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# INFINITE-HORIZON LORENTZ TUBES AND GASES: RECURRENCE AND ERGODIC PROPERTIES

MARCO LENCI AND SERGE TROUBETZKOY

ABSTRACT. We construct classes of two-dimensional aperiodic Lorentz systems that have infinite horizon and are ‘chaotic’, in the sense that they are (Poincaré) recurrent, uniformly hyperbolic, ergodic, and the first-return map to any scatterer is  $K$ -mixing. In the case of the Lorentz tubes (i.e., Lorentz gases in a strip), we define general measured families of systems (*ensembles*) for which the above properties occur with probability 1. In the case of the Lorentz gases in the plane, we define families, endowed with a natural metric, within which the set of all chaotic dynamical systems is uncountable and dense.

MSC 2010: 37D50, 37A40, 60K37, 37B20, 36A25.

## 1. INTRODUCTION

A Lorentz system is a dynamical system of a point particle moving inertially in an unbounded domain (in this paper we consider only planar domains) endowed with an infinite number of locally convex scatterers. When the particle hits a scatterer, which is regarded as infinitely massive, it undergoes an elastic collision: the angle of reflection equals the angle of incidence.

The most popular such system is undoubtedly the Lorentz gas, devised by Lorentz in 1905 [Lo] to study the dynamics of an electron in a crystal; the term ‘gas’ was introduced later in the century, when versions of the Lorentz model were used to give a statistical description of the motion of a molecule in a gas.<sup>1</sup>

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<sup>1</sup>To our knowledge, the first appearance of this model within the scope of the kinetic theory of gases is in a 1932 textbook of theoretical physics [J]. The model is used briefly for the estimation of the number of collisions per unit time of a molecule in a gas; however, no mention of Lorentz is made. The first occurrence of the phrase ‘Lorentz gas’ in the scientific literature seems to date back to 1941 [G]. Curiously, the first occurrence of the phrase ‘Lorentzian gas’ that we are aware of is found in the article preceding [G] in the same issue of the same journal [HIV]. Finally, it is interesting to notice that, although Lorentz’s original papers [Lo] are

In the past century, Lorentz systems have been preferred models in the fields of statistical physics, optics, acoustics, and generally anywhere the diffusive properties of a chaotic motion were to be investigated. Throughout this history, for reasons of mathematical convenience, the models that were studied most often and most deeply were *periodic* (i.e., the configuration of scatterers was invariant for the action of a discrete group of translations) and with *finite horizon* (i.e., the free flight was bounded above). Only recently have aperiodic Lorentz systems come to the fore [Le1, Le2, DSV, CLS, SLDC]. However, while there is a considerable literature on infinite-horizon periodic Lorentz gases, almost nothing is known, at least to these authors, on aperiodic Lorentz systems with infinite horizon.

In this note we consider 2D Lorentz gases and also Lorentz tubes, that is, Lorentz systems confined to a strip of  $\mathbb{R}^2$  [CLS, SLDC]. We construct billiards that have infinite horizon and possess the ergodic properties that one would expect of these chaotic systems, such as ergodicity and strong mixing properties. For these infinite-measure preserving dynamical systems, it turns out that (Poincaré) recurrence is not only a necessary but also a sufficient condition for ergodicity; this is a consequence of the hyperbolic structure that our systems can be shown to have [Le1, CLS]. As for mixing, since a universally accepted definition of mixing is not available in infinite ergodic theory (see, e.g., the discussion in [Le3]), we characterize this aspect of the dynamics by proving that certain first-return maps are  $K$ -automorphisms (which implies strong mixing). This is again a consequence of recurrence and hyperbolicity.

Another important question that we aim to discuss is that of the typicality of such ergodic properties. One would expect that, when the effective dimension is one (Lorentz tubes) or two (Lorentz gases), recurrence, and all the stochastic properties that it entails, hold for “most” systems (and one would expect the former case to be more easily worked out than the latter).

We address this question by introducing the following two classes of billiards: A strip, respectively the plane, is partitioned into infinitely many congruent *cells*. In each cell we place a configuration of semi-dispersing scatterers, i.e., a finite union of piecewise smooth closed sets, whose smooth boundary components are seen as convex from the exterior. The set of all the *global* configurations of scatterers gives rise to a family of dynamical systems of the same type. A natural distance

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about electrons in a metal and not molecules in a gas, he treats the problem by deriving and solving a Boltzmann-like transport equation; see also [K].

between two configurations can be defined, which makes the above set a metric space. Furthermore, if the configuration is chosen according to a probability law, the family becomes a measured family, or an *ensemble*, of dynamical systems. This structure is often referred to as a *quenched random dynamical system*.

In the case of the tubes we give fairly explicit sufficient conditions for the above-mentioned ergodic properties, thus proving that, for many reasonable random laws on the “disorder” (including all non-degenerate Bernoulli measures), such properties hold almost surely in the ensemble; we state these results in Section 2. In the harder case of the gases, we prove that the set of the ergodic systems is uncountable and dense within the whole space; this is described in Section 3. Outlines of the proofs are given in Section 4.

It is worthwhile to mention that the billiards we construct have infinite but *locally finite* horizon, that is, though the free flight has no upper bound, no straight line exists that intersects no scatterers.

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## 2. LORENTZ TUBES

Let  $C$  denote the unit square, which will be henceforth referred to as *cell*. Let  $G^1$  denote an open segment along one of the sides of  $C$ , say the left one, and  $G^2$  the corresponding segment on the opposite side (via the natural orthogonal projection).  $G^1$  and  $G^2$  are called the *gates* of  $C$ .

A *local configuration of scatterers* is a “fat” closed subset  $\Gamma \subset C$  (this means that  $\Gamma$  is the closure of its interior) such that:

- (A1)  $\partial\Gamma$  is made up of a finite number of  $C^3$ -smooth curves  $\gamma_i$ , which may only intersect at their endpoints ( $\gamma_i$  is always considered a closed set).
- (A2) Either  $\gamma_i$  is a segment (and we call it *flat* boundary) or its curvature is everywhere positive—this means that  $\gamma_i$  bends towards the inside of  $\Gamma$  (in which case we call  $\gamma_i$  a *dispersing* boundary).
- (A3)  $\partial C \setminus (G^1 \cup G^2) \subset \partial\Gamma$ ; and  $G^1, G^2$  do not intersect  $\partial\Gamma$ .
- (A4) The angle formed by  $\gamma_i$  and each intersecting  $\gamma_j$  (or  $G^j$ ) is non-zero.

We consider a finite number  $\Gamma^1, \Gamma^2, \dots, \Gamma^m$  of such local configurations (see Figs. 1 and 2).

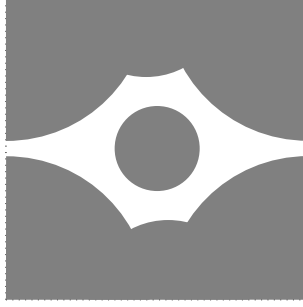


FIGURE 1. A blocking cell for the Lorentz tube

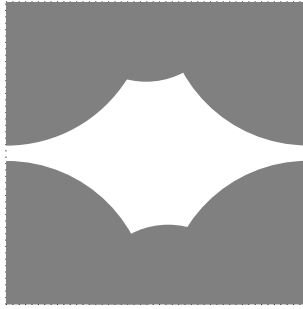


FIGURE 2. A non-blocking cell for the Lorentz tube

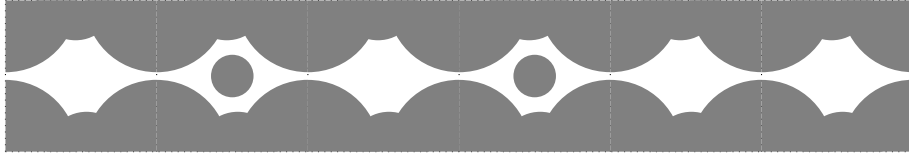


FIGURE 3. A Lorentz tube

A Lorentz tube (LT) is a chain of cells  $C_n$  ( $n \in \mathbb{Z}$ ), such that  $G_n^2$ , the right gate of  $C_n$ , coincides with  $G_{n+1}^1$ , the left gate of  $C_{n+1}$ . More precisely, call  $C_n := [n, n+1] \times [0, 1]$  the particular copy of  $C$  immersed in  $\mathbb{R}^2$  as indicated, and denote  $\Omega := \{1, 2, \dots, m\}$ . Then, for  $\ell := (\ell_n)_{n \in \mathbb{Z}} \in \Omega^{\mathbb{Z}}$ , we define the *billiard table*

$$(1) \quad \mathcal{Q} = \mathcal{Q}_\ell := \bigcup_{n \in \mathbb{Z}} C_n \setminus \Gamma_n^{\ell_n},$$

where  $\Gamma_n^{\ell_n}$  is the configuration  $\Gamma^{\ell_n}$  translated to the cell  $C_n$  (see Fig. 3). The collection  $(\Gamma_n^{\ell_n})_{n \in \mathbb{Z}}$ —equivalently  $\ell$ —is called the *global configuration of scatterers*.

We henceforth say ‘the LT  $\ell$ ’ to mean both the table  $\mathcal{Q}_\ell$  and the billiard dynamics defined on it. By this we mean, precisely, the dynamical system  $(\mathcal{M}_\ell, T_\ell, \mu_\ell)$ —more concisely,  $(\mathcal{M}, T, \mu)$ —where:

- $\mathcal{M}$  is the collection of all the *line elements* of the dynamics, i.e., all the position-velocity pairs  $(q, v)$ , where  $q \in \partial\mathcal{Q}$  and  $v$  is a unit vector based in  $q$  and pointing toward the interior of  $\mathcal{Q}$ .  $(q, v)$  is meant to represent the dynamical variables of the particle right after a collision ( $v$  can be chosen unitary because in this Hamiltonian system the conservation of energy equals the conservation of speed).
- $T$  is the map that takes  $(q, v)$  into the next post-collisional line element  $(q', v')$ , along the billiard trajectory of  $(q, v)$ ; this map fails to be well defined only at a negligible set of line elements, which are called *singular*, cf. below.  $T$  is usually called the (*standard*) *billiard map*.
- $\mu$  is the invariant measure induced on the Poincaré section  $\mathcal{M}$  by the Liouville measure; it is well known that  $d\mu(q, v) = \langle n_q, v \rangle dq dv$ , where  $n_q$  is inner unit normal to  $\partial\mathcal{Q}$  in  $q$ . It is easy to verify that the set of all singular points in phase space is null w.r.t.  $\mu$ . Finally, as it is evident,  $\mu(\mathcal{M}) < \infty$  if and only if the total length of  $\partial\mathcal{Q}$  is finite.

Notice that, by the definition of  $\mathcal{Q}$ , a trajectory that intersects a gate  $G_n^i$  crosses it. (The trajectory that intersects the gate non-transversally is either singular or will be considered as such.)

We assume that there are two types of local configurations: the *blocking* configurations, corresponding to the index set  $\Omega_B := \{1, 2, \dots, m'\}$  ( $m' < m$ ), and the *non-blocking* configurations, corresponding to the set  $\Omega_{NB} := \{m'+1, m'+2, \dots, m\}$ . They verify the following conditions (see Figs. 1 and 2):

- (A5) If  $a \in \Omega_B$ , any billiard trajectory that enters a cell with configuration  $\Gamma^a$  must collide with a dispersing boundary before leaving it (cf. Fig. 1).
- (A6) If  $a \in \Omega_{NB}$ , the only trajectories that do not have dispersing collisions within a cell with configuration  $\Gamma^a$  are those that go from  $G^1$  to  $G^2$ , or viceversa. For these trajectories, however, the number of flat collisions is bounded above (cf. Fig. 2).

Clearly an LT  $\ell \in \Omega_B^{\mathbb{Z}}$  has finite horizon, i.e., the free flight between two successive collisions has an upper bound. In fact, the *hyperbolic horizon* is finite as well, whereby we mean that the free flight

between two successive *dispersing* collisions is bounded above (this latter, stronger, property is more essential for semi-dispersing billiards). On the contrary, an arbitrary LT in  $\Omega^{\mathbb{Z}}$  may have neither property.

A word  $a_1 a_2 \cdots a_l$ , with  $a_i \in \Omega$ , is called a *factor* of  $\ell \in \Omega^{\mathbb{Z}}$  if there exists  $n$  such that  $\ell_{n+i} = a_i$ , for  $i = 1, 2, \dots, l$ . The factor is called blocking (respectively, non-blocking) if all the  $a_i$  belong to  $\Omega_B$  (respectively,  $\Omega_{NB}$ ); it is called constant if they are all equal. The positive integer  $l$  is called the length of the factor.

The technique used by Troubetzkoy in [Tr1], when applied to the present models, easily implies that any LT in  $\Omega^{\mathbb{Z}}$  which has arbitrarily long blocking constant factors (both forwards and backwards) is recurrent. Furthermore, Cristadoro, Lenci and Seri have shown that, for LTs in  $\Omega_B^{\mathbb{Z}}$ , ergodicity and recurrence are equivalent [CLS], and they imply  $K$ -mixing for suitable return maps (this last result is actually stated in [SLDC] but its proof applies as well to the models of [CLS]).

Our results on the Lorentz tubes are a combination and an extension of these ideas. To describe them we introduce some notation that will appear obscure at first, but will be explained momentarily. For fixed  $\ell$ , define  $g_0^+ := g_0^- := 0$  and, recursively for  $j > 0$ ,

$$(2) \quad g_{j+1}^{\pm} := \min \left\{ k \in \mathbb{Z}^+ \mid \ell_{\pm \sum_{i=0}^j g_i^{\pm} \pm k} \in \Omega_B \right\}.$$

In other words,  $g_j^+ - 1$  (respectively,  $g_j^- - 1$ ) is the length of the  $j^{\text{th}}$  non-blocking factor to the right (respectively, to the left) of the cell  $C_0$  (with the convention that between two blocking cells there is a non-blocking factor of length 0). Notice that, if  $\ell_0 \in \Omega_B$ , this coding reflects exactly the sequence of non-blocking factors of  $\ell$ . Otherwise, there is a little difference which, see Remark 2 below, does not affect the upcoming statement.

**Theorem 1.** *Assume (A1)-(A6). For any  $\ell \in \Omega^{\mathbb{Z}}$  which has arbitrarily long blocking constant factors, both forward and backwards, and such that both sequences  $(g_j^{\pm})_{j \in \mathbb{N}}$  grow at most like a power-law, the corresponding dynamical system  $(\mathcal{M}, T, \mu)$  is:*

- (a) *uniformly hyperbolic, in the sense that local stable and unstable manifolds exist at a.e. point of  $\mathcal{M}$ , and the corresponding ( $T$ -invariant) laminations are absolutely continuous w.r.t.  $\mu$  (see, e.g., [Le1] for details); also, the contraction (respectively expansion) coefficient of  $T^n$ , along the stable (respectively unstable) direction, is bounded below by  $C\lambda^n$ , where  $C$  and  $\lambda$  are uniform constants;*

- (b) *recurrent in the sense of Poincaré, i.e., given a measurable  $A \subset \mathcal{M}$ , the orbit of a.e. point in  $A$  comes back to  $A$  infinitely many times.*
- (c) *ergodic, that is, if  $A$  is  $T$ -invariant modulo  $\mu$ , then either  $A$  or its complement has measure zero.*

Furthermore, the first-return map to any smooth component of  $\partial\mathcal{Q}$ , of the type  $\gamma_i$  as in (A1), is  $K$ -mixing.

**Remark 2.** *In the above theorem, both hypotheses are shift-invariant in  $\Omega^{\mathbb{Z}}$ , as a relocation of the origin on  $\ell$  will produce at most a shift in  $(g_j^{\pm})$  and a change of the first few terms. Therefore, when convenient—to do away with the problem mentioned in the previous paragraph—we make the convention that any  $\ell$  is shifted to the left the minimum amount of times for  $C_0$  to have a blocking configuration.*

An LT as described in Theorem 1 need not have infinite horizon. To start with, it is necessary that the non-blocking configurations let free trajectories through, something that is not postulated in (A6). But more is needed as well. For instance, if one designs the local configurations  $\Gamma^a$ , with  $a \in \Omega_{NB}$ , so that, for any  $l \in \mathbb{Z}^+$ , all non-blocking factors of length  $l$  admit a free flight of length  $\geq l$  (and this is easy, cf. Fig. 2), then a necessary and sufficient condition for an LT to have infinite horizon is that at least one of the two sequences  $(g_j^{\pm})$  is unbounded. Or one might ask for something less: for example, that just the *constant* non-blocking factors admit long free flights. This is enough to guarantee that very many LTs have an infinite horizon, cf. Corollary 3.

An important question is whether the LTs to which Theorem 1 applies are typical in  $\Omega^{\mathbb{Z}}$ . This of course depends on the definition of ‘typical’. One strong notion of typicality is the measure-theoretic notion, provided a probability measure  $\Pi$  is put on  $\Omega^{\mathbb{Z}}$  (endowed with the natural  $\sigma$ -algebra generated by the cylinders). In this case,  $(\Omega^{\mathbb{Z}}, \Pi)$  becomes a measured family (in jargon, an *ensemble*) of dynamical systems, which we call *quenched random Lorentz tube*.

Many reasonable measures  $\Pi$  ensure that the assertions of Theorem 1 hold  $\Pi$ -almost surely. Here is an example:

**Corollary 3.** *Let  $(p_1, p_2, \dots, p_m)$  be a stochastic vector, with  $p_a > 0$ , and let  $\Pi$  be the Bernoulli measure on  $\Omega^{\mathbb{Z}}$  relative to that vector (i.e., the unique measure for which  $\Pi(\ell_n = a) = p_a$ , for all  $n$ ). Then  $\Pi$ -a.e. LT in  $\Omega^{\mathbb{Z}}$  has the properties stated in Theorem 1. Furthermore, if any non-blocking constant factor of length  $l$  ( $\forall l \in \mathbb{Z}^+$ ) admits a free flight of length  $\geq l$ , then a.e. LT has infinite horizon as well.*





FIGURE 4. A blocking cell for the Lorentz gas

**Proof of Corollary 3.** Let  $p_B := \sum_{a=1}^{m'} p_a$  and  $p_{NB} := \sum_{a=m'+1}^m p_a$ . By hypothesis,  $p_B, p_{NB} \in (0, 1)$ . It is evident that, w.r.t.  $\Pi$ , the random variables  $(g_j^\pm)$  are i.i.d., with distribution  $\Pi(g_j^\pm = k) = p_B(p_{NB})^{k-1}$  ( $k \in \mathbb{Z}^+$ ).

We claim that, for  $\Pi$ -a.e.  $\ell$ , there exists  $K = K(\ell)$  such that  $g_j^\pm \leq Kj$ . In fact, observe that

$$(3) \quad \Pi(g_j^\pm \geq j) = (p_{NB})^{j-1}.$$

This implies that the probabilities of the ‘events’  $\{g_j^\pm \geq j\}$  form a summable sequence. Thus, by Borel-Cantelli, for a.e.  $\ell$ , the inequality  $g_j^\pm/j \geq 1$  is verified only for a finite number of  $j$ ’s. Setting  $K := \max_j(g_j^\pm/j)$  proves the claim.

To finish the proof of Corollary 3, observe that, for a non-degenerate Bernoulli measure, a.e.  $\ell$  contains arbitrarily long blocking and non-blocking constant factors.  $\square$

### 3. LORENTZ GASES

We now turn to the truly two-dimensional case. We tile the plane with  $\mathbb{Z}^2$  copies of the unit square  $C$ . In this case  $C$  is endowed with four gates:  $G^1$  and  $G^2$ , congruent and opposite open segments (say, the left and the right gate, respectively);  $G^3$  and  $G^4$ , again congruent and opposite open segments (the lower and the upper gates).

Apart from this, we have the same structure as in Section 2: a finite number of local configurations indexed by the set  $\Omega = \Omega_B \cup \Omega_{NB}$ , where  $\Omega_B$  denotes the blocking and  $\Omega_{NB}$  the non-blocking configurations, as in Figs. 4 and 5. Again, these sets verify six assumptions, (A1), (A2), (A4), (A5), and the counterparts of (A3) and (A6):

(A3')  $\partial C \setminus (G^1 \cup \dots \cup G^4) \subset \partial\Gamma$ ; and  $G^1, \dots, G^4$  do not intersect  $\partial\Gamma$ .

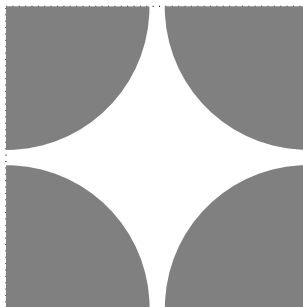


FIGURE 5. A non-blocking cell for the Lorentz gas

(A6') If  $a \in \Omega_{NB}$ , the only trajectories that do not have dispersing collisions within a cell with configuration  $\Gamma^a$  are those that go from  $G^1$  to  $G^2$ , or viceversa; or from  $G^3$  to  $G^4$ , or viceversa. For these trajectories the number of flat collisions is bounded above (cf. Fig. 5).

A global configuration is the collection  $\ell := (\ell_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^2} \in \Omega^{\mathbb{Z}^2}$ , with  $\mathbf{n} := (n_1, n_2)$ , and the billiard table is

$$(4) \quad \mathcal{Q} = \mathcal{Q}_\ell := \bigcup_{\mathbf{n} \in \mathbb{Z}^2} C_{\mathbf{n}} \setminus \Gamma_{\mathbf{n}}^{\ell_{\mathbf{n}}},$$

where  $C_{\mathbf{n}} := [n_1, n_1 + 1] \times [n_2, n_2 + 1]$ , and  $\Gamma_{\mathbf{n}}^{\ell_{\mathbf{n}}}$  is the configuration  $\Gamma_{\mathbf{n}}^{\ell_{\mathbf{n}}}$  translated to  $C_{\mathbf{n}}$ . We call  $\mathcal{Q}$  (and the billiard map thereon) a Lorentz gas (LG)—see a realization in Fig. 6.

A word  $a_1 a_2 \cdots a_l$  is called a *horizontal factor* of  $\ell \in \Omega^{\mathbb{Z}^2}$  if there exists  $\mathbf{n} = (n_1, n_2)$  such that  $\ell_{n_1+i, n_2} = a_i$ , for  $i = 1, 2, \dots, l$ . The analogous definition is given for a vertical factor. As in Section 2, a factor is called non-blocking if  $a_i \in \Omega_{NB}$  and constant if  $a_i = a$ , for all  $i$ .

Since we are interested in infinite-horizon billiards, we discuss a sufficient condition to obtain this property. It will be seen in Section 4 that the LGs we construct possess arbitrarily long, both horizontal and vertical, non-blocking constant factors. They have infinite horizon if all non-blocking, say, horizontal factors of length  $l$  ( $\forall l \in \mathbb{Z}^+$ ) admit a free flight of length  $\geq l$ .

In the two-dimensional setting no one has succeeded in proving that recurrence is a typical property in a measure-theoretic sense (this is actually an important open problem, cf. [Le2, CD]). The known results are with respect to a topological notion of typicality. For this, we make

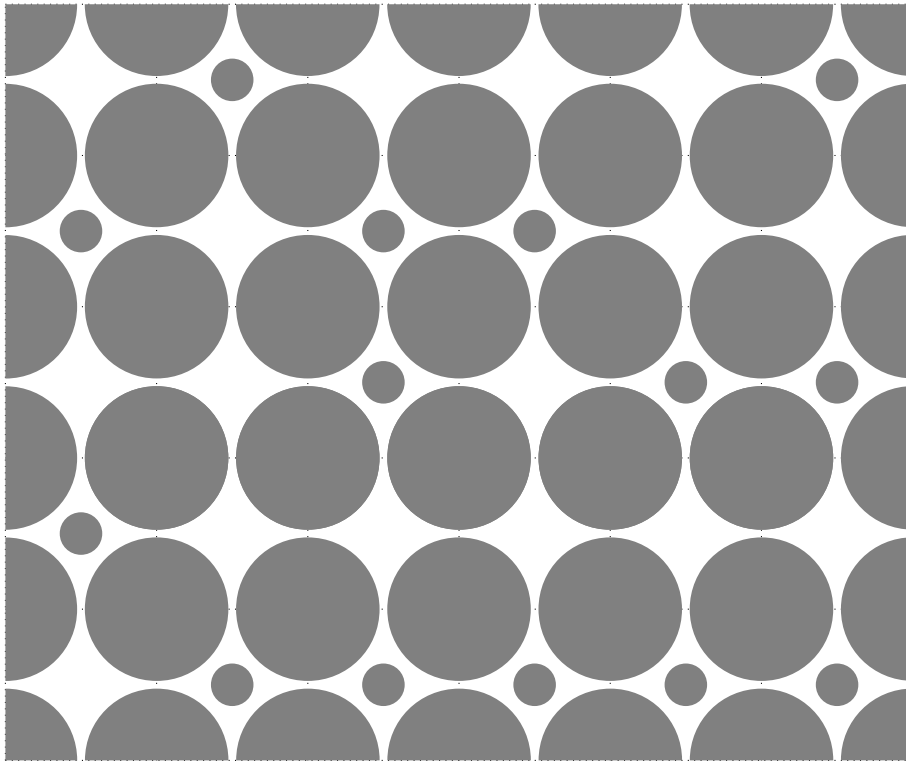


FIGURE 6. A Lorentz gas

$\Omega^{\mathbb{Z}^2}$  a metric space by endowing it with the distance

$$(5) \quad \text{dist}(\ell, \ell') := \sum_{\mathbf{n} \in \mathbb{Z}^2} 2^{-|\mathbf{n}_1| - |\mathbf{n}_2|} |\ell_{\mathbf{n}} - \ell'_{\mathbf{n}}|.$$

It was shown by Lenci that, relative to the above metric, the Baire-typical LG in  $\Omega_B^{\mathbb{Z}^2}$  is recurrent [Le2] and ergodic [Le1]. On the other hand, Troubetzkoy has shown that, under the assumption that long factors of non-blocking configurations admit long free flights, the Baire-typical LG in  $\Omega^{\mathbb{Z}^2}$  is recurrent and has infinite horizon [Tr2]. Once again, we combine and extend these ideas to show that a great number of infinite-horizon LGs are recurrent and chaotic in the sense of Theorem 1.

**Theorem 4.** *Assuming (A1), (A2), (A3'), (A4), (A5), (A6'), the metric space  $\Omega^{\mathbb{Z}^2}$  contains a dense uncountable set of LGs that are hyperbolic, recurrent and ergodic in the sense of Theorem 1, and such that the return map to any smooth component of the type  $\gamma_i$  is  $K$ -mixing. Furthermore, if, for all  $l \in \mathbb{Z}^+$ , any non-blocking horizontal constant*

factor of length  $l$  admits a free flight of length  $\geq l$ , then those LGs have infinite horizon as well.

#### 4. PROOFS

**4.1. Sketch of the proof of Theorem 1.** Let us fix an LT as in the statement of Theorem 1. Its recurrence is proved essentially in the same way as for the staircase billiards of [Tr1]. For the sake of completeness, we give an outline of the argument.

Here and in the remainder we are going to need a notation for the portion of the phase space pertaining to the cell  $C_n$ :

$$(6) \quad \mathcal{M}_n := \{(q, v) \in \mathcal{M} \mid q \in C_n\}.$$

It is clear by the hypotheses that the LT contains, both forward and backwards, arbitrarily long constant factors of the *same* blocking configuration, say  $\Gamma^1$ . In this paragraph and the next, when we say blocking factor, we will always mean a factor of configurations  $\Gamma^1$ . Consider a blocking factor beginning with the cell  $C_{n_1}$  and ending with the cell  $C_{n_2}$  ( $n_2 > n_1$ ). Let us call  $A_{n_1}^{n_2}$  the set of all the line elements of  $\mathcal{M}_{n_1}$  whose trajectories cross the factor rightwards, before coming back to  $C_{n_1}$ . Likewise, let  $A_{n_2}^{n_1}$  be the set of all the line elements of  $\mathcal{M}_{n_2}$  whose trajectories cross the factor leftwards, before coming back to  $C_{n_2}$ . Now fix  $k \in \mathbb{Z}^+$ . Since an LT made up entirely of cells  $\Gamma^1$  is recurrent, there exists a positive integer  $l_k$  such that, if the length of the factor (namely,  $n_2 - n_1 + 1$ ) is bigger than or equal to  $l_k$ , both  $\mu(A_{n_1}^{n_2})$  and  $\mu(A_{n_2}^{n_1})$  are smaller than  $1/k$ .

Consider a wandering set  $W$  and, for any  $n \in \mathbb{Z}$ , set  $W_n := W \cap \mathcal{M}_n$ .  $W_n$  is also a wandering set. For all  $k \in \mathbb{Z}^+$ , there exist a blocking factor of length  $\geq l_k$  (say, from the cell  $C_{n_1}$  to the cell  $C_{n_2}$ ) to the right of  $C_n$  and a blocking factor of length  $\geq l_k$  (say, from  $C_{n_3}$  to  $C_{n_4}$ ) to the left of  $C_n$ . By definition, the orbits of the points of  $W_n$  are all disjoint and unbounded, which means they must intersect either  $A_{n_1}^{n_2}$  or  $A_{n_4}^{n_3}$  in distinct points. Therefore, using the invariance of the measure,  $\mu(W_n) \leq \mu(A_{n_1}^{n_2} \cup A_{n_4}^{n_3}) = 2/k$ . Since  $k$  and  $n$  are arbitrary,  $W$  is a null set.

Turning to the hyperbolicity and ergodicity, once the recurrence is known, one proceeds as in [Le1] or [CLS]. (A fairly accurate summary of the whole proof is given in [SLDC], although that article refers to LTs in dimension higher than two.) Here we limit ourselves to mentioning which parts of the proof can be worked out using the standard techniques for classical hyperbolic billiards [KS, LW] and which ones need to be adapted to our particular infinite-measure system.

Focusing on the hyperbolicity first, it can be seen that all the arguments used in the proof of Theorem 1(a), with one exception, are local, that is, depend on the value of  $T$  (its invariant cones, its distortion coefficients, etc.) on a neighborhood of a given point, or a given orbit. In other words, they cannot distinguish between a finite-measure semi-dispersing billiard—for which everything works well [KS]—or an infinite-measure one.

The only argument that is not local is the one whereby  $\mu$ -almost all orbits stay sufficiently far away from the singular points of  $T$  (so that the singularities of the map do not interfere with the construction of the local stable and unstable manifolds). The singular points are organized in smooth curves, which are called *singularity lines*. Each such line corresponds to all the line elements whose first collision occur at a given corner of  $\partial\mathcal{Q}$ , or tangentially to a certain smooth component of it. Hence, there are a countable number of singularity lines, but, by (A1), only a finite number of them are relative to a single cell  $C_n$  (indeed, that number is uniformly bounded in  $n$ ). It turns out that, if Lemma 5 below holds, almost all orbits approach the singular set—which we henceforth denote  $\mathcal{S}$ —not faster than a negative power-law in time (where by time we mean ‘number of collisions’). This is enough to make the sought argument work [Le1, CLS].

Moving on to statement (c), as explained in [Le1], one can exploit a suitable Local Ergodicity Theorem for billiards (say, the version of [LW]), which uses only local arguments except in its most delicate part, the so-called Tail Bound. It turns out, however, that a version of the Tail Bound can be proved for our LTs too, if the following result holds (cf. Sec. 3 of [SLDC]).

**Lemma 5.** *For  $t \in \mathbb{Z}^+$ , let  $\mathcal{S}^t$  denote the singular set for the map  $T^t$ , i.e.,  $\mathcal{S}^t := \bigcup_{j=0}^{t-1} T^{-j}(\mathcal{S})$ . There exist constants  $C, \alpha > 0$  such that, for all  $n \in \mathbb{Z}$  and  $t \in \mathbb{Z}^+$ ,  $\mathcal{M}_n \cap \mathcal{S}^t$  is made up of at most  $Ct^\alpha$  smooth curves.*

*Proof.* Without loss of generality, we may specialize to  $n = 0$  (a translation of the tube does not change our argument—cf. Remark 2). Lemma 5 is equivalent to proving that the phase space orbits of all the points in  $\mathcal{M}_0$ , as a whole, can only intersect  $Ct^\alpha$  singularity lines within time  $t$ . By (A5), the configuration space trajectories of length  $t$  of these points cannot go further left than the cell  $C_{g_t^-}$  or further right than  $C_{g_t^+}$ . The corresponding phase space orbits are thus contained in  $\bigcup_{n=g_t^-}^{g_t^+} \mathcal{M}_n$ . Since, by hypothesis, the number of sets in that union

grows like a power-law in  $t$  and, as recalled earlier, each set contains a finite number of singularity lines, we have concluded the proof.  $\square$

Once we have local ergodicity (all but countably many points in phase space have a neighborhood contained in one ergodic component), global ergodicity is easily shown. Also, the assertion about the  $K$ -mixing first-return map is proved as in [SLDC], Sec. 4.

**4.2. Sketch of the proof of Theorem 4.** Let us endow  $\mathbb{Z}^2$  with the norm  $\|\mathbf{n}\| = \|(n_1, n_2)\| := |n_1| + |n_2|$ . For  $j \in \mathbb{Z}^+$ , set  $D_j := \{\mathbf{n} \in \mathbb{Z}^2 \mid \|\mathbf{n}\| = j^2\}$  (this set resembles the border of a rhombus in  $\mathbb{Z}^2$ ). Given  $k \in \mathbb{Z}^+$ , define

$$(7) \quad Z_i := \mathbb{Z}^2 \setminus \bigcup_{j \geq i} D_j$$

and

$$(8) \quad \mathcal{L}_i := \left\{ \ell = (\ell_{\mathbf{n}}) \in \Omega^{\mathbb{Z}^2} \mid \ell_{\mathbf{n}} = 1, \forall \mathbf{n} \notin Z_i \right\}.$$

In other words,  $\mathcal{L}_i$  is the set of all the global configurations which have “blocking circles” (namely, circles filled with cells of type  $\Gamma^1$ ) at all radii  $j^2$ , with  $j \geq i$ . Clearly,  $\mathcal{L}_i \cong \Omega^{Z_i}$  in a natural sense. In each  $\mathcal{L}_i$  we apply the method of [Tr2] to construct a  $G_\delta$  dense set  $\mathcal{R}_i$  of recurrent Lorentz gases. Let us sketch this method.

In what follows, whenever we mention circles, balls, annuli, we will always mean circles, balls, annuli in  $\mathbb{Z}^2$ , relative to the norm  $\|\cdot\|$  and centered in the origin. Denote by  $\xi$  a configuration of cells in a ball of  $Z_i$ , equivalently, a vector of  $\Omega^B$ , where  $B$  is the restriction to  $Z_i$  of a ball in  $\mathbb{Z}^2$ ;  $B$  will be referred to as the *support* of  $\xi$ . Clearly, there are countably many such (finite) configurations, so we can index them as  $(\xi_k)_{k \in \mathbb{Z}^+}$ . For each such  $k$ , let us construct a finite configuration  $\eta_k$  such that:

- the support of  $\eta_k$  is a ball of radius  $\rho > \rho_1$ , where  $\rho_1$  is radius of the support of  $\xi_k$ , and the restriction of  $\eta_k$  to the support of  $\xi_k$  is  $\xi_k$ ;
- there exists a positive integer  $\rho_2 \in (\rho_1, \rho)$  such that, if  $\rho < \|\mathbf{n}\| \leq \rho_2$ ,  $\ell_{\mathbf{n}} = 1$ , i.e., there is a “blocking annulus” (of type  $\Gamma^1$ ) of radii  $\rho_1, \rho_2$ ;  $\rho_2$  must be so large that the phase space measure of all the line elements based in the inner circle of the annulus, whose trajectories reach the outer circle before coming back to inner circle, does not exceed  $1/k$ ;
- $\rho - \rho_2 \geq k$  and, for  $\mathbf{n} \notin Z_i$  with  $\rho_2 < \|\mathbf{n}\| \leq \rho$ ,  $\ell_{\mathbf{n}} = m$ , i.e., the outer part of the configuration  $\eta_k$  (safe for the blocking circles

$D_j$ , which do not belong to  $Z_i$ ) is a non-blocking annulus (of type  $\Gamma^m$ ) with thickness  $\geq k$ .

Then, let us denote by  $\mathcal{A}_i^k$  the cylinder in  $\mathcal{L}_i$  defined by all the configurations in  $Z_i$  that coincide with  $\eta_k$  on its support. It is not hard to show that  $\mathcal{A}_i^k$  is open w.r.t. the metric (5). Hence

$$(9) \quad \mathcal{R}_i := \bigcap_{n \in \mathbb{Z}^+} \bigcup_{k \geq n} \mathcal{A}_i^k$$

is a  $G_\delta$  set that is clearly dense in  $\mathcal{L}_i$ . The recurrence of any LG  $\ell \in \mathcal{R}_i$  is proved essentially as in Section 4.1, by showing that, if  $W$  is a wandering set, then  $\mu(W \cap \mathcal{M}_{\mathbf{n}}) \leq 1/k$ , for all  $\mathbf{n} \in \mathbb{Z}^2$  and  $k$  large enough. Also, the construction of the non-blocking annulus in each  $\eta_k$  and the hypothesis on the non-blocking horizontal constant factors (cf. Theorem 4) imply that  $\ell$  has infinite horizon.

The presence of the blocking circles  $D_j$  is necessary to ensure that the set of all the trajectories with initial positions, say, in  $C_{\mathbf{n}}$ , stay confined, within time  $t$ , to a portion of the LG that comprises at most  $Ct^\alpha$  cells, for some  $C, \alpha > 0$ . (Here one uses (A6') as well.) This makes the equivalent of Lemma 5 hold, which in turn yields hyperbolicity, ergodicity and the other statements of Theorem 4.

Finally,  $\mathcal{R} := \bigcup_i \mathcal{R}_i$  is a dense uncountable set of LGs in  $\Omega^{\mathbb{Z}^2}$  that possess all the sought properties.

## REFERENCES

- [CD] N. CHERNOV, D. DOLGOPYAT, *Hyperbolic billiards and statistical physics*, International Congress of Mathematicians. Vol. II, 1679–1704, Eur. Math. Soc., Zürich, 2006.
- [CLS] G. CRISTADORO, M. LENCI, M. SERI, *Recurrence for quenched random Lorentz tubes*, Chaos 20 (2010), 023115.
- [DSV] D. DOLGOPYAT, D. SZÁSZ, T. VARJÚ, *Limit theorems for locally perturbed planar Lorentz processes*, Duke Math. J. 148 (2009), 459–499.
- [G] K. E. GREW, *Thermal diffusion in hydrogen-deuterium mixtures*, Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. 178 (1941), 390–399.
- [HIV] H. R. HEATH, T. L. IBBS, N. E. WILD, *The diffusion and thermal diffusion of hydrogen-deuterium, with a note on the thermal diffusion of hydrogen-helium*, Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. 178 (1941), 380–389.
- [J] G. JOOS, *Lehrbuch der Theoretischen Physik*, Akademische Verlagsgesellschaft, Leipzig, 1932.
- [KS] A. KATOK, J.-M. STRELCYN (in collaboration with F. LEDRAPPIER AND F. PRZYTYCKI), *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lectures Notes in Mathematics 1222, Springer-Verlag, Berlin-New York, 1986.

- [K] A. J. KOX, *H. A. Lorentz's contribution to kinetic gas theory*, Ann. Sci. 47 (1990), 591-606.
- [Le1] M. LENCI, *Aperiodic Lorentz gas: recurrence and ergodicity*, Ergodic Theory Dynam. Systems 23 (2003), 869–883.
- [Le2] M. LENCI, *Typicality of recurrence for Lorentz gases*, Ergodic Theory Dynam. Systems 26 (2006), 799–820.
- [Le3] M. LENCI, *On infinite-volume mixing*, Comm. Math. Phys. 298 (2010), 485-514.
- [LW] C. LIVERANI, M. WOJTKOWSKI, *Ergodicity in Hamiltonian systems*, in: Dynamics Reported: Expositions in Dynamical Systems (N.S.), vol. 4, Springer-Verlag, Berlin, 1995.
- [Lo] H. A. LORENTZ, *The motion of electrons in metallic bodies I, II, and III*, Koninklijke Akademie van Wetenschappen te Amsterdam, Section of Sciences, 7 (1905), 438–453, 585–593, 684–691.
- [SLDC] M. SERI, M. LENCI, M. DEGLI ESPOSTI, G. CRISTADORO, *Recurrence and higher ergodic properties for quenched random Lorentz tubes in dimension bigger than two*, preprint (2010), [arXiv:1011.6414v1](https://arxiv.org/abs/1011.6414v1).
- [Tr1] S. TROUBETZKOY, *Recurrence in generic staircases*, preprint (2010), [arXiv:1005.0292v2](https://arxiv.org/abs/1005.0292v2) (to appear in Discr. Cont. Dynam. Systems).
- [Tr2] S. TROUBETZKOY, *Typical recurrence for the Ehrenfest wind-tree model*, J. Statist. Phys. 141 (2010), 60–67.

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