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Some Remarks on Relations between Proofs and Games

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1 Abstract

This paper aims at studying relations between proof systems and games in a given logic and at analyzing what can be the interest and limits of a game formulation as an alternative semantic framework for modelling proof search and also for understanding relations between logics. In this perspective, we firstly study proofs and games at an abstract level which is neither related to a particular logic nor adopts a specific focus on their relations. Then, in order to instantiate such an analysis, we describe a dialogue game for intuitionistic logic and emphasize the adequateness between proofs and winning strategies in this game. Finally, we consider how games can be seen to provide an alternative formulation for proof search and we stress on the possible mix of logical rules and search strategies inside games rules. We conclude on the merits and limits of the game semantics as a tool for studying logics, validity in these logics and some relations between them.

2 Proofs and Games

In this section, we present a common terminology to present both proof systems and games at a relatively abstract level. Our aim consists in obtaining tools on which bridges can be built between the proof-theoretical approach and the game semantics approach in establishing the (universal) validity of logical formulæ. We explain how proofs and games can be viewed as complementary notions. We illustrate how proof trees in calculi correspond to winning strategies in games and vice-versa.

2.1 Rule instances in deduction systems

We start with the definition of deduction system: it is composed of a set of *statements* denoted stm and a set of *(rule) instances* denoted inst. The notion of statement is basic and depends on the particular deduction system which is currently considered. It intuitively represents objects that can either be valid or invalid. An instance is a pair $([p_1, \dots, p_n], c)$ in $\mathsf{stm}^\star \times \mathsf{stm}$ composed of a finite list $[p_1, \dots, p_n]$ of *premises* and a *conclusion* c. Both $p_1, \dots, p_n$ and c are statements. An instance is usually denoted:

$$\frac{p_1 \quad \dots \quad p_n}{c}$$

In case $n = 0$, i.e., when the list of premises is empty, the instance is called an *axiom*. Remark that inst is not the set of all instances but only a selection of some instances. This set is usually described by *logical rules* which generate instances by substitution of their parameters (see below). Let us illustrate these points through examples.

2.1.1 Examples of deduction systems

Hilbert type systems for classical logic. In this case, statements are just formulæ and stm is the set of formulæ of classical propositional logic. The set of instances is composed of axioms like for example $([\,], (A \wedge B) \supset A)$ (for any formulæ A and B) and the instances of the *modus ponens rule* $([A, A \supset B], B)$ (for any formulæ A and B) denoted by:

$$\frac{}{(A \wedge B) \supset A}\,\langle \mathrm{Ax}_1 \rangle \qquad\qquad \frac{A \qquad A \supset B}{B}\,\langle \mathrm{MP} \rangle$$

The axiom $\langle \mathrm{Ax}_1 \rangle$ has two parameters (A and B) and we denote by $\langle \mathrm{Ax}_1 \rangle(A, B)$ the instance displayed. For rule $\langle \mathrm{MP} \rangle$, the parameters are A and B and the instance displayed is denoted $\langle \mathrm{MP} \rangle(A, B)$. We might omit the value of parameters when they are obvious.

Gentzen sequent calculus for intuitionistic logic. The statements are sequents of the form $A_1, \dots, A_k \vdash B$, where $A_1, \dots, A_k, B$ are formulæ of intuitionistic logic. $A_1, \dots, A_k \vdash B$ denotes a pair composed of a *multiset of hypotheses* $A_1, \dots, A_k$ and a *conclusion* B.[1] In the case of the sequent calculus, instances look like

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\,\langle \vee_L \rangle \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\langle \vee_{R1} \rangle \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\langle \vee_{R2} \rangle$$

[1] Note that in this case, the term conclusion is used both as a qualifier for formulæ and for sequents/statements. It is up to the reader to distinguish those two notions depending on the context.

$$\frac{\Gamma, A \supset B \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}\ \langle\supset_L\rangle \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \langle\supset_R\rangle$$

where Γ is an arbitrary multiset of formulæ and A, B, C are arbitrary formulæ. These instances are denoted $\langle\vee_L\rangle(\Gamma, A, B, C)$, $\langle\vee_{R1}\rangle(\Gamma, A, B)$, $\langle\vee_{R2}\rangle(\Gamma, A, B)$, $\langle\supset_L\rangle(\Gamma, A, B, C)$ and $\langle\supset_R\rangle(\Gamma, A, B)$ respectively.

2.1.2 Validity, derivability and proofs

A *validity* $V \in \mathbb{P}(\mathsf{stm})$ is the set of statements which are supposed to be valid. Statements which do not belong to V are considered as invalid. The validity V is said to be *closed w.r.t. an instance* $([p_1, \ldots, p_n], c)$ if $p_1, \ldots, p_n \subseteq V$ implies $c \in V$. In other words, the conclusion must be valid when the premises are. Usually, validities are defined through semantic means, i.e., by interpreting statements as semantical objects in some mathematical universe.

Any deduction system generates a particular validity called *derivability* which is the least validity closed under all the instances in the set inst. It is possible to provide an extensional definition of the notion of derivability through the notion of proof. A *proof tree* is a finite ordered tree labeled with statements such that for any node labeled by c, with a list of sons labeled by $p_1, \ldots, p_n$ respectively, the inclusion $([p_1, \ldots, p_n], c) \in \mathsf{inst}$ holds. Thus a proof tree is build by composing instances, starting from axioms such as the following tree for Gentzen sequent systems:

$$\dfrac{\dfrac{}{A, A \supset B \vdash A}\ \mathrm{Ax} \qquad \dfrac{}{A, B \vdash B}\ \mathrm{Ax}}{A, A \supset B \vdash B}\ \supset_L$$

When c appears as the label of the root node of a proof tree T, we say that T is a *proof of* c. The notion of proof characterizes the notion of derivability because it can be shown that the statements which are derivable are exactly those which have a proof. Thus exhibiting a proof is a necessary and sufficient condition to establish derivability and the notions of derivability and provability are one and the same.

The notion of *underivability*, i.e., the opposite of derivability, can also be defined extensively through the concept of *refutation*. A refutation tree is also a tree labeled with statements but contrary to proof trees, they can be infinite both in width or depth. The condition is that for each node labeled with c in the refutation tree, there is a one to one correspondance between the sons of c and thoses instances in inst which have c as conclusion and that each son is labeled with one of the premises of its corresponding instance (see [17] for more detailed presentation).

2.2 Games and strategies

Now let us define the notion of *game* within this setting. A game has two players: the proponent P and the opponent O. Intuitively, the proponent P tries to *prove* (aka *defend* in game terminology) statements and O tries to *refute* (aka *attack*) the application of rule instances.

Each step of the game is a play by either P or O: first, the player *receives* the previous move, then he makes some *choice* (maybe according to a predefined strategy, see later), and finally *moves* by transmitting its choice to the other player. P plays instances in inst whereas O plays statements in stm.

Formally, a game is a strictly alternating sequence of plays by P and O which is non-empty, either finite or infinite, of the form:

$$\mathsf{O} \to c_0 \to \mathsf{P} \to i_1 \to \mathsf{O} \to p_1 \to \ldots \to p_{k-1} \to \mathsf{P} \to i_k \to \mathsf{O} \to p_k \to \ldots$$

and which verifies the following rules:

1. the first move is by O which transmits a statement $c_0 \in \mathsf{stm}$ to P;

2. when P receives a move $c \in \mathsf{stm}$ from O, he choose an instance $i \in \mathsf{inst}$ for which c is the conclusion, i.e., i must be of the form $([\ldots], c)$, and plays i;

3. when O receives a move $i \in \mathsf{inst}$ for P, he choose a statement $p \in \mathsf{stm}$ which is one of the premises of i, i.e., i must be of the form $([\ldots, p, \ldots], \ldots)$, and plays p.

A *game for the statement* c is a game where the first move (played by O) is the statement c. A P-*game* is a (finite) game where the last move is played by P and an O-*game* is a (finite) game where the last move is played by O. An ∞-*game* is an infinite game. So each game is either a P-game, an O-game or an ∞-game and these three cases are mutually exclusive.

A *game is finished* if it cannot be extended anymore, i.e., either it is an ∞-game or when it is finite, the last player's move made it impossible for the other player to move further.

A *game is won by* P when it is a finished P-game. Thus it is a finite game and O cannot move anymore because the game cannot be extended. This happens when and only when the last move of P is an axiom. A game won by P is also called a *winning* P-*game*. Here is an example of winning P-game in the Gentzen system for intuitionistic logic:

$$\mathsf{O} \to A, A \supset B \vdash B \to \mathsf{P} \to \langle \supset_L \rangle \to \mathsf{O} \to A, B \vdash B \to \mathsf{P} \to \langle \mathrm{Ax} \rangle \to \mathsf{O}$$

A *game is won by* O when it is finished and not won by P, i.e., either it is an ∞-game or it is an O-game where P cannot move further, i.e., O

played a statement which the conclusion of no instance in inst. A game won by O is also called a *winning* O*-game*.

Supposing that X and Y are logical variables, here is a finite winning O-game:

$$\mathsf{O} \to X \vdash X \vee Y \to \mathsf{P} \to \langle \vee_{R2} \rangle \to \mathsf{O} \to X \vdash Y \to \mathsf{P}$$

and an infinite winning O-game:

$$\mathsf{O} \to X \supset Y \vdash X \to \mathsf{P} \to \langle \supset_L \rangle \to \mathsf{O} \to X \supset Y \vdash X \to \mathsf{P} \to \cdots \texttt{cycle}$$

A *strategy for* P is a procedure that tells P how to move when it is up to him to play, but not necessarily in all circumstances. Thus a strategy can be defined as a partial function from O-games to inst. And a *strategy for* O is a partial function from P-games to stm.

A *game is played according to a strategy* φ *for* P if every move of P is determined by the strategy: for every partial game (i.e., prefix of the alternating list of plays) of the game of the form

$$\mathsf{O} \to c_0 \to \mathsf{P} \to i_1 \to \mathsf{O} \to p_1 \to \ldots \to p_{k-1} \to \mathsf{P} \to i_k \to \mathsf{O}$$

we have

$$\varphi(\mathsf{O} \to c_0 \to \mathsf{P} \to i_1 \to \mathsf{O} \to p_1 \to \ldots \to p_{k-1} \to \mathsf{P}) = i_k$$

When a strategy determines the behavior of P, bifurcations might still occur depending on the behavior of O. A strategy for P can be represented by a tree where branches represent the games played according to the strategy (see Figure 1 for an example).

A strategy φ is a *winning strategy for* P *on* c if φ is a strategy for P such that every finished game for c played according to φ is won by P. In other words, whatever O does, every game for c in which the behavior of P is determined by φ must stop on a move from P which prevents any further move from O.

It is also possible to define *winning strategies for* O *on* c as determinations of the behavior of O that either lead to finished O-games (where P cannot move further) or develop ∞-games where P and O play indefinitely.

2.3 Proofs vs. Games

The key result that relates proofs and games is the following:

There is a winning strategy for P *on* c *if and only if* c *has a proof.*

The demonstration is done by showing how proofs and strategies correspond to each other.[2] We do not give the details here but illustrate this result with an example.

[2]Dually, it is possible to show that there is a winning strategy for O on c if and only if c has a refutation, see [17].

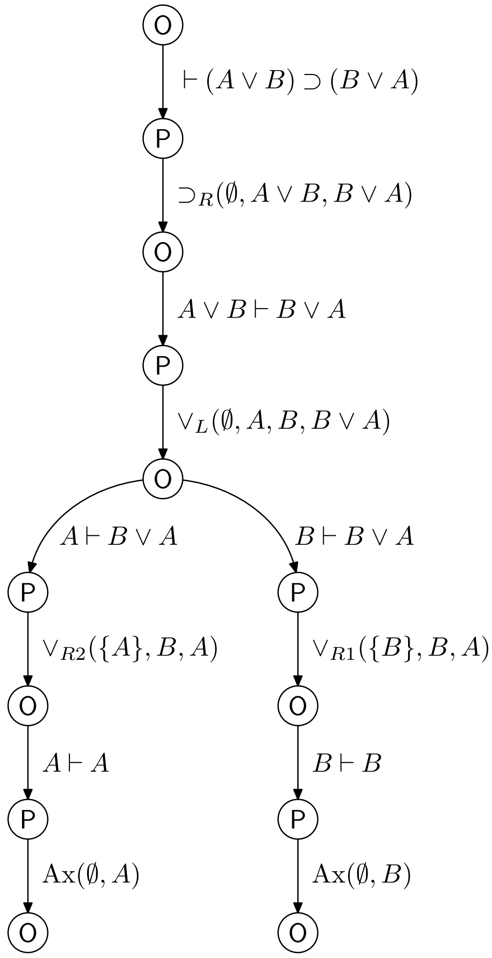

Figure 1: A winning strategy for P on the sequent $\vdash (A \vee B) \supset (B \vee A)$.

In Figure 1, we consider the sequent $\vdash (A \vee B) \supset (B \vee A)$ and we display a winning strategy for P on this input. The strategy g is a tree composed of two branches with a bifurcation. It is winning for the sequent $\vdash (A \vee B) \supset (B \vee A)$ because the two outcomes are winning for P and no other sequence of moves by O could have lead to another outcome.

If we compare the strategy with the following Gentzen proof of the same sequent, the correspondece between the proof branches and the branches of the strategy tree is obvious, considering that the proof tree is upside-down compared to the strategy tree:

$$
\cfrac{\cfrac{\cfrac{}{A \vdash A}\,\mathrm{Ax}}{A \vdash B \vee A}\,\vee_{R2} \qquad \cfrac{\cfrac{}{B \vdash B}\,\mathrm{Ax}}{B \vdash B \vee A}\,\vee_{R1}}{\cfrac{A \vee B \vdash B \vee A}{\vdash (A \vee B) \supset (B \vee A)}\,\supset_R}\,\vee_L
$$

2.4 From IL deduction systems to Game rules

Now let us concentrate on the games generated by the Gentzen system for intuitionistic logic. Let us consider the two rules for the intuitionistic implication $\supset$:

$$
\frac{\Gamma, A \supset B \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}\,\langle\supset_L\rangle \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\langle\supset_R\rangle
$$

If we focus on rule $\langle\supset_R\rangle$, it has three parameters: Γ, A and B and the rule generates a set of instances that have two premises when these parameters are substituted with values. Rules represent sets of instances of a particular shape.

Now if we interpret the rule $\langle\supset_L\rangle$ in terms of choices of moves for either P or O, it corresponds to the following sequence in the game:

- if P receives the sequent $\Gamma, A \supset B \vdash C$ and chooses to play the move $\langle\supset_L\rangle(\Gamma, A, B, C)$ (aka attack $A \supset B$ in game terminology);
- then O can either play $\Gamma, A \supset B \vdash A$ (aka attack A) or play $\Gamma, B \vdash C$ (aka grant B and attack C).

For rule $\langle\supset_R\rangle$, we obtain:

- if P receives the sequent $\Gamma \vdash A \supset B$ and chooses to play the move $\langle\supset_R\rangle(\Gamma, A, B)$ (aka defend $A \supset B$);
- then O must play $\Gamma, A \vdash B$ (aka grant A and attack B).

At this step we aim at focusing on the development of game rules from proof systems in intuitionistic logic. In this perspective, the next section is devoted to particular games, called *dialogue games*, introduced by Lorenzen [19]. Here we consider such games for intuitionistic logic through a recent formulation by Fermüller et al [11]. Let us mention that similar work has been done for intermediate logics, like Gödel-Dummett logics, fuzzy logics and multi-valued logics [9, 11, 5, 10].

3 Dialogue Games

Logical dialogue games have many forms and versions nowadays. The main formulations are the one in Blass-Abramsky style [1, 4] in which logical connectives are game combinators, and the one in Lorenzen style that is based on idealized confrontational dialogues. They all refer in different ways to Lorenzen's idea to identify the logical validity of a formula A with the existence of a winning strategy for a proponent P in a idealized dialogue in which P tries to show A against systematic doubts by an opponent O [19]. This idea was first rigorously developed in order to provide an alternative characterization of intuitionistic logic in [7]. Here we aim at presenting a version of dialogue games, provided by Fermüller in [9, 11], that is well suited for showing the relationship between such games and Gentzen systems and that is equivalent to other versions of dialogue games for intuitionistic logic.

Here an atomic formula (atom) is either a propositional formula or $\perp$ (falsum). Compound formulæ are built from atoms using the connectives $\wedge$, $\vee$, $\supset$; $\neg A$ abbreviates $A \supset \perp$. Moreover the signs ?, $?_{Left}?_{Right}$ can be stated by players P and O as shown below.

Dialogues games are characterized by two kinds of rules: logical ones and structural ones. The *logical dialogue rules* define how to attack a compound formula and how to defend against such an attack. They are summarized in the following table:

X:	attack from Y	defense of X
$A \wedge B$	$?_{Left}$ or $?_{Right}$ (Y chooses)	A or B according the choice of Y
$A \vee B$	?	A or B (X chooses)
$A \supset B$	A	B
$\neg A$	A	No defense

Note that X and Y can be each one either P or O, according to the state of game. Then both players may launch attacks and defend against attacks during the course of a dialogue.

In this context a *dialogue* is a sequence of *moves* which are either attacking or defending statements following the logical rules. Each dialogue refers to a finite multiset of formulæ that are *initially granted* by O and to an initial formula to be defended by P. Moves can be seen as state transitions. In any state of the dialogue the formulæ initially granted or stated by O are called *granted formulæ* at this state. The last formula stated by P and that either already has been attacked or must be attacked in O's next move is called *active formula*. Then in each state of the dialogue we can associate a so-called *dialogue sequent* $\Pi \vdash A$ where Π denotes the granted formulæ and A the active formula.

The *structural dialogue rules* regulate the succession of moves. A number of different systems of structural rules have been proposed and analyzed in

[8, 16]. Here these rules are the following:

- **Start:** O starts by attacking P's initial formula;
- **Alternate:** moves strictly alternate between players O and P;
- **Atom:** atomic formulas (including $\perp$) can neither by attacked or defended by P;
- **E-rule:** each move of O reacts directly to the immediately preceding move by P. It means that if P attacks a granted formula the O's next move either defends this formula or attacks the formula used by P to launch this attack. If P's last move was a defending one then O has to attack immediately the formula stated by P in that defense move.

Quite a number of different systems of structural rules have been proposed for dialogue games in the literature (see [16] for more details). To complete this presentation we give the *winning conditions* (for P):

- **W:** O has attacked a formula that has already been granted, either initially or in an alter move, by O;
- **W$\perp$:** O has granted $\perp$.

A *dialogue tree dt* for $\Pi \vdash C$ is a rooted directed tree with nodes labelled with dialogue sequents and edges corresponding to moves, such that each branch of *dt* is a dialogue with initially granted formula Π and initial formula C. Then the nodes of such trees correspond to states of a dialogue. We have two kinds of nodes, namely P-nodes and O-nodes, depending if it is P's or O's turn to move at the corresponding state.

A finite dialogue tree is a *winning strategy* (for the player P) if the following conditions hold:

1. Every P-node has at most one successor O-node.
2. All leaf nodes are P-nodes in which the winning conditions for P are fullfiled.
3. Every O-node has a successor node for each move by O that is a permissible continuation of the dialogue at this stage.

In this context a dialogue game can be viewed as a state transition system where moves in the dialogue correspond to transitions between O-nodes and P-nodes. A dialogue then is a possible trace in the system and a winning strategy can be obtained by a systematic "unraveling" of all possible traces. More details are given in [11].

Let us illustrate in Figure 2 the dialogue games defined by Fermüller on the example of the intuitionistic formula $(A \vee B) \supset (B \vee A)$. The figure

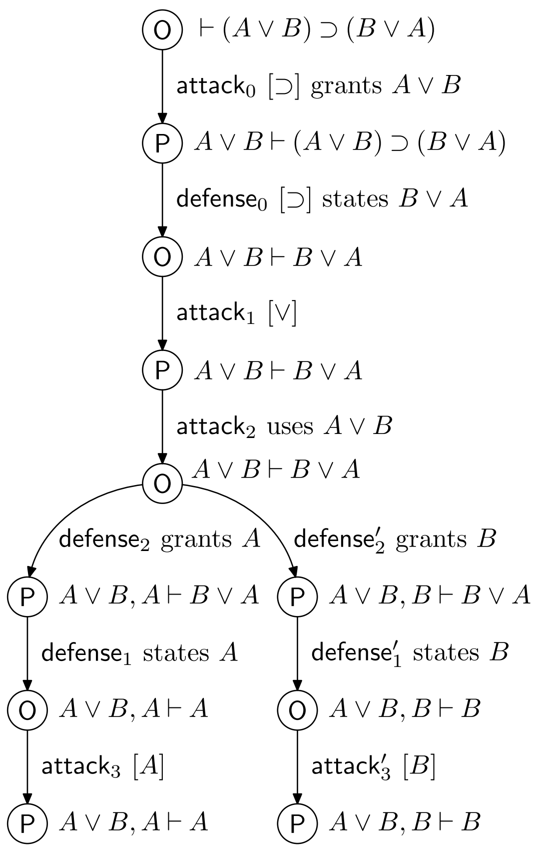

Figure 2: A winning strategy for P in Fermüller's dialogue games.

displays a winning strategy for P which ends on the winning condition that O attacks a formula he has already granted (attack_3 and attack'_3).

Having defined such dialogue games for intuitionistic logic, a key step consists in proving their adequateness, namely their soundness and completeness. Such a proof is given by showing that winning strategies can be transformed into proofs in a sequent calculus for intuitionistic logic, and vice-versa. All details about these dialogue games and its adequateness proof can be found in [9, 11].
In the next section we aim at studying the real interest of dialogue games for proof search. We have illustrated, through the above dialogue games for intuitionistic logic, the connections between sequent proofs and winning strategies. Many works have been devoted to proof search in intuitionistic logic following different formalisms like standard calculus [6] or sequent calculus with labels [2] or constraints [24] with a focus on the capture of semantics or algorithmic properties, like termination or non-duplication of formulæ [13, 18]. In front of such works and results the real interest and pos-

itive impact of dialogue games for proof search in intuitionistic logic cannot be claimed and merit more studies.

4 Dialogue Games and Proof search

Dialogue games seem to be another way to mimic what happens in some sequent calculi but it is not clear that this formalism is an appropriate formalism in the perspective of improvements in proof search procedures. It is the case in intuitionistic logic but what about other logics?

A parallel version of dialogue games for intuitionistic logic has been shown to be adequate for a number of intermediate logics [9]. In this case the soundness and completeness proofs are based on the relations between so-called hypersequent proofs and winning strategies for parallel dialogue games. Hypersequent calculi have been proposed as a flexible type of proof system for many logics. However, the relation between hypersequent proofs and the semantics of the logics is much less clear than in the case of classical and intuitionistic sequents. The hypersequent calculi formulated for intermediate logics like Gödel-Dummett logics (LC) do not directly relate to a semantic foundation of these logics.

In this context, hypersequents have a strong relation to dialogue games that constitute an alternative to standard semantics. Then the proposal of parallel dialogue games appears as semantic foundations for proof search. It allows to understand the claim that LC is related to parallel programs but mainly to study relations between different intermediate logics through game rules. It could be seen also a tool for studying models of parallel proof search but also as a formalism for modelling proof search and to directly represent important proof search strategies.

It appears that logical rules and strategies can be mixed in game rules. It could limit some choices in the proof search process, simplify this one but with an underlying problem that is: what about the adequateness with logical rules in sequent formalisms?

In order to illustrate this point of mixing rules and strategies into game rules we come back to intuitionistic logic and consider a related work on dialogue games which has been developed during these last years by Rahman and his school for a lot of systems of formal logic [21, 22].[3]

Here we stress on four specific rules in this intuitionistic dialogue:

(a) **Atoms**: every player can assert atomic formulæ, but P cannot assume the atomic formula q if q has not before been conceded by O.

[3] Thanks a lot to Shahid Rahman whose correspondence was a great encouragement for this inquiry.

(b) **Elimination of negation**: if one player X asserts $\neg\varphi$, the attack of the other player Y must be φ.

(c) **Intuitionistic Round Closure Rule**: whenever player X is to play, he can attack any move of Y in so far as the other rules let him do so, or defend against *the last attack* of Y (the attack in the game with the highest rank), provided he has not already defended against it. A player may postpone a defense as long as there are attacks he can put forth.

(d) **Intuitionistic No-Delaying-Tactics Rule**: if O has introduced an atomic formula which can be now used by P, then P may perform a repetition of an attack. No other type of repetition is allowed [15].

The rule (a) is in relation to the intuitionistic sequent calculus where the right part of the sequent must be a single formula and not a disjunction of formulæ. That is why if atomic formulæ cannot be attacked, they *are used* as attacks by O. For example if the previous P's formula is the sequent $p \vdash q$, then O attacks by playing *via* the atom p and the attack is $\vdash p$.

Two comments on the rule (b): suppose that $\varphi = p$ (that φ is atomic) and that O asserts $\neg p$, then P cannot attack $\neg p$ if O has not conceded p before. Moreover this rule on negation corresponds to the intuitionistically admissible rule $\dfrac{\Gamma \vdash \neg A}{\Gamma, A \vdash}$

The structural rule (c) must be put in contrast with the *Classical Round Closure Rule*: whenever player X is to play, he can attack any move of Y in so far as the other rules let him do so, or defend against any attack of Y (even the ones against which he has already defended). In other terms, players can play again earlier defenses (which makes sense when another move is available). Of course this rule is not compatible with the "intuitionistic closure round rule" [15].

The sequent calculus can both express and justify the previous dialogical rules. The rules (c) and (d) are based on the specific feature of intuitionistic logic which contains non-invertible rules. Let us recall that an inference rule is called "invertible", if the validity of the conclusion of the rule implies the validity of all the premises of the rule. The difference between invertible rules and non-invertible rules is crucial for proof search [17] and influences the intuitionistic search of proof. To show it, we are going to give two examples.

Intuitionistic dialogue game for the Law of Excluded Middle (LEM)[4]

[4]Compare with [9], section 4. The tableau should be read like that: at the top of the player X there is the formula that he receives; his action on the formula is explained in the box and can be seen on the right of the arrow below. At the top of Y there is that attack or defense, received by Y and to which he replies, and so on.

$$\begin{tikzcd}
\mathsf{O} \quad \vdash A \lor \neg A \arrow[d, "\mathsf{attack}_0\ {[\lor]}"] \\
\mathsf{P} \arrow[d, "\mathsf{defense}_0\ {[\lor]}\ \text{states}\ \neg A"] \\
\mathsf{O} \quad \vdash \neg A \arrow[d, "\mathsf{attack}_1\ {[\neg]}"] \\
\mathsf{P} \quad A \vdash
\end{tikzcd}$$

The previous dialogue game proves that the LEM is not intuitionistically valid. It can be understood as a picture of the following failed search for a proof in LJ sequent calculus for intuitionistic logic, read from the bottom to the top:

$$\dfrac{\dfrac{A \vdash}{\vdash \neg A}\ \neg_r}{\vdash A \lor \neg A}\ \lor_r^r$$

But the proof search in LJ could also give the following failed proof attempt:

$$\dfrac{\dfrac{\neg A \vdash}{\vdash A}\ \neg_l}{\vdash A \lor \neg A}\ \lor_r^l$$

The replacement the *Intuitionistic Closure Rule Round* by the *Classical* one would allow the revision of the defense of P and then the dialogue game would prove the LEM:

$$\begin{tikzcd}
\mathsf{O} \quad \vdash A \lor \neg A \arrow[d, "\mathsf{attack}_0\ {[\lor]}"] \\
\mathsf{P} \arrow[d, "\mathsf{defense}_0\ {[\lor]}\ \text{states}\ \neg A"] \\
\mathsf{O} \quad \vdash \neg A, A \lor \neg A \arrow[d, "\mathsf{attack}_1\ {[\neg]}\ \text{grants}\ A"] \\
\mathsf{P} \quad A \vdash A \lor \neg A \arrow[d, "\text{revision of}\ \mathsf{defense}_0\ {[\lor]}\ \text{states}\ A"] \\
\mathsf{O} \quad A \vdash A, A \lor \neg A
\end{tikzcd}$$

Notice that $\vdash A \vee \neg A$ is proved in three lines only in the LK sequent calculus for classical logic:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Ax}}{\vdash A, \neg A}\,\neg_r}{\vdash A \vee \neg A}\,\vee_r$$

The key point to explain the difference between the LK proof of the validity of the LEM in classical logic and the disproof of its validity in intuitionistic logic lies in the fact that the LK rule

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,\langle \vee_r \rangle$$

is invertible, while the LJ rules

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\langle \vee_r^l \rangle \quad \text{and} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\langle \vee_r^r \rangle$$

are not invertible; and that crucial difference between LK (for classical logic) and LJ (for intuitionistic logic) explains that the derivation of the LEM as well as the elimination of the double negation are possible in LK, but not in LJ. That point is translated in dialogue games *à la Rahman*, by the difference of two structural rules. We are going to conclude this section by an example slightly more difficult, but more interesting.

The dialogue game of Figure 3 proves that the double negation of the LEM is intuitionistically valid. It is the mirror image of the following derivation in LJ, read from the top to the bottom:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Ax}}{A \vdash A \vee \neg A}\,\vee_r^l}{\neg(A \vee \neg A), A \vdash}\,\neg_l}{\neg(A \vee \neg A) \vdash \neg A}\,\neg_r}{\neg(A \vee \neg A) \vdash A \vee \neg A}\,\vee_r^r}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}\,\neg_l}{\neg(A \vee \neg A) \vdash}\,\text{contraction}_l}{\vdash \neg\neg(A \vee \neg A)}\,\neg_r$$

The interest of the previous proof is to shed light on the last structural rule of the dialogue game, which allows the repetition of attack if an atomic formula has been conceded by O. When that proof is read from the bottom

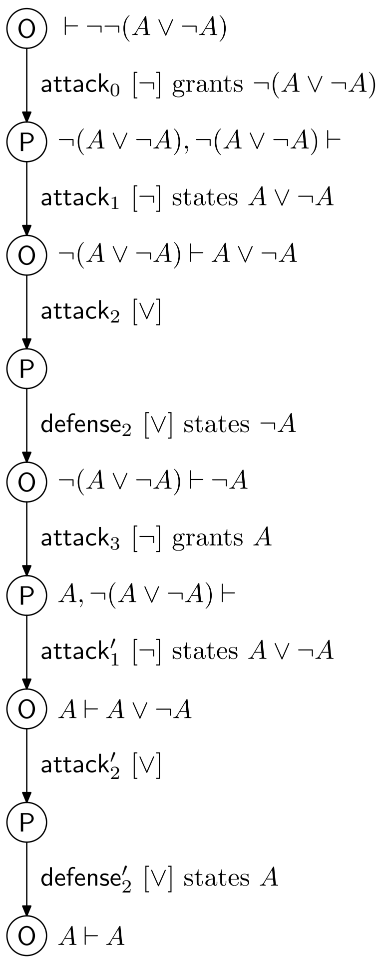

Figure 3: Dialogue games as proof of $\neg\neg(A \vee \neg A)$.

to the top, one sees that the repetition of the attack can be made at the penultimate step, just before the Axiom which replies on the attack on the disjunction of the consequent (and which of course does not appear in the sequent calculus style). Notice that the Weakening (resp. the Contraction) rule of the sequent calculus is the rule thanks to which the first attack of P is possible. Contraction and Weakening are the structural rules allowing the duplication or the omission of the hypothesis in the search proof. The fallacious impression that the last structural rule of the intuitionistic dialogue game is maybe *ad hoc*,[5] disappears when one understands that this structural dialogical rule is closely related with Weakening and Contraction

[5]Because it make possible the proof of the intuitionistic validity of every formula like $\neg\neg(A \vee \neg B)$, i.e., every formula belonging to the famous Gödel-Gentzen translation phenomena.

which concern the "management of resources" inside the search of proofs in sequent calculus [17, 12].

5 Concluding Remarks

The relation between dialogue games and sequent calculus shows how some strategies of proof are already contained in logical rules and structural rules. It shows also that if dialogue games may appear easier to learn and to use, that method of logic is not deeply different from other methods like tableaux methods or sequent calculi. The game semantics in deduction systems does not escape to the logico-mathematical necessity. The trip in the dialogical games is like a linguistic journey to a country where similar laws exist as elsewhere, but expressed differently.

From this perspective the term "dialogical logic" can be seen as an ambigous or misleading expression.[6] The impressive work of Rahman and his school shows that in fact the dialogue games are like a general method of logic allowing to translate a number of different systems of logic. But one must draw a distinction between the ways of expressing rules and calculi, and the logical systems themselves. Strictly speaking there is no "dialogical logic", but there are different dialogue games according different logical systems in different logics. In some cases, dialogue games is a formalism to study relations between some logics and can appear as a semantic framework to study logics.

Another false impression, most seriously flawed, could have its cause in the ease with which it is possible in dialogue games, of passing from intuitionistic tests of validity to classical proofs [12]. This elegant feature of the dialogical method might be a trap for a philosopher who could imagine that one gets, via the dialogue games, a unified logical theory apt to prove both the theorems of intuitionistic logic and the theorems of classical logic:

> *The semantics stays the same, only the rules change. The relationship between logical systems become more transparent (see [14] p. 260).*

Unfortunately, too strong an enthusiasm for dialogue games can lead to abandoning of what semantics really is from a logical point of view. The semantics changes with the change of structural rules, and the relations between logical systems become more transparent only if one keeps in mind what the reasons of the rules are and which rules can be changed in order to change of system of deduction. For example, dialogue games do not change the fact that the semantics of intuitionistic logic and the semantics of the classical logic are different, both in the philosophical sense and in the

[6]See the title of the papers [14, 15] and of the book [12].

algebraic sense of the word: if one deals with the latter sense of "semantics", in classical logic we adopt an algebra of truth functions over the set of two truth values, and that is not true for the intuitionistic logic. [7] last, from the "philosophical" sense of semantics, one cannot forget that the classical logic gives up the constructivist meaning that the intuitionistic school assigns to the logical connectives. Thus, the slogan of Quine remains relevant: "*change of logic, change of subject*" [20], chap. 6.

To conclude this paper, we can wonder the pedagogical analogy that Rahman uses sometimes between logic and the chess play. This analogy seems in agreement with a radical anti-realism and with a "pragmatism" as philosophy of logic: it seems possible to change indefinitely the structural rules of logic, and to indefinitely combine these rules in order to play with systems of deduction. The method of logical games gave birth to the motto on the character "pragmatic" and "dynamic" of logic, which is akin to a post-Wittgensteinian philosophy of logic and to a supplementary critic against Platonism. This article has not even touched the profound and difficult debate on the nature of logic. But if the translation of logical proofs into games is an interesting method or tool, it leaves that philosophical question still open.

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[7] [3], p. 195, proposition 5.2: "No truth-functional, faithful n-valued logic, for any fixed, finite n, can provide a semantics appropriate for intuitionistic propositional logic.". See also [23], pp. 106-107, for the distinction between the "philosophical" sense of "semantics" and the "algebraic" sense of that word.

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