array}$$

The result follows.  $\square$

**Conjecture.**  $\text{Cb}(a/B) \subseteq \Sigma\text{cl}(\text{Cb}(a/\Sigma\text{cl}(B)))$ .

If this conjecture were true, strong and ordinary  $\Sigma$ -basedness and  $\Sigma$ -CM-triviality would obviously coincide. Since we have not been able to show this, we weaken our definition of ampleness.

**Definition 4.10.**  $\Phi$  is *weakly  $n$ - $\Sigma$ -ample* if there are tuples  $a_0, \dots, a_n$ , where  $a_n$  is a tuple of realizations of partial types in  $\Phi$  over  $A$ , with

- (1)  $a_n \not\downarrow_A a_0$ .
- (2)  $a_{i+1} \downarrow_{\Sigma\text{cl}(Aa_i)} a_0 \dots a_{i-1}$  for  $1 \leq i < n$ .
- (3)  $\text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \text{bdd}(A)$ .
- (4)  $\Sigma\text{cl}(Aa_0 \dots a_{i-1}a_i) \cap \Sigma\text{cl}(Aa_0 \dots a_{i-1}a_{i+1}) = \Sigma\text{cl}(Aa_0 \dots a_{i-1})$  for  $1 \leq i < n$ .

Note that (3) implies that  $\text{tp}(a_0/A)$  is foreign to  $\Sigma$  by Fact 3.17. If  $\Sigma$  is the family of bounded partial types, then weak and ordinary  $n$ - $\Sigma$ -ampleness just equal  $n$ -ampleness.

**Remark 4.11.** A  $n$ - $\Sigma$ -ample family of types is weakly  $n$ - $\Sigma$ -ample. If  $\Sigma'$  is  $\Sigma$ -analysable, then a weakly  $n$ - $\Sigma$ -ample family is weakly  $n$ - $\Sigma'$ -ample, and in particular  $n$ -ample.

*Proof:* As in Remark 4.3, if  $a_0, \dots, a_n$  witness  $n$ - $\Sigma$ -ampleness over some set  $A$ , put  $a'_n = a_n$  and  $a'_i = \text{Cb}(a'_{i+1}/\Sigma\text{cl}(Aa_i))$ . Then  $a'_0, \dots, a'_n$  witness weak  $n$ - $\Sigma$ -ampleness over  $A' = \text{bdd}(Aa'_0) \cap \Sigma\text{cl}(Aa'_1)$ . Similarly, if  $a_0, \dots, a_n$  witness weak  $n$ - $\Sigma$ -ampleness over  $A$ , then  $a'_0, \dots, a'_n$  witness weak  $n$ - $\Sigma'$ -ampleness over  $A' = \text{bdd}(Aa'_0) \cap \Sigma'\text{cl}(Aa'_1)$ , as condition (4) follows from

$$a'_i \downarrow_{Aa'_0 \dots a'_{i-1}} \Sigma\text{cl}(Aa_0 \dots a_{i-1})$$

and Lemma 3.18 applied with respect to  $\Sigma'$ -closure.  $\square$

**Lemma 4.12.** (1)  $\Phi$  is strongly  $\Sigma$ -based iff  $\Phi$  is not weakly 1- $\Sigma$ -ample.

(2)  $\Phi$  is strongly  $\Sigma$ -CM-trivial iff  $\Phi$  is not weakly 2- $\Sigma$ -ample.

*Proof:* This is similar to the proof of Lemma 4.7, so we shall be concise.

(1) Suppose  $\Phi$  is strongly  $\Sigma$ -based and consider  $a_0, a_1, A$  with

$$\text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \text{bdd}(A).$$

Put  $a = a_1$  and  $B = Aa_0$ . By strong  $\Sigma$ -basedness

$$\text{Cb}(a/B) \subseteq \Sigma\text{cl}(Aa) \cap \text{bdd}(B) = \text{bdd}(A),$$

whence  $a_1 \downarrow_A a_0$ , so  $\Phi$  is not weakly 1- $\Sigma$ -ample.

Conversely, if  $\Phi$  is not strongly  $\Sigma$ -based, let  $a, A, B$  be a counterexample. Put  $a_0 = \text{Cb}(a_1/B)$  and  $a_1 = a$ . Then  $a_0 \notin \Sigma\text{cl}(Aa_1)$ . Now take  $A' = \text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1)$ . Clearly  $A' = \text{bdd}(A'a_0) \cap \Sigma\text{cl}(A'a_1)$ . Suppose  $a_1 \downarrow_{A'} a_0$ . Since  $a_0 = \text{Cb}(a_1/B)$  implies  $a_1 \downarrow_{a_0} A'$ , we obtain

$$a_0 \subseteq \text{bdd}(A') \subseteq \Sigma\text{cl}(Aa_1),$$

a contradiction. Hence  $a_0, a_1, A'$  witness weak 1- $\Sigma$ -ampleness of  $\Phi$ .

(2) Suppose  $\Phi$  is strongly  $\Sigma$ -CM-trivial and consider  $a_0, a_1, a_2, A$  with

$$\begin{aligned} \text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) &= \text{bdd}(A), \\ \Sigma\text{cl}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0a_2) &= \Sigma\text{cl}(Aa_0), \quad \text{and} \end{aligned}$$

$$a_2 \downarrow_{\Sigma\text{cl}(Aa_1)} a_0.$$

Put  $a = a_2$ ,  $B = a_0$  and  $C = a_0a_1$ . Then  $\text{Cb}(a/\Sigma\text{cl}(AC)) \subseteq \Sigma\text{cl}(Aa_1)$  by Fact 3.18. Moreover

$$\Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) = \Sigma\text{cl}(AB),$$

whence by strong  $\Sigma$ -CM-triviality

$$\begin{aligned} \text{Cb}(a/AB) &\subseteq \Sigma\text{cl}(A, \text{Cb}(a/AC)) \cap \text{bdd}(AB) \\ &\subseteq \Sigma\text{cl}(Aa_1) \cap \text{bdd}(Aa_0) = \text{bdd}(A). \end{aligned}$$

Hence  $a_2 \downarrow_A a_0$ , so  $\Phi$  is not 2- $\Sigma$ -ample.

Conversely, if  $\Phi$  is not strongly  $\Sigma$ -CM-trivial, let  $a, A, B, C$  be a counterexample. Put

$$a_0 = \text{Cb}(a/AB), \quad a_1 = \text{Cb}(a/\Sigma\text{cl}(AC)), \quad a_2 = a,$$

$$A' = \text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1).$$

Then  $a_2 \downarrow_{\Sigma\text{cl}(A'a_1)} a_0$  and  $a_0 \notin \Sigma\text{cl}(Aa_1)$ ; moreover

$$\text{bdd}(A'a_0) \cap \Sigma\text{cl}(A'a_1) = \text{bdd}(A').$$

Now  $a_2 \downarrow_{a_0} AB$  implies

$$\Sigma\text{cl}(A'a_0a_2) \downarrow_{\Sigma\text{cl}(A'a_0)} \Sigma\text{cl}(AB)$$

by Fact 3.18, whence as before

$$\Sigma\text{cl}(A'a_0a_2) \cap \Sigma\text{cl}(A'a_0a_1) = \Sigma\text{cl}(A'a_0).$$

Suppose  $a_2 \downarrow_{A'} a_0$ . Then  $a_0 \subseteq \text{bdd}(A') \subseteq \Sigma\text{cl}(Aa_1)$ , a contradiction. Hence  $a_0, a_1, a_2, A'$  witness weak 2- $\Sigma$ -ampleness of  $\Phi$ .  $\square$

**Lemma 4.13.** *If  $\Phi$  is not (weakly)  $n$ - $\Sigma$ -ample, neither is the family of  $\emptyset$ -conjugates of  $\text{tp}(a/A)$  for any  $a \in \Sigma\text{cl}(\bar{a}A)$ , where  $\bar{a}$  is a tuple of realizations of partial types in  $\Phi$  over  $A$ .*

*Proof:* Suppose the family of  $\emptyset$ -conjugates of  $\text{tp}(a/A)$  is  $n$ - $\Sigma$ -ample, as witnessed by  $a_0, \dots, a_n$  over some parameters  $B$ . There is a tuple  $\bar{a}$  of realizations of partial types in  $\Phi$  over some  $\emptyset$ -conjugates of  $A$  inside  $B$  such that  $a_n \in \Sigma\text{cl}(\bar{a}B)$ ; we may choose it such that

$$\bar{a} \downarrow_{a_n B} a_0 \dots a_{n-1}.$$

Then  $\bar{a} \downarrow_{a_{n-1}a_n B} a_0 \dots a_{n-2}$ , and hence

$$\bar{a} \downarrow_{\Sigma\text{cl}(a_{n-1}a_n B)} a_0 \dots a_{n-2}.$$

As  $a_n \downarrow_{\Sigma\text{cl}(a_{n-1}B)} a_0 \dots a_{n-2}$  implies

$$\Sigma\text{cl}(a_{n-1}a_n B) \downarrow_{\Sigma\text{cl}(a_{n-1}B)} a_0 \dots a_{n-2}$$



by Fact 3.18, we get

$$\bar{a} \downarrow_{\Sigma\text{cl}(a_{n-1}B)} a_0 \dots a_{n-2}.$$

We also have  $\bar{a} \downarrow_{a_0 \dots a_{n-2} a_n B} a_{n-1}$ , whence

$$\Sigma\text{cl}(a_0 \dots a_{n-2} \bar{a} B) \downarrow_{\Sigma\text{cl}(a_0 \dots a_{n-2} a_n B)} \Sigma\text{cl}(a_0 \dots a_{n-2} a_{n-1} B);$$

since  $\Sigma$ -closure is boundedly closed,

$$\begin{aligned} \Sigma\text{cl}(a_0 \dots a_{n-2} \bar{a} B) \cap \Sigma\text{cl}(a_0 \dots a_{n-2} a_{n-1} B) \\ \subseteq \Sigma\text{cl}(a_0 \dots a_{n-2} a_n B) \cap \Sigma\text{cl}(a_0 \dots a_{n-2} a_{n-1} B) \\ = \Sigma\text{cl}(a_0 \dots a_{n-2} B). \end{aligned}$$

Finally,  $\bar{a} \downarrow_{\Sigma\text{cl}(B)} a_0$  would imply  $\Sigma\text{cl}(\bar{a} B) \downarrow_{\Sigma\text{cl}(B)} a_0$  by Fact 3.18, and hence  $a_n \downarrow_{\Sigma\text{cl}(B)} a_0$ , a contradiction. Thus  $\bar{a} \not\downarrow_{\Sigma\text{cl}(B)} a_0$ , and  $a_0, \dots, a_{n-1}, \bar{a}$  witness  $n$ - $\Sigma$ -ampleness of  $\Phi$  over  $B$ , a contradiction.

Now suppose  $a_0, \dots, a_n$  witness weak  $n$ - $\Sigma$ -ampleness over  $B$ , and choose  $\bar{a}$  as before. Then (2) and (4) from the definition follow as above. Suppose  $\bar{a} \downarrow_B a_0$ . Since  $\text{tp}(a_0/B)$  is foreign to  $\Sigma$ , so is  $\text{tp}(a_0/B\bar{a})$ . Then  $a_0 \downarrow_{B\bar{a}} \Sigma\text{cl}(B\bar{a})$  by Fact 3.17, whence  $a_0 \downarrow_B a_n$ , a contradiction. Thus  $\bar{a} \not\downarrow_B a_0$  and we have (1).

Finally (3) is trivial unless  $n = 1$ . In that case  $\bar{a} \downarrow_{Ba_1} a_0$  implies  $\Sigma\text{cl}(B\bar{a}) \downarrow_{\Sigma\text{cl}(Ba_1)} \text{bdd}(Ba_0)$ , whence

$$\text{bdd}(Ba_0) \cap \Sigma\text{cl}(B\bar{a}) \subseteq \text{bdd}(Ba_0) \cap \Sigma\text{cl}(Ba_1) = \text{bdd}(B).$$

So  $a_0, \dots, a_{n-1}, \bar{a}$  witness weak  $n$ - $\Sigma$ -ampleness of  $\Phi$  over  $B$ , again a contradiction.  $\square$

**Lemma 4.14.** *Suppose  $B \downarrow_A a_0 \dots a_n$ . If  $a_0, \dots, a_n$  witness (weak)  $n$ - $\Sigma$ -ampleness over  $A$ , they witness (weak)  $n$ - $\Sigma$ -ampleness over  $B$ .*

*Proof:* Clearly  $B \downarrow_{a_0 \dots a_{i-1} A} a_0 \dots a_{i+1} A$ , so Lemma 3.19 yields

$$\Sigma\text{cl}(Ba_0 \dots a_{i-1} a_i) \cap \Sigma\text{cl}(Ba_0 \dots a_{i-1} a_{i+1}) = \Sigma\text{cl}(Ba_0 \dots a_{i-1})$$

for  $i < n$  in the ordinary and for  $1 \leq i < n$  in the weak case. In the latter, since  $\text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \text{bdd}(A)$  by assumption, Lemma 3.19 also gives

$$\text{bdd}(Ba_0) \cap \Sigma\text{cl}(Ba_1) = \text{bdd}(B).$$

Next,  $a_{i+1} \downarrow_{Aa_0 \dots a_i} B$ , whence  $a_{i+1} \downarrow_{\Sigma \text{cl}(Aa_0 \dots a_i)} \Sigma \text{cl}(Ba_i)$  by Lemma 3.18. Now  $a_{i+1} \downarrow_{\Sigma \text{cl}(Aa_i)} a_0 \dots a_{i-1}$  implies  $a_{i+1} \downarrow_{\Sigma \text{cl}(Aa_i)} \Sigma \text{cl}(Aa_0 \dots a_i)$ , whence

$$a_{i+1} \downarrow_{\Sigma \text{cl}(Ba_i)} a_0 \dots a_{i-1}$$

for  $1 \leq i < n$  by transitivity.

Finally,  $a_n \downarrow_{\Sigma \text{cl}(A)} \Sigma \text{cl}(B)$  by Fact 3.18, so  $a_n \downarrow_{\Sigma \text{cl}(B)} a_0$  would imply  $a_n \downarrow_{\Sigma \text{cl}(A)} a_0$ , a contradiction. Hence  $a_n \not\downarrow_{\Sigma \text{cl}(B)} a_0$ . In the weak case,  $a_n \downarrow_A B$  and  $a_n \not\downarrow_A a_0$  yield directly  $a_n \not\downarrow_B a_0$ .  $\square$

**Lemma 4.15.** *Let  $\Psi$  be an  $\emptyset$ -invariant family of types. If  $\Phi$  and  $\Psi$  are not (weakly)  $n$ - $\Sigma$ -ample, neither is  $\Phi \cup \Psi$ .*

*Proof:* Suppose  $\Phi \cup \Psi$  is weakly  $n$ - $\Sigma$ -ample, as witnessed by  $a_0, \dots, a_n = bc$  over some parameters  $A$ , where  $b$  and  $c$  are tuples of realizations of partial types in  $\Phi$  and  $\Psi$ , respectively. As  $\Psi$  is not  $n$ - $\Sigma$ -ample, we must have  $c \not\downarrow_A a_0$ . Put  $a'_0 = \text{Cb}(bc/a_0A)$ . Then  $\text{tp}(a'_0/A)$  is internal in  $\text{tp}(b/A)$  by Corollary 2.4. Put

$$a'_n = \text{Cb}(a'_0/a_nA).$$

Then  $\text{tp}(a'_n/A)$  is  $\text{tp}(a'_0/A)$ -internal and hence  $\text{tp}(b/A)$ -internal. Note that  $a_n \not\downarrow_A a_0$  implies  $a_n \not\downarrow_A a'_0$ , whence

$$a'_n \not\downarrow_A a'_0 \quad \text{and} \quad a'_n \not\downarrow_A a_0.$$

Moreover  $a'_n \in \text{bdd}(Aa_n)$ , so  $a_0, \dots, a_{n-1}, a'_n$  witness weak  $n$ - $\Sigma$ -ampleness over  $A$ .

As  $\text{tp}(a'_n/A)$  is  $\text{tp}(b/A)$ -internal, there is  $B \downarrow_A a'_n$  and a tuple  $\bar{b}$  of realizations of  $\text{tp}(b/A)$  with  $a'_n \in \text{dcl}(B\bar{b})$ . We may assume

$$B \downarrow_{a'_n} Aa_0 \dots a_{n-1},$$

whence  $B \downarrow_A a_0 \dots a_{n-1}a'_n$ . Hence  $a_0, \dots, a_{n-1}, a'_n$  witness weak  $n$ - $\Sigma$ -ampleness over  $B$  by Lemma 4.14. As  $a'_n \in \text{dcl}(B\bar{b})$ , this contradicts non weak  $n$ - $\Sigma$ -ampleness of  $\Phi$  by Lemma 4.13.

The proof in the ordinary case is analogous, replacing  $A$  by  $\Sigma \text{cl}(A)$ .  $\square$

**Corollary 4.16.** *For  $i < \alpha$  let  $\Phi_i$  be an  $\emptyset$ -invariant family of partial types. If  $\Phi_i$  is not (weakly)  $n$ - $\Sigma$ -ample for all  $i < \alpha$ , neither is  $\bigcup_{i < \alpha} \Phi_i$ .*

*Proof:* This just follows from the local character of forking and Lemma 4.15.  $\square$

**Lemma 4.17.** *If the family of  $\emptyset$ -conjugates of  $\text{tp}(a/A)$  is not (weakly)  $n$ - $\Sigma$ -ample and  $a \perp A$ , then  $\text{tp}(a)$  is not (weakly)  $n$ - $\Sigma$ -ample.*

*Proof:* Suppose  $\text{tp}(a)$  is (weakly)  $n$ - $\Sigma$ -ample, as witnessed by  $a_0, \dots, a_n$  over some parameters  $B$ , where  $a_n = (b_i : i < k)$  is a tuple of realizations of  $\text{tp}(a)$ . For each  $i < k$  choose  $B_i \perp_{b_i} (B, a_0 \dots a_n, B_j : j < i)$  with  $B_i b_i \equiv Aa$ . Then  $B_i \perp b_i$ , whence  $(B_i : i < k) \perp Ba_0 \dots a_n$ . Then  $a_0, \dots, a_n$  witness (weak)  $n$ - $\Sigma$ -ampleness over  $(B, B_i : i < k)$  by Lemma 4.14, a contradiction, since  $\text{tp}(b_i/B_i)$  is an  $\emptyset$ -conjugate of  $\text{tp}(a/A)$  for all  $i < k$ .  $\square$

**Remark 4.18.** In fact, in the above Lemma it suffices to merely assume that the single type  $\text{tp}(a/A)$  is not (weakly)  $n$ - $\Sigma$ -ample in the theory  $T(A)$ , using Corollary 4.16. It follows that ampleness is preserved under adding and forgetting parameters.

**Corollary 4.19.** *Let  $\Psi$  be an  $\emptyset$ -invariant family of types. If  $\Psi$  is  $\Phi$ -internal and  $\Phi$  is not (weakly)  $n$ - $\Sigma$ -ample, neither is  $\Psi$ .*

*Proof:* Immediate from Lemmas 4.13 and 4.17.  $\square$

**Theorem 4.20.** *Let  $\Psi$  be an  $\emptyset$ -invariant family of types. If  $\Psi$  is  $\Phi$ -analysable and  $\Phi$  is not (weakly)  $n$ - $\Sigma$ -ample, neither is  $\Psi$ .*

*Proof:* Suppose  $\Psi$  is (weakly)  $n$ - $\Sigma$ -ample, as witnessed by  $a_0, \dots, a_n$  over some parameters  $A$ , where  $a_n$  is a tuple of realizations of  $\Psi$ . Put  $a'_n = \ell_1^\Phi(a_n/\Sigma\text{cl}(A) \cap \text{bdd}(Aa_n))$ . Then  $a_n$  and  $a'_n$  are domination-equivalent over  $\Sigma\text{cl}(A) \cap \text{bdd}(Aa_n)$  by Theorem 3.5; moreover  $a_n$  and hence  $a'_n$  are independent of  $\Sigma\text{cl}(A)$  over  $\Sigma\text{cl}(A) \cap \text{bdd}(Aa_n)$  by Fact 3.17, so  $a_n$  and  $a'_n$  are domination-equivalent over  $\Sigma\text{cl}(A)$ . Thus  $a_0, \dots, a'_n$  witness non- $\Sigma$ -ampleness over  $A$ , contradicting Corollary 4.19.

For the weak case we put  $a'_n = \ell_1^\Phi(a_n/A)$ . So  $a_n$  and  $a'_n$  are domination-equivalent over  $A$ , whence  $a'_n \not\perp_A a_0$ . Thus  $a_0, \dots, a'_n$  witness weak non- $\Sigma$ -ampleness over  $A$ , contradicting again Corollary 4.19.  $\square$

## 5. ANALYSABILITY OF CANONICAL BASES

As an immediate Corollary to Theorem 4.20, we obtain the following:

**Theorem 5.1.** *Suppose every type in  $T$  is non-orthogonal to a regular type, and let  $\Sigma$  be the family of all  $n$ -ample regular types. Then  $T$  is non weakly  $n$ - $\Sigma$ -ample.*

*Proof:* A non  $n$ -ample type is not weakly  $\Sigma$ -ample by Remark 4.11. So all regular types are not weakly  $n$ - $\Sigma$ -ample. But every type is analysable in regular types by the non-orthogonality hypothesis.  $\square$

**Corollary 5.2.** *Suppose every type in  $T$  is non-orthogonal to a regular type. Then  $\text{tp}(\text{Cb}(a/b)/a)$  is analysable in the family of all non one-based regular types, for all  $a, b$ .*

*Proof:* This is just Theorem 5.1 for  $n = 1$ .  $\square$

Note that a forking extension of a non one-based regular type of infinite rank may be one-based.

**Remark 5.3.** In fact, the proof shows more. For every type  $p$  let  $\Sigma(p)$  be the collection of types in  $\Sigma$  not foreign to  $p$ . Then  $\text{tp}(\text{Cb}(a/b)/a)$  is analysable in  $\Sigma(\text{tp}(\text{Cb}(a/b)))$ . In particular, if  $\text{tp}(a)$  or  $\text{tp}(b)$  has rank less than  $\omega^\alpha$ , so does  $\text{tp}(\text{Cb}(a/b))$ . Hence  $\text{tp}(\text{Cb}(a/b)/a)$  is analysable in the family of all non one-based regular types of rank less than  $\omega^\alpha$ .

Corollary 5.2 is due to Zoé Chatzidakis for types of finite SU-rank in simple theories [5, Proposition 1.10]. In fact, she even obtains  $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$  to be analysable in the family of non one-based types of rank 1, and to decompose in components, each of which is analysable in a non-orthogonality class of regular types. In infinite rank, one has to work modulo types of smaller rank. So let  $\Sigma_\alpha$  be the collection of all partial types of SU-rank  $< \omega^\alpha$ , and  $\mathcal{P}_\alpha$  be the family of non  $\Sigma_\alpha$ -based types of SU-rank  $\omega^\alpha$ .

**Theorem 5.4.** *Let  $b = \text{Cb}(a/\Sigma_\alpha \text{cl}(b))$  be such that  $\text{SU}(b) < \omega^{\alpha+1}$  for some ordinal  $\alpha$  and let  $A = \Sigma_\alpha \text{cl}(b) \cap \Sigma_\alpha \text{cl}(a)$ . Then  $\text{tp}(b/A)$  is  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable.*

*Proof:* Firstly, if  $a \in \Sigma_\alpha \text{cl}(b)$  then  $a = b \in A$ . Similarly, if  $b \in \Sigma_\alpha \text{cl}(a)$  then  $b \in A$ ; in both cases  $\text{tp}(b/A)$  is trivially  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable. Hence we may assume  $a \notin \Sigma_\alpha \text{cl}(b)$  and  $b \notin \Sigma_\alpha \text{cl}(a)$ .

Suppose towards a contradiction that the result is false and consider a counterexample  $a, b$  with  $\text{SU}(b)$  minimal modulo  $\omega^\alpha$  and then  $\text{SU}(b/\Sigma_\alpha \text{cl}(a))$  being maximal modulo  $\omega^\alpha$ . Note that this implies

$$\omega^\alpha \leq \text{SU}(b/a) \leq \text{SU}(b/A) \leq \text{SU}(b) < \omega^{\alpha+1}.$$

Clearly (after adding parameters) we may assume  $A = \Sigma_\alpha \text{cl}(\emptyset)$ . Then for any  $c$  the type  $\text{tp}(c)$  is  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable iff  $\text{tp}(c/A)$  is.

**Claim.** *We may assume  $a = \text{Cb}(b/\Sigma_\alpha \text{cl}(a))$ .*

*Proof of Claim:* Put  $\tilde{a} = \text{Cb}(b/\Sigma_\alpha \text{cl}(a))$  and  $\tilde{b} = \text{Cb}(\tilde{a}/\Sigma_\alpha \text{cl}(b))$ . Then  $\tilde{a} \in \Sigma_\alpha \text{cl}(a)$  and  $a \downarrow_{\tilde{a}} b$ . Hence  $\Sigma_\alpha \text{cl}(b) = \Sigma_\alpha \text{cl}(\tilde{b})$  by [23, Lemma 3.5.8], and  $\text{tp}(\tilde{b})$  is not  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable either. Thus the pair  $\tilde{a}, \tilde{b}$  also forms a counterexample. Moreover,  $\text{SU}(b)$  equals  $\text{SU}(\tilde{b})$  modulo  $\omega^\alpha$  and  $\text{SU}(b/\Sigma_\alpha \text{cl}(a)) = \text{SU}(b/\Sigma_\alpha \text{cl}(\tilde{a}))$  equals  $\text{SU}(\tilde{b}/\Sigma_\alpha \text{cl}(\tilde{a}))$  modulo  $\omega^\alpha$ .  $\square$

Since  $a$  is definable over a finite part of a Morley sequence in  $\text{lstp}(b/a)$  by supersimplicity of  $\text{tp}(b)$ , we see that  $\text{SU}(a) < \omega^{\alpha+1}$ . On the other hand,  $a \notin \Sigma_\alpha \text{cl}(b)$  implies  $\text{SU}(a/b) \geq \omega^\alpha$ .

Let  $\hat{a} \subseteq \text{bdd}(a)$  and  $\hat{b} \subseteq \text{bdd}(b)$  be maximal  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable. Then  $a \notin \Sigma_\alpha \text{cl}(\hat{a})$  and  $b \notin \Sigma_\alpha \text{cl}(\hat{b})$ , and  $\text{tp}(a/\hat{a})$  and  $\text{tp}(b/\hat{b})$  are foreign to  $\Sigma_\alpha \cup \mathcal{P}_\alpha$ . Since  $\text{Cb}(\hat{a}/b)$  and  $\text{Cb}(\hat{b}/a)$  are  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable, we obtain

$$a \downarrow_{\hat{a}} \hat{b} \quad \text{and} \quad b \downarrow_{\hat{b}} \hat{a}.$$

**Claim.**  $\text{tp}(b/\hat{b})$  and  $\text{tp}(a/\hat{a})$  are both  $\Sigma_\alpha$ -based.

*Proof of Claim:* Let  $\Phi$  be the family of  $\Sigma_\alpha$ -based types of SU-rank  $\omega^\alpha$ . Then  $\text{tp}(a/\hat{a})$  is  $(\Sigma_\alpha \cup \mathcal{P}_\alpha \cup \Phi)$ -analysable, but foreign to  $\Sigma_\alpha \cup \mathcal{P}_\alpha$ . Put  $a_0 = \ell_1^\Phi(a/\hat{a})$  and  $b_0 = \ell_1^\Phi(b/\hat{b})$ . Then  $a \sqsubseteq_{\hat{a}} a_0$  and  $b \sqsubseteq_{\hat{b}} b_0$  by Lemma 3.5(3); as  $a \downarrow_{\hat{a}} \hat{b}$  and  $b \downarrow_{\hat{b}} \hat{a}$  we even have  $a \sqsubseteq_{\hat{a}\hat{b}} a_0$  and  $b \sqsubseteq_{\hat{a}\hat{b}} b_0$ . Since  $a \not\downarrow_{\hat{a}\hat{b}} b$  we obtain  $a_0 \not\downarrow_{\hat{a}\hat{b}} b_0$ . Moreover,  $\text{tp}(a_0/\hat{a})$  and  $\text{tp}(b_0/\hat{b})$  are  $\Sigma_\alpha$ -based by Theorem 4.20 (or [24, Theorem 11]).

On the other hand, as  $a_0 \not\downarrow_{\hat{b}} b_0$ , we see that  $b' = \text{Cb}(a_0/\Sigma_\alpha \text{cl}(b_0))$  is not contained in  $\hat{b}$  and hence is not  $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable. So  $a_0, b'$  is another counterexample; by minimality of SU-rank  $b$  and  $b'$  have the same SU-rank modulo  $\omega^\alpha$ , whence  $b \in \Sigma_\alpha \text{cl}(b_0)$ . Hence  $\text{tp}(b/\hat{b})$  is  $\Sigma_\alpha$ -based, as is  $\text{tp}(a/\hat{a})$  since  $a = \text{Cb}(b/a)$  and  $a \downarrow_{\hat{a}} \hat{b}$ .  $\square$

**Claim.**  $\Sigma_\alpha \text{cl}(a, \hat{b}) = \Sigma_\alpha \text{cl}(b, \hat{a}) = \Sigma_\alpha \text{cl}(a, b)$ .

*Proof of Claim:* As  $\text{tp}(a/\hat{a})$  is  $\Sigma_\alpha$ -based, we have

$$a \downarrow_{\Sigma_\alpha \text{cl}(a) \cap \Sigma_\alpha \text{cl}(\hat{a}b)} \hat{a}b,$$

whence

$$\Sigma_\alpha \text{cl}(a) \downarrow_{\Sigma_\alpha \text{cl}(a) \cap \Sigma_\alpha \text{cl}(\hat{a}b)} b$$

by Fact 3.18. Thus  $a = \text{Cb}(b/\Sigma_\alpha \text{cl}(a)) \in \Sigma_\alpha \text{cl}(\hat{a}b)$ . Similarly  $b \in \Sigma_\alpha \text{cl}(\hat{b}a)$ .  $\square$

Let now  $(b)^\frown(b_j : j < \omega)$  be a Morley sequence in  $\text{tp}(b/a)$  and let  $\hat{b}_j$  represent the part of  $b_j$  corresponding to  $\hat{b}$ . Then  $(\hat{b}_j : j < \omega)$  is a Morley sequence in  $\text{tp}(\hat{b}/\hat{a})$  since  $a \perp_{\hat{a}} \hat{b}$ . As  $\text{SU}(\hat{b}) < \infty$  there is some minimal  $m < \omega$  such that  $\hat{a} = \text{Cb}(\hat{b}/\hat{a}) \in \Sigma_{\alpha\text{cl}}(\hat{b}, \hat{b}_j : j < m)$ . Then  $m > 0$ , as otherwise  $\Sigma_{\alpha\text{cl}}(b) = \Sigma_{\alpha\text{cl}}(\hat{a}, b) \ni a$ , which is impossible. Moreover,  $a \in \Sigma_{\alpha\text{cl}}(\hat{a}, b_j)$  for all  $j < m$  by invariance and hence,  $a \in \Sigma_{\alpha\text{cl}}(\hat{b}, b_j : j < m)$ .

Put  $b' = \text{Cb}(b_j : j < m/\Sigma_{\alpha\text{cl}}(b))$ . Then  $(b_j : j < m) \perp_{b'\hat{b}} \Sigma_{\alpha\text{cl}}(b)$ , so by Fact 3.18

$$\Sigma_{\alpha\text{cl}}(\hat{b}, b_j : j < m) \perp_{\Sigma_{\alpha\text{cl}}(b', \hat{b})} \Sigma_{\alpha\text{cl}}(b).$$

Then  $a \perp_{\Sigma_{\alpha\text{cl}}(b', \hat{b})} \Sigma_{\alpha\text{cl}}(b)$ , so  $b = \text{Cb}(a/\Sigma_{\alpha\text{cl}}(b)) \in \Sigma_{\alpha\text{cl}}(b', \hat{b})$ . As  $b \notin \Sigma_{\alpha\text{cl}}(\hat{b})$  we obtain  $b' \notin \Sigma_{\alpha\text{cl}}(\hat{b})$ .

**Claim.**  $\text{tp}(b'/\Sigma_{\alpha\text{cl}}(b') \cap \Sigma_{\alpha\text{cl}}(b_j : j < m))$  is not  $(\Sigma_{\alpha} \cup \mathcal{P}_{\alpha})$ -analysable.

*Proof of Claim:* Note first that  $(b_j : j < m) \perp_a b$  implies

$$\Sigma_{\alpha\text{cl}}(b_j : j < m) \perp_{\Sigma_{\alpha\text{cl}}(a)} \Sigma_{\alpha\text{cl}}(b)$$

by Fact 3.18, whence

$$\Sigma_{\alpha\text{cl}}(b') \cap \Sigma_{\alpha\text{cl}}(b_j : j < m) \subseteq \Sigma_{\alpha\text{cl}}(b) \cap \Sigma_{\alpha\text{cl}}(a) = \Sigma_{\alpha\text{cl}}(\emptyset).$$

As  $b \in \Sigma_{\alpha\text{cl}}(b', \hat{b})$  and  $\text{tp}(b/\hat{b})$  is not  $(\Sigma_{\alpha} \cup \mathcal{P}_{\alpha})$ -analysable, neither is  $\text{tp}(b'/\hat{b})$ , nor *a fortiori*  $\text{tp}(b'/\Sigma_{\alpha\text{cl}}(\emptyset))$ .  $\square$

As  $b' = \text{Cb}(b_j : j < m/\Sigma_{\alpha\text{cl}}(b'))$ , the pair  $(b_j : j < m), b'$  forms another counterexample. By minimality  $\text{SU}(b)$  equals  $\text{SU}(b')$  modulo  $\omega^{\alpha}$ , which implies  $\Sigma_{\alpha\text{cl}}(b) = \Sigma_{\alpha\text{cl}}(b')$ .

As  $\text{tp}(b_j/\hat{b}_j)$  is foreign to  $\Sigma_{\alpha} \cup \mathcal{P}_{\alpha}$  and  $\hat{b}$  is  $(\Sigma_{\alpha} \cup \mathcal{P}_{\alpha})$ -analysable, we obtain  $\hat{b} \perp_{(\hat{b}_j : j < m)} (b_j : j < m)$  and hence by Fact 3.18

$$\hat{b} \perp_{\Sigma_{\alpha\text{cl}}(\hat{b}_j : j < m)} \Sigma_{\alpha\text{cl}}(b_j : j < m).$$

On the other hand, as  $\hat{a} \in \Sigma_{\alpha\text{cl}}(\hat{b}, \hat{b}_j : j < m)$  but  $\hat{a} \notin \Sigma_{\alpha\text{cl}}(\hat{b}_j : j < m)$  by minimality of  $m$ , we get

$$\text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(\hat{b}_j : j < m)) >_{\alpha} \text{SU}(\hat{b}/\hat{a}, \Sigma_{\alpha\text{cl}}(\hat{b}_j : j < m)),$$

where the index  $\alpha$  indicates modulo  $\omega^{\alpha}$ .

Moreover, as  $\hat{b} \perp_{\hat{a}} a$  we get  $\hat{b} \perp_{\Sigma_{\alpha\text{cl}}(\hat{a})} \Sigma_{\alpha\text{cl}}(a)$ , i.e.  $\text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(\hat{a})) = \text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(a))$ . Since  $\Sigma_{\alpha\text{cl}}(b) = \Sigma_{\alpha\text{cl}}(b')$  and  $b \in \Sigma_{\alpha\text{cl}}(\hat{a}\hat{b})$  we obtain

$$\begin{aligned} \text{SU}(b'/\Sigma_{\alpha\text{cl}}(b_j : j < m)) &=_{\alpha} \text{SU}(b/\Sigma_{\alpha\text{cl}}(b_j : j < m)) \\ &\geq_{\alpha} \text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(b_j : j < m)) =_{\alpha} \text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(\hat{b}_j : j < m)) \\ &>_{\alpha} \text{SU}(\hat{b}/\hat{a}, \Sigma_{\alpha\text{cl}}(\hat{b}_j : j < m)) =_{\alpha} \text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(\hat{a})) \\ &=_{\alpha} \text{SU}(\hat{b}/\Sigma_{\alpha\text{cl}}(a)) =_{\alpha} \text{SU}(b/\Sigma_{\alpha\text{cl}}(a)), \end{aligned}$$

contradicting the maximality of  $\text{SU}(b/\Sigma_{\alpha\text{cl}}(a))$  modulo  $\omega^{\alpha}$ . This finishes the proof.  $\square$

As a corollary we obtain Chatzidakis' Theorem for the finite SU-rank case, apart from the decomposition in orthogonal components:

**Corollary 5.5.** *Let  $b = \text{bdd}(\text{Cb}(a/b))$  be such that  $\text{SU}(b) < \omega$ . Then  $\text{tp}(b/\text{bdd}(b) \cap \text{bdd}(a))$  is analysable in the family of all non one-based types of SU-rank 1.*

## 6. APPLICATIONS AND THE CANONICAL BASE PROPERTY

In this section and the next,  $\Sigma^{nob}$  will be the family of non one-based regular types (seen as partial types). For the applications one would like (and often has) more than mere  $\Sigma^{nob}$ -basedness of canonical bases:

**Definition 6.1.** A theory  $T$  has the *Canonical Base Property CBP* if  $\text{tp}(\text{Cb}(a/b)/a)$  is almost  $\Sigma^{nob}$ -internal for all  $a, b$ .

**Remark 6.2.** In other words, in view of Corollary 5.2 a theory has the CBP if for all  $a, b$  the type  $\text{tp}(\text{Cb}(a/b)/a)$  is  $\Sigma^{nob}$ -flat.

It had been conjectured that all supersimple theories of finite rank have the CBP, but there is a probable counter-example due to Hrushovski.

**Remark 6.3.** Chatzidakis has shown for types of finite SU-rank that the CBP implies that even  $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$  is almost  $\Sigma^{nob}$ -internal [5, Theorem 1.15].

**Example.** The CBP holds for types of finite rank in

- Differentially closed fields in characteristic 0 [21].
- Generic difference fields [21, 5].
- Compact complex spaces [4, 7, 20].

Moreover, in those cases we have a good knowledge of the non one-based types.

Kowalski and Pillay [15, Section 4] have given some consequences of  $\Sigma$ -basedness in the context of groups. In fact, they work in a theory with the CBP, but they remark that their results hold, with  $\Sigma$ -*analysable* instead of *almost  $\Sigma$ -internal*, in all stable  $\Sigma$ -based theories.

**Fact 6.4.** *Let  $G$  be a hyperdefinable  $\Sigma$ -based group in a stable theory.*

- (1) *If  $H \leq G$  is connected with canonical parameter  $c$ , then  $\text{tp}(c)$  is  $\Sigma$ -analysable.*
- (2)  *$G/Z(G)$  is  $\Sigma$ -analysable.*

An inspection of their proof shows that mere simplicity of the ambient theory is sufficient, replacing centers by approximate centers and connectivity by local connectivity. Recall that the *approximate center* of a group  $G$  is

$$\tilde{Z}(G) = \{g \in G : [G : C_G(g)] < \infty\}.$$

A subgroup  $H \leq G$  is *locally connected* if for all group-theoretic or model-theoretic conjugates  $H^\sigma$  of  $H$ , if  $H$  and  $H^\sigma$  are commensurate, then  $H = H^\sigma$ . Locally connected subgroups and their cosets have canonical parameters; every subgroup is commensurable with a unique minimal locally connected subgroup, its *locally connected component*. For more details about the approximate notions, the reader is invited to consult [23, Definition 4.4.9 and Proposition 4.4.10].

**Proposition 6.5.** *Let  $G$  be a hyperdefinable  $\Sigma$ -based group in a simple theory.*

- (1) *If  $H \leq G$  is locally connected with canonical parameter  $c$ , then  $\text{tp}(c)$  is  $\Sigma$ -analysable.*
- (2)  *$G/\tilde{Z}(G)$  is  $\Sigma$ -analysable.*

*Proof:*

- (1) Take  $h \in H$  generic over  $c$  and  $g \in G$  generic over  $c, h$ . Let  $d$  be the canonical parameter of  $gH$ . Then  $\text{tp}(gh/g, c)$  is the generic type of  $gH$ , so  $d$  is interbounded with  $\text{Cb}(gh/g, c)$ . By  $\Sigma$ -basedness,  $\text{tp}(d/gh)$  is  $\Sigma$ -analysable. But  $c \in \text{dcl}(d)$ , so  $\text{tp}(c/gh)$  is  $\Sigma$ -analysable, as is  $\text{tp}(c)$  since  $c \perp gh$ .
- (2) For generic  $g \in G$  put

$$H_g = \{(x, x^g) \in G \times G : x \in G\},$$

and let  $H_g^{lc}$  be the locally connected component of  $H_g$ . Then  $g\tilde{Z}(G)$  is interbounded with the canonical parameter of  $H_g^{lc}$ , so  $\text{tp}(g\tilde{Z}(G))$  is  $\Sigma$ -analysable, as is  $G/\tilde{Z}(G)$ .  $\square$



**Theorem 6.6.** *Let  $G$  be an  $\emptyset$ -hyperdefinable  $\Sigma$ -based group in a simple theory. If  $G$  is supersimple or type-definable, there is a normal nilpotent  $\emptyset$ -hyperdefinable subgroup  $N$  such that  $G/N$  is almost  $\Sigma$ -internal. In particular, a supersimple or type-definable group  $G$  in a simple theory has a normal nilpotent hyperdefinable subgroup  $N$  such that  $G/N$  is almost  $\Sigma^{\text{nob}}$ -internal.*

*Proof:*  $G/\tilde{Z}(G)$  is  $\Sigma$ -analysable by Proposition 6.5. Hence there is a continuous sequence

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_\alpha \triangleright \tilde{Z}(G)$$

of normal  $\emptyset$ -hyperdefinable subgroups such that successive quotients  $Q_i = G_i/G_{i+1}$  are  $\Sigma$ -internal for all  $i < \alpha$ , and  $G_\alpha/\tilde{Z}(G)$  is bounded.

Now  $G$  acts on every quotient  $Q_i$ . Let

$$N_i = \{g \in G : [Q_i : C_{Q_i}(g)] < \infty\}$$

be the approximate stabilizer of  $Q_i$  in  $G$ , again an  $\emptyset$ -hyperdefinable subgroup. If  $(q_j : j < \kappa)$  is a long independent generic sequence in  $Q_i$  and  $g, g'$  are two elements of  $G$  which have the same action on all  $q_j$  for  $j < \kappa$ , there is some  $j_0 < \kappa$  with  $q_{j_0} \perp g, g'$ . Since  $g^{-1}g'$  stabilizes  $q_{j_0}$  it lies in  $N_i$ , and  $gN_i$  is determined by the sequence  $(q_j, q_j^g : j < \kappa)$ . Thus  $G/N_i$  is  $Q_i$ -internal, whence  $\Sigma$ -internal.

Put  $N = \bigcap_{i < \alpha} N_i$ . Since  $\prod_{i < \alpha} G/N_i$  projects definably onto  $G/N$ , the latter quotient is also  $\Sigma$ -internal. In order to finish it now suffices to show that  $N$  is virtually nilpotent. In particular, we may assume that  $N$  is  $\emptyset$ -connected.

Consider the approximate ascending central series  $\tilde{Z}_i(N)$ . Note that  $N$  centralizes  $G_\alpha/\tilde{Z}(G)$  by  $\emptyset$ -connectivity. Moreover,  $N$  approximatively stabilizes all quotients  $(G_i \cap N)/(G_{i+1} \cap N)$ . Hence, if  $G_{i+1} \cap N \leq \tilde{Z}_j(N)$ , then  $G_i \cap N \leq \tilde{Z}_{j+1}(N)$ . If  $G$  is supersimple, we may assume that all the  $Q_i$  are unbounded, so  $\alpha$  is finite and  $N = \tilde{Z}_{\alpha+2}(N)$ . In the type-definable case, note that  $\tilde{Z}_i(N)$  is relatively  $\emptyset$ -definable by [23, Lemma 4.2.6]. So by compactness the least ordinal  $\alpha_i$  with  $G_{\alpha_i} \cap N \leq \tilde{Z}_i(N)$  must be a successor ordinal, and  $\alpha_{i+1} \leq \alpha_i - 1 < \alpha_i$ . Hence the sequence must stop and there is  $k < \omega$  with  $N = \tilde{Z}_k(N)$ . But then  $N$  is nilpotent by [23, Proposition 4.4.10.3].  $\square$

**Remark 6.7.** In a similar way one can show that if  $G$  acts definably and faithfully on a  $\Sigma$ -analysable group  $H$  and  $H$  is supersimple or type-definable, then there is a hyperdefinable normal nilpotent subgroup  $N \triangleleft G$  such that  $G/N$  is almost  $\Sigma$ -internal.

## 7. FINAL REMARKS

We have seen that for (weak)  $\Sigma$ -ampleness only the first level of an element is important. However, the difference between strong  $\Sigma^{nob}$ -basedness and the CBP is precisely the possible existence of a second (or higher)  $\Sigma^{nob}$ -level of  $\text{Cb}(a/b)$  over  $a$ , i.e. its non  $\Sigma^{nob}$ -flatness.

A possible approach to the CBP could be to replace  $\Sigma^{nob}$ -closure by  $\Sigma_1^{nob}$ -closure. In fact it is possible to define a corresponding notion of  $\Sigma_1$ -ampleness, and to prove an analogue of Theorem 4.20. However, since  $\Sigma_1$ -closure is not a closure operator, the equivalence between  $\Sigma_1^{nob}$ -basedness (i.e. the CBP) and non 1- $\Sigma_1^{nob}$ -ampleness breaks down. So far we have not found a way around this.

Another approach could be to try using Corollary 6.6 in the applications, rather than Fact 6.4 (or Proposition 6.5), but we have not looked into this.

Finally, it might be interesting to look for a variant of ampleness which does take all levels into account, as one might hope to obtain stronger structural consequences.

## REFERENCES

- [1] Prerna Bihani Juhlin. *Fine structure of dependence in superstable theories of finite rank*, PhD thesis, University of Notre Dame, Indiana, 2010.
- [2] Steven Buechler and Colleen Hoover. *The classification of small types of rank  $\omega$* , J. Symb. Logic 66:1884–1898, 2001.
- [3] Steven Buechler. *Vaught’s conjecture for superstable theories of finite rank*, Ann. Pure Appl. Logic 155:135–172, 2008.
- [4] Frédéric Campana. *Algébricité et compacité dans l’espace des cycles d’un espace analytique complexe*, Math. Ann. 251:7–18, 1980.
- [5] Zoé Chatzidakis. *A note on canonical bases and modular types in supersimple theories*, preprint, September 2002.
- [6] David Evans. *Ample dividing*, J. Symb. Logic 68:1385–1402, 2003.
- [7] Akira Fujiki. *On the Douady space of a compact complex space in the category  $\mathcal{A}$* , Nagoya Math. J. 85:189–211, 1982.
- [8] Peter Hall. *Some sufficient conditions for a group to be nilpotent*, Ill. J. Math. 2:787–801, 1958.
- [9] Peter Hall and Brian Hartley. *The stability group of a series of subgroups*, Proc. Lond. Math. Soc., III. Ser. 16:1–39, 1966.
- [10] Ehud Hrushovski. *Locally modular regular types*. In: *Classification Theory, Proceedings, Chicago 1985* (ed. John Baldwin). Springer-Verlag, Berlin, D, 1985.
- [11] Ehud Hrushovski. *Contributions to stable model theory*. PhD Dissertation, 1986.

- [12] Ehud Hrushovski. *The Manin-Mumford conjecture and the model theory of difference fields*, Ann. Pure Appl. Logic 112:43–115, no. 1, 2001.
- [13] Ehud Hrushovski. *A new strongly minimal set*, Ann. Pure Appl. Logic 62:147–166, 1993.
- [14] Leo Kaloujnine. *ber gewisse Beziehungen zwischen einer Gruppe und ihren Automorphismen*. In: *Bericht ber die Mathematiker-Tagung in Berlin, Januar 1953*, 164172. Deutscher Verlag der Wissenschaften, Berlin, 1953.
- [15] Piotr Kowalski and Anand Pillay. *Quantifier elimination for algebraic D-groups*, Trans. Amer. Math. Soc. 358:167–181, 2005.
- [16] Anand Pillay. *The geometry of forking and groups of finite Morley rank*, J. Symb. Logic 60:1251–1259, 1995.
- [17] Anand Pillay. *Geometric stability theory*. Oxford Logic Guides 32. Oxford University Press, Oxford, GB, 1996.
- [18] Anand Pillay. *A note on CM-triviality and the geometry of forking*, J. Symb. Logic 65:474–480, 2000.
- [19] Anand Pillay. *Notes on analysability and canonical bases*. e-print available at <http://www.math.uiuc.edu/People/pillay/remark.zoe.pdf>, 2001.
- [20] Anand Pillay. *Model-theoretic consequences of a theorem of Campana and Fujiki*, Fund. Math. 174(2):187–192, 2002.
- [21] Anand Pillay and Martin Ziegler. *Jet spaces of varieties over differential and difference fields*, Selecta Math. (N.S.) 9:579–599, 2003.
- [22] Frank O. Wagner. *Stable Groups*. LMS Lecture Note Series 240. Cambridge University Press, Cambridge, UK, 1997.
- [23] Frank O. Wagner. *Simple Theories*. Mathematics and Its Applications 503. Kluwer Academic Publishers, Dordrecht, NL, 2000.
- [24] Frank O. Wagner. *Some remarks on one-basedness*, J. Symb. Logic 69:34–38, 2004.

UNIVERSITAT DE BARCELONA; DEPARTAMENT DE LÒGICA, HISTÒRIA I FILOSOFIA DE LA CIÈNCIA, MONTALEGRE 6, 08001 BARCELONA, SPAIN

UNIVERSITÉ DE LYON; CNRS; UNIVERSITÉ LYON 1; INSTITUT CAMILLE JORDAN UMR5208, 43 BD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

*E-mail address:* `dpalacin@ub.edu`

*E-mail address:* `wagner@math.univ-lyon1.fr`