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Discrete surfaces and infinite smooth words*

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Abstract In the present paper, we study the $(1,1,1)$-discrete surfaces introduced in [Jam04]. In [Jam04], the $(1,1,1)$-discrete surfaces are not assumed to be connected. In this paper, we prove that assuming connectedness is not restrictive, in the sense that, any two-dimensional coding of a $(1,1,1)$-discrete surface is the two-dimensional coding of both connected and simply connected ones. In the second part of this paper, we investigate a particular class of discrete surfaces: those generated by infinite smooth words. We prove that the only smooth words generating such surfaces are $K_{\{1,3\}}$, $1K_{\{1,3\}}$ and $2K_{\{1,3\}}$, where $K_{\{1,3\}} = 3331113331313331113331331 3\ldots$ is the generalized Kolakoski's word on the alphabet $\{1,3\}$.

Résumé Dans cet article, nous étudions les $(1,1,1)$-surfaces discrètes introduites dans [Jam04]. Dans l'article [Jam04], les surfaces ne sont pas supposées discrètes. Nous montrons dans cet article qu'il n'est pas restrictif de faire une telle supposition et que tout codage bidimensionel d'une $(1,1,1)$-surface discrète code à la fois une surface connexe et une surface simplement connexe. La seconde partie de cet article est consacrée à l'étude des surfaces discrètes engendrées par des mots lisses. Nous démontrons que les seuls mots lisses engendrant de telles surfaces sont les mots $K_{\{1,3\}}$, $1K_{\{1,3\}}$ et $2K_{\{1,3\}}$, où $K_{\{1,3\}} = 3331113331313331113331331 3\ldots$ est le mot de Kolakoski généralisé sur l'alphabet $\{1,3\}$.

1 Introduction

A wide literature has been devoted to the study of Sturmian words, that is, the infinite words over a binary alphabet which have $n+1$ factors of length n [Lot02]. These words are also equivalently defined as a discrete approximation of a line with irrational slope. Then, many attempts have been done to generalize this class of infinite words to two-dimensional words. For instance, in [Vui98,BV00,ABS04], it is shown that the orbit of an element $\mu \in [0,1[$ under the

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action of two rotations codes a discrete plane. Furthermore, the problem of one or two-dimensional words characterizing discrete lines or planes is investigated in [BV00,Lot02,ABS04,BT04,DGK03]. In [Jam04], the author introduces the $(1,1,1)$-discrete surfaces as a quite natural generalization of the discrete planes of [BV00,ABS04] and shows how to decide whether a given two-dimensional sequence over the three-letter alphabet $\{1,2,3\}$ codes a $(1,1,1)$-discrete surface.

In this paper, we study the connectedness and the simple connectedness of the $(1,1,1)$-discrete surfaces and we show that, given a two-dimensional sequence u over the three-letter alphabet $\{1,2,3\}$, then u codes $(1,1,1)$-discrete surface if and only if u codes a connected one and a simply connected one. Secondly, we study the $(1,1,1)$-discrete surfaces associated with smooth words as in for instance [BL02,BLL02,BBLP03] for the case of 2-letters alphabets and [BBC04] for arbitrary alphabets. These surfaces have local geometric properties and we give an explicit description of the associated smooth words.

This paper is organized as follows. In Section 2, we recall the basic notions concerning $(1,1,1)$-discrete surfaces and the combinatorics of two-dimensional words over a finite alphabet. In Section 3, we prove the first main result of this paper, namely

Theorem. *Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$ be a two-dimensional sequence. The following assertions are equivalent:*

i) the sequence u codes a $(1,1,1)$-discrete surface;
ii) the sequence u codes a connected and simply connected $(1,1,1)$-discrete surface.

We also prove that any connected surface coded by an element of $\{1,2,3\}^{\mathbb{Z}^2}$ is simply connected. Finally, in Section 4, we demonstrate the second main result of this paper, that is:

Theorem. *Let w be a smooth word over the alphabet $\{1,2,3\}$. The tiling $T(w)$ associated to w is a piece of a discrete surface if and only if $w \in \{K_{\{1,3\}}, 1K_{\{1,3\}}, 2K_{\{1,3\}}\}$ where $K_{\{1,3\}}$ is the generalized Kolakoski's word.*

2 Basic notions

2.1 Discrete surfaces

In this section we recall the basic notions concerning $(1,1,1)$-discrete surfaces and discrete planes.

Let $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ denotes the canonical basis of the Euclidean space $\mathbb{R}^3$. An element of $\mathbb{Z}^3$ is called a *voxel*. The fundamental unit cube $\mathcal{C}$ is the set defined by:

$$\mathcal{C} = \left\{x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \mid (x_1,x_2,x_3) \in [0,1]^3\right\}.$$

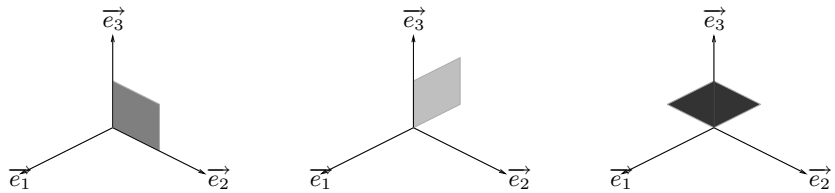
Let $\overrightarrow{x} \in \mathbb{Z}^3$. The set $\overrightarrow{x} + \mathcal{C}$ is called the unit cube pointed by $\overrightarrow{x}$.

Let $\mathcal{P}$ be the plane with equation $(\overrightarrow{v}, \overrightarrow{x}) = \mu$ with $\overrightarrow{v} \in \mathbb{R}^3_+$, $\mu \in \mathbb{R}$ and $(\overrightarrow{v}, \overrightarrow{x}) = v_1x_1 + v_2x_2 + v_2x_3$ denoting the usual scalar product of the vectors

$\overrightarrow{v}$ and $\overrightarrow{x}$. Let $\mathcal{C}_{\mathcal{P}}$ be the union of the unit cubes pointed by a voxel $x \in \mathbb{Z}^3$ and intersecting the open half-space $(\overrightarrow{v}, \overrightarrow{x}) < \mu$. We call discrete plane associated to $\mathcal{P}$ the set $\mathfrak{P}_{\mathcal{P}} = \overline{\mathcal{C}_{\mathcal{P}}} \setminus \overset{\circ}{\mathcal{C}}_{\mathcal{P}}$ of $\mathcal{C}_{\mathcal{P}}$, where $\overline{\mathcal{C}_{\mathcal{P}}}$ (resp. $\overset{\circ}{\mathcal{C}}_{\mathcal{P}}$) is the closure (resp. the interior) of the set $\mathcal{C}_{\mathcal{P}}$ in $\mathbb{R}^3$, provided with its usual topology.

Let us now define the three *fundamental faces* (see Figure 1):

$$\begin{aligned} E_1 &= \{x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \mid (x_2, x_3) \in [0,1[^2\}, \\ E_2 &= \{-x_1\overrightarrow{e_1} + x_3\overrightarrow{e_3} \mid (x_1, x_3) \in [0,1[^2\}, \\ E_3 &= \{-x_1\overrightarrow{e_1} - x_2\overrightarrow{e_2} \mid (x_1, x_2) \in [0,1[^2\}. \end{aligned}$$



(a) Face of type 1. (b) Face of type 2 (c) Face of type 3

Fig. 1. The three fundamental faces.

Let $\overrightarrow{x} \in \mathbb{Z}^3$ and $k \in \{1,2,3\}$. The set $\overrightarrow{x} + E_k$ is called a *pointed face of type* k. The vector $\overrightarrow{x}$ is called the *distinguished vertex* of $\overrightarrow{x} + E_k$.

Let $\pi : \mathbb{R}^3 \longrightarrow \{\overrightarrow{x} \in \mathbb{R}^3 \mid (\overrightarrow{e_1}+\overrightarrow{e_2}+\overrightarrow{e_3}, \overrightarrow{x}) = 0\}$ be the orthogonal projection map onto the plane $\mathcal{P}_0$ with equation $(\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}, \overrightarrow{x}) = 0$.

The following properties hold:

Theorem 1. [ABS04] *Let $\mathfrak{P}_{\mathcal{P}}$ be a discrete plane and let $\mathcal{V}_{\mathcal{P}}$ be the set of vertices of $\mathfrak{P}_{\mathcal{P}}$. We suppose that $\mathcal{P}$ admits a normal vector $\overrightarrow{v} \in \mathbb{R}^3_+$.*

1. *The set $\mathfrak{P}_{\mathcal{P}}$ is partitioned by pointed faces.*
2. *The restriction maps $\pi_{|\mathcal{V}_{\mathcal{P}}} : \mathcal{V}_{\mathcal{P}} \longrightarrow \pi\left(\mathbb{Z}^3\right)$ and $\pi_{|\mathfrak{P}_{\mathcal{P}}} : \mathfrak{P}_{\mathcal{P}} \longrightarrow \left\{\overrightarrow{x} \in \mathbb{R}^3 \mid (\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}, \overrightarrow{x}) = 0\right\}$ are bijective.*

Let us now define the $(1,1,1)$-discrete surfaces as follows:

Definition 1. [Jam04] *A disjoint union $\mathfrak{S} \subseteq \mathbb{R}^3$ of pointed faces is a* $(1,1,1)$-discrete surface *if the map*

$$\begin{array}{rcl} \pi_{|\mathfrak{S}} : \mathfrak{S} & \longrightarrow & \mathcal{P}_0 \\ \overrightarrow{x} & \mapsto & \pi(\overrightarrow{x}) \end{array}$$

is a bijection (see Figure 2). The set $\mathcal{V}_{\mathfrak{S}} = \mathfrak{S} \cap \mathbb{Z}^3$ is called the set of vertices *of $\mathfrak{S}$.*

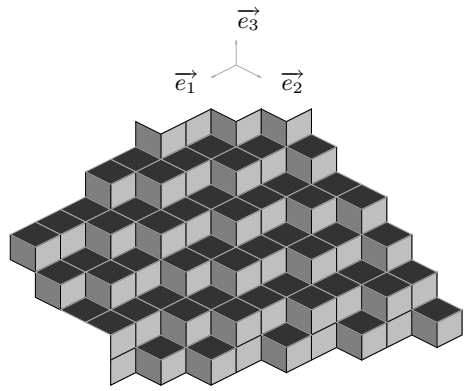


Fig. 2. A piece of a discrete surface

From now on, to reduce the notation, we will use the terminology *discrete surface* instead of $(1,1,1)$-discrete surface.

Before associating a two-dimensional coding over the three letter-alphabet $\{1,2,3\}$ to any discrete surface $\mathfrak{S}$, let us recall a technical lemma:

Lemma 1. *Let $\mathfrak{S}$ be a discrete surface. The following properties hold:*

i) The map

$$\begin{array}{rccc} \pi_{|\mathcal{V}_{\mathcal{S}}} : & \mathfrak{S} & \longrightarrow & \Gamma = \pi\left(\mathbb{Z}^3\right) \\ & \overrightarrow{x} & \mapsto & \pi(\overrightarrow{x}) \end{array}$$

is a bijection.

ii) Each vertex $\overrightarrow{x}$ of $\mathcal{V}_{\mathcal{P}}$ is the distinguished vertex of one and only one pointed face.

We can now associate a two-dimensional coding over the three letter-alphabet $\{1,2,3\}$ to any discrete surface $\mathfrak{S}$ as follows: let $\Gamma = \pi(\mathbb{Z}^3) = \mathbb{Z}\pi(\overrightarrow{e_1}) \oplus \mathbb{Z}\pi(\overrightarrow{e_2})$. We identify Γ and $\mathbb{Z}^2$ by the lattice isomorphism

$$\begin{array}{rccc} \Phi : & \mathbb{Z}^2 & \longrightarrow & \Gamma \\ & (m,n) & \mapsto & m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2}). \end{array}$$

To any discrete surface $\mathfrak{S}$, we associate the two-dimensional coding $u \in \{1,2,3\}^{\mathbb{Z}^2}$ defined by: $\forall (m,n) \in \mathbb{Z}^2$, $\forall k \in \{1,2,3\}$,

$$u_{m,n} = k \text{ if and only if } \pi^{-1}_{|\mathcal{V}_{\mathcal{S}}}(m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2})) \text{ is of type } k \text{ in } \mathfrak{S}.$$

In other words, $u_{m,n} = k$ if the pre-image of the points $m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2})$ is of type k in $\mathfrak{S}$.

A natural question is: given a two-dimensional sequence $u \in \{1,2,3\}^{\mathbb{Z}^2}$, does u code a discrete surface ? Before answering this question, we need several notions concerning combinatorics of two-dimensional words over a finite alphabet.

2.2 Basic notions on two-dimensional sequences over a finite alphabet

In this section, we recall some basic notions of combinatorics on two-dimensional words over a finite alphabet (see for instance [GR97]).

Let Σ be a finite alphabet. Let Ω be a finite subset of $\mathbb{Z}^2$. A function $w : \Omega \longrightarrow \Sigma$ is called a *finite pointed pattern over the alphabet* Σ.

A *shape* $\overline{\Omega}$ of $\mathbb{Z}^2$ is the equivalence class of $\Omega \subseteq \mathbb{Z}^2$ for the following equivalence relation:

$$\Omega \sim \Omega' \iff \exists (v_1, v_2) \in \mathbb{Z}^2,\ \Omega_1 = \Omega_2 + (v_1, v_2).$$

Let Ω be a finite subset of $\mathbb{Z}^2$. A *finite pattern* of shape $\overline{\Omega}$ is the equivalence class of a finite pointed pattern $w : \Omega \longrightarrow \Sigma$ for the following equivalence relation: for all pair $w : \Omega \longrightarrow \Sigma$ and $w' : \Omega \longrightarrow \Sigma$ of finite pointed patterns, $w \sim w'$ if and only if there exists an element $(v_1, v_2) \in \mathbb{Z}^2$ such that:

$$\Omega = \Omega' + (v_1, v_2) \text{ and } \forall (m, n) \in \Omega,\ w_{m,n} = {w'}_{m+v_1, n+v_2}.$$

In order to simplify the notation, from now on, we will denote the finite patterns w instead of $\overline{w}$ and we will denote the shapes Ω instead of $\overline{\Omega}$.

Let $u \in \Sigma^{\mathbb{Z}^2}$ be a two-dimensional sequence and let $w : \Omega \longrightarrow \Sigma$ be a finite pointed pattern. An *occurrence* of w in u is an element $(m_0, n_0) \in \mathbb{Z}^2$ such that for all $(m, n) \in \Omega$, $w_{m,n} = u_{m_0+m, n_0+n}$. The set of finite patterns occurring in u is called the *language* of u and is denoted $\mathcal{L}(u)$. Given a shape Ω, the set of finite patterns with shape Ω occurring in u is called the *Ω-language* of u and is denoted by $\mathcal{L}_\Omega(u)$.

Let Ω be a shape. The *Ω-complexity map of* u is the function $p_\Omega : \Sigma^{\mathbb{Z}^2} \longrightarrow \mathbb{N} \cup \{\infty\}$ defined as follows:

$$\begin{array}{rcl} p_\Omega : \Sigma^{\mathbb{Z}^2} & \longrightarrow & \mathbb{N} \cup \{\infty\} \\ u & \mapsto & |\mathcal{L}_\Omega(u)|, \end{array}$$

where $|\mathcal{L}_\Omega(u)|$ is the cardinality of the set $\mathcal{L}_\Omega(u)$.

2.3 Recognition of discrete surfaces

Let $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$. In this section we investigate the following question: does u code a discrete surface? Let us first introduce the pointed hooks and the hook shape of u.

Definition 2. *The* hook shape *is the equivalence class of the sets* $\{(m, n); (m, n+1); (m+1, n+1)\}$*, for* $(m, n) \in \mathbb{Z}^2$*, for the relation* $\sim$ *in Section 2.2.*

The following theorem holds:

Theorem 2. [Jam04] *Let* $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$*. Then* u *codes a discrete surface if and only if the hook-language of* u *is included in the following set of patterns (see Figure 4).*

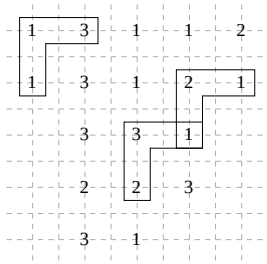


Fig. 3. Examples of hook words occurring in a sequence $u \in \{1,2,3\}^{\mathbb{Z}^2}$.

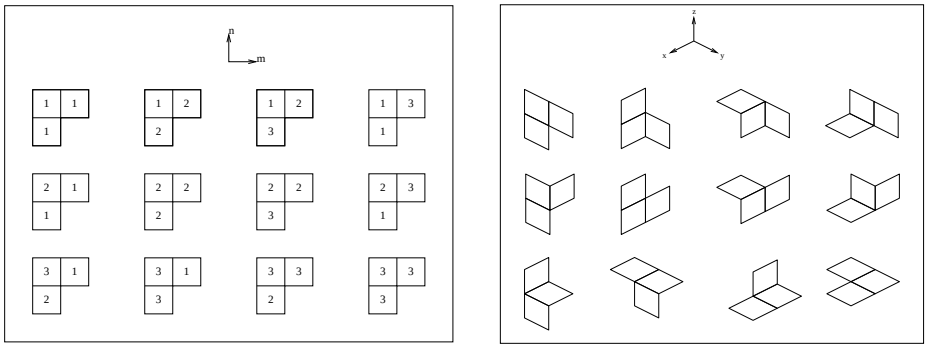


Fig. 4. Left: The permitted hook-words. Right: The 3-dimensional representation of the permitted hook-words.

A drawback of the previous definition of discrete surface is to be non-intuitive. For instance, let us consider a discrete plane $\mathfrak{P}_{\mathcal{P}}$. By construction, $\mathfrak{P}_{\mathcal{P}}$ is connected and simply connected, that is, it does not contain any hole. Let $\overrightarrow{x} \in \mathcal{V}_{\mathcal{P}}$ be a vertex of $\mathfrak{P}_{\mathcal{P}}$ of type k. Then, $\mathfrak{P} \setminus \{\overrightarrow{x} + E_k\} \cup \{(\overrightarrow{x} + \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}) + E_k\}$ is still a discrete surface.

In the following section, we show that assuming the discrete surfaces to be connected is not a restriction. More precisely, we prove that any sequence $u \in \{1,2,3\}^{\mathbb{Z}^2}$ which codes a discrete surface codes a connected and simply connected one.

3 The connected discrete surfaces

In this section, we investigate the discrete surfaces introduced in [Jam04] and prove that, for any discrete surface $\mathfrak{S}$, there exists a connected discrete surface $\mathfrak{S}$ with the same two-dimensional coding.

Theorem 3. *Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$ be a two-dimensional sequence. The following assertions are equivalent:*

i) the sequence u codes a discrete surface;
ii) the sequence u codes a connected discrete surface.

Let $x = m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2})$, with $(m,n) \in \mathbb{Z}^2$. Using the previously introduced identification of Γ and $\mathbb{Z}^2$ (see Section 2), we will denote $u_{\overrightarrow{x}}$ instead of $u_{m,n}$.

Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$ be the two-dimensional coding of a discrete surface, that is,

$$\bigcup_{\overrightarrow{x} \in \Gamma} \left(\overrightarrow{x} + \pi(E_{u_{\overrightarrow{x}}})\right),$$

where E_i is a fundamental face ($i \in \{1,2,3\}$), is a partition of the plane $\mathcal{P}_0$ (see [Jam04]). Let $\overrightarrow{x} \in \Gamma$ and $\overrightarrow{y} \in \Gamma$. We define a partial order relation $\xrightarrow{u}$ over Γ as follows: for all $\overrightarrow{x} \in \Gamma$, for all $\overrightarrow{y} \in \Gamma$, $\overrightarrow{x} \xrightarrow{u} \overrightarrow{y}$ if and only $\overrightarrow{y}$ is on the boundary of $\overrightarrow{x} + \pi(E_{u_{\overrightarrow{x}}})$.

Lemma 2. *Let $\overrightarrow{x} \in \Gamma$. Then $\overrightarrow{x} \xrightarrow{u} \overrightarrow{x} - \pi(\overrightarrow{e_1})$ and $\overrightarrow{x} \xrightarrow{u} \overrightarrow{x} - \pi(\overrightarrow{e_1}) - \pi(\overrightarrow{e_2})$ (an illustration of this property is given in Figure 5).*

Proof. This is immediatly deduced from $\pi(\overrightarrow{e_3}) = -(\pi(\overrightarrow{e_1}) + \pi(\overrightarrow{e_2}))$ and from the definitions of E_1, E_2 and E_3 and the projection map π. ■

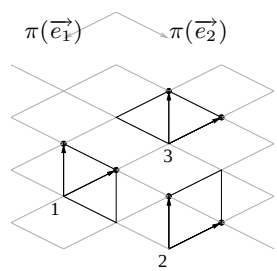


Fig. 5. Lattice representation of the partial order relation $\xrightarrow{u}$.

Finally, Theorem 3 is a direct consequence on the following lemma.

Lemma 3. *Let $\overrightarrow{x} \in \mathbb{Z}^3$ and let $\overrightarrow{l} = \pi(\overrightarrow{x}) - \pi(\overrightarrow{e_1}) - \pi(\overrightarrow{e_2}) \in \Gamma$ and $\overrightarrow{r} = \pi(\overrightarrow{x}) - \pi(\overrightarrow{e_1}) \in \Gamma$ be the left and the right targets of the arrows with source on $\pi(\overrightarrow{x})$ in the graph of Figure 5. Then, the set*

$$\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right) + \left(\overrightarrow{y} + E_{u_{\overrightarrow{l}}}\right) + \left(\overrightarrow{z} + E_{u_{\overrightarrow{r}}}\right),$$

with

$$\begin{aligned} \overrightarrow{y} = \overrightarrow{x} + \overrightarrow{e_3} \text{ and } \overrightarrow{z} = \overrightarrow{x} + \overrightarrow{e_2} + \overrightarrow{e_3} &\text{ if } u_{\pi(\overrightarrow{x})} = 1, \\ \overrightarrow{y} = \overrightarrow{x} + \overrightarrow{e_3} \text{ and } \overrightarrow{z} = \overrightarrow{x} - \overrightarrow{e_1} &\text{ if } u_{\pi(\overrightarrow{x})} = 2, \\ \overrightarrow{y} = \overrightarrow{x} - \overrightarrow{e_1} - \overrightarrow{e_2} \text{ and } \overrightarrow{z} = \overrightarrow{x} - \overrightarrow{e_1} &\text{ if } u_{\pi(\overrightarrow{x})} = 3. \end{aligned}$$

is connected and, in each previous case, $\pi(\overrightarrow{y}) = \overrightarrow{l}$ and $\pi(\overrightarrow{z}) = \overrightarrow{r}$.

Proof. In each case, one can verify that $\{\overrightarrow{y}, \overrightarrow{z}\} \subseteq \overline{\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right)}$. For instance, let us suppose that $u_{\pi(\overrightarrow{x})} = 1$. Then,

$$\overline{\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right)} = \overrightarrow{x} + \left\{x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \mid (x_2, x_3) \in [0,1]^2\right\},$$

and $\overrightarrow{y} = \overrightarrow{x} + \overrightarrow{e_3} \in \overline{\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right)}$ (see Figure 6). Idem for $\overrightarrow{z} = \overrightarrow{x} + \overrightarrow{e_2} + \overrightarrow{e_3}$. Finally, $\pi(\overrightarrow{y}) = \pi(\overrightarrow{x} + \overrightarrow{e_3}) = \pi(\overrightarrow{x}) - \pi(\overrightarrow{e_1}) - \pi(\overrightarrow{e_2}) = \pi(\overrightarrow{l})$. ■

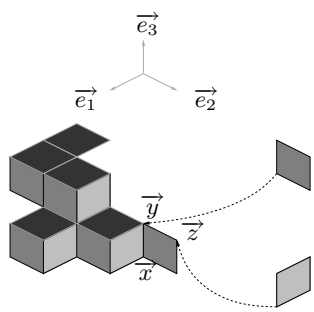


Fig. 6. Computing a connected surface by induction.

We can now prove Theorem 3:

Sketch of the proof. For all $\overrightarrow{x}_0 \in \Gamma$ and $r \in \mathbb{R}_+$, let us denote $B_\Gamma(\overrightarrow{x}_0, r) = \{m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2}) \in \Gamma \mid \max\{|m|, |n|\} < r\}$.

Let $\mathfrak{S}_1 = E_{u_{\overrightarrow{0}}}$. Then, the set $\mathfrak{S}_1$ is connected, $\pi(\mathfrak{S}_1 \cap \mathbb{Z}^3) = B_\Gamma(\overrightarrow{0}, 1)$ and for all $\overrightarrow{x} \in \mathbb{Z}^3 \cap \mathfrak{S}_1$, $\overrightarrow{x}$ is of type $u_{\pi(\overrightarrow{x})}$. Let $r \in \mathbb{N}^\star$ and let us suppose that $\mathfrak{S}_r$ is a connected disjoint union of pointed faces such that $\pi(\mathfrak{S}_r) \cap \mathbb{Z}^3 = B_\Gamma(\overrightarrow{0}, r)$ and for all $\overrightarrow{x} \in \mathbb{Z}^3 \cap \mathfrak{S}_r$, $\overrightarrow{x}$ is of type $u_{\pi(\overrightarrow{x})}$. Then, with Lemma 3 and the connectedness of the relation $\xrightarrow{u}$ (see Figure 7), one can be convinced that it is possible to build, by induction on the sets $\left(B_\Gamma(\overrightarrow{0}, r)\right)_{r \in \mathbb{N}^\star}$, a connected union $\mathfrak{S}_{r+1}$ of pointed faces such that $\pi(\mathfrak{S}_{r+1} \cap \mathbb{Z}^3) = B_\Gamma(\overrightarrow{0}, r+1)$. Indeed, for any element $\overrightarrow{y} \in B_\Gamma(\overrightarrow{0}, r+1) \setminus B_\Gamma(\overrightarrow{0}, r)$, there exists an element $\overrightarrow{x} \in B_\Gamma(\overrightarrow{0}, r)$ such that $\overrightarrow{x} \xrightarrow{u} \overrightarrow{y}$. We thus obtain an increasing sequence $(\mathfrak{S}_r)_{r \in \mathbb{N}^\star}$ of connected unions of pointed faces such that, for all $r \in \mathbb{N}^\star$, $\pi_{|\mathfrak{S}_r} : \mathfrak{S}_r \longrightarrow \mathcal{P}$ is an injection(remind that we assume u to code a discrete surface). Let

$$\mathfrak{S} = \bigcup_{r \in \mathbb{N}^\star} \mathfrak{S}_r.$$

The set $\mathfrak{S}$ is connected (it is an increasing sequence of connected sets) and $\pi : \mathfrak{S} \longrightarrow \mathcal{P}$ is injective. Finally, since $\pi(\mathfrak{S} \cap \mathbb{Z}^3) = \Gamma$ and u codes a discrete

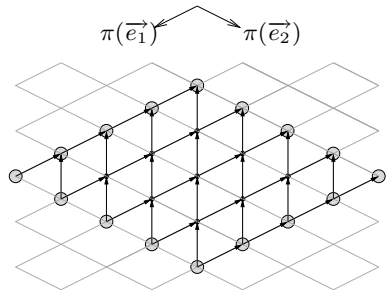


Fig. 7. A part of the graph of the relation $\xrightarrow{u}$.

surface, that is,

$$\bigcup_{\overrightarrow{x}\in\mathfrak{S}\cap\mathbb{Z}^3}\left(\pi(\overrightarrow{x})+\pi(E_{u_{\pi(\overrightarrow{x})}})\right)=\mathcal{P}_0,$$

we conclude that $\pi:\mathfrak{S}\longrightarrow\mathcal{P}_0$ is surjective. ■

An other interesting property of two-dimensional sequences coding discrete surfaces is:

Theorem 4. *Let $u\in\{1,2,3\}^{\mathbb{Z}^2}$ and let $\mathfrak{S}$ be a connected discrete surface coded by u. Then $\mathfrak{S}$ is simply connected, that is, $\mathfrak{S}$ admits no hole.*

Sketch of the proof. Let $(\mathfrak{S}_r)_{r\in\mathbb{N}^\star}$ be the sequence computed in the proof of Theorem 3. Then the following assertion holds:

$$\forall r\in\mathbb{N}^\star,\ \overline{\mathfrak{S}_r}\subseteq\mathfrak{S}_{r+2}.$$

Then,

$$\mathfrak{S}=\bigcup_{r\in\mathbb{N}^\star}\overline{\mathfrak{S}_r}$$

and $\mathfrak{S}$ is a union of closed sets. On can notice that each set $B(\overrightarrow{x},r)=\{\overrightarrow{y}\in\mathbb{R}^3\mid ||\overrightarrow{x}-\overrightarrow{y}||_\infty<r\}$, with $r\in\mathbb{R}_+^\star$, intersects at most a finite number of closed pointed faces. Hence, it becomes easy to show that $\mathfrak{S}$ is closed. Since $\pi:\mathbb{R}^3\longrightarrow\mathcal{P}_0$ is continuous, it follows that $\pi_{|\mathfrak{S}}:\mathfrak{S}\longrightarrow\mathcal{P}_0$ is continuous. It remains to show that $\pi_{|\mathfrak{S}}^{-1}:\mathcal{P}_0\longrightarrow\mathfrak{S}$ is continuous. It follows from the fact that $\pi:\mathbb{R}^3\longrightarrow\mathcal{P}_0$ is a closed map. Finally, we have proved that $\pi_{|\mathfrak{S}}:\mathfrak{S}\longrightarrow\mathcal{P}$ is an homeomorphism. Finally, $\mathfrak{S}$ is simply connected, since $\mathcal{P}$ is simply connected. ■

4 Discrete surfaces generated by smooth words

In this section, we first recall some notions of combinatorics on words over arbitrary alphabets, as defined in [BBC04]. Then, we study the discrete surfaces which are generated by a specific class of words, the right infinite smooth words

over the alphabet {1,2,3}. We prove that there are only three such discrete surfaces.

Let us consider a finite *alphabet* Σ of *letters*. A *right infinite word* is a sequence $w \in \Sigma^{\mathbb{N}}$. Every word $w \in \Sigma^{\mathbb{N}}$ can be uniquely written as a product of factors as follows:

$$w = {\alpha_1}^{e_1} {\alpha_2}^{e_2} {\alpha_3}^{e_3} \dots$$

with $e_j \in \mathbb{N}^\star$ and $\alpha_i \neq \alpha_{i+1}$. Hence, the *run-length encoding* defined by:

$$\begin{array}{rccc} \Delta : & \Sigma^{\mathbb{N}} & \longrightarrow & \mathbb{N}^{\mathbb{N}} \\ & w = {\alpha_1}^{e_1} {\alpha_2}^{e_2} {\alpha_3}^{e_3} \dots & \mapsto & e_1 e_2 e_3 \dots, \end{array}$$

is well defined on $\Sigma^{\mathbb{N}}$.

Given a non-empty finite subset Σ of $\mathbb{N}$, we define the *right infinite smooth words* over Σ as the words which are invariant under the action of the Δ operator. More precisely, the set $\mathcal{K}_\Sigma$ of the right infinite smooth words over Σ is defined as follows:

$$\mathcal{K}_\Sigma = \left\{ w \in \Sigma^{\mathbb{N}} \mid \forall k \in \mathbb{N}, \Delta^k(w) \in \Sigma^{\mathbb{N}} \right\}.$$

Example 1. If $\Sigma = \{1, 2\}$, the operator Δ as two fixpoints, namely

$$\Delta(K_{\{1,2\}}) = K_{\{1,2\}}, \qquad \Delta(1 \cdot K_{\{1,2\}}) = 1 \cdot K_{\{1,2\}},$$

where $K_{\{1,2\}}$ is the well-known Kolakoski word [Kol66], whose first terms are:

$$K_{\{1,2\}} = 221121221221121122121121221121121221221121221211211 22\dots$$

Given a smooth words w over a finite alphabet Σ, we define the *tiling associated to* w (see [BBLP03]) as the two-dimensional sequence $(T(w)_{m,n})_{(m,n)\in\mathbb{N}^2}$ as follows:

$$\forall m \in \mathbb{N},\ (T(w)_{m,\bullet}) = \Delta^m(w).$$

In other words, for any $m \in \mathbb{N}$, the m-th line of $(T(w)_{m,n})_{(m,n)\in\mathbb{N}^2}$ is the right infinite word $\Delta^m(w)$.

Let us now state the main result of this section:

Theorem 5. *Let w be a smooth word over the alphabet $\{1, 2, 3\}$. The tiling $T(w)$ associated to w is a piece of a discrete surface if and only if $w \in \{K_{\{1,3\}}, 1K_{\{1,3\}}, 2K_{\{1,3\}}\}$.*

Proof. Using the permitted hook-words of Figure 4 and the smoothness condition, that is,

$$\forall m \in \mathbb{N},\ T(w)_{m+1,\bullet} = \Delta\left(T(w)_{m,\bullet}\right),$$

an exhaustive inspection gives that $T(w)$ must start by one of the patterns of Figure 8.

Clearly, the other 5 patterns are excluded because they do not respect the smoothness condition. We proceed by exhaustive inspection. Let us for instance investigate the first case (see Figure 9). In the two first extensions of the ini-

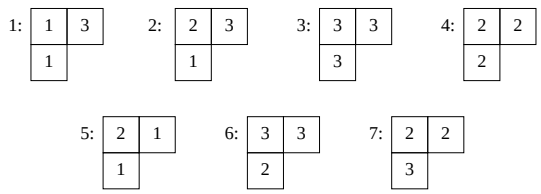


Fig. 8. The possible starting patterns.

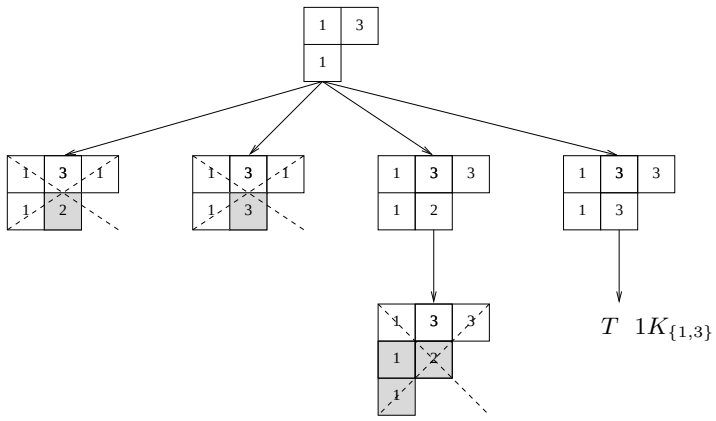


Fig. 9. The different extensions in the first case.

tial word, the smoothness condition does not hold. In the third extension, the smoothness condition provides a forbidden pattern. In the last extension, we obtain the tiling associated to the word $1K_{\{1,3\}}$.

The other cases can be treated in the same way. For instance, we obtain the tiling $T\left(2K_{\{1,3\}}\right)$ in the second case, and the tiling $T\left(K_{\{1,3\}}\right)$ in the third case. None of the other cases leads to a discrete surface (for a complete proof, see Appendix A). ■

Since the identification of the three fundamental faces to the letters 1, 2 and 3 is arbitrary, a natural question arises: what is the action of permutation on the coding alphabet? Does the result still hold? If not, for each permutation, what are the smooth tilings that describe a discrete surface? This will be described in a forthcoming paper.

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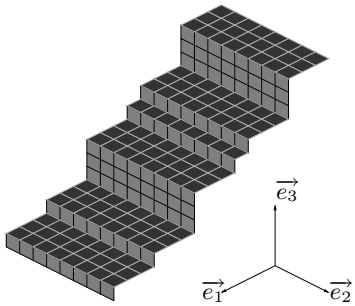

Fig. 10. A connected discrete surface associated to the word $K_{\{1,3\}}$.

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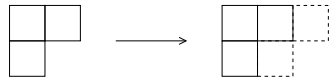
A Appendix

In this appendix, we give the detailled proof of Theorem 5. The next definitions are necessary.

Every hook pattern may be extended on the right by using the set of permitted hook-words of Figure 4. This process is identified by

p: 

Example 2. The pattern

3	3
3	

can be extended on the right by using patterns in $\mathcal{S}$ in only 2 ways:

3	3
3	

$\longrightarrow$

3	3	1
3	3	

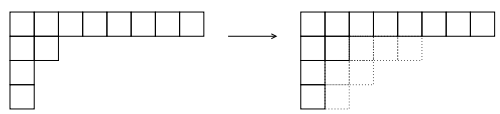
or

3	3
3	

$\longrightarrow$

3	3	3
3	3	

Every tiling may be completed by using the smoothness property. This process is identified by

s: 

and is defined by $R_i = D_r(R_{i-1})$, where R_i denotes the i-th row of the tiling and D_r denotes the right derivative of a smooth word (see [BMP04]).

1	2	3	3	2	2	1	3
1	1						
2							
2							

(a)

1	2	3	3	2	2	1	3
1	1	2	2	1			
2	2						
2							

(b)

Fig. 11. Example of completion with the smoothness property.

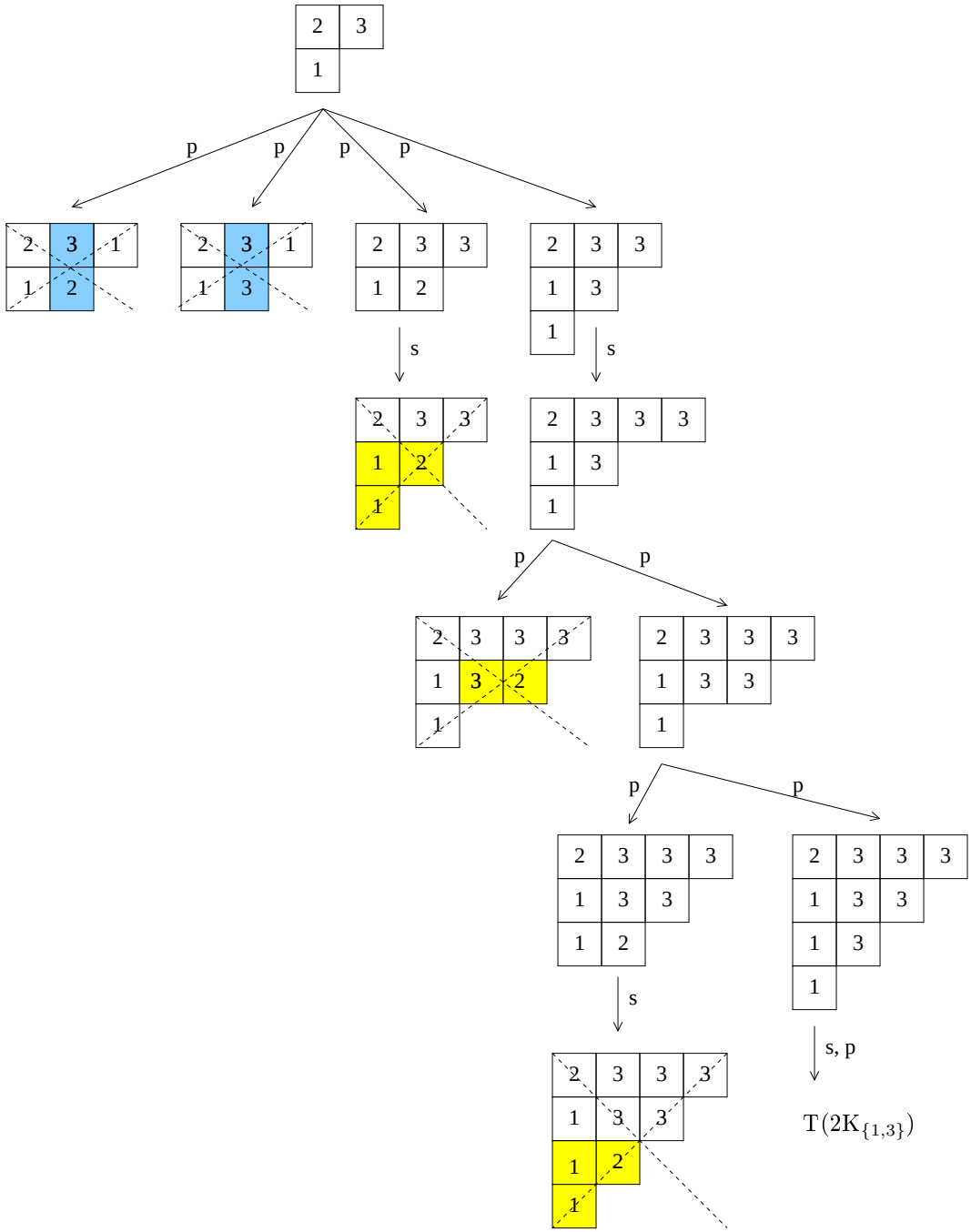
Example 3. Using the smoothness property, the tiling in Figure 11 A) is extended to the tiling in Figure 11 B).

Proof of Theorem 5. Using the smoothness condition, it follows that the tilings start with a pattern from Figure 8. Clearly, the other 5 patterns are excluded because they do not respect the smoothness condition. We proceed by

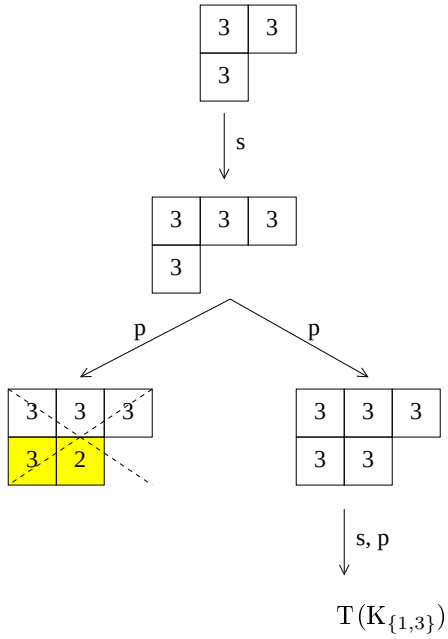
exhaustive inspection.

Case 1: Already done in the paper.

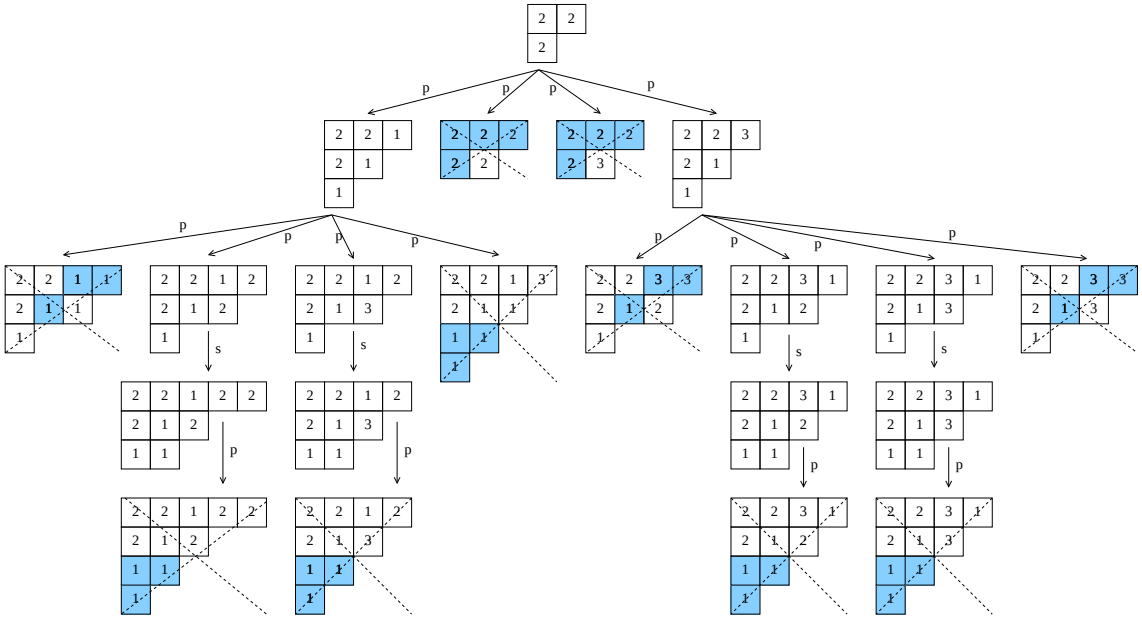
Case 2:


p
s
s, p
$T(2K_{\{1,3\}})$

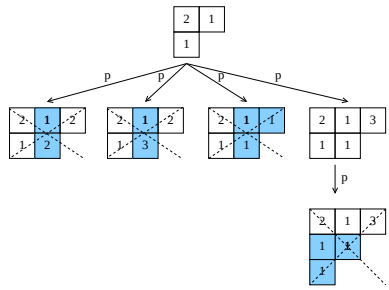
Case 3:



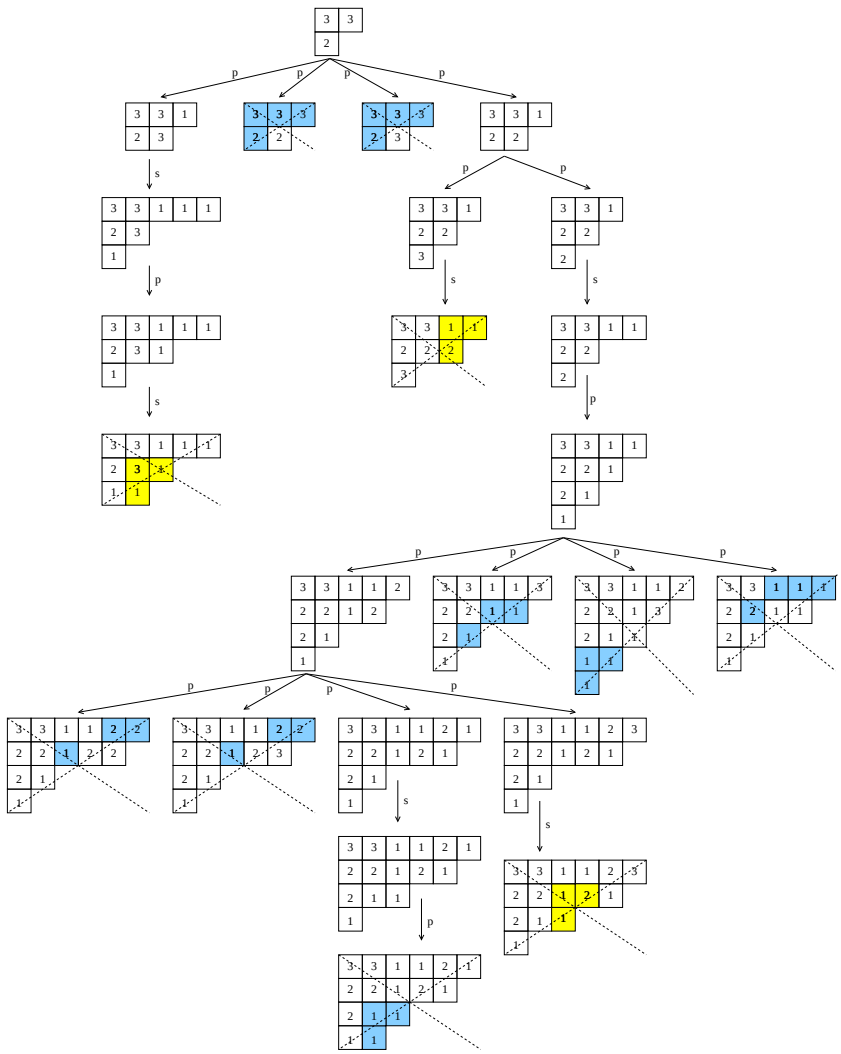
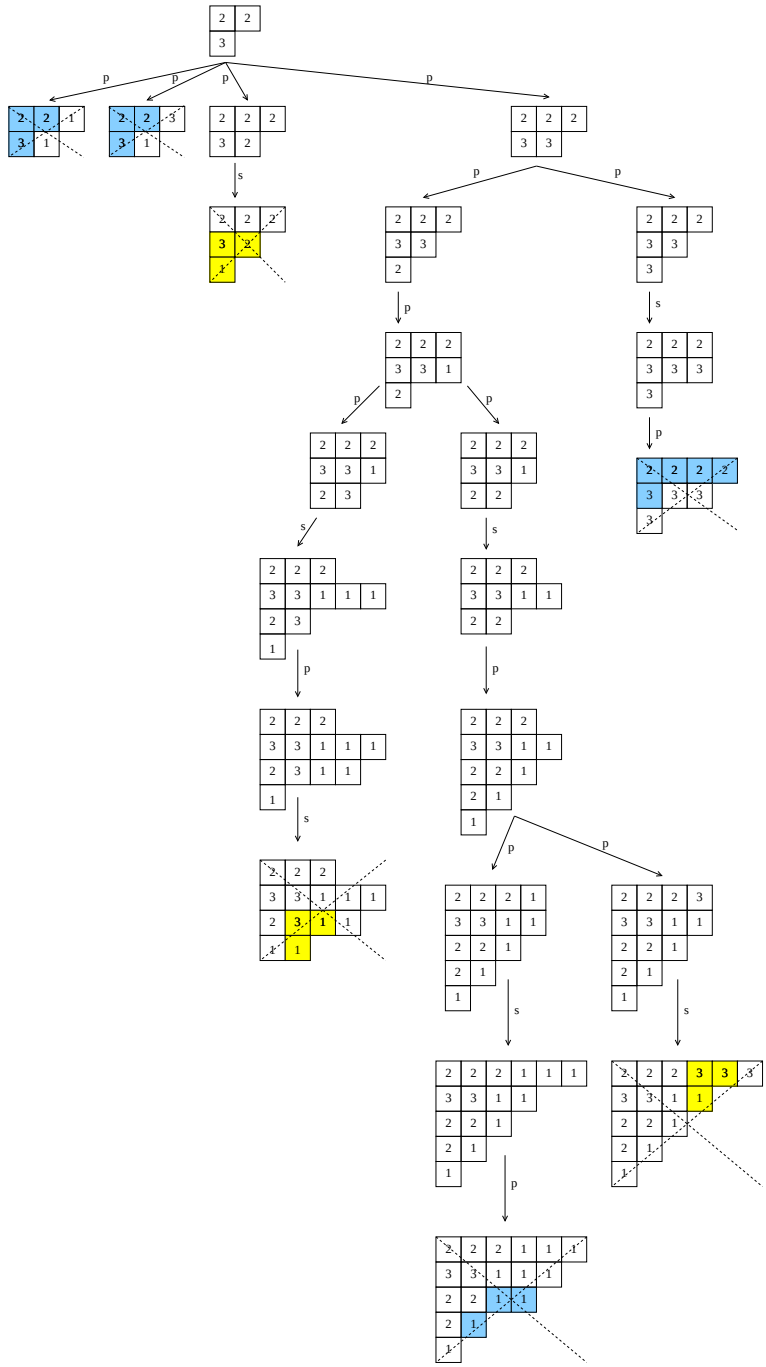
Case 4:



Case 5:



Case 6:



Case 7: