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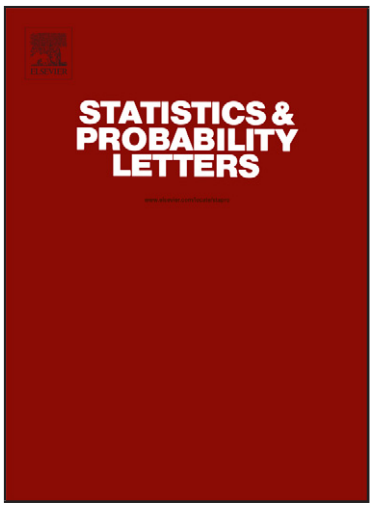

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A Class of Controlled Bisexual Branching Processes with Mating Depending on the Number of Progenitor Couples

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Abstract

We introduce a class of bisexual branching processes where a control is achieved on the number of couples and the mating depends on the number of progenitor couples. We establish several probabilistic properties and some results about its extinction probability.

Keywords: branching processes; bisexual processes; controlled processes.

1. Introduction

In order to describe the probabilistic evolution of two-sex populations, several classes of bisexual processes have been investigated, see e.g. Alsmeyer and Rösler (1996), Bruss (1984), Daley (1968), González *et al.* (2000, 2001), Molina *et al.* (2002) or Xing and Wang (2005). We refer the reader to Hull (2003) or Haccou *et al.* (2005)

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for surveys about these processes. However, the range of bisexual models investigated is not large enough in order to get an optimum modelling in some populations. By environmental, social, or other factors, it could be advisable to carry out a control on the number of couples (female-male mating units) in the population. Furthermore it is reasonable to admit that, in different generations, the same number of females and males gives rise to different number of couples. In an attempt to contribute some solution to these situations, in this work we introduce the class of controlled bisexual branching processes with mating depending on the number of progenitor couples and we provide some theoretical results about it. In Section 2, we give its mathematical formal description and its intuitive interpretation. Section 3 is devoted to determining several probabilistic properties about the model and Section 4 deals with its extinction probability.

2. The model

We introduce the controlled bisexual process with mating depending on the number of progenitor couples as a two-sex model $\{(F_n, M_n)\}_{n\geq 1}$ initiated with $Z_0^* = N^* \geq 1$ couples and defined, for $n \in \mathbb{Z}^+$, in the recursive form:

$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n^*} (f_{n,i}, m_{n,i}), Z_{n+1} = L_{Z_n^*}(F_{n+1}, M_{n+1}), Z_{n+1}^* = \phi(Z_{n+1}) \quad (1)$$

where $\{(f_{n,i}, m_{n,i})\}_{n\geq 0; i\geq 1}$ is a sequence of independent and identically distributed non-negative, integer-valued random vectors; $\{L_k\}_{k\geq 0}$ is a sequence of non-negative real functions on $\mathbb{R}^+ \times \mathbb{R}^+$, for each k fixed L_k is assumed to be monotonic non-decreasing in each argument, integer-valued on the integers, $L_k(x, 0) = L_k(0, y) = 0, x, y \in \mathbb{R}^+$; and ϕ is a non-negative real function on $\mathbb{R}^+$ such that $\phi(0) = 0$ and $\phi(k) \in \mathbb{Z}_0^+$, $k \in \mathbb{Z}_0^+$, with $\mathbb{Z}^+$ and $\mathbb{R}^+$ denoting the non-negative integer and real num-

bers, respectively, and $\mathbb{Z}_0^+ = \mathbb{Z}^+ - \{0\}$. Intuitively, $(f_{n,i}, m_{n,i})$ represents the number of females and males descending from the ith couple of the nth generation. Hence (F_{n+1}, M_{n+1}) is the number of females and males in the $(n+1)$th generation, which form Z_{n+1} couples through the mating function $L_{Z_n^*}$, where $Z_n^* = \phi(Z_n)$ denotes the number of progenitor couples, namely, the couples that after the control governed by ϕ really participated in the reproduction. We remark that, if $\phi(Z_n) > Z_n$ then $\phi(Z_n) - Z_n$ couples are introduced in the population; if $\phi(Z_n) < Z_n$ then $Z_n - \phi(Z_n)$ couples leave the population; and no control is made if $\phi(Z_n) = Z_n$.

It is easy to verify that $\{(Z_{n-1}, F_n, M_n)\}_{n\geq 1}$, $\{Z_n\}_{n\geq 1}$ and $\{Z_n^*\}_{n\geq 0}$ are homogeneous Markov chains with 0 being an absorbing state for $\{Z_n\}_{n\geq 1}$ and $\{Z_n^*\}_{n\geq 0}$. Moreover its theoretical interest, the bisexual model introduced in this work has several practical implications in population dynamics. For example, in phenomena concerning to inhabit or re-inhabit environments with animal species which have sexual reproduction, the probable evolution of the numbers of females, males, originated couples, and progenitor couples, may be described in term of this model. Indeed, the motivation behind the process (1) is the interest in developing stochastic models to describe probabilistically such situations.

Remark 2.1 *The class of processes given in (1) includes some models investigated in the bisexual branching process literature. Indeed, when $\phi(x) = x, x \geq 0$ one obtains the class of bisexual models introduced in Molina et al. (2002). We remark that if ϕ is non-decreasing then one has, for each $k \in \mathbb{Z}^+$, that $L_k^* = \phi \circ L_k$ is a mating function. In such a particular case, the model (1) may be considered under the perspective of the class of processes studied in Molina et al. (2002) but losing the information provided by the sequence $\{Z_n\}_{n\geq 1}$. In this context, the theory developed for this*

class of processes is completed with the results established in the present work for ϕ superadditive. Furthermore, if ϕ is the identity and $L_k(x,y) = L(x,y), x, y \in \mathbb{R}^+, k \in \mathbb{Z}^+$ then the class (1) is reduced to the class of models given in Daley (1968).

We will consider the following working assumptions about the mating functions:

(A1): For each $k \in \mathbb{Z}^+$, L_k is a superadditive function, namely, for $n \geq 2$

$$L_k\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right) \geq \sum_{i=1}^{n} L_k(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+, i = 1, \ldots, n.$$

(A2): For each $x, y \in \mathbb{R}^+$ fixed, the sequence $\{L_k(x,y)\}_{k\geq 0}$ is non-decreasing.

Assumption $(A1)$ expresses the fact that $\sum_{i=1}^{n} x_i$ females and $\sum_{i=1}^{n} y_i$ males coexisting together will form a number of couples greater than or equal to the total number of couples produced from n groups of x_i females and y_i males, $i = 1, \ldots, n$, living separately. Assumption $(A2)$ represents the usual fact of many biological populations in which the number of matings in certain generation depends on the number of progenitor couples in the previous one in such a way that the mating is promoted if such a number grows. Some sequences $\{L_k\}_{k\geq 0}$ verifying $(A1)$ and $(A2)$ are, for instance: (a) $L_k(x,y) = x\min\{k,y\}$; (b) $L_k(x,y) = \min\{x, ky\}$; or (c) $L_k(x,y) = \min\{x,y\}$ if $k \leq k_0$ or $x\min\{k,y\}$ if $k > k_0$ (restricted to non-negative integers) where $k_0 \in \mathbb{Z}_0^+$ is fixed.

3. Probabilistic properties

Let us denote by $f(s,t) = E[s^{f_{0,1}} t^{m_{0,1}}]$, $h_n(s,t) = E[s^{F_n} t^{M_n}]$ and $g_n^*(s) = E[s^{Z_n^*}]$, $s, t \in [0,1]$, $n \in \mathbb{Z}_0^+$. Clearly $g_0^*(s) = s^{N^*}$.

Proposition 3.1 *For $s, t \in [0,1]$ and $n \in \mathbb{Z}^+$, $h_{n+1}(s,t) = g_n^*(f(s,t))$.*

Proof. It is clear that $h_1(s,t) = g_0^*(f(s,t)), s,t \in [0,1]$. For $n \in \mathbb{Z}_0^+$

$$\begin{aligned} h_{n+1}(s,t) &= E\left[E[s^{F_{n+1}}t^{M_{n+1}} \mid Z_n]\right] = \sum_{k=0}^{\infty} E\left[s^{\sum_{i=1}^{\phi(k)} f_{n,i}} t^{\sum_{i=1}^{\phi(k)} m_{n,i}}\right] P(Z_n = k) \\ &= \sum_{k=0}^{\infty} (f(s,t))^{\phi(k)} P(Z_n = k) g_n^*(f(s,t)), \quad s,t \in [0,1]. \end{aligned}$$ ∎

Let us write $\mu = E[(f_{0,1}, m_{0,1})]$, $\Sigma = Cov[(f_{0,1}, m_{0,1})]$, $\mu_n = E[(F_n, M_n)]$ and $\Sigma_n = Cov[(F_n, M_n)]$, $n \in \mathbb{Z}_0^+$. As a consequence of Proposition 3.1, $\mu_{n+1} = E[Z_n^*]\mu$ and $\Sigma_{n+1} = E[Z_n^*]\Sigma + Var[Z_n^*]\mu'\mu$, $n \in \mathbb{Z}^+$, where μ' denotes the transpose of μ.

Let $Y_k = L_k(f_{0,1}, m_{0,1})$ and $Y_k^* = \phi(Y_k)$, $k \in \mathbb{Z}^+$ and let us denote by $\pi_k(s) = E[s^{Y_k}]$, $\pi_k^*(s) = E[s^{Y_k^*}]$, and $g_{n+1}(s) = E[s^{Z_{n+1}}]$, $s \in [0,1]$, $k, n \in \mathbb{Z}^+$. Clearly $Y_0 = Y_0^* = 0$ hence $\pi_0(s) = \pi_0^*(s) = 1$.

Proposition 3.2 *Assume* $(A1)$. *For* $s \in [0,1]$ *and* $n \in \mathbb{Z}^+$

(i) $g_{n+1}(s) \leq E\left[\left(\pi_{Z_n^*}(s)\right)^{Z_n^*}\right]$.

(ii) If ϕ *is non-increasing on* $(0,\infty)$ *then* $g_{n+1}^*(s) \geq E\left[(\pi_{Z_n^*}^*(s))^{Z_n^*}\right]$.

(iii) If ϕ *is superadditive then* $g_{n+1}^*(s) \leq E\left[(\pi_{Z_n^*}^*(s))^{Z_n^*}\right]$.

Proof. *(i)* Using $(A1)$ one has that $g_1(s) \leq E[(\pi_{Z_0^*}(s))^{Z_0^*}], s \in [0,1]$ and

$$\begin{aligned} E[s^{Z_{n+1}} \mid Z_n = k] &= E\left[s^{L_{\phi(k)}\left(\sum_{i=1}^{\phi(k)} f_{n,i}, \sum_{i=1}^{\phi(k)} m_{n,i}\right)}\right] \leq E\left[s^{\sum_{i=1}^{\phi(k)} L_{\phi(k)}(f_{n,i}, m_{n,i})}\right] \\ &= E\left[s^{Y_{\phi(k)}}\right]^{\phi(k)} = \left(\pi_{\phi(k)}(s)\right)^{\phi(k)}, \quad k \in \mathbb{Z}^+, n \in \mathbb{Z}_0^+. \end{aligned}$$

(ii) Let $\phi(\mathbb{Z}^+) = \{\phi(k), k \in \mathbb{Z}^+\}$. By $(A1)$ and the fact that ϕ is non-increasing on $(0,\infty)$, one obtains for $s \in [0,1]$ and $n \in \mathbb{Z}^+$

$$\begin{aligned} g^*_{n+1}(s) &= \sum_{j\in\phi(\mathbb{Z}^+)} E\left[s^{\phi\left(L_j\left(\sum_{i=1}^j f_{n,i},\sum_{i=1}^j m_{n,i}\right)\right)}\right] P(Z^*_n=j) \\ &\geq \sum_{j\in\phi(\mathbb{Z}^+)} E\left[s^{\sum_{i=1}^j \phi(L_j(f_{n,i},m_{n,i}))}\right] P(Z^*_n=j) \\ &= \sum_{j\in\phi(\mathbb{Z}^+)} \left(\pi^*_j(s)\right)^j P(Z^*_n=j) = E\left[(\pi^*_{Z^*_n}(s))^{Z^*_n}\right]. \end{aligned}$$

(iii) It is proved in a similar way that (ii) using the fact that ϕ is non-decreasing. ∎

By Proposition 3.2, $E[Z_{n+1}] \geq E[Z^*_n E[Y_{Z^*_n}]]$ and $E[Z^*_{n+1}] \leq$ (or $\geq$)$E[Z^*_n E[Y^*_{Z^*_n}]]$, $n \in \mathbb{Z}^+$ if ϕ is non-increasing (or superadditive). Also, assuming that $Y_j \geq Y_1$, $j \in \mathbb{Z}^+_0$, one deduces that $\pi_j(s) \leq \pi_1(s), s \in [0,1], j \in \mathbb{Z}^+_0$ and, again from Proposition 3.2, $g_{n+1}(s) \leq g^*_n(\pi_1(s))$ and $g^*_{n+1}(s) \geq$ (or $\leq$)$g^*_n(\pi^*_1(s))$ if ϕ is non-increasing (or superadditive), $s \in [0,1], n \in \mathbb{Z}^+$. Thus, for $n \in \mathbb{Z}^+$, $E[Z_{n+1}] \geq E[Z^*_n]E[Y_1]$ and

$$E[Z^*_{n+1}] \leq (\text{or } \geq) N^*(E[Y^*_1])^{n+1} \text{if} \phi \text{is non-increasing (or superadditive)} \qquad (2)$$

Let $T_n = \sum_{i=1}^n Z_i, n \in \mathbb{Z}^+_0$ and $T^*_n = \sum_{i=0}^n Z^*_i, n \in \mathbb{Z}^+$ and let us denote by $G_n(s,t) = E[s^{T_n}t^{Z_n}]$, $G^*_n(s,t) = E[s^{T^*_n}t^{Z^*_n}]$ and $H_n(s,t) = E[s^{T_n}t^{Z^*_n}]$, $s,t \in [0,1]$, $n \in \mathbb{Z}^+_0$. Clearly $G_1(s,t) = g_1(st)$ and $G^*_0(s,t) = (st)^{N^*}$, $s,t \in [0,1]$.

Proposition 3.3 *Assume* $(A1)$ *and* $Y_j \geq Y_1, j \in \mathbb{Z}^+_0$. *For* $s,t \in [0,1]$

(i) $G_{n+1}(s,t) \leq H_n(s,\pi_1(st))$, $n \in \mathbb{Z}^+_0$.

(ii) If ϕ *is non-increasing on* $(0,\infty)$ *then* $G^*_{n+1}(s,t) \geq G^*_n(s,\pi^*_1(st))$, $n \in \mathbb{Z}^+$.

(iii) If ϕ *is superadditive then* $G^*_{n+1}(s,t) \leq G^*_n(s,\pi^*_1(st))$, $n \in \mathbb{Z}^+$.

Proof. *(i)* First note that, from $(A1)$ and the fact that $Y_j \geq Y_1, j \in \mathbb{Z}^+_0$, one deduces that $E[s^{Z_{n+1}} \mid \mathcal{F}_n] \leq (\pi_1(s))^{Z^*_n}$ almost surely, where $\mathcal{F}_n = \sigma(Z_1,\ldots,Z_n)$, $n \in \mathbb{Z}^+_0$, hence $G_{n+1}(s,t) = E\left[s^{T_n}E[(st)^{Z_{n+1}} \mid \mathcal{F}_n]\right] \leq E\left[s^{T_n}(\pi_1(st))^{Z^*_n}\right]$.

(ii) Again by $(A1)$ and taking into account that ϕ is non-increasing, one derives that $E[s^{Z^*_{n+1}} \mid \mathcal{F}^*_n] \geq (\pi^*_1(s))^{Z^*_n}$ almost surely, where $\mathcal{F}^*_n = \sigma(Z^*_0, \ldots, Z^*_n)$, $n \in \mathbb{Z}^+$, hence $G^*_{n+1}(s,t) = E\left[s^{T^*_n} E[(st)^{Z^*_{n+1}} \mid \mathcal{F}^*_n]\right] \geq E\left[s^{T^*_n} \pi^*_1(st)^{Z^*_n}\right]$.

(iii) It is proved in a similar manner that (ii) by using that ϕ is superadditive.

By Proposition 3.3(i), $E[T_{n+1}] \geq E[Y_1] \sum_{i=1}^{n} E[Z^*_i] + E[Z_1]$, $n \in \mathbb{Z}^+_0$ and, using Proposition 3.3(ii) and (2), assuming ϕ non-increasing, one deduces that

$$E[T^*_{n+1}] \leq N^*(n+2) \text{if} \alpha = 1 \text{or} \leq N^*(1-\alpha^{n+2})(1-\alpha)^{-1} \text{if} \alpha \neq 1 \tag{3}$$

where $\alpha = E[Y^*_1]$. The reverse is obtained in (3) when ϕ is superadditive. Notice that, if $N^* = 1$, then one has that $Y_1 = Z_1$ hence that $\pi_1(s) = g_1(s), \pi^*_1(s) = g^*_1(s), s \in [0,1]$, and $\alpha = E[\phi(L_1(f_{0,1}, m_{0,1}))]$.

We now study some stochastic monotony properties about $\{Z_n\}_{n\geq 1}$, $\{Z^*_n\}_{n\geq 0}$, $\{F_n\}_{n\geq 1}$ and $\{M_n\}_{n\geq 1}$.

Proposition 3.4 *Assume* $(A2)$ *and* ϕ *non-increasing on* $(0,\infty)$. *Then for* $k_i \in \mathbb{Z}^+$, $j_i \in \phi(\mathbb{Z}^+)$, $i = 1,2$, *with* $k_1 < k_2$, $j_1 < j_2$, $y \in \mathbb{R}$, *and* $n \in \mathbb{Z}^+_0$, $P(Z_{n+1} \leq y \mid Z_n = k_2) \geq P(Z_{n+1} \leq y \mid Z_n = k_1)$ *and* $P(Z^*_{n+1} \leq y \mid Z^*_n = j_2) \geq P(Z^*_{n+1} \leq y \mid Z^*_n = j_1)$.

Proof. Using $(A2)$, the fact that $\phi(k_2) \leq \phi(k_1)$, and that $L_{\phi(k_1)}$ is monotonic non-decreasing in each argument, one derives for $y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$

$$P(Z_{n+1} > y \mid Z_n = k_2) \leq P\left(L_{\phi(k_1)}\left(\sum_{i=1}^{\phi(k_2)} f_{n,i}, \sum_{i=1}^{\phi(k_2)} m_{n,i}\right) > y\right)$$

$$\leq P\left(L_{\phi(k_1)}\left(\sum_{i=1}^{\phi(k_1)} f_{n,i}, \sum_{i=1}^{\phi(k_1)} m_{n,i}\right) > y\right) = P(Z_{n+1} > y \mid Z_n = k_1).$$

On the other hand, for $y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$

$$P(Z^*_{n+1} > y \mid Z^*_n = j_2) \leq P\left(\phi\left(L_{j_1}\left(\sum_{i=1}^{j_2} f_{n,i}, \sum_{i=1}^{j_2} m_{n,i}\right)\right) > y\right)$$
$$\leq P\left(\phi\left(L_{j_1}\left(\sum_{i=1}^{j_1} f_{n,i}, \sum_{i=1}^{j_1} m_{n,i}\right)\right) > y\right) = P(Z^*_{n+1} > y \mid Z^*_n = j_1). \quad \blacksquare$$

It is easy to verify that similar results with the reverse can be established, along the line of Proposition 3.4, if ϕ is non-decreasing. Let $\{(F^{(i)}_n, M^{(i)}_n)\}_{n\geq 1}$, $i \in \mathbb{Z}^+_0$ be independent versions of $\{(F_n, M_n)\}_{n\geq 1}$ and let us denote by $\{Z^{(i)}_n\}_{n\geq 1}$ and $\{Z^{*(i)}_n\}_{n\geq 0}$ their associated sequences of originated and progenitor couples, respectively, with $Z^{*(i)}_0 = 1,\ i \in \mathbb{Z}^+_0$.

Proposition 3.5 *Assume* $(A1)$, $(A2)$, *and* ϕ *superadditive. Then, for* $k \in \mathbb{Z}^+$ *and* $y \in \mathbb{R}$, $P(Z^*_{n+k} \leq y) \leq P(\sum_{i=1}^{Z^*_k} Z^{*(i)}_n \leq y)$, $n \in \mathbb{Z}^+$ *and* $P(F_{n+k} \leq y) \leq P(\sum_{i=1}^{Z^*_k} F^{(i)}_n \leq y)$ *(similarly* $P(M_{n+k} \leq y) \leq P(\sum_{i=1}^{Z^*_k} M^{(i)}_n \leq y)$ *),* $n \in \mathbb{Z}^+_0$.

Proof. First, we shall prove that for $n, k, j \in \mathbb{Z}^+$ and $y \in \mathbb{R}$,

$$P(Z^*_{k+n+1} \leq y \mid Z^*_{n+k} = j) \leq P\left(\sum_{i=1}^{Z^*_k} Z^{*(i)}_{n+1} \leq y \,\middle|\, \sum_{i=1}^{Z^*_k} Z^{*(i)}_n = j\right) \tag{4}$$

In fact, using $(A2)$, the superadditivity of ϕ and $(A1)$, one obtains

$$P\left(\sum_{i=1}^{Z^*_k} Z^{*(i)}_{n+1} \leq y \,\middle|\, \sum_{i=1}^{Z^*_k} Z^{*(i)}_n = j\right) \geq$$

$$P\left(\phi\left(L_{\sum_{i=1}^{Z^*_k} Z^{*(i)}_n}\left(\sum_{i=1}^{Z^*_k}\sum_{l=1}^{Z^{*(i)}_n} f^{(i)}_{n,l}, \sum_{i=1}^{Z^*_k}\sum_{l=1}^{Z^{*(i)}_n} m^{(i)}_{n,l}\right)\right) \leq y \,\middle|\, \sum_{i=1}^{Z^*_k} Z^{*(i)}_n = j\right) =$$

$$P\left(\phi\left(L_j\left(\sum_{l=1}^{j} f_{k+n,l}, \sum_{l=1}^{j} m_{k+n,l}\right)\right) \leq y\right) = P(Z^*_{k+n+1} \leq y \mid Z^*_{n+k} = j).$$

Note that for $n = 0$, using that $Z^{*(i)}_0 = 1$, it is clear that $P(Z^*_k \leq y) = P(\sum_{i=1}^{Z^*_k} Z^{*(i)}_0 \leq y)$. Assume that $P(Z^*_{k+n} \leq y) \leq P(\sum_{i=1}^{Z^*_k} Z^{*(i)}_n \leq y)$. Then, considering that ϕ is

non-decreasing, that for $n, k \in \mathbb{Z}^+$, $y \in \mathbb{R}$ fixed, $\{P(Z^*_{k+n+1} \leq y \mid Z^*_{n+k} = j)\}_{j\geq 0}$ is a non-increasing sequence, the induction hypothesis, Lemma 1 (see Appendix) and (4), one deduces that

$$P(Z^*_{k+n+1} \leq y) \leq \sum_{j=0}^{\infty} P(Z^*_{n+k+1} \leq y \mid Z^*_{n+k} = j) P\left(\sum_{i=1}^{Z^*_k} Z^{*(i)}_n = j\right)$$
$$\leq \sum_{j=0}^{\infty} P\left(\sum_{i=1}^{Z^*_k} Z^{*(i)}_{n+1} \leq y \mid \sum_{i=1}^{Z^*_k} Z^{*(i)}_n = j\right) P\left(\sum_{i=1}^{Z^*_k} Z^{*(i)}_n = j\right) = P\left(\sum_{i=1}^{Z^*_k} Z^{*(i)}_{n+1} \leq y\right).$$

Using a similar reasoning are proved the other inequalities. ∎

Remark 3.1 *Under assumptions in Proposition 3.5, it can be also proved that:*

(i) If $\phi(Z_n) \leq Z_n$, $n \in \mathbb{Z}^+_0$, i.e. emigration of couples is produced in each generation, then $P(Z_{n+k} \leq y) \leq P(\sum_{i=1}^{Z^*_k} Z^{*(i)}_n \leq y)$, $y \in \mathbb{R}^+$, $n \in \mathbb{Z}^+$.

(ii) If $\phi(Z_n) \geq Z_n$, $n \in \mathbb{Z}^+_0$, i.e. immigration of couples is produced in each generation, then $P(Z^*_{n+k} \leq y) \leq P(\sum_{i=1}^{Z_k} Z^{(i)}_n \leq y)$, $P(F_{n+k} \leq y) \leq P(\sum_{i=1}^{Z_k} F^{(i)}_n \leq y)$ and $P(M_{n+k} \leq y) \leq P(\sum_{i=1}^{Z_k} M^{(i)}_n \leq y)$, $y \in \mathbb{R}^+$, $n \in \mathbb{Z}^+$.

4. Extinction probability

It is clear that the process (1) will become extinct if, in some generation, there are not any progenitor couples. Let us write by $q_{N^*} = P(Z^*_n \to 0 \mid Z^*_0 = N^*)$ the extinction probability when the process is initiated with N^* couples. Let us introduce the sequences $\{\widetilde{Z}_n\}_{n\geq 0}$, $\{\overline{Z}_n\}_{n\geq 0}$ and $\{\widehat{Z}_n\}_{n\geq 0}$ where $\widetilde{Z}_{n+1} = \sum_{i=1}^{\widetilde{Z}_n} L_1(f_{n,i}, m_{n,i})$, $\overline{Z}_{n+1} = \sum_{i=1}^{\overline{Z}_n} \phi(L_1(f_{n,i}, m_{n,i}))$ and $\widehat{Z}_{n+1} = \sum_{i=1}^{\phi(\widehat{Z}_n)} L_1(f_{n,i}, m_{n,i})$, $n \in \mathbb{Z}^+$, with $\widetilde{Z}_0 = \overline{Z}_0 = N^*$, and $\widehat{Z}_0 = N_0 \geq 1$. We assume that N_0 is such that $\phi(N_0) \leq N^*$. Note that $\{\widetilde{Z}_n\}_{n\geq 0}$ and $\{\overline{Z}_n\}_{n\geq 0}$ are asexual Galton-Watson processes with offspring distributions as the laws of $Y_1 = L_1(f_{0,1}, m_{0,1})$ and $Y^*_1 = \phi(Y_1)$, respectively. On the other hand,

$\{\widehat{Z}_n\}_{n\geq 0}$ is an asexual controlled Galton-Watson process with offspring distribution as the law associated to Y_1.

Proposition 4.1 *Assume* $(A1)$ *and* $(A2)$. *If any of the following conditions holds:*

(a) ϕ *is non-decreasing,* $\phi(k) \geq k$, $k \in \mathbb{Z}_0^+$ *and* $E[Y_1] > 1$.

(b) ϕ *is superadditive and* $E[Y_1^*] > 1$.

(c) ϕ *is superadditive and* $\sup_{k>0} k^{-1}\phi(k) > E[Y_1]^{-1}$.

Then $q_{N^*} < 1$, $N^* \in \mathbb{Z}_0^+$.

Proof. Suppose that condition (a) is satisfied. We shall prove, by induction on n, that $Z_n^* \geq \widetilde{Z}_n$, $n \in \mathbb{Z}^+$. In fact, for $n = 0$, $Z_0^* = \widetilde{Z}_0 = N^* \geq 1$. Assume that $Z_n^* \geq \widetilde{Z}_n$, then, by $(A1)$ and $(A2)$,

$$Z_{n+1}^* \geq \phi\left(\sum_{i=1}^{Z_n^*} L_1(f_{n,i}, m_{n,i})\right) \geq \phi\left(\sum_{i=1}^{\widetilde{Z}_n} L_1(f_{n,i}, m_{n,i})\right) \geq \widetilde{Z}_{n+1}.$$

Thus $q_{N^*} \leq \widetilde{q}_{N^*} = P(\widetilde{Z}_n \to 0 | \widetilde{Z}_0 = N^*)$ and, from Galton-Watson process theory, using the fact that $E[Y_1] > 1$, one deduces that $\widetilde{q}_{N^*} < 1$ and therefore $q_{N^*} < 1$. We now assume condition (b) holds. It is sufficient to verify that $Z_n^* \geq \overline{Z}_n$, $n \in \mathbb{Z}^+$. Again by induction, it is clear that $Z_0^* = \overline{Z}_0 = N^*$. Suppose that $Z_n^* \geq \overline{Z}_n$, then from $(A1)$ and $(A2)$,

$$Z_{n+1}^* \geq \sum_{i=1}^{Z_n^*} \phi\left(L_1(f_{n,i}, m_{n,i})\right) \geq \sum_{i=1}^{\overline{Z}_n} \phi\left(L_1(f_{n,i}, m_{n,i})\right) \overline{Z}_{n+1}.$$

Thus $q_{N^*} \leq \overline{q}_{N^*} = P(\overline{Z}_n \to 0 \mid \overline{Z}_0 = N^*)$ and, taking into account that $E[Y_1^*] > 1$, the result is derived. Finally, if (c) is satisfied then, considering that $\phi(0) = 0$ and $\phi(k) \in \mathbb{Z}_0^+$, $k \in \mathbb{Z}_0^+$, one obtains that $q_{N^*} = P(Z_n \to 0 \mid Z_0^* = N^*)$. To complete the

proof it is sufficient to verify that $Z_n \geq \widehat{Z}_n, n \in \mathbb{Z}_0^+$. Now, by induction on n, if $n = 1$, using $(A1)$, $(A2)$ and the fact that $\phi(N_0) \leq N^*$, $Z_1 = L_{N^*}\left(\sum_{i=1}^{N^*} f_{0,i}, \sum_{i=1}^{N^*} m_{0,i}\right) \geq \sum_{i=1}^{\phi(N_0)} L_1(f_{0,i}, m_{0,i}) = \widehat{Z}_1$. Assume that $Z_n \geq \widehat{Z}_n$. Taking into account that ϕ is superadditive and therefore non-decreasing, one has that $Z_n^* \geq \phi(\widehat{Z}_n)$ hence

$$Z_{n+1} \geq \sum_{i=1}^{Z_n^*} L_{Z_n^*}(f_{n,i}, m_{n,i}) \geq \sum_{i=1}^{Z_n^*} L_1(f_{n,i}, m_{n,i}) \geq \sum_{i=1}^{\phi(\widehat{Z}_n)} L_1(f_{n,i}, m_{n,i}) = \widehat{Z}_{n+1}.$$

Therefore $q_{N^*} \leq \widehat{q}_{N^*} = P(\widehat{Z}_n \to 0 \mid \widehat{Z}_0 = N^*)$. Now, using that ϕ is superadditive and $\sup_{k>0} k^{-1}\phi(k) > E[Y_1]^{-1}$, one deduces that $\widehat{q}_{N^*} < 1$ and consequently $q_{N^*} < 1$.

∎

Remark 4.1 *The concept of the mean growth rate per couple introduced in Bruss (1984) can be extended in a natural form to the class of processes (1). In fact, for each $j \in \phi(\mathbb{Z}_0^+) = \phi(Z^+) - \{0\}$, we define the mean growth rate per progenitor couple as the expected value $R_j = E[Z_n^{*-1} Z_{n+1}^* \mid Z_n^* = j]$. Note that, assuming $(A1)$, $(A2)$ and ϕ superadditive, using a similar reasoning to that one used in Molina et al. (2002), Theorem 3.2, one obtains that, if the offspring mean vector μ is a positive interior point of $\mathbb{R}^+ \times \mathbb{R}^+$ such that $\lim_{j \nearrow \infty, j \in \phi(\mathbb{Z}_0^+)} j^{-1}\phi(L_j(j\mu)) < \infty$ then $q_{N^*} = 1, \quad N^* \in \mathbb{Z}_0^+$ if and only if $R \leq 1$ where $R = \sup_{j \in \phi(\mathbb{Z}_0^+)} R_j$.*

Appendix

Lemma 1. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n), (u_1, \ldots, u_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $k = 1, \ldots, n$ and $u_1 \geq \ldots \geq u_n \geq 0$. Then $\sum_{i=1}^{n} u_i x_i \leq \sum_{i=1}^{n} u_i y_i$.

Proof. Let $t_i = \sum_{j=1}^{i} x_j, s_i = \sum_{j=1}^{i} y_j, i = 1, \ldots, n$. Clearly $t_i \leq s_i, i = 1, \ldots, n$, hence it is sufficient to verify that $\sum_{i=1}^{n-1} (u_i - u_{i+1}) t_i + u_n t_n \leq \sum_{i=1}^{n-1} (u_i - u_{i+1}) s_i + u_n s_n$. Now, this inequality holds because $u_i - u_{i+1} \geq 0, i = 1, \ldots, n$ and $u_n \geq 0$. ∎

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