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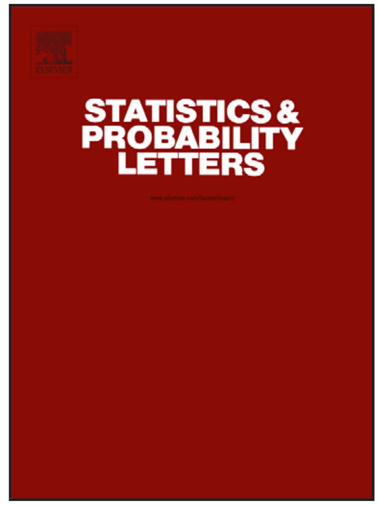

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Central Limit Theorem by Moments

René BLACHER

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René BLACHER
Laboratoire LMC, BP 53 38.041 Grenoble Cedex 9 FRANCE

Summary : In a previous central limit theorem by moments, it has been proved that the moments converge to those of the normal distribution if the moments of sums are asymptotically independent (cf Theoreme de la limite centrale par les moments, Blacher R. Cpte rendus Acad. Sci Paris. t-311- I, 465-468). . In this paper we generalize this result by adding a negligible sequence to these sums. So, we can prove that the moments of some functionals of strong mixing sequences converge.

KeyWords : Central Limit Theorem, moments, strongly mixing sequence, higher order correlation coefficients.

1 Introduction

Let $\{X_n\}$ be a sequence of random variables defined on a probability space (Ω, A, P) such that, for all $n \in N$, $E\{X_n\} = 0$ and $0 < E\{|X_n|^k\} < \infty$ for all $k \in N$.

On decompose $X_1 + X_2 + ... + X_n$ en $X_1 + X_2 + ... + X_{u(n)}$, $X_{u(n)+1} + X_{u(n)+2} + ... + X_{u(n)+t(n)}$ and $X_{u(n)+t(n)+1} + X_{u(n)+t(n)+2} + ... + X_{u(n)+t(n)+u(n)}$, where $u(n) + t(n) + u(n) = n$.

Notations 1.1 *: We denote by $\kappa(n) \in N$, an increasing sequence such that $\kappa(1) = 0$, $\kappa(n) \leq n$ and $\kappa(n)/n \to 0$ as $n \to \infty$. We define the sequences u(n) and t(n) by : u(1)=1, $u(n) = max\{m \in N^* \big| 2m + \kappa(m) \leq n\}$ and t(1)=0, t(n) = n-2u(n) if $n \geq 2$. Moreover, we simplify u(n) and t(n) as u and t.*

Let $\sigma(u)^2$ be the variance of X_1+X_2+...+X_u . One sets $S_u = \frac{X_1+X_2+.........+X_u}{\sigma(u)}$, $\xi_u = \frac{X_{u+1}+X_{u+2}+.........+X_{u+t}}{\sigma(u)}$ and $S'_u = \frac{X_{u+t+1}+X_{u+t+2}+.........+X_{u+t+u}}{\sigma(u)}$. If t(n)=0, we set $\xi_u = 0$. Moreover, one tolerates the notation $S_n = \frac{X_1+X_2+.........+X_n}{\sigma(n)}$.

We denote also by $\sigma^o(t)^2$ the variance of $X_{u+1} + X_{u+2} + ... + X_{u+t}$.

In [?] , one has proved that $u \to \infty$, $n/u \to 2$ and $t/u \to 0$ as $n \to \infty$.

Moreover, if $\kappa(n) \to \infty$, $t \to \infty$ as $n \to \infty$. Finally, ξ_u is negligible if $\sigma^o(t)/\sigma(u) \to 0 : E\{(\xi_u)^2\} \to 0$.

Notations 1.2 *: We define conditions H_{mS} and H_{mI} by the following way.*

$H_{mS} : \forall p \in \mathbf{N}$, $E\big\{(S_u + v_u)^p\big\} - E\big\{(S_u + v_u)^p\big\} \to 0$ *as* $n \to \infty$.

$H_{mI} : \forall (p,q) \in \mathbf{N^2}$, $E\big\{(S_u + v_u)^p(S'_u + v'_u)^q\big\} - E\big\{(S_u + v_u)^p\big\}E\big\{(S'_u + v'_u)^q\big\} \to 0$ *as* $n \to \infty$,

where $\{v_u\}$ and $\{v'_u\}$ are two sequences of random variables such that $E\{(v_u)^{2k}\} + E\{(v'_u)^{2k}\} \to 0$ as $n \to \infty$ for all $k \in N$.

Now, we want to study the MCLT (Moment Central Limit theorem) for $\{X_n\}$. Generally, one assumes that $\{X_n\}$ is a martingale or is strong mixing. Then, one proves that under some assumptions, moments converge or are bounded : cf Bernstein, [?] , Brown [?] , Eissein-Janson [?], Hernndorf [?], Birkel [?], Krugov [?], Mairoboda [?], Yokohama ([?],[?]), Cox-Kim [?], Ibragimov-Lifshits [?], Soulier [?], Rozovsky [?].

So one can deduce that H_{mI} holds. For example, if $\{X_n\}$ is φ-mixing, H_{mI} holds if $t(n) \to \infty$ as $n \to \infty$ enough slowly (cf [?],[?]). If $\{X_n\}$ is strong mixing, H_{mI} holds if $E\{|S_n|^p\}$ is bounded for all $p \in N$. Of course, if $\{X_n\}$ is strictly stationary, H_{mS} holds.

Now, the martingale condition is a strong condition. It is the same for strong mixing conditions : even they does not hold for some AR(1) processes. Then, some authors have introduce weaker hypotheses : Versik Ornstein ([?], [?]), Withers [?], Cogburn [?] Rosenblatt [?], Pinskers [?]. But they did not studied the moments's convergence. Now another look is possible : one can define asymptotical independence conditions by using moments. In this way, one can use higher order correlation coefficients (cf Lancaster [?], Blacher [?]). Then, in [?] we have turned the convergence of moments into an equivalent condition on these coefficients. These results have led to the introduction of the condition H_{mI} : that is, rather than to assume that $\{X_n\}$ is strong mixing, we choose H_{mI} as asymptotical independence assumptions. Then, under this assumption, $S_n \to^M N(0,1)$, that is, for all $p \in N$, $E\{(S_n)^p\} \to \mu_p$ as $n \to \infty$, the moment of order p of N(0,1).

Théorème 1 *: Assume that $E\{|\xi_u|^k\} \to 0$ as $n \to \infty$ for all $k \in N$. Assume that H_{mS} and H_{mI} hold. Then, $S_n \to^M N(0,1)$.*

In [?], we have proved this theorem when $v_u = v'_u = 0$. In section III, we prove theorem **??** for any sequences v_u and v'_u . In II-a, we explain why we need this generalization. In II-b, we apply these results to functionals of IID sequences. In II-C we prove the MCLT for some functionals of strong mixing sequences or more general sequences.

2 Moments Central Limit Theorems for functionals of some strictly stationary sequences

2.1 Discussion

Notations 2.1 *: Let Ψ_t be a strictly stationary process. Assume that $X_t = \sum_{i=0}^{\infty} k_{i+1}(\Psi_{t+i})$, where the k_i's are measurable functions such that $E\{k_i(\Psi_1)\} = 0$ and $0 < E\{k_i(\Psi_1)^{2p}\} < \infty$ for all $i \in N$ and all $p \in N$.*

We set $M_n = \frac{1}{\sigma(n)} \sum_{r=1}^{n} \sum_{s=1}^{r} k_s(\Psi_r)$ and $r_n = \frac{1}{\sigma(n)} \sum_{r=1}^{\infty} \sum_{s=r+1}^{r+n} k_s(\Psi_{n+r})$. Then, one can write $S_n = M_n + r_n$ with $E\{M_n\} = E\{r_n\} = 0$.

Of course, $S_{u(n)} = M_{u(n)} + r_{u(n)}$. Moreover, one can write $S'_{u(n)} = M'_{u(n)} + r'_{u(n)}$ with $M'_u = \frac{1}{\sigma(u)} \sum_{r=1}^{u} \sum_{s=1}^{r} k_s(\Psi_{r+u+t})$ and $r'_u = \frac{1}{\sigma(u)} \sum_{r=1}^{\infty} \sum_{s=r+1}^{r+u} k_s(\Psi_{u+r+u+t})$.

Now one wants to know if the MCLT hold. For example let $\{\Theta_t\}$ be an IID sequence of random variables. Assume that $\Psi_t = \Theta_t$ and that, for all $p \in N$, $E\{|r_n|^p\} \to 0$ as $n \to \infty$.

In order to prove the MCLT, one can try to apply the results of [21], i.e. theorem **??** with $v_u = v'_u = 0$: we write

$$E\{(S_u)^p(S'_u)^q\} = E\{(M_u)^p\}E\{(M'_u)^q\} + \sum_{s \neq q, r \neq p} A_{r,s} E\big\{(M_u)^r(r_u)^{p-r}(M'_u)^s(r'_u)^{q-s}\big\}.$$

By Holder Inequality, $E\big\{(M_u)^r(r_u)^{p-r}(M'_u)^s(r'_u)^{q-s}\big\}$ converges to 0 if $E\{(M_u)^r\}$ is bounded. Then, in order to use the theorem **??** with $v_u = v'_u = 0$, we have to prove that $E\{(M_u)^r\}$ is bounded. It is also difficult as well to prove that $E\{(S_n)^r\}$ is bounded. But, if one uses theorem **??** with $v_u = -r_u$ and $v'_u = -r'_u$, H_{mI} holds obviously : $E\big\{(S_u+v_u)^p(S'_u+v'_u)^q\big\} = E\big\{(S_u+v_u)^p\big\}E\big\{(S'_u+v'_u)^q\big\}$. For this reason we generalize theorem **??** to the case where v_u and v'_u are not equal to 0.

2.2 II-b) Functional of some IID sequences.

Proposition 2.1 *: Assume that $\Psi_t = \Theta_t$ and that $\Big|\sum_{s=0}^{\infty} \sum_{i=s}^{\infty} k_{i+1}(\Psi_1)\Big| < C < \infty$. Then, $S_n \to^M N(0,1)$.*

Proof : First, we need the following lemma.

Lemma 2.2 *: Assume that $E\big\{k_i(\Psi_r)k_j(\Psi_s)\big\} = 0$ for all $(i,j,r,s) \in N^4$, $r \neq s$. Assume $\sum_{s=0}^{\infty} \Big|\sum_{i=s}^{\infty} k_{i+1}(\Psi_1)\Big| < C < \infty$ and that, in $L^2(\Omega)$, $\sum_{i=1}^{r} k_i(\Psi_1) \to K$ as $r \to \infty$ where $\int K^2.dP > 0$. Then, $\sigma(n) \to \infty$ and $sup|r_n| \to 0$.*

Proof : There exists N_0 such that $E\Big\{\Big(\sum_{r=1}^{n} \sum_{s=1}^{r} k_s(\Psi_r)\Big)^2\Big\} = \sum_{r=1}^{n} E\Big\{\Big(\sum_{s=1}^{r} k_s(\Psi_r)\Big)^2\Big\} \geq (n/2)\int K^2.dP$ if $n \geq N_0$. Moreover, $\sigma(n)|r_n| \leq 2C$. ★

We choose $\kappa(n) = 0$. Because $\sigma(n) \to \infty$, $E\{|\xi_u|^p\} \to 0$ as $n \to \infty$ for all $p \in N$. Moreover, $E\{(M_u)^p\} = E\{(M'_u)^p\}$ and $E\{(M_u)^p(M'_u)^q\} = E\{(M_u)^p\}E\{(M'_u)^q\}$. Then we apply theorem **??** with $v_u = -r_u$ and $v'_u = -r'_u$. ★

Example 2.3 *: Assume that Θ_1 has the standard normal distribution and that $k_i(\Psi_1) = i^{-3}sin(2\pi\Theta_1)$. Then, $S_n \to^M N(0,1)$.*

One can also prove the asymptotic normality in some cases where r_n is not bounded .

Example 2.4 : **AR(1)Process**. *Assume $X_t = \sum_{i=0}^{\infty} a_i\Theta_{t+i}$ where $0 < a < 1$. Then, $M_n = \frac{1}{\sigma(n)}\sum_{j=1}^{n} \frac{1-a^j}{1-a}\Theta_j$ and $r_n = \frac{1}{\sigma(n)}\sum_{j=n+1}^{\infty} \frac{a^{j-n}(1-a^n)}{1-a}\Theta_j$. Therefore, for all $r \in N$, $\kappa u_r(r_u) = \frac{\kappa u_r(\Theta_1)}{\sigma(u)^r}\sum_{j=u+1}^{\infty}\left(\frac{a^{j-u}(1-a^u)}{1-a}\right)^r \to 0$, where κu_r is the r-th cumulant. Therefore, $E\{|r_u|^p\} \to 0$ for all $p \in N^*$.*

*Then, the condition H_{mI} is satisfied for any sequence $\{u(n)\}$ defined in notation **??**. Then, $S_n \to^M N(0,1)$.*

Recall, there exists AR(1) processes which are not strongly mixing : e.g. a=1/2, $P\{\Theta_t = 1/2\} = P\{\Theta_t = -1/2\} = 1/2$(cf [**?**] p 360-362). Then, these processes satisfy H_{mI} but are not strongly mixing. Moreover, $S_n \to^M N(0,1)$.

2.3 Functional of some strong mixing sequences

Strong mixing sequences can often be approached by functional of IID sequences. For example, let $\tau_t = \sum_{i=0}^{\infty} b^{i+1}h_i(\overline{\Theta_{t+i}})$, where $|h_i(\overline{\Theta_1})| \leq 1$, $|b| \leq 1/2$ and where $\overline{\Theta_t}$ is an IID sequence independent of Θ_t.

For example, if $h_i(\overline{\Theta_1}) = a^i\Theta_1$, τ_t can be strong mixing or not. Then, one can prove MCLT even for functionals of some non strong mixing sequence.

Example 2.5 *: Assume that $k_i(\Psi_t) = i^{-5/2}g_i(\Theta_t).sin(e(i)\tau_t)$ where $E\{g_i(\Theta_1)\} = 0$ and $|e(i)| \leq 2\pi$ for all $i \in N$. Assume that $\sum_{s=0}^{\infty}\Big|\sum_{i=s}^{\infty} k_{i+1}(\psi_1)| < C < \infty$.*

*Then, by lemma **??**, $\sigma(n) \to \infty$ and $sup|r_n| \to 0$ as $n \to \infty$. Moreover, $|\tau_t| \leq 1$ and $\tau_t = \tau_{t,u-t} + b^{u-t+1}\rho$ where $|\rho| \leq 1$ and $\tau_{t,u-t} = \sum_{i=0}^{u-t} b^{i+1}h_i(\overline{\Theta_{t+i}})$. Therefore, $sin(e(s)\tau_r) = sin(e(s)\tau_{r,u-r}) + 2\pi b^{u-r+1}\rho^*$ where $|\rho^*| \leq 1$.*

*Then, $M_u = \sigma(u)^{-1}\sum_{r=1}^{u}\sum_{s=1}^{r} s^{-5/2}g_s(\Theta_r)sin(e(s)\tau_{r,u-r}) + r_u^*$, where $Sup|r_u^*| \to 0$ as $n \to \infty$. Then, one applies theorem **??** with $v_u = -r_u - r_u^*$ and a similar decomposition for M'_u : $E\big\{(S_u + v_u)^p(S'_u + v'_u)^q\big\} = E\big\{(S_u + v_u)^p\big\}E\big\{(S'_u + v'_u)^q\big\}$. One deduces that $S_n \to N(0,1)$.*

By using theorem **??**, the MCLT can be proved for many other functionals, in particular , if $\{\Psi_t\}$ is a q-dependent sequence.

3 Proof of theorem ??

At first, for example, by Schwarz inequality, $E\{S_u\xi_u\}^2 \leq E\{S_u^2\}E\{\xi_u^2\} \leq E\{\xi_u^2\} = o(1)$ where "o" is the classical "o" : $o(1) \to 0$ as $n \to \infty$.

We deduce that $\sigma(n)^2 = \sigma(u)^2 E\{(S_u + v_u + S'_u + v'_u + \xi_u - v_u - v'_u)^2\} = \sigma(u)^2[2E\{(S_u)^2\} + o(1)]$. Then, for all $h \in N$, $\sigma(u)^h/\sigma(n)^h = 2^{-h/2} + o(1)$.

For all $p \in N$, we set $M_n^p = E\{(S_n)^p\}$.Clearly, $M_n^0 = 1$,$M_n^1 = 0$ and $M_n^2 = 1$.

Let $h \in N$, $2 \leq h \leq 2k$. Assume that, for all $p \in N$, $p \leq 2h-2$, $M_n^p \to \mu_p$, the moment or order p of N(0,1) .

By Holder inequality, $|E\{(S_u)^r(v_u)^{p-r}\}| \leq E\{|S_u|^p\}^{r/p}E\{|v_u|^p\}^{(p-r)/p}$ if $r < p$. Therefore, if $r < p \leq 2h-2$, $\big|E\{(S_u)^r(v_u)^{p-r}\}\big| = o(1)$. If $r < 2h$, $\big|E\{(S_u)^r(v_u)^{2h-r}\}\big| = M_s^{2h}o(1)$, where $M_s^{2h}(n) = sup\big\{M_{u(n)}^{2h}, 1\big\}$.

Then, by using H_{mI} and H_{mS} ,

$$E\big\{(S_u + v_u + S'_u + v'_u)^{2h}\big\}$$

$$= E\{(S_u+v_u)^{2h}\}+E\{(S'_u+v'_u)^{2h}\}+\sum_{m=1}^{2h-1}\frac{2h!}{m!(2h-m)!}E\{(S_u+v_u)^{2h-m}\}E\{(S'_u+v'_u)^m\}+o(1)$$

$$= 2E\{(S_u+v_u)^{2h}\}+\sum_{m=2}^{2h-2}\frac{2h!}{m!(2h-m)!}E\{(S_u+v_u)^{2h-m}\}E\{(S_u+v_u)^m\}+o(1)$$

$$= 2E\{S_u^{2h}\}+2\sum_{r=0}^{2h-1}\frac{2h!}{r!(2h-r)!}E\{S_u^r\}E\{v_u^{2h-r}\}+\sum_{m=2}^{2h-2}\frac{2h!}{m!(2h-m)!}E\{S_u^{2h-m}\}E\{S_u^m\}+o(1)$$

$$= 2M_u^{2h} + M_s^{2h}o(1) + \sum_{m=2}^{2h-2}\frac{2h!}{m!(2h-m)!}\mu_{2h-m}\mu_m + o(1)$$

$$\leq 2M_u^{2h} + M_s^{2h}|o(1)| + 2(2^{h-1}-1)\mu_{2h} + |o(1)|.$$

Moreover, if $m \geq 1$,

$$|E\{(S_u + v_u + S'_u + v'_u)^{2h-m}(\xi_u - v_u - v'_u)^m\}|$$

$$\leq |E\{(S_u + v_u + S'_u + v'_u)^{2h}\}|^{(2h-m)/(2h)}|E\{(\xi_u - v_u - v'_u)^{2h}\}|^{m/(2h)}$$

$$\leq |o(1)|\big|2M_s^{2h} + M_s^{2h}|o(1)| + 2(2^{h-1}-1)\mu_{2h} + |o(1)|\big|^{(2h-m)/(2h)}$$

$$\leq |o(1)|M_s^{2h} + |o(1)|.$$

Therefore,

$$M_n^{2h} = \frac{\sigma(u)^{2h}}{\sigma(n)^{2h}} E\{(S_u + v_u + S'_u + v'_u + \xi_u - v_u - v'_u)^{2h}\}$$

$$= (2^{-h} + o(1)) \sum_{m=0}^{2h} \frac{2h!}{m!(2h-m)!} E\{(S_u + v_u + S'_u + v'_u)^{2h-m}(\xi_u - v_u - v'_u)^m\}$$

$$\leq 2^{1-h}\big|M_s^{2h} + M_s^{2h}|o(1)| + (2^{h-1}-1)\mu_{2h} + |o(1)|\big| + |o(1)|M_s^{2h} + |o(1)|$$

$$\leq 2^{1-h}\big|M_s^{2h} + (2^{h-1}-1)\mu_{2h}\big|[1 + \epsilon(n)],$$

where $\epsilon(n) \to 0$ as $n \to \infty$.

The same equality has been obtained in the proof of [**?**] when $v_u = v'_u = 0$. We deduce by the same way that M_n^{2h} is bounded. Therefore, $M_n^{2h} = 2^{1-h}\left[M_u^{2h} + (2^{h-1}-1)\mu_{2h}\right] + o(1)$. We deduce the convergence of M_n^{2h-1} and M_n^{2h} to μ_{2h-1} and μ_{2h}. $\bigstar$

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