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INFINITE-HORIZON LORENTZ TUBES AND GASES: RECURRENCE AND ERGODIC PROPERTIES

MARCO LENCI AND SERGE TROUBETZKOY

Abstract. We construct classes of two-dimensional aperiodic Lorentz systems that have infinite horizon and are 'chaotic', in the sense that they are (Poincaré) recurrent, uniformly hyperbolic and ergodic, and the first-return map to any scatterer is K-mixing. In the case of the Lorentz tubes (i.e., Lorentz gases in a strip), we define general measured families of systems (*ensembles*) for which the above properties occur with probability 1. In the case of the Lorentz gases in the plane, we define families, endowed with a natural metric, within which the set of all chaotic dynamical systems is uncountable and dense.

MSC 2010: 37D50, 37A40, 60K37, 37B20, 36A25.

1. Introduction

A Lorentz system is a dynamical system of a point particle moving inertially in an unbounded domain (in this paper we consider only planar domains) endowed with an infinite number of dispersing (i.e., locally convex) scatterers. When the particle hits a scatterer, which is regarded as infinitely massive, it undergoes an elastic collision: the angle of reflection equals the angle of incidence.

The most popular such system is undoubtedly the Lorentz gas, devised by Lorentz in 1905 [Lo] to study the dynamics of an electron in a crystal; the term 'gas' was introduced later in the century, when versions of the Lorentz model were used to give a statistical description of the motion of a molecule in a gas.[1]

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[1]To our knowledge, the first appearance of this model within the scope of the kinetic theory of gases is in a 1932 textbook of theoretical physics [J]. The model is used briefly for the estimation of the number of collisions per unit time of a molecule in a gas; however, no mention of Lorentz is made. The first occurrence of the phrase 'Lorentz gas' in the scientific literature seems to date back to 1941 [Gr]. Curiously, the first occurrence of the phrase 'Lorentzian gas' that we are aware of is found in the article preceding [Gr] in the same issue of the same journal [HIV]. Finally, it is interesting to notice that, although Lorentz's original papers [Lo] are

In the past century, Lorentz systems have been preferred models in the fields of statistical physics, optics, acoustics, and generally anywhere the diffusive properties of a chaotic motion were to be investigated. Throughout this history, for reasons of mathematical convenience, the models that were studied most often and most deeply were *periodic* (i.e., the configuration of scatterers was invariant for the action of a discrete group of translations) and with *finite horizon* (i.e., the free flight was bounded above). Only recently have aperiodic Lorentz systems come to the fore [Le1, Le2, DSV, CLS, SLDC, Tr2] (within the scope of dynamical systems, that is, aperiodic models had already been studied in other contexts; e.g., [Ga, BBS, BBP]). However, while there is a considerable literature on infinite-horizon periodic Lorentz gases, almost nothing is known, at least to these authors, on aperiodic Lorentz systems with infinite horizon.

In this note we consider 2D Lorentz gases and also Lorentz tubes, that is, Lorentz systems confined to a strip of $\mathbb{R}^2$ [CLS, SLDC]. We construct billiards that have infinite horizon and possess the ergodic properties that one would expect of these chaotic systems, such as ergodicity and strong mixing properties. For these infinite-measure preserving dynamical systems, it turns out that (Poincaré) recurrence is not only a necessary but also a sufficient condition for ergodicity; this is a consequence of the hyperbolic structure that our systems can be shown to have [Le1, CLS]. As for mixing, since a universally accepted definition of mixing is not available in infinite ergodic theory (see, e.g., the discussion in [Le3]), we characterize this aspect of the dynamics by proving that certain first-return maps are K-automorphisms (which implies strong mixing). This is again a consequence of recurrence and hyperbolicity.

Another important question that we aim to discuss is that of the typicality of such ergodic properties. One would expect that, when the effective dimension is one (Lorentz tubes) or two (Lorentz gases), recurrence, and all the stochastic properties that it entails, hold for "most" systems (and one would expect the former case to be more easily worked out than the latter).

We address this question by introducing the following two classes of billiards. A strip and the plane are partitioned into infinitely many congruent *cells*. In each cell we place a configuration of *dispersing* scatterers, i.e., a finite union of piecewise smooth closed sets, whose smooth boundary components are seen as convex from the exterior.

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about electrons in a metal and not molecules in a gas, he treats the problem by deriving and solving a Boltzmann-like transport equation; see also [K].

The set of all the *global* configurations of scatterers gives rise to a family of dynamical systems of the same type. A natural distance between two configurations can be defined, which makes the above set a metric space. Furthermore, if the configuration is chosen according to a probability law, the family becomes a measured family, or an *ensemble*, of dynamical systems. This structure is often referred to as a *quenched random dynamical system*.

In the case of the tubes we give fairly explicit sufficient conditions for the above-mentioned ergodic properties, thus proving that, for many reasonable random laws on the "disorder" (including all non-degenerate Bernoulli measures), such properties hold almost surely in the ensemble; we state these results in Section 2. For the harder case of the gases, we prove that the set of the ergodic systems is uncountable and dense within the whole space; this is described in Section 3. Outlines of the proofs are given in Section 4.

It is worthwhile to mention that the billiards we construct have infinite but *locally finite* horizon, that is, though the free flight has no upper bound, no straight line exists that intersects no scatterers.

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2. Lorentz tubes

Let C denote the unit square, which will be henceforth referred to as the *cell*. Let G^1 denote an open segment along one of the sides of C, say the left one, and G^2 the corresponding segment on the opposite side (via the natural orthogonal projection). G^1 and G^2 are called the *gates* of C.

A *local configuration of scatterers* is a "fat" closed subset $\Gamma \subset C$ (this means that Γ is the closure of its interior) such that:

(A1) $\partial\Gamma$ is made up of a finite number of C^3-smooth curves γ_i, which may only intersect at their endpoints (γ_i is always considered a closed set).

(A2) $\partial C \setminus (G^1 \cup G^2) \subset \partial\Gamma$; and G^1, G^2 do not intersect $\partial\Gamma$.

(A3) Either γ_i is part of ∂C or its curvature is bounded below by a positive constant (with the convention that positive curvature means that γ_i bends towards the inside of Γ).

(A4) The angle formed by γ_i and each intersecting γ_j (or G^j) is non-zero.

Remark 1. *Observe that* (A1)-(A3) *imply that the flat parts of* $\partial\Gamma$ *are 'external' boundaries, in the sense that they can never be reached by a particle (the contrary would violate the hypothesis on the "fatness" of* Γ*). So we are dealing with dispersing, not semi-dispersing, billiards.*

We consider a finite number $\Gamma^1, \Gamma^2, \ldots, \Gamma^m$ of such local configurations (see Figs. 1 and 2).

FIGURE 1. A blocking cell for the Lorentz tube

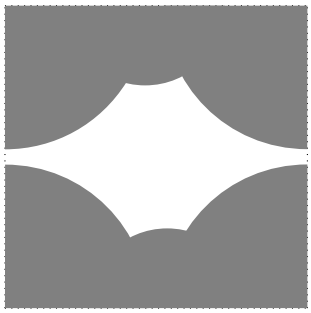

FIGURE 2. A non-blocking cell for the Lorentz tube

A Lorentz tube (LT) is a chain of cells C_n ($n \in \mathbb{Z}$) such that G^2_n, the right gate of C_n, coincides with G^1_{n+1}, the left gate of C_{n+1}. More precisely, call $C_n := [n, n+1] \times [0, 1]$ the particular copy of C immersed in $\mathbb{R}^2$ as indicated, and denote $\Omega := \{1, 2, \ldots, m\}$. Then, for $\ell := (\ell_n)_{n\in\mathbb{Z}} \in \Omega^{\mathbb{Z}}$, we define the *billiard table*

$$\mathcal{Q} = \mathcal{Q}_\ell := \bigcup_{n\in\mathbb{Z}} C_n \setminus \Gamma^{\ell_n}_n, \tag{1}$$

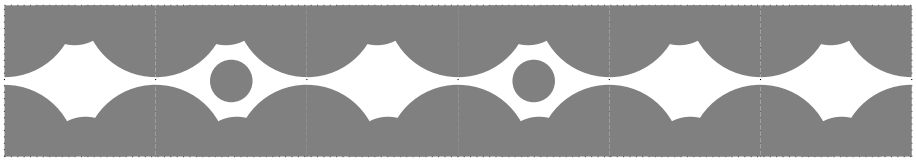

FIGURE 3. A Lorentz tube

where $\Gamma_n^{\ell_n}$ is the configuration Γ^{ℓ_n} translated to the cell C_n (see Fig. 3). The collection $(\Gamma_n^{\ell_n})_{n\in\mathbb{Z}}$—equivalently ℓ—is called the *global configuration of scatterers*.

We henceforth say 'the LT ℓ' to mean both the table $\mathcal{Q}_\ell$ and the billiard dynamics defined on it. By this we mean, precisely, the dynamical system $(\mathcal{M}_\ell, T_\ell, \mu_\ell)$—more concisely, $(\mathcal{M}, T, \mu)$—where:

- $\mathcal{M}$ is the collection of all the *line elements* of the dynamics, i.e., all the position-velocity pairs (q, v), where $q \in \partial\mathcal{Q}$ and v is a unit vector based in q and pointing toward the interior of $\mathcal{Q}$. (q, v) is meant to represent the dynamical variables of the particle right after a collision (v can be chosen unitary because in this Hamiltonian system the conservation of energy equals the conservation of speed).
- T is the map that takes (q, v) into the next post-collisional line element (q', v'), along the billiard trajectory of (q, v); this map fails to be well defined only at a negligible set of line elements, which are called *singular*, cf. below. T is usually called the *(standard) billiard map*.
- μ is the invariant measure induced on the Poincaré section $\mathcal{M}$ by the Liouville measure; it is well known that $d\mu(q, v) = \langle n_q, v\rangle dqdv$, where n_q is inner unit normal to $\partial\mathcal{Q}$ in q. It is easy to verify that the set of all singular points in phase space is null w.r.t. μ Finally, as is evident, $\mu(\mathcal{M}) < \infty$ if and only if the total length of $\partial\mathcal{Q}$ is finite.

Notice that, by the definition of $\mathcal{Q}$, a trajectory that intersects a gate G_n^i crosses it. (The trajectory that runs along the segment G_n^i will always be considered singular.)

We assume that there are two types of local configurations: the *blocking* configurations, corresponding to the index set $\Omega_B := \{1, 2, \ldots, m'\}$ ($m' < m$), and the *non-blocking* confugurations, corresponding to the set $\Omega_{NB} := \{m' + 1, m' + 2, \ldots, m\}$. The former type verifies the following condition:

(A5) If $a \in \Omega_B$, any billiard trajectory that enters a cell with configuration Γ^a must experience a collision before leaving it (Fig. 1).

An example of a configuration Γ^a, with $a \in \Omega_{NB}$, is shown in Fig. 2.

Clearly, an LT $\ell \in \Omega_B^{\mathbb{Z}}$ has finite horizon, i.e., the free flight between two successive collisions has an upper bound. An arbitrary LT in $\Omega^{\mathbb{Z}}$ might not have this property.

A word $a_1 a_2 \cdots a_l$, with $a_i \in \Omega$, is called a *factor* of $\ell \in \Omega^{\mathbb{Z}}$ if there exists n such that $\ell_{n+i} = a_i$, for $i = 1, 2, \ldots, l$. The factor is called blocking (respectively, non-blocking) if all the a_i belong to Ω_B (respectively, Ω_{NB}); it is called constant if they are all equal. The positive integer l is called the length of the factor.

The technique used by Troubetzkoy in [Tr1], when applied to the present models, easily implies that any LT in $\Omega^{\mathbb{Z}}$ which has arbitrarily long blocking constant factors (both forwards and backwards) is recurrent. Furthermore, Cristadoro, Lenci and Seri have shown that, for LTs in $\Omega_B^{\mathbb{Z}}$, ergodicity and recurrence are equivalent [CLS], and they imply K-mixing for suitable return maps (this last result is actually stated in [SLDC] but its proof applies as well to the models of [CLS]).

Our results on the Lorentz tubes are a combination and an extension of these ideas. To describe them we introduce some notation that will appear obscure at first, but will be explained shortly. For fixed ℓ, define $g_0^+ := g_0^- := 0$ and, recursively for $j > 0$,

$$g_{j+1}^{\pm} := \min \left\{ k \in \mathbb{Z}^+ \;\middle|\; \ell_{\pm \sum_{i=0}^{j} g_i^{\pm} \pm k} \in \Omega_B \right\}. \tag{2}$$

In other words, $g_j^+ - 1$ (respectively, $g_j^- - 1$) is the length of the jth non-blocking factor to the right (respectively, to the left) of the cell C_0 (with the convention that between two blocking cells there is a non-blocking factor of length 0). Notice that, if $\ell_0 \in \Omega_B$, this coding reflects exactly the sequence of non-blocking factors of ℓ. Otherwise, there is a little difference which, see Remark 3 below, does not affect the upcoming statement.

Theorem 2. *Assume* (A1)-(A5). *For any $\ell \in \Omega^{\mathbb{Z}}$ which has arbitrarily long blocking constant factors, both forward and backwards, and such that both sequences $(g_j^{\pm})_{j \in \mathbb{N}}$ grow at most like a power-law, the corresponding dynamical system $(\mathcal{M}, T, \mu)$ is:*

(a) uniformly hyperbolic, in the sense that local stable and unstable manifolds exist at a.e. point of $\mathcal{M}$, and the corresponding (T-invariant) laminations are absolutely continuous w.r.t. μ (see, e.g., [Le1] *for details); the contraction coefficient of T^n, along the stable direction, is bounded above by $C\lambda^n$ and its expansion coefficient, along the unstable direction, is bounded below by $C^{-1}\lambda^{-n}$, where $C > 0$ and $\lambda \in (0, 1)$ are uniform constants;*

(b) *recurrent in the sense of Poincaré, i.e., given a measurable* $A \subset \mathcal{M}$*, the orbit of a.e. point in* A *comes back to* A *infinitely many times;*
(c) *ergodic, that is, if* $T(A) = A \bmod \mu$*, then either* A *or its complement has measure zero.*

Furthermore, the first-return map to any smooth component of $\partial\mathcal{Q}$*, of the type* γ_i *as in* (A1)*, is K-mixing.*

Remark 3. *In the above theorem, both hypotheses are shift-invariant in* $\Omega^{\mathbb{Z}}$*, as a relocation of the origin on* ℓ *will produce at most a shift in* $(g_j^{\pm})$ *and a change of the first few terms. Therefore, when convenient—to do away with the problem mentioned in the previous paragraph—we make the convention that any* ℓ *is shifted to the left the minimum amount of times for* C_0 *to have a blocking configuration.*

An LT as described in Theorem 2 need not have infinite horizon. To start with, it is necessary that the non-blocking configurations let free trajectories through, something that was not postulated. But more is needed as well. For instance, if one designs the local configurations Γ^a, with $a \in \Omega_{NB}$, so that, for any $l \in \mathbb{Z}^+$, all non-blocking factors of length l admit a free flight of length $\geq l$ (and this is easy, cf. Fig. 2), then a necessary and sufficient condition for an LT to have infinite horizon is that at least one of the two sequences $(g_j^{\pm})$ is unbounded. Or one might ask for something less: for example, that just the *constant* non-blocking factors admit long free flights. This is enough to guarantee that very many LTs have an infinite horizon, cf. Corollary 4.

An important question is whether the LTs to which Theorem 2 applies are typical in $\Omega^{\mathbb{Z}}$. This of course depends on the definition of 'typical'. One strong notion of typicality is the measure-theoretic notion, provided a probability measure Π is put on $\Omega^{\mathbb{Z}}$ (endowed with the natural σ-algebra generated by the cylinders). In this case, $(\Omega^{\mathbb{Z}}, \Pi)$ becomes a measured family (in jargon, an *ensemble*) of dynamical systems, which we call *quenched random Lorentz tube.*

Many reasonable measures Π ensure that the assertions of Theorem 2 hold Π-almost surely. Here is an example:

Corollary 4. *Let* $(p_1, p_2, \ldots, p_m)$ *be a stochastic vector, with* $p_a > 0$*, and let* Π *be the Bernoulli measure on* $\Omega^{\mathbb{Z}}$ *relative to that vector (i.e., the unique measure for which* $\Pi(\ell_n = a) = p_a$*, for all* n*). Then* Π*-a.e. LT in* $\Omega^{\mathbb{Z}}$ *has the properties stated in Theorem 2. Furthermore, if any non-blocking constant factor of length* l *(*$\forall l \in \mathbb{Z}^+$*) admits a free flight of length* $\geq l$*, then a.e. LT has infinite horizon as well.*

FIGURE 4. A blocking cell for the Lorentz gas

Proof of Corollary 4. Let $p_B := \sum_{a=1}^{m'} p_a$ and $p_{NB} := \sum_{a=m'+1}^{m} p_a$. By hypothesis, $p_B, p_{NB} \in (0,1)$. It is evident that, w.r.t. Π, the random variables $(g_j^{\pm})$ are i.i.d., with distribution $\Pi(g_j^{\pm} = k) = p_B(p_{NB})^{k-1}$ $(k \in \mathbb{Z}^+)$.

We claim that, for Π-a.e. ℓ, there exists $K = K(\ell)$ such that $g_j^{\pm} \leq Kj$. In fact, observe that

$$\Pi(g_j^{\pm} \geq j) = (p_{NB})^{j-1}. \tag{3}$$

This implies that the probabilities of the 'events' $\{g_j^{\pm} \geq j\}$ form a summable sequence. Thus, by Borel-Cantelli, for a.e. ℓ, the inequality $g_j^{\pm}/j \geq 1$ is verified only for a finite number of j's. Setting $K := \max_j(g_j^{\pm}/j)$ proves the claim.

To finish the proof of Corollary 4, observe that, for a non-degenerate Bernoulli measure, a.e. ℓ contains arbitrarily long blocking and non-blocking constant factors. □

3. Lorentz gases

We now turn to the truly two-dimensional case. We tile the plane with $\mathbb{Z}^2$ copies of the unit square C. In this case C is endowed with four gates: G^1 and G^2, congruent and opposite open segments (say, the left and the right gate, respectively); G^3 and G^4, again congruent and opposite open segments (the lower and the upper gates).

Apart from this, we have the same structure as in Section 2: a finite number of local configurations indexed by the set $\Omega = \Omega_B \cup \Omega_{NB}$, where Ω_B denotes the blocking and Ω_{NB} the non-blocking configurations, as in Figs. 4 and 5. Again, these sets verify five assumptions, (A1), (A3), (A4), (A5), and the counterpart of (A2):

(A2′) $\partial C \setminus (G^1 \cup \ldots \cup G^4) \subset \partial\Gamma$; and $G^1, \ldots, G^4$ do not intersect $\partial\Gamma$.

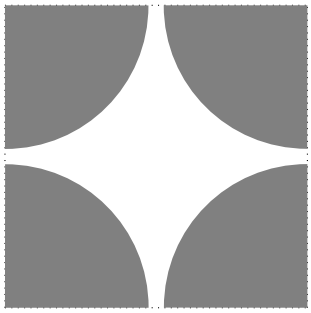

FIGURE 5. A non-blocking cell for the Lorentz gas

A global configuration is the collection $\ell := (\ell_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2} \in \Omega^{\mathbb{Z}^2}$, with $\mathbf{n} := (n_1, n_2)$, and the billiard table is

$$\mathcal{Q} = \mathcal{Q}_\ell := \bigcup_{\mathbf{n}\in\mathbb{Z}^2} C_{\mathbf{n}} \setminus \Gamma^{\ell_{\mathbf{n}}}_{\mathbf{n}}, \tag{4}$$

where $C_{\mathbf{n}} := [n_1, n_1+1] \times [n_2, n_2+1]$, and $\Gamma^{\ell_{\mathbf{n}}}_{\mathbf{n}}$ is the configuration $\Gamma^{\ell_{\mathbf{n}}}$ translated to $C_{\mathbf{n}}$. We call $\mathcal{Q}$ (and the billiard map thereon) a Lorentz gas (LG)—see a realization in Fig. 6.

A word $a_1 a_2 \cdots a_l$ is called a *horizontal factor* of $\ell \in \Omega^{\mathbb{Z}^2}$ if there exists $\mathbf{n} = (n_1, n_2)$ such that $\ell_{n_1+i, n_2} = a_i$, for $i = 1, 2, \ldots, l$. The analogous definition is given for a vertical factor. As in Section 2, a factor is called non-blocking if $a_i \in \Omega_{NB}$ and constant if $a_i = a$, for all i.

Since we are interested in infinite-horizon billiards, we discuss a sufficient condition for obtaining this property. It will be seen in Section 4 that the LGs that we construct possess arbitrarily long, both horizontal and vertical, non-blocking constant factors. They have infinite horizon if all non-blocking, say, horizontal factors of length l ($\forall l \in \mathbb{Z}^+$) admit a free flight of length $\geq l$.

In the two-dimensional setting no one has succeeded in proving that recurrence is a typical property in a measure-theoretic sense (this is actually an important open problem, cf. [Le2, CD]). The known results are with respect to a topological notion of typicality. For this, we make $\Omega^{\mathbb{Z}^2}$ a metric space by endowing it with the distance

$$\operatorname{dist}(\ell, \ell') := \sum_{\mathbf{n}\in\mathbb{Z}^2} 2^{-|n_1|-|n_2|} \left|\ell_{\mathbf{n}} - \ell'_{\mathbf{n}}\right|. \tag{5}$$

It was shown by Lenci that, relative to the above metric, the Baire-typical LG in $\Omega_B^{\mathbb{Z}^2}$ is recurrent [Le2] and ergodic [Le1]. On the other

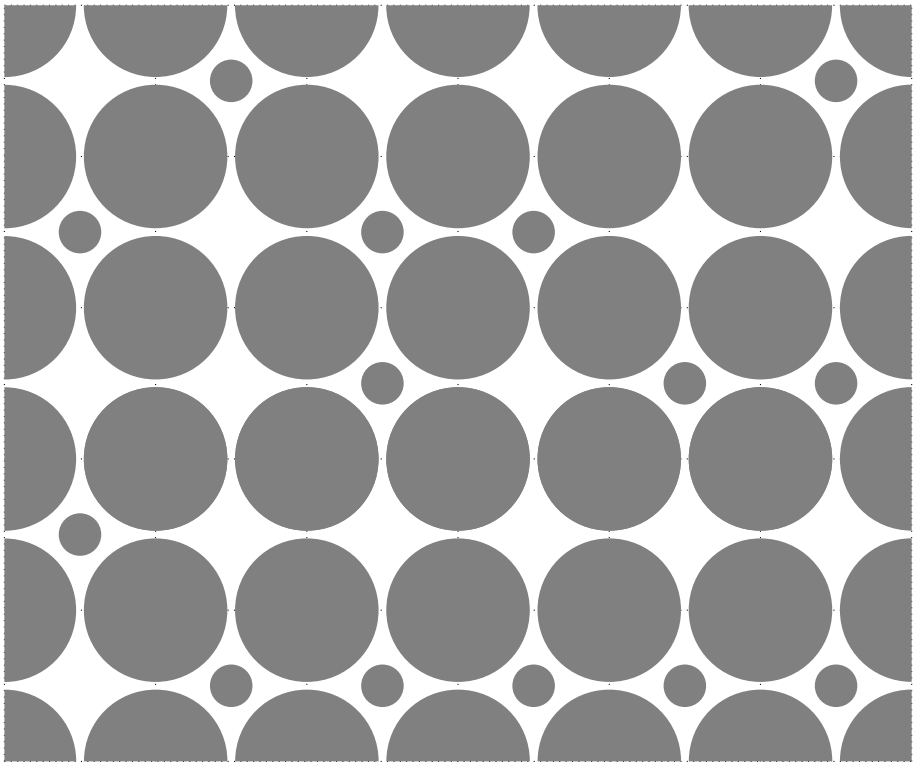

FIGURE 6. A Lorentz gas

hand, Troubetzkoy has shown that, under the assumption that long factors of non-blocking configurations admit long free flights, the Baire-typical LG in $\Omega^{\mathbb{Z}^2}$ is recurrent and has infinite horizon [Tr2]. Once again, we combine and extend these ideas to show that a great number of infinite-horizon LGs are recurrent and chaotic in the sense of Theorem 2.

Theorem 5. *Assuming* (A1), (A2′), (A3)-(A5), *the metric space* $\Omega^{\mathbb{Z}^2}$ *contains a dense uncountable set of LGs that are hyperbolic, recurrent and ergodic in the sense of Theorem 2, and such that the return map to any smooth component of the type* γ_i *is K-mixing. Furthermore, if, for all* $l \in \mathbb{Z}^+$, *any non-blocking horizontal constant factor of length* l *admits a free flight of length* $\geq l$, *then those LGs have infinite horizon as well.*

4. PROOFS

4.1. **Sketch of the proof of Theorem 2.** Let us fix an LT as in the statement of Theorem 2. Its recurrence is proved essentially in

the same way as for the staircase billiards of [Tr1]. For the sake of completeness, we give an outline of the argument.

Here and in the remainder we are going to need a notation for the portion of the phase space pertaining to the cell C_n:

$$\mathcal{M}_n := \{(q, v) \in \mathcal{M} \mid q \in C_n\}\,. \tag{6}$$

It is clear by the hypotheses that the LT contains, both forward and backwards, arbitrarily long constant factors of the *same* blocking configuration, say Γ^1. In this paragraph and the next, when we say blocking factor, we will always mean a factor of configurations Γ^1. Consider a blocking factor beginning with the cell C_{n_1} and ending with the cell C_{n_2} ($n_2 > n_1$). Let us call $A_{n_1}^{n_2}$ the set of all the line elements of $\mathcal{M}_{n_1}$ whose trajectories visit a cell to the right of C_{n_2}, before coming back to C_{n_1}. Likewise, let $A_{n_2}^{n_1}$ be the set of all the line elements of $\mathcal{M}_{n_2}$ whose trajectories visit a cell to the left of C_{n_1}, before coming back to C_{n_2}. Now fix $k \in \mathbb{Z}^+$. Since an LT made up entirely of cells Γ^1 is recurrent, there exists a positive integer l_k such that, if the length of the factor (namely, $n_2 - n_1 + 1$) is bigger than or equal to l_k, both $\mu(A_{n_1}^{n_2})$ and $\mu(A_{n_2}^{n_1})$ are smaller than $1/k$.

Consider a wandering set W and, for any $n \in \mathbb{Z}$, set $W_n := W \cap \mathcal{M}_n$. W_n is also a wandering set. For all $k \in \mathbb{Z}^+$, there exist a blocking factor of length $\geq l_k$ (say, from the cell C_{n_1} to the cell C_{n_2}) to the right of C_n and a blocking factor of length $\geq l_k$ (say, from C_{n_3} to C_{n_4}) to the left of C_n. By definition, the orbits of the points of W_n are all disjoint and unbounded, which means they must intersect either $A_{n_1}^{n_2}$ or $A_{n_4}^{n_3}$ in distinct points. Therefore, using the invariance of the measure, $\mu(W_n) \leq \mu(A_{n_1}^{n_2} \cup A_{n_4}^{n_3}) = 2/k$. Since k and n are arbitrary, W is a null set.

Turning to the hyperbolicity and ergodicity, once the recurrence is known, one proceeds as in [Le1] or [CLS]. (A fairly accurate summary of the whole proof is given in [SLDC], although that article refers to LTs in dimension higher than two.) Here we limit ourselves to mentioning which parts of the proof can be worked out using the standard techniques for classical hyperbolic billiards [KS, LW] and which ones need to be adapted to our particular infinite-measure system.

Focusing on the hyperbolicity first, it can be seen that all the arguments used in the proof of Theorem 2*(a)*, with one exception, are local, that is, depend on the value of T (its invariant cones, its distortion coefficients, etc.) on a neighborhood of a given point, or a given orbit. In other words, they cannot distinguish between a finite-measure dispersing billiard—for which everything works well [KS]—or an infinite-measure one.

The only argument that is not local is the one whereby μ-almost all orbits stay sufficiently far away from the singular points of T (so that the singularities of the map do not interfere with the construction of the local stable and unstable manifolds). The singular points are organized in smooth curves, called *singularity lines*. Each such line corresponds to all the line elements whose first collision occurs at a given corner of $\partial\mathcal{Q}$, or tangentially to a certain smooth component of it. Hence, there are a countable number of singularity lines. They can be counted (or at least overestimated) in a way that, in each region of the phase space $\mathcal{M}_\gamma := \{(q,v) \in \mathcal{M} \mid q \in \gamma\}$, where γ is a smooth portion of the boundary as in (A1), there are at most two singularity lines for every source of singularity (a vertex or a tangency) "seen" from γ. (The factor 2 comes from the fact that there are two ways to be tangent to a smooth boundary, one for each orientation.) Also, the length of each singularity line is bounded above by a universal constant having to do with the size of the cell.

Let us indicate with $\mathcal{S}$ the singular set of T, that is, the union of all the singularity lines described above. It turns out that, if Lemma 6 below holds, almost all orbits approach $\mathcal{S}$ no faster than a negative power-law in time (where by time we mean 'number of collisions'). This is enough to make the sought argument work [Le1, CLS].

Moving on to statement *(c)*, as explained in [Le1], one can exploit a suitable Local Ergodicity Theorem for billiards (say, the version of [LW]), which uses only local arguments except in its most delicate part, the so-called Tail Bound. It turns out, however, that a version of the Tail Bound can be proved for our LTs too, if the following result holds (cf. Sec. 3 of [SLDC]).

Lemma 6. *There exist constants $C, \alpha > 0$ such that, for any $n \in \mathbb{Z}$, any $U \subseteq \mathcal{M}_n$, and any $t \in \mathbb{Z}^+$, $\bigcup_{j=0}^t T^j(U)$ intersects at most Ct^α singularity lines from $\mathcal{S}$. (In more descriptive terms, each trajectory with initial conditions in U may approach at most Ct^α singularity lines of T, in time t, and this bound is uniform in U, if U is not too large.)*

Proof. Let us define $f_j^\pm := \pm\sum_{i=0}^j g_i^\pm$, so that f_j^+ is the location of the blocking cell to the right of the jth non-blocking factor, on the right side of the tube w.r.t. C_0; while f_j^- is the location of the blocking cell to the left of the jth non-blocking factor, on the left side of the tube; cf. definition (2). It follows from the hypotheses of Theorem 2 that both sequences $(|f_j^\pm|)_{j\in\mathbb{N}}$ are bounded by a power-law in j.

Coming to the statement of the lemma, we may assume that $U = \mathcal{M}_0$ (in fact, proving the result for $U' := \mathcal{M}_n$ automatically proves it for

U; then, choosing $n = 0$ is no loss of generality because a translation of the tube does not change our argument—cf. Remark 3).

By (A5), the configuration space trajectories of length t, with initial conditions in $\mathcal{M}_0$, cannot go further left than the cell $C_{f_t^-}$ or further right than $C_{f_t^+}$. The corresponding phase space orbits are thus contained in

$$A_t := \bigcup_{n=f_t^-}^{f_t^+} \mathcal{M}_n. \tag{7}$$

A line element in A_t can "see" at most those cells that range from $C_{f_{t+1}^-}$ to $C_{f_{t+1}^+}$. Therefore, the number of singularity lines of T in each set of the type $\mathcal{M}_\gamma \subset A_t$ ($\mathcal{M}_\gamma$ was defined earlier) does not exceed $C'(f_{t+1}^+ - f_{t+1}^- + 1)$, for some $C' > 0$. But there are at most $C''(f_t^+ - f_t^- + 1)$ sets of that type. The product of these two estimates, which, as shown earlier, grows no faster than a power-law in t, is an upper bound for the number of singularity lines from $\mathcal{S}$ in A_t. □

Once we have local ergodicity (all but countably many points in phase space have a neighborhood contained in one ergodic component), global ergodicity is easily shown. Also, the assertion about the K-mixing first-return map is proved as in [SLDC], Sec. 4.

4.2. **Sketch of the proof of Theorem 5.** Let us endow $\mathbb{Z}^2$ with the norm $\|\mathbf{n}\| = \|(n_1, n_2)\| := |n_1| + |n_2|$. For $j \in \mathbb{Z}^+$, set $D_j := \{\mathbf{n} \in \mathbb{Z}^2 \mid \|n\| = j^2\}$ (this set resembles the border of a rhombus in $\mathbb{Z}^2$). Given $i \in \mathbb{Z}^+$, define

$$Z_i := \mathbb{Z}^2 \setminus \bigcup_{j \geq i} D_j \tag{8}$$

and

$$\mathcal{L}_i := \left\{ \ell = (\ell_{\mathbf{n}}) \in \Omega^{\mathbb{Z}^2} \;\middle|\; \ell_{\mathbf{n}} = 1, \forall \mathbf{n} \notin Z_i \right\}. \tag{9}$$

In other words, $\mathcal{L}_i$ is the set of all the global configurations which have "blocking circles" (namely, circles filled with cells of type Γ^1) at all radii j^2, with $j \geq i$. Clearly, $\mathcal{L}_i \cong \Omega^{Z_i}$ in a natural sense. Note that $\mathcal{L}_i \subset \mathcal{L}_{i+1}$ and that $\bigcup_i \mathcal{L}_i$ is dense in $\Omega^{\mathbb{Z}^2}$. In each $\mathcal{L}_i$ we apply the method of [Tr2] to construct a G_δ dense set $\mathcal{R}_i$ of recurrent Lorentz gases. Let us sketch this method.

In what follows, whenever we mention circles, balls, annuli, we will always mean circles, balls, annuli in $\mathbb{Z}^2$, relative to the norm $\|\cdot\|$ and centered in the origin. Fix i and denote by ξ a configuration of cells in a ball of Z_i, equivalently, a vector of Ω^B, where B is the restriction to

Z_i of a ball in $\mathbb{Z}^2$; B will be referred to as the *support* of ξ. Clearly, there are countably many such (finite) configurations, so we can index them as $(\xi_k)_{k\in\mathbb{Z}^+}$ (the dependence on i is suppressed). For each such k, let us construct a finite configuration η_k such that:

- the support of η_k is a ball of radius $\rho > \rho_1$, where ρ_1 is radius of the support of ξ_k, and the restriction of η_k to the support of ξ_k is ξ_k;
- there exists a positive integer $\rho_2 \in (\rho_1, \rho)$ such that, if $\rho_1 < \|\mathbf{n}\| \le \rho_2$, $\ell_{\mathbf{n}} = 1$, i.e., there is a "blocking annulus" (of type Γ^1) of radii ρ_1, ρ_2; ρ_2 must be so large that the phase space measure of all the line elements based in the inner circle of the annulus, whose trajectories reach the outer circle before coming back to inner circle, does not exceed $1/k$;
- $\rho - \rho_2 \ge k$ and, for $\mathbf{n} \notin Z_i$ with $\rho_2 < \|\mathbf{n}\| \le \rho$, $\ell_{\mathbf{n}} = m$, i.e., the outer part of the configuration η_k (except for the blocking circles D_j, which do not belong to Z_i) is a non-blocking annulus (of type Γ^m) with thickness $\ge k$.

Then, let us denote by $\mathcal{A}_i^k$ the cylinder in $\mathcal{L}_i$ defined by all the configurations in Z_i that coincide with η_k on its support. It is not hard to show that $\mathcal{A}_i^k$ is open in $\mathcal{L}_i$ w.r.t. the metric (5). Hence

$$\mathcal{R}_i := \bigcap_{n\in\mathbb{Z}^+} \bigcup_{k\ge n} \mathcal{A}_i^k \tag{10}$$

is a G_δ set that is clearly dense in $\mathcal{L}_i$. The recurrence of any LG $\ell \in \mathcal{R}_i$ is proved essentially as in Section 4.1, by showing that, if W is a wandering set, then $\mu(W \cap \mathcal{M}_{\mathbf{n}}) \le 1/k$, for all $\mathbf{n} \in \mathbb{Z}^2$ and k large enough. Also, the construction of the non-blocking annulus in each η_k and the hypothesis on the non-blocking horizontal constant factors (cf. Theorem 5) imply that ℓ has infinite horizon.

The presence of the blocking circles D_j is necessary to ensure that the set of all the trajectories with initial positions, say, in $C_{\mathbf{n}}$, stay confined, within time $t \in \mathbb{N}$, to a portion of the LG that comprises at most Ct^α cells, for some $C, \alpha > 0$. This makes the equivalent of Lemma 6 hold, which in turn yields hyperbolicity, ergodicity and the other statements of Theorem 5.

Finally, $\mathcal{R} := \bigcup_i \mathcal{R}_i$ is a dense uncountable set of LGs in $\Omega^{\mathbb{Z}^2}$ that possess all the sought properties.

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